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BOUNDS FOR EIGENVALUES OF NONSINGULAR $\mathcal{H}$-TENSOR*

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Abstract. The bounds for the Z-spectral radius of nonsingular $\mathcal{H}$-tensor, the upper and lower bounds for the minimum H-eigenvalue of nonsingular (strong) $\mathcal{M}$-tensor are studied in this paper. Sharper bounds than known bounds are obtained. Numerical examples illustrate that our bounds give tighter bounds.

Dedicated to Professor Ravindra B. Bapat on the occasion of his 60th birthday

Key words. $\mathcal{M}$-tensor, $\mathcal{H}$-tensor, Z-spectral radius, Minimum H-eigenvalue.

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1. Introduction.

Eigenvalue problems of higher order tensors have become an important topic in applied mathematics branch, numerical multilinear algebra, and it has a wide range of practical applications [2, 3, 4, 1, 8, 12, 13, 14, 15, 16, 19, 20, 21].

A tensor can be regarded as a higher-order generalization of a matrix. Let $\mathbb{C}$ (respectively, $\mathbb{R}$) be the complex (respectively, real) field. An m-order n-dimensional square tensor $\mathcal{A}$ with n^m entries can be defined as follows,

$$\mathcal{A} = (a_{i_1 i_2 \dots i_m}), \quad a_{i_1 i_2 \dots i_m} \in \mathbb{C}, \quad 1 \le i_1, i_2, \dots, i_m \le n.$$

Let $\mathcal{A}$ be an m-order n-dimensional tensor, and $x \in \mathbb{C}^n$. Then

$$\mathcal{A}x^m = \sum_{i_1, i_2, \dots, i_m = 1}^{n} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}, \tag{1.1}$$

and $\mathcal{A}x^{m-1}$ is a vector in $\mathbb{C}^n$, with its ith component defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, i_3, \dots, i_m = 1}^{n} a_{i i_2 \dots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}.$$

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Let r be a positive integer. Then $x^{[r]} = [x_1^r, x_2^r, \ldots, x_n^r]^\top$ is a vector in $\mathbb{C}^n$, with its ith component defined by x_i^r.

The following two definitions were first introduced and studied by Qi and Lim, respectively.

Definition 1.1. **([8, 11, 14])** *Let $\mathcal{A}$ be an m-order n-dimensional real tensor. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A}$, if it satisfies the equation*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

We call (λ, x) an H-eigenpair, if both λ and x are real.

Definition 1.2. **([8, 11, 14])** *Let $\mathcal{A}$ be an m-order n-dimensional real tensor. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenvalue and E-eigenvector (or simply E-eigenpair) of $\mathcal{A}$, if they satisfy the equation*

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^\top x = 1. \end{cases}$$

We call (λ, x) a Z-eigenpair, if both λ and x real. Here $x^\top$ denotes the transpose of x.

In [8], He and Huang presented the definition of the Z-spectral radius of $\mathcal{A}$ as follows.

Definition 1.3. **([1, 8])** *Suppose that $\mathcal{A}$ is an m-order n-dimensional real tensor. Let $\sigma(\mathcal{A})$ denote the Z-spectrum of $\mathcal{A}$ by the set of all Z-eigenvalues of $\mathcal{A}$. Assume that $\sigma(\mathcal{A}) \neq \emptyset$. Then the Z-spectral radius of $\mathcal{A}$ is denoted by*

$$\rho(\mathcal{A}) = \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

Particularly, if $\mathcal{A}$ is an m-order n-dimensional nonnegative tensor, then

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

Recently, many contributions have been made on the bounds of the spectral radius of nonnegative tensor in [1, 10, 13, 14]. Similarly, bounds for the Z-spectral radius were given in [8] for the $\mathcal{H}$-tensors. Also, in [7], He and Huang obtained the upper and lower bounds for the minimum H-eigenvalue of nonsingular (strong) $\mathcal{M}$-tensors. In this paper, our purpose is to propose sharper bounds for the Z-spectral radius of nonsingular $\mathcal{H}$-tensors and for the minimum H-eigenvalue of nonsingular (strong) $\mathcal{M}$-tensors.

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2. Preliminaries. We start this section with some fundamental notions and properties on tensors. An m-order n-dimensional tensor $\mathcal{A}$ is called *nonnegative* ([2, 3, 9, 16, 20, 21]), if each entry is nonnegative. Similar to Z-matrices, we denote tensors with all non-positive off-diagonal entries by $\mathcal{Z}$-tensors. The m-order n-dimensional *identity tensor*, denoted by $\mathcal{I} = (\delta_{i_1 i_2 \dots i_m})$, is the tensor with entries

$$\delta_{i_1 i_2 \dots i_m} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

The tensor $\mathcal{D} = (d_{i_1 i_2 \dots i_m})$ is the *diagonal tensor* of $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$, if

$$\begin{cases} d_{i_1 i_2 \dots i_m} = a_{i_1 i_2 \dots i_m}, & i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.1. **([18])** *Let $\mathcal{A}$ and $\mathcal{B}$ be two m-order n-dimensional tensors. If there exists matrices P and Q of n-order with $P\mathcal{I}Q = \mathcal{I}$ such that $\mathcal{B} = P\mathcal{A}Q$, then we say that the two tensors are similar.*

Let the tensor $\mathcal{F}$ be associated with an undirected d-partite graph $G(\mathcal{F}) = (V, E(\mathcal{F}))$, the vertex set of which is the disjoint union $V = \bigcup_{j=1}^{d} V_j$, with $V_j = [m_j], j \in [d]$. The edge $(i_k, i_l) \in V_k \times V_l, k \neq l$ belongs to $E(\mathcal{F})$ if and only if $f_{i_1, i_2, \dots, i_d} > 0$ for some $d-2$ indices $i_1, \dots, i_d \setminus \{i_k, i_l\}$. The tensor $\mathcal{F}$ is called *weakly irreducible* if the graph $G(\mathcal{F})$ is connected. We call $\mathcal{F}$ *irreducible* if for each proper nonempty subset $\emptyset \neq I \subsetneqq V$, the following condition holds: let $J := V \setminus I$. Then there exists $k \in [d], i_k \in I \cap V_k$ and $i_j \in J \cap V_j$ for each $j \in [d] \setminus \{k\}$ such that $f_{i_1, \dots, i_d} > 0$. This definition of irreducibility agrees with [2, 13].

Friedland et al. [6] showed that if $\mathcal{F}$ is irreducible then $\mathcal{F}$ is *weakly irreducible* and presented the following results.

Lemma 2.2. **([6])** *If the nonnegative tensor $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is weakly irreducible. For $m = 2$, $\mathcal{A}$ is irreducible if and only if $\mathcal{A}$ is weakly irreducible.*

Lemma 2.2 illustrates that a nonnegative irreducible tensor must be weakly irreducible. For a general tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$, $a_{i_1 i_2 \dots i_m} \in \mathbb{C}$, we can draw the following conclusion.

Lemma 2.3. *If a tensor $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is weakly irreducible. For $m = 2$, $\mathcal{A}$ is irreducible if and only if $\mathcal{A}$ is weakly irreducible.*

Proof. Let $\mathcal{A} = \mathcal{D} - \mathcal{E}$, where $\mathcal{D}$ is the diagonal tensor of $\mathcal{A}$. If $\mathcal{A}$ is irreducible, it is equivalent that $\mathcal{E}$ is irreducible. Note that $|\mathcal{E}|$ is a nonnegative tensor, by Lemma 2.2, $|\mathcal{E}|$ is weakly irreducible, and then $\mathcal{A}$ is weakly irreducible. Similar to the proof of [6], we can get case $m = 2$. $\square$

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LEMMA 2.4. (**[14]**) *The product of the eigenvalues λ_i of tensor $\mathcal{A}$ is equal to* $\det(\mathcal{A})$*, that is,*

$$\det(\mathcal{A}) = \prod_{i=1}^{n(m-1)^{n-1}} \lambda_i.$$

We call tensor $\mathcal{A}$ is *nonsingular*, if $\det(\mathcal{A}) \neq 0$.

DEFINITION 2.5. (**[5, 23]**) *We call a tensor $\mathcal{A}$ an $\mathcal{M}$-tensor, if there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $\eta \geq \rho(\mathcal{B})$ such that*

$$\mathcal{A} = \eta\mathcal{I} - \mathcal{B}.$$

If $\eta > \rho(\mathcal{B})$ then $\mathcal{A}$ is called a nonsingular (strong) $\mathcal{M}$-tensor.

In [23], Zhang et al. obtained the following result for the H-eigenvalues of a nonsingular (strong) $\mathcal{M}$-tensor.

LEMMA 2.6. (**[23]**) *Let $\mathcal{A}$ be a nonsingular (strong) $\mathcal{M}$-tensor and $\tau(\mathcal{A})$ denote the minimal value of the real part of all eigenvalues of $\mathcal{A}$. Then $\tau(\mathcal{A}) > 0$ is an H-eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector. If $\mathcal{A}$ is weakly irreducible $\mathcal{Z}$-tensor, then $\tau(\mathcal{A}) > 0$ is the unique eigenvalue with a positive eigenvector.*

Yang and Yang [20], Yuan and You [22] showed that if

$$\mathcal{B} = D^{-(m-1)}\mathcal{A}D^{(m-1)}, \tag{2.1}$$

where D is a diagonal nonsingular matrix, then $\mathcal{A}$ and $\mathcal{B}$ are *similar*. It is easy to see that the similarity relation is an equivalent relation, and similar tensors have the same characteristic polynomials, and thus they have the same spectrum (as a multi-set).

Now, we introduce the comparison tensor of any tensor $\mathcal{A}$.

DEFINITION 2.7. (**[5]**) *Let $\mathcal{A} = (a_{i_1\ldots i_m})$ be an m-order and n-dimensional tensor. We call a tensor $\mathcal{M}(\mathcal{A}) = (m_{i_1i_2\ldots i_m})$ the comparison tensor of $\mathcal{A}$ if*

$$m_{i_1i_2\ldots i_m} = \begin{cases} \ \ |a_{i_1i_2\ldots i_m}|, & (i_1i_2\ldots i_m) = (i_1i_1\ldots i_1), \\ -|a_{i_1i_2\ldots i_m}|, & (i_1i_2\ldots i_m) \neq (i_1i_1\ldots i_1). \end{cases}$$

In the following, some basic definitions are given, which will be used in the subsequent discussion. In [5], Ding et al. extended H-matrices to $\mathcal{H}$-tensors as follows.

DEFINITION 2.8. (**[5]**) *We call a tensor $\mathcal{A}$ an $\mathcal{H}$-tensor, if its comparison tensor is an $\mathcal{M}$-tensor; we call it as a nonsingular $\mathcal{H}$-tensor, if its comparison tensor is a nonsingular $\mathcal{M}$-tensor.*

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Very recently, Kannan et al. [17] established some properties of strong $\mathcal{H}$-tensors and general $\mathcal{H}$-tensors.

REMARK 2.9. *It follows from definition* 2.8 *that an $\mathcal{M}$-tensor is an $\mathcal{H}$-tensor and a nonsingular $\mathcal{M}$-tensor is a nonsingular $\mathcal{H}$-tensor.*

DEFINITION 2.10. **([5])** *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. $\mathcal{A}$ is quasi-diagonally dominant, if there exists a positive vector $x=(x_1,x_2,\dots,x_n)^\top$ such that*

$$|a_{ii\dots i}|x_i^{m-1} \geq \sum_{(i_2i_3\dots i_m)\neq(ii\dots i)} |a_{ii_2\dots i_m}|x_{i_2}x_{i_3}\dots x_{i_m}, \quad i=1,2,\dots,n. \tag{2.2}$$

If the strict inequality holds in (2.2) for all i, $\mathcal{A}$ is called quasi-strictly diagonally dominant.

LEMMA 2.11. **([5])** *A tensor $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor if and only if it is quasi-strictly diagonally dominant.*

3. Bounds for the spectral radius of $\mathcal{H}$-tensors. In this section, we present some bounds for the Z-spectral radius of $\mathcal{H}$-tensors. For convenience, let $N=\{1,2,\dots,n\}$. We denote by $R_i(\mathcal{A})$ and $R(\mathcal{A})$ the sum of the ith row and the maximal row sum of $\mathcal{A}$, respectively, i.e.,

$$R_i(\mathcal{A}) = \sum_{i_2,i_3,\dots,i_m=1}^{n} |a_{ii_2\dots i_m}|, \quad R(\mathcal{A}) = \max_i R_i(\mathcal{A}).$$

In [1], Chang, Pearson, and Zhang have given the following bounds for the Z-eigenvalues of an m-order n-dimensional tensor $\mathcal{A}$.

LEMMA 3.1. **([1])** *Let $\mathcal{A}$ be an m-order and n-dimensional tensor with $\sigma(\mathcal{A})\neq\emptyset$. Then*

$$\rho(\mathcal{A}) \leq \sqrt{n}\max_{i\in N}\sum_{i_2,i_3,\dots,i_m=1}^{n} |a_{ii_2\dots i_m}| = \sqrt{n}R(\mathcal{A}).$$

For positively homogeneous operators, Song and Qi [19] established the relationship between the Gelfand formula and the spectral radius, as well as the upper bound of the spectral radius. Following the Corollary 4.5 in [19], He and Huang [8] presented the following lemma.

LEMMA 3.2. **([8, 19])** *Let $\mathcal{A}$ be an m-order and n-dimensional tensor with $\sigma(\mathcal{A})\neq\emptyset$. Then*

$$\rho(\mathcal{A}) \leq \max_{i\in N}\sum_{i_2,i_3,\dots,i_m=1}^{n} |a_{ii_2\dots i_m}| = R(\mathcal{A}).$$

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Based on the above lemma, we obtain some upper bounds for the Z-spectral radius when $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor as follows.

THEOREM 3.3. *Let $\mathcal{A}$ be an m-order and n-dimensional nonsingular $\mathcal{H}$-tensor with $\sigma(\mathcal{A}) \neq \emptyset$. Then*

$$\rho(\mathcal{A}) \leq 2 \max_{i \in N} |a_{ii\ldots i}|.$$

Proof. Since $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor, there exists a positive diagonal matrix $X =$diag$(x_1, x_2, \ldots, x_n)$ such that $\mathcal{A}X^{(m-1)}$ is strictly diagonally dominant. Then

$$X^{-(m-1)}\mathcal{A}X^{(m-1)},$$

is also strictly diagonally dominant, i.e.,

$$|a_{ii\ldots i}| > \sum_{\substack{(i_2,i_3,\ldots,i_m)\\ \neq(i,i,\ldots,i)}} \frac{|a_{ii_2\ldots i_m}|x_{i_2}x_{i_3}\ldots x_{i_m}}{x_i^{m-1}},\ i \in N.$$

Because $X^{-(m-1)}\mathcal{A}X^{(m-1)}$ and $\mathcal{A}$ are similar, it follows that

$$\begin{aligned}
\rho(\mathcal{A}) &= \rho(X^{-(m-1)}\mathcal{A}X^{(m-1)}) \leq R(X^{-(m-1)}\mathcal{A}X^{(m-1)}) \\
&= \max_i \sum_{i_2,i_3,\ldots,i_m=1}^{n} \frac{|a_{ii_2\ldots i_m}|x_{i_2}x_{i_3}\ldots x_{i_m}}{x_i^{m-1}} \\
&= \max_i(|a_{ii\ldots i}| + \sum_{\substack{(i_2,i_3,\ldots,i_m)\\ \neq(i,i,\ldots,i)}} \frac{|a_{ii_2\ldots i_m}|x_{i_2}x_{i_3}\ldots x_{i_m}}{x_i^{m-1}}) \\
&< 2\max_i |a_{ii\ldots i}|.
\end{aligned}$$

□

By the above theorem, the following corollary can be obtained easily.

COROLLARY 3.4. *If $\mathcal{A}$ is an m-order and n-dimensional nonsingular $\mathcal{H}$-tensor with $\sigma(\mathcal{A}) \neq \emptyset$, then*

$$\rho(\mathcal{A}) \leq \min\left\{R(\mathcal{A}), 2\max_{i\in N}|a_{ii\ldots i}|\right\}.$$

COROLLARY 3.5. *If $\mathcal{A}$ is an m-order and n-dimensional nonsingular $\mathcal{M}$-tensor with $\sigma(\mathcal{A}) \neq \emptyset$, then*

$$\rho(\mathcal{A}) \leq \min\left\{R(\mathcal{A}), 2\max_{i\in N} a_{ii\ldots i}\right\}.$$

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REMARK 3.6. *In fact, the bound of Theorem 3.3 is not better than the bound in Lemma 3.2 for diagonally dominant $\mathcal{H}$-tensors. However, by Lemma 2.11 we know that $\mathcal{H}$-tensors are not necessary diagonally dominant. Thus, the bound given in Theorem 3.3 is sharper than the one given in Lemma 3.2 for non-diagonally dominant $\mathcal{H}$-tensors. The following example illustrates the same.*

EXAMPLE 3.7. *Let $\mathcal{A} = (a_{ijk})$ be an 3-order 2-dimension tensor with the form,*

$$\begin{array}{llll} a_{111} = 1.1, & a_{112} = -1, & a_{121} = -1, & a_{122} = 1, \\ a_{211} = -1, & a_{221} = 1, & a_{212} = 1, & a_{222} = 1.1. \end{array}$$

It is easy to check that $\mathcal{A}$ is quasi-strictly diagonally dominant and then $\mathcal{A}$ is an nonsingular $\mathcal{H}$-tensor. By Lemma 3.1, we have,

$$\rho(\mathcal{A}) \leq \sqrt{n} R(\mathcal{A}) = 5.7974.$$

By Lemma 3.2, we obtain the upper bound,

$$\rho(\mathcal{A}) \leq R(\mathcal{A}) = 4.1.$$

Now from Theorem 3.3, we have the following bound:

$$\rho(\mathcal{A}) \leq 2 \max_i |a_{iii}| = 2.2.$$

Obviously, the bound given in Theorem 3.3 is sharper than those given in Lemma 3.2 and Lemma 3.1.

4. Bounds for the minimum eigenvalue of $\mathcal{M}$-tensors. In this section, we consider the minimum H-eigenvalue of $\mathcal{M}$-tensors. We adopt the following notation throughout this section. We define a nonnegative matrix $M(\mathcal{A})$, where

$$(M(\mathcal{A}))_{ij} = \begin{cases} r_i(\mathcal{A}), & i = j, \\ a_{ij\ldots j}, & i \neq j. \end{cases} \qquad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\ldots i_m}=0 \\ \delta_{ji_2\ldots i_m}=0}} r_i(\mathcal{A}) - |a_{ij\ldots j}|,$$

and

$$\triangle_{ij}(\mathcal{A}) = [a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A})]^2 - 4a_{ij\ldots j} r_j(\mathcal{A}),$$

with

$$r_i(M(\mathcal{A})) = \sum_{j\neq i} M(\mathcal{A})_{ij}, \qquad \tilde{r}_i(\mathcal{A}) = r_i(\mathcal{A}) - r_i(M(\mathcal{A})),$$

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and

$$\tilde{\triangle}_{ij}(\mathcal{A}) = [a_{ii\ldots i} - a_{jj\ldots j} + \tilde{r}_i(\mathcal{A})]^2 + 4r_i(M(\mathcal{A}))r_j(\mathcal{A}).$$

LEMMA 4.1. *Let $\mathcal{A}$ be a weakly irreducible $\mathcal{M}$-tensor and $t_i = \sum\limits_{k\neq i,j} |a_{ik\ldots k}|$, $i \in N$.*

(1) *If $0 \leq t_i \leq 2[a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A}) - 2r_j(\mathcal{A})]$, $i, j \in N$, then $\triangle_{ij}(\mathcal{A}) \geq \tilde{\triangle}_{ij}(\mathcal{A})$.*

(2) *If $t_i \geq 2[a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A}) - 2r_j(\mathcal{A})]$, $i, j \in N$, then $\triangle_{ij}(\mathcal{A}) \leq \tilde{\triangle}_{ij}(\mathcal{A})$.*

Proof. For convenience, denote $a = a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A})$, notice that $\tilde{r}_i(\mathcal{A}) = r_i^j(\mathcal{A}) - t_i$. Thus

$$\begin{array}{rcl} \triangle_{ij}(\mathcal{A}) - \tilde{\triangle}_{ij}(\mathcal{A}) & = & a^2 - 4a_{ij\ldots j}r_j(\mathcal{A}) - (a - t_i)^2 - 4[(t - a_{ij\ldots j})r_j(\mathcal{A})] \\ & = & -t_i^2 + 2[a - 2r_j(\mathcal{A})]t_i. \end{array}$$

The equation $-t_i^2 + 2[a - 2r_j(\mathcal{A})]t_i = 0$ has two roots $t_{i_1} = 0$ and $t_{i_2} = 2[a - 2r_j(\mathcal{A})]$. Therefore, if $0 \leq t_i \leq 2[a - 2r_j(\mathcal{A})]$. Thus $\triangle_{ij}(\mathcal{A}) \geq \tilde{\triangle}_{ij}(\mathcal{A})$, and if $t_i \geq 2[a - 2r_j(\mathcal{A})]$, then $\triangle_{ij}(\mathcal{A}) \leq \tilde{\triangle}_{ij}(\mathcal{A})$. ∎

In [7], He and Huang gave the following bounds for the minimum H-eigenvalue of irreducible $\mathcal{M}$-tensors.

LEMMA 4.2. **([7])** *Let $\mathcal{A}$ be an irreducible $\mathcal{M}$-tensor. Then $\tau(\mathcal{A}) \leq \min\limits_{i\in N}\{a_{ii\ldots i}\}$.*

LEMMA 4.3. **([7])** *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be an irreducible $\mathcal{M}$-tensor. Then*

$$\begin{aligned} &\min_{\substack{i,j\in N\\ j\neq i}} \tfrac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - r_i^j(\mathcal{A}) - \triangle_{ij}^{\frac{1}{2}}(\mathcal{A})\right\} \leq \tau(\mathcal{A}) \leq \\ &\max_{\substack{i,j\in N\\ j\neq i}} \tfrac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - r_i^j(\mathcal{A}) - \triangle_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}. \end{aligned} \tag{4.1}$$

For the weakly irreducible $\mathcal{M}$-tensor, we have a result similar to that of Lemma 4.2 in the following.

LEMMA 4.4. *Let $\mathcal{A}$ be a weakly irreducible $\mathcal{M}$-tensor. Then $\tau(\mathcal{A}) \leq \min\limits_{i\in N}\{a_{ii\ldots i}\}$.*

Proof. The proof is similar to that of Theorem 2.1 in [7], and omit it. ∎

Based on the above lemma, we derive the bounds for the minimum H-eigenvalue of weakly irreducible $\mathcal{M}$-tensors as follows.

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THEOREM 4.5. *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be a weakly irreducible $\mathcal{M}$-tensor. Then*

$$\begin{aligned} &\min_{\substack{i,j\in N\\ j\neq i}} \frac{1}{2}\left\{a_{ii\ldots i}+a_{jj\ldots j}-\tilde{r}_i(\mathcal{A})-\tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A})\right\} \leq \tau(\mathcal{A}) \leq \\ &\max_{\substack{i,j\in N\\ j\neq i}} \frac{1}{2}\left\{a_{ii\ldots i}+a_{jj\ldots j}-\tilde{r}_i(\mathcal{A})-\tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}. \end{aligned} \tag{4.2}$$

Proof. Let $x>0$ be an eigenvector of $\mathcal{A}$ corresponding to $\tau(\mathcal{A})$. i.e.,

$$\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}. \tag{4.3}$$

Suppose that

$$x_t \geq x_s \geq \max_{i\in N}\{x_i : i\neq t, i\neq s\}.$$

From (4.3), we have

$$[\tau(\mathcal{A}) - a_{tt\ldots t}]x_t^{m-1} = \sum_{\substack{\delta_{ii_2\ldots i_m}=0\\ (i_2 i_3\ldots i_m)\neq(jj\ldots j)}} a_{ti_2\ldots i_m}x_{i_2}x_{i_3}\ldots x_{i_m} + \sum_{j\neq t} a_{tj\ldots j}x_j^{m-1}.$$

Taking modulus in the above equation and using the triangle inequality gives,

$$\begin{aligned} |\tau(\mathcal{A}) - a_{tt\ldots t}|x_t^{m-1} &\leq \sum_{\substack{\delta_{ii_2\ldots i_m}=0\\ (i_2 i_3\ldots i_m)\neq(jj\ldots j)}} |a_{ti_2\ldots i_m}|x_{i_2}x_{i_3}\ldots x_{i_m} + \sum_{j\neq t}|a_{tj\ldots j}|x_j^{m-1} \\ &\leq \sum_{\substack{\delta_{ii_2\ldots i_m}=0\\ (i_2 i_3\ldots i_m)\neq(jj\ldots j)}} |a_{ti_2\ldots i_m}|x_t^{m-1} + \sum_{j\neq t}|a_{tj\ldots j}|x_s^{m-1} \\ &= \tilde{r}_t(\mathcal{A})x_t^{m-1} + r_t(M)x_s^{m-1}. \end{aligned}$$

Note that $\tau(\mathcal{A}) \leq a_{tt\ldots t}$, and

$$[a_{tt\ldots t} - \tau(\mathcal{A})]x_t^{m-1} \leq \tilde{r}_t(\mathcal{A})x_t^{m-1} + r_t(M)x_s^{m-1}.$$

Equivalently

$$[a_{tt\ldots t} - \tau(\mathcal{A}) - \tilde{r}_t(\mathcal{A})]x_t^{m-1} \leq r_t(M)x_s^{m-1}. \tag{4.4}$$

From (4.3), we also obtain

$$[a_{ss\ldots s} - \tau(\mathcal{A})]x_s^{m-1} \leq r_s(\mathcal{A})x_t^{m-1}. \tag{4.5}$$

Multiplying inequalities (4.4) with (4.5), we have

$$[a_{tt\ldots t} - \tau(\mathcal{A}) - \tilde{r}_t(\mathcal{A})][a_{ss\ldots s} - \tau(\mathcal{A})]x_t^{m-1}x_s^{m-1} \leq r_t(M)r_s(\mathcal{A})x_s^{m-1}x_t^{m-1}.$$

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Note that $x_t^{m-1} x_s^{m-1} < 0$, and

$$[a_{tt\ldots t} - \tau(\mathcal{A}) - \tilde{r}_t(\mathcal{A})][a_{ss\ldots s} - \tau(\mathcal{A})] \leq r_t(M) r_s(\mathcal{A}). \tag{4.6}$$

This is

$$\tau(\mathcal{A})^2 - [a_{tt\ldots t} + a_{ss\ldots s} - \tilde{r}_t(\mathcal{A})]\tau(\mathcal{A}) - r_t(M) r_s(\mathcal{A}) + [a_{tt\ldots t} - \tilde{r}_t(\mathcal{A})] a_{ss\ldots s} \leq 0.$$

Note that

$$[a_{tt\ldots t} + a_{ss\ldots s} - \tilde{r}_t(\mathcal{A})]^2 - 4[a_{ss\ldots s} - \tilde{r}_t(\mathcal{A})] a_{tt\ldots t} = [a_{tt\ldots t} - a_{ss\ldots s} + \tilde{r}_t(\mathcal{A})]^2.$$

This gives the following bound for $\tau(\mathcal{A})$,

$$\begin{aligned} \tau(\mathcal{A}) &\geq \tfrac{1}{2}\{a_{tt\ldots t} + a_{ss\ldots s} - \tilde{r}_t(\mathcal{A}) - \triangle_{ts}^{\frac{1}{2}}(\mathcal{A})\} \\ &\geq \min_{\substack{i,j \in N \\ j \neq i}} \tfrac{1}{2}\{a_{ii\ldots i} + a_{jj\ldots j} - \tilde{r}_i(\mathcal{A}) - \tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A})\}. \end{aligned}$$

On the other hand, let

$$x_l \leq x_u \leq \min_{i \in N}\{x_i : i \neq t, i \neq s\}.$$

From (4.3), we have

$$(a_{uu\ldots u} - \tau(\mathcal{A})) x_u^{m-1} = - \sum_{\delta_{ui_2\ldots i_m} = 0} a_{ui_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m} \geq r_u(\mathcal{A}) x_l^{m-1}. \tag{4.7}$$

and

$$\begin{aligned} (a_{ll\ldots l} - \tau(\mathcal{A})) x_l^{m-1} &= - \sum_{\substack{\delta_{li_2\ldots i_m} = 0 \\ (i_2 i_3 \ldots i_m) \neq (jj\ldots j)}} a_{li_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m} - \sum_{j \neq l} a_{lj\ldots j} x_j^{m-1} \\ &\geq \tilde{r}_l(\mathcal{A}) x_l^{m-1} + r_l(M) x_u^{m-1}. \end{aligned}$$

Then

$$[a_{ll\ldots l} - \tau(\mathcal{A}) - \tilde{r}_l(\mathcal{A})] x_l^{m-1} \geq r_l(M) x_u^{m-1}. \tag{4.8}$$

Multiplying inequalities (4.7) with (4.8), we have

$$[a_{uu\ldots u} - \tau(\mathcal{A})][a_{ll\ldots l} - \tau(\mathcal{A}) - \tilde{r}_l(\mathcal{A})] \geq r_l(M) r_u(\mathcal{A}). \tag{4.9}$$

Inequality (4.9) is equivalent to

$$\tau(\mathcal{A})^2 - [a_{ll\ldots l} + a_{uu\ldots u} - \tilde{r}_l(\mathcal{A})]\tau(\mathcal{A}) - r_l(M) r_u(\mathcal{A}) + [a_{ll\ldots l} - \tilde{r}_l(\mathcal{A})] a_{uu\ldots u} \geq 0.$$

This gives the following bound for $\tau(\mathcal{A})$,

$$\begin{aligned} \tau(\mathcal{A}) &\leq \tfrac{1}{2}\left\{a_{ll\ldots l} + a_{uu\ldots u} - \tilde{r}_l(\mathcal{A}) - \triangle_{lu}^{\frac{1}{2}}(\mathcal{A})\right\} \\ &\leq \max_{\substack{i,j \in N \\ j \neq i}} \tfrac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - \tilde{r}_j(\mathcal{A}) - \tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}. \end{aligned}$$

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This completes the proof. □

In what follows, we will show the bounds in Theorem 4.5 are tighter and sharper than those of Lemma 4.3.

THEOREM 4.6. *Under the conditions of Lemma* 4.1*. If*

$$0 \le t_i \le 2\left[a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A}) - 2r_j(\mathcal{A})\right],\ i,j \in N,$$

then

$$\begin{aligned}&\min_{\substack{i,j\in N\\ j\ne i}} \tfrac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - r_i^j(\mathcal{A}) - \triangle_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}\\ &\le \min_{\substack{i,j\in N\\ j\ne i}} \tfrac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - \tilde{r}_i(\mathcal{A}) - \tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}.\end{aligned}$$

Proof. From the Lemma 4.1, if $0 \le t_i \le 2[a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A}) - 2r_j(\mathcal{A})$, then $\triangle_{ij}(\mathcal{A}) \ge \tilde{\triangle}_{ij}(\mathcal{A})$. Note that $\tilde{r}_i(\mathcal{A}) = r_i^j(\mathcal{A}) - t_i$, and then

$$a_{ii\ldots i} + a_{jj\ldots j} - r_i^j(\mathcal{A}) - \triangle_{ij}^{\frac{1}{2}}(\mathcal{A}) \le a_{ii\ldots i} + a_{jj\ldots j} - \tilde{r}_i(\mathcal{A}) - \tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A}),\ i,j \in N,$$

which implies that

$$\begin{aligned}&\min_{\substack{i,j\in N\\ j\ne i}} \tfrac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - r_i^j(\mathcal{A}) - \triangle_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}\\ &\le \min_{\substack{i,j\in N\\ j\ne i}} \tfrac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - \tilde{r}_i(\mathcal{A}) - \tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}.\end{aligned}$$

□

REMARK 4.7. *From Theorem* 4.6*, we can see that the lower bound of* $\tau(\mathcal{A})$ *in Theorem* 4.5 *is sharper than those of Lemma* 4.3*, if*

$$0 \le t_i \le 2\left[a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A}) - 2r_j(\mathcal{A})\right],\ i,j \in N.$$

THEOREM 4.8. *Under the conditions of Lemma* 4.1*. If*

$$t_i \ge 2\left[a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A}) - 2r_j(\mathcal{A})\right] + 1,\ i,j \in N,$$

then

$$\begin{aligned}&\max_{\substack{i,j\in N\\ j\ne i}} \tfrac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - \tilde{r}_i(\mathcal{A}) - \tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}\\ &\le \max_{\substack{i,j\in N\\ j\ne i}} \tfrac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - r_i^j(\mathcal{A}) - \triangle_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}.\end{aligned}$$

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Proof. From the proof of Lemma 4.1, we know that

$$\triangle_{ij}(\mathcal{A}) - \tilde{\triangle}_{ij}(\mathcal{A}) + t_i = -t_i^2 + 2[(a - 2r_j(\mathcal{A})) + 1]t_i. \tag{4.10}$$

Because equation (4.10) has two roots $t_{i_1} = 0$ and $t_{i_2} = 2[a - 2r_j(\mathcal{A})] + 1$. Therefore, if $t_{i_2} \geq 2(a - 2r_j(\mathcal{A})) + 1$, then

$$\triangle_{ij}(\mathcal{A}) \leq \tilde{\triangle}_{ij}(\mathcal{A}) - t_i.$$

Note that $\tilde{r}_i(\mathcal{A}) = r_i^j(\mathcal{A}) - t_i$, we have

$$\tilde{r}_i(\mathcal{A}) + \tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A}) \geq r_i^j(\mathcal{A}) + \triangle_{ij}^{\frac{1}{2}}(\mathcal{A}).$$

Hence

$$\begin{aligned} &\max_{\substack{i,j\in N\\ j\neq i}} \frac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - \tilde{r}_i(\mathcal{A}) - \tilde{\triangle}_{ij}^{\frac{1}{2}}(\mathcal{A})\right\} \\ &\leq \max_{\substack{i,j\in N\\ j\neq i}} \frac{1}{2}\left\{a_{ii\ldots i} + a_{jj\ldots j} - r_i^j(\mathcal{A}) - \triangle_{ij}^{\frac{1}{2}}(\mathcal{A})\right\}. \end{aligned}$$

□

REMARK 4.9. *From Theorem* 4.8*, we can see that the upper bound of* $\tau(\mathcal{A})$ *in Theorem* 4.5 *is sharper than those in Lemma* 4.3*, if* $t_i \geq 2[a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A}) - 2r_j(\mathcal{A})] + 1,\ i, j \in N.$

REMARK 4.10. *Since* $t_i \geq 0$*, if* $0 \leq t_i \leq 2[a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A}) - 2r_j(\mathcal{A})]$ *for some* i*, and* $t_i \geq 2[a_{ii\ldots i} - a_{jj\ldots j} + r_i^j(\mathcal{A}) - 2r_j(\mathcal{A})] + 1$ *for some other* i*, we can see that the upper and lower bounds of* $\tau(\mathcal{A})$ *in Theorem* 4.5 *are tighter than those of Lemma* 4.3*. The following example shows this.*

EXAMPLE 4.11. *Let* $\mathcal{A} = (a_{ijk})$ *be an* 4*-order* 3*-dimension tensor with the form,*

$$a_{111} = a_{222} = 5,\ a_{333} = a_{444} = 4,\ a_{ijj} = -1,\ i \neq j,$$

$$a_{121} = -0.5,\ a_{212} = -1,\ a_{ijk} = 0,\ \text{otherwise}.$$

By Lemma 4.3, we have the bound

$$0.2614 \leq \tau(\mathcal{A}) \leq 1.5635.$$

We have our new bounds from Theorem 4.5.

$$0.7251 \leq \tau(\mathcal{A}) \leq 1.2769.$$

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5. Conclusion. In this paper, the Z-spectral radius for nonsingular $\mathcal{H}$-tensor and the minimum H-eigenvalue of nonsingular (strong) $\mathcal{M}$-tensor are studied. Furthermore, we prove that the results of this paper are sharper than those of [1, 8] and [7].

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