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BOUNDS FOR NUMBERS OF GENERATORS OF COHEN-MACAULAY IDEALS

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Let $(R, \underline{m})$ be a local Cohen-Macaulay ring of dimension d and multiplicity $e(R) = e$. A natural question to ask about an $\underline{m}$-primary ideal I is whether there is any relation between the number of generators of I and the least power t of $\underline{m}$ contained in I. (t will be called the nilpotency degree of R/I) It is quite straight forward to obtain a bound for $v(I)$, the number of generators in a minimal basis of I, in terms of t and e. However, there are several interesting applications. The first is the existence of a bound for the number of generators of any Cohen-Macaulay ideal I, i.e. any ideal I such that R/I is Cohen-Macaulay, in terms of $e(R/I)$, $e(R)$ and height I. The second application is a bound in terms of d and e for the reduction exponent of $\underline{m}$.

1. $\underline{m}$-primary ideals. In this section we will use only the standard facts about the existence and properties of superficial elements. However, later we will need a result stronger than the usual existence theorem for these elements so we take this opportunity to recall the definition and prove this special form of the existence theorem.

DEFINITION. Let $(R, \underline{m})$ be a local ring. An element x in $\underline{m}$ is superficial for $\underline{m}$ if there is an integer $c > 0$ such that

$$(\underline{m}^n : x) \cap \underline{m}^c = \underline{m}^{n-1} \quad \text{for all} \quad n > c .$$

It is a standard fact that x is superficial for $\underline{m}$ if and only if there is an integer $c > 0$ such that $0 \neq \bar{X} \in \underline{m}/\underline{m}^2 = G_1$ and

$$(0 : \bar{x}G) \cap G_n = 0$$

for $n \geqq c$ where $G_n = m^n/m^{n+1}$, and $G = G_0 \oplus G_1 \oplus \cdots$.

LEMMA 1.1. *Let $(R, \underline{m})$ be a local ring with $R/\underline{m}$ infinite. Let $I, J_1, \cdots, J_s$ be distinct ideals of R which are also distinct from $\underline{m}$. Then there is an element x in R such that*

(1) $x \notin J_i$, $\quad i = 1, \cdots, s$

(2) *x is a superficial element for $\underline{m}$*

and

(3) *the image of x in a superficial element for $\underline{m}/I$.*

Proof. Let $G = R/\underline{m} \oplus \underline{m}/\underline{m}^2 \oplus \cdots$ and $\bar{G} = R/\underline{m} \oplus \underline{m}/\underline{m}^2 + I \oplus$

$\underline{m}^2 + I/\underline{m}^3 + I \oplus \cdots$. Then $\bar{G} = G/K$ where K is a homogeneous ideal of G. Let $\operatorname{Ass} G = \{P_1, \cdots P_t\}$, $\operatorname{Ass} \bar{G} = \{Q_1/K, \cdots, Q_l/K\}$, where the Q_i are primes in G. Suppose that $P_t = \underline{m}/\underline{m}^2 \oplus \underline{m}^2/\underline{m}^3 \oplus \cdots$ and $Q_l = P_t/K$. The following subspaces are all proper $R/\underline{m}$-subspaces of $\underline{m}/\underline{m}^2$:

$$J_i + \underline{m}^2/\underline{m}^2,\ i = 1, \cdots, n;\ P_i \cap (\underline{m}/\underline{m}^2)\ ,\quad i = 1, \cdots, t-1$$
$$Q_i \cap (\underline{m}/\underline{m}^2),\ i = 1, \cdots, l-1;\ I + \underline{m}^2/\underline{m}^2\ .$$

Since $R/\underline{m}$ is infinite there is a nonzero $\bar{x}$ in $\underline{m}/\underline{m}^2$ such that $\bar{x}$ is not in any of these subspaces. We claim that if x is any element of $\underline{m}$ which maps to $\bar{x}$, then x is superficial for $\underline{m}$ and the image of x in R/I is superficial for $\underline{m}/I$. We need $(0\colon G\bar{x}) \cap \underline{m}^n/\underline{m}^{n+1} = 0$ and $(0\colon \bar{G}(x + I/I)) \cap \underline{m}^n + I/\underline{m}^{n+1} + I = 0$ for large n. Let $0 = N_1 \cap \cdots \cap N_{t-1} \cap N_t$ with N_i P_i-primary be a primary decomposition of 0 in G. Then $(0\colon G\bar{x}) \subseteqq N_1 \cap \cdots \cap N_{t-1}$. But $(\underline{m}/\underline{m}^2)^c \subseteqq N_t$ for some c, hence $(0\colon G\bar{x}) \cap \underline{m}^n/\underline{m}^{n+1} = 0$ for $n \geqq c$. The same reasoning shows that the image of x in R/I is a superficial element for $\underline{m}/I$.

THEOREM 1.2. *Let $(R, \underline{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let be an $\underline{m}$-primary ideal and t the nilpotency degree of R/I. Then*

$$v(I) \leqq t^{d-1}e(R) + d - 1\ .$$

Proof. The proof is by induction on d. If $d = 1$, the theorem is well-known (cf. [6] or [7]) but we include a proof for completeness. We may assume that $R/\underline{m}$ is infinite so that $\underline{m}$ has a superficial element x which is also a nonzero divisor. Since $d = 1$, $x\underline{m}^n = \underline{m}^{n+1}$ for some $n > 0$. We have $\lambda(R/xR) = \lambda(I/xI) = e(R)$, where $\lambda(B)$ denotes the length of an R-module B. The exact sequence

$$0 \longrightarrow \underline{m}I/xI \longrightarrow I/xI \longrightarrow I/\underline{m}I \longrightarrow 0$$

gives $\lambda(I/\underline{m}I) = \lambda(I/xI) - \lambda(\underline{m}I/xI) = e(R) - \lambda(\underline{m}I/xI)$.

Assume $d > 1$. Again assuming that $R/\underline{m}$ is infinite as we may, there is a nonzero divisor x such that x is a superficial element for $\underline{m}$. Pass to the $d - 1$ dimensional Cohen-Macaulay ring R/x^t. I/x^t is $\underline{m}/x^t$-primary so, by induction,

$$v(I/x^t) \leqq t^{d-2}e(R/x^t) + d - 2\ .$$

Hence $v(I) \leqq v(I/x^t) + 1 \leqq t^{d-2}te(R) + d - 1$.

REMARKS. 1. If $(R, \underline{m})$ is regular local and $d \geqq 2$ then $v(I) \leqq gt^{d-2} + d - 1$, where g is the degree of I, i.e. $I \subseteqq \underline{m}^g \backslash \underline{m}^{g+1}$.

2. (1.2) generalizes a result of Abhyankar [1]. In [1], Abhyankar

shows that the Cohen-Macaulay hypothesis is necessary.

If $(R, \underline{m})$ is a d-dimensional local ring, the Hilbert function H of $\underline{m}$ is defined as follows:

$$H(n) = v(\underline{m}^n) = \lambda(\underline{m}^n/\underline{m}^{n+1})$$

for integers $n \geqq 0$. For large n, $H(n)$ is a polynomial of degree $d-1$. If we apply (1.2) to $I = \underline{m}^n$ we obtain, for Cohen-Macaulay rings, a polynomial of degree $d-1$ which bounds $H(n)$ for all n.

COROLLARY 1.3. *Let $(R, \underline{m})$ be a Cohen-Macaulay local ring of multiplicity e and dimension $d > 0$. Then, if $P(n)$ is the polynomial, $P(n) = en^{d-1} + d - 1$, $H(n) \leqq P(n)$ for all $n > 0$.*

Using a trick of Kirby [4] we have

COROLLARY 1.4. *Let I be an ideal ideal of a d-dimensional Cohen-Macaulay local ring $(R, \underline{m})$, $d > 0$. Define the Artin-Rees number of I, $a(I)$, to be the least integer a such that $I \cap \underline{m}^a \subseteqq I\underline{m}$. Then*

$$v(I) \leqq a^{d-1}e(R) + d - 1 .$$

Proof. $I/I\underline{m} = I/I\underline{m} + I \cap \underline{m}^a = I/I \cap I\underline{m} + \underline{m}^a \cong I + \underline{m}^a/I\underline{m} + \underline{m}^a$. Hence $v(I) \leqq v(I + \underline{m}^a) = \lambda(I + \underline{m}^a/\underline{m}(I + \underline{m}^a))$. So by (1.2), $v(I) \leqq a^{d-1}e(R) + d - 1$.

Note that if R/I is Cohen-Macaulay, then $a(I) \leqq e(R/I) + 1$ but in general $a(I)$ is not bounded by $e(R/I)$.

2. Applications. If $(R, \underline{m})$ is a local ring, an ideal I is a Cohen-Macaulay ideal if R/I is a Cohen-Macaulay ring.

THEOREM 2.1. *Let $(R, \underline{m})$ be a d-dimensional Cohen-Macaulay local ring. Let I be a Cohen-Macaulay ideal of height $h > 0$. Then*

$$v(I) \leqq e(R/I)^{h-1}e(R) + h - 1 .$$

Proof. We may assume $R/\underline{m}$ is infinite. The proof is by induction on $s = \dim R/I$. If $s = 0$ then by (1.2) it is sufficient to note that $e(R/I) = \lambda(R/I) \geqq$ nilpotency degree of R/I.

Assume $s > 0$. By (1.1) there is a nonzero divisor x in $\underline{m}$ such that x is superficial for $\underline{m}$, and the image of x in R/I is a nonzero divisor in R/I and is a superficial element for $\underline{m}/I$. We pass to the $d-1$ dimensional Cohen-Macaulay ring R/x and to the height h Cohen-Macaulay ideal $(I, x)/x$. By induction $v(I) = v((I, x)/x) \leqq e(R/(I, x))^{h-1}e(R/x) + h - 1 = e(R/I)^{h-1}e(R) + h - 1$.

REMARKS 1. For height 1 ideals I, (2.1) gives Rees' theorem [6] stating that $v(I) \leqq e(R)$. If height $I = 2$, Rees [6] has the result $v(I) \leqq e(R) + e(R/I)$ which gives a better bound than (2.1) except when R or R/I is regular.

2. If R is an equicharacteristic regular local ring, Becker [2] has results similar to (2.1).

Another application of (1.2) gives a bound for what we will call the reduction exponent $r(\underline{m})$ of $\underline{m}$, where $\underline{m}$ is the maximal ideal of a d-dimensional Cahen-Macaulay local ring $(R, \underline{m})$. Assume that $R/\underline{m}$ is infinite. $r(\underline{m})$ is the least integer r such that there exists a system of parameters $\bar{x}_1, \cdots, \bar{x}_d$ of degree 1 in $G = R/\underline{m} \oplus \underline{m}/\underline{m}^2 \oplus \cdots$ with $\mathscr{M}^r \subseteqq (\bar{x}_1, \cdots, \bar{x}_d)$ where $\mathscr{M} = \underline{m}/\underline{m}^2 \oplus \underline{m}^2/\underline{m}^3 \oplus \cdots$.

THEOREM 2.2. *Let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring of multiplicity $e(R)$ with $R/\underline{m}$ infinite and $d > 0$. Then*

$$r(\underline{m}) \leqq d!\, e(R) - 1 .$$

Proof. In our case the main theorem of Eakin and Sathaye in [3] states that $H(n) < \binom{n+d}{d}$ implies $r(\underline{m}) \leqq n$. By Corollary 1.3, $H(n) \leqq n^{d-1}e(R) + d - 1$. So it is sufficient to note that

$$(d!\, e(R))^{d-1}e(R) + d - 1 < \binom{d!\, e(R) + d}{d} .$$

This generalizes a result in [7] where R was assumed to be of dimension 1.

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