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BOUNDS FOR THE CASTELNUOVO-MUMFORD REGULARITY

M. BRODMANN AND T. GÖTSCH

ABSTRACT. We extend the "linearly exponential" bound for the Castelnuovo-Mumford regularity of a graded ideal in a polynomial ring $K[x_1, \dots, x_r]$ over a field (established by Galligo and Giusti in characteristic 0 and recently, by Caviglia-Sbarra for abitrary K) to graded submodules of a graded module over a homogeneous Cohen-Macaulay ring $R = \oplus_{n\geq 0} R_n$ with artinian local base ring R_0. As an application we get a "linearly exponential" bound for the Castelnuovo-Mumford regularity of a graded R-module M in terms of the degrees which occur in a minimal free presentation of M.

1. Introduction. The first result on Castelnuovo-Mumford regularity, proved a long time before this notion even was created, is a bounding result: Castelnuovo's "bound on the regularity of the vanishing ideal of a projective space curve" (cf [**6**]).

Similarly, the classical controversy around the "problem of the finitely many steps" (cf [**12, 13**]) which grew out of Hilbert's "Syzygientheorie", also may be understood as the question for a regularity bound: Do the degrees which occur in a minimal free presentation of a (finitely generated) graded module (over a polynomial ring over a field) bound the Castelnuovo-Mumford regularity of this module (cf 6.6 for more details)?

When Mumford introduced the notion of "Castelnuovo regularity" (cf [**16**]) he first proved a bounding result which is of basic significance for the construction of Hilbert- and Picard schemes.

Since then, the search for regularity bounds has become a theme of constant interest, motivated by the crucial rôle played by these

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bounds for the foundational and the computational aspect of algebraic geometry.

In the present paper, we take up this theme.

Our aim is to establish an upper bound for the Castelnuovo-Mumford regularity of a graded submodule of a graded Cohen-Macaulay module over a homogeneous noetherian ring with artinian base ring. This generalizes a bounding result for graded ideals of a polynomial ring over a field, recently proved by Caviglia-Sbarra. Moreover, it improves the existing regularity bounds for graded modules over polynomial rings over a field given in terms of the degrees and the size of the presenting matrix.

Let r be an integer > 1, let $K[\underline{x}] = K[x_1, \dots, x_r]$ be a polynomial ring over a field K and let $\mathfrak{a} \subseteq K[\underline{x}]$ be a graded ideal. Let $\mathrm{reg}(\mathfrak{a})$ denote the Castelnuovo-Mumford regularity of $\mathfrak{a}$ and let $d(\mathfrak{a})$ be the generating degree of $\mathfrak{a}$ (for the definitions see 2.3 B), C) and 2.2 C)). Recently Caviglia and Sbarra [**7**] have shown that

$$\mathrm{reg}(\mathfrak{a}) \leq (2d(\mathfrak{a}))^{2^{r-2}}. \tag{1.1}$$

Previously, this estimate was known to be true only in characteristic 0, by results of Galligo [**9**] and Giusti [**10**] (cf also Bayer-Mumford [**1**]). According to Mayr-Meyer [**15**], the "linearly exponential bound" (1.1) is "close to being sharp": Namely, for each $r > 1$ there is an ideal $\mathfrak{a}_r \subseteq \mathbb{C}[x_1, \cdots, x_r]$ such that $d(\mathfrak{a}_r) = 4$ and $\mathrm{reg}(\mathfrak{a}_r) \geq 2^{2^{(r-2)/10}}$, (cf [**1**]).

We shall generalize the estimate (1.1) to the situation where $K[\underline{x}]$ is replaced by a graded homomorphic image V of a graded Cohen-Macaulay module U over a homogeneous noetherian ring $R = \oplus_{n \geq 0} R_n$ with artinian local base ring R_0, and where $\mathfrak{a}$ is replaced by a graded proper submodule M of V. We shall establish the following bounding result, in which $d(T)$ is used to denote the generating degree of a graded R-module T (cf 2.2 C)):

Let $\mathfrak{b} \subseteq R$ be a graded ideal with $\mathfrak{b}V \subseteq M$ and $M \underset{R}{:} V \subseteq \sqrt{\mathfrak{b}}$. Let $d \geq d(M)$ and $t \geq \max\{1, d(\mathfrak{b})\}$. Set $s := \dim(V/M)$ and $c := \dim(U) - s$. Moreover, let $b := \mathrm{beg}(U)$ be the beginning of U (cf 2.2 A)) and let $e(U)$ be the multiplicity of U. Then (cf 5.3)

$$\text{(1.2)} \qquad \operatorname{reg}(M) \le \begin{cases} \operatorname{reg}(V) + (t-1)c + 1, & \text{if } s = 0; \\ [\max\{d, \operatorname{reg}(V) + (t-1)c + 1\} & \\ \quad + e(U)t^c - b]^{2^{s-1}} + b, & \text{if } s > 0. \end{cases}$$

As an application of this, we may generalize the bound (1.1) as follows:

Let $\mathfrak{a}$ be a proper graded ideal of positive height in a homogeneous Cohen-Macaulay ring $R = \oplus_{n\ge 0} R_n$ of dimension $r > 1$ with artinian local base ring R_0. Then (cf 5.6):

$$\text{(1.3)} \qquad \operatorname{reg}(\mathfrak{a}) \le [\operatorname{reg}(R) + d(\mathfrak{a})(1 + e(R))]^{2^{r-2}}.$$

If R is as above, if V is a graded R-module generated by $\mu(< \infty)$ homogeneous elements and $M \subseteq V$ is a graded submodule, another application of (1.2) is (cf 6.1)

$$\text{(1.4)} \quad \operatorname{reg}(M) \le [\max\{d(M), \operatorname{reg}(V) + 1\} + (\mu + 1)e(R) - \alpha]^{2^{r-1}} + \alpha,$$

where $\alpha := \min\{\operatorname{beg}(V), \operatorname{reg}(V) - \operatorname{reg}(R)\}$. This latter estimate brings us back to the roots of computational algebraic geometry: to the "*Problem of the Finitely Many Steps*" (cf [**12, 13**]). From (1.4) we namely may conclude, that there is a "*linearly exponential bound*" for the regularity of a graded module M in terms of the discrete data of a minimal free presentation of M. More precisely, let $R = \oplus_{n\ge 0} R_n$ be a homogeneous Cohen-Macaulay ring of dimension $r > 0$ such that R_0 is artinian. Let

$$\oplus_{j=1}^{\nu} R(-b_j) \to \oplus_{i=1}^{\mu} R(-a_i) \to M \to 0$$

be an exact sequence of graded R-modules with integers $b_1 \le b_2 \le \cdots \le b_\nu, a_1 \le b_1$ and $a_1 \le a_2 \le \cdots \le a_\mu$. Finally, let $\mu^* := \{\sup\{i \in \{1, \cdots, \mu\} \big| a_i \le b_\nu\}$. Then (cf 6.3)

$$\text{(1.5)} \qquad \operatorname{reg}(M) \le \max\{a_\mu + \operatorname{reg}(R), [b_\nu + \operatorname{reg}(R) + 1 \\ + (\mu^* + 1)e(R) - a_1]^{2^{r-1}} + a_1 - 1\}.$$

In the special case where $R = K[x_1, \cdots, x_r]$ is a polynomial ring over a field we get (cf 6.5)

$$\text{(1.6)} \qquad \operatorname{reg}(M) \le \max\{a_\mu, [b_\nu + \mu^* + 2 - a_1]^{2^{r-1}} + a_1 - 1\}.$$

This is a considerable improvement with respect to the "squarely exponential" estimate, which is obtained by an iterated application of the "Hermann-Hentzelt bound", and the "factorially exponential" estimate obtained by the "generalized Bayer-Mumford bound" (cf 6.6).

Our approach to the main estimate relies on a detailed analysis of the behavior of local cohomology with support in the irrelevant ideal R_+ with respect to filter-regular sequences (cf 3.1, 3.2 for the definition of this concept). Our first step is to bound the ends of R_+-torsion modules in an appropriate way (cf 3.8). The second step is to establish a bound on the length of "filter-kernels" (cf 4.6). Then we combine these two estimates with a generalized version of the regularity criterion of Bayer-Stillman [**2**] in order to get the final estimate (1.2), (cf. 5.1, 5.2). Using a different approach Chardin-Fall-Nagel [**8**] independently obtained similar bounds.

2. Preliminaries. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers. If $\mathcal{S} \subseteq \mathbb{R}$ is a set of real numbers we form the supremum $\sup(\mathcal{S})$, respectively the infimum $\inf(\mathcal{S})$ in $\mathbb{R} \cup \{\pm\infty\}$ under the convention that $\sup(\varnothing) = -\infty$ and $\inf(\varnothing) = \infty$.

2.1. *Notation and Conventions.* A) Throughout this paper, let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a homogeneous noetherian ring, that is a $\mathbb{N}_0$-graded ring with noetherian base ring R_0 such that $R = R_0[f_1, f_2 \cdots, f_r]$ with finitely many elements $f_1, f_2, \cdots, f_r \in R_1$

B) By R^h we denote the set $\bigcup_{n\in\mathbb{N}_0} R_n$ of *homogeneous elements of* R and by R^h_+ the set $\bigcup_{n\in\mathbb{N}} R_n$ of *homogeneous elements of positive degree.* Moreover we introduce the *irrelevant ideal* $R_+ := \oplus_{n\in\mathbb{N}} R_n$ of R.

C) If $x_1, \cdots, x_r$ are indeterminates, the *polynomial ring* $R_0[\underline{x}] := R_0[x_1, \cdots, x_r]$ is furnished with its *standard grading* and thus is homogeneous.

D) We say that R_0 *has infinite residue fields* if $R_0/\mathfrak{m}_0$ is an infinite field for each maximal ideal $\mathfrak{m}_0$ of R_0.

2.2. Definition. A) Let $T = \oplus_{n\in\mathbb{Z}} T_n$ be a graded R-module. We define the *beginning* and the *end* of T, respectively, by

$$\operatorname{beg}(T) := \inf\{n \in \mathbb{Z} \big| T_n \neq 0\}; \quad \operatorname{end}(T) := \sup\{n \in \mathbb{Z} \big| T_n \neq 0\}.$$

B) Let T be as in part A) and let $m \in \mathbb{Z}$. We define the m-th *left-truncation* and the m-th *right-truncation* of T, respectively, by

$$T_{\geq m} := \oplus_{n \geq m} T_n \; ; \quad T_{\leq m} := \oplus_{n \leq m} T_n.$$

As R is $\mathbb{N}_0$-graded, $T_{\geq m}$ is a graded R-submodule of T.

C) Let T be as above. We denote the *generating degree* of T by $d(T)$, thus

$$d(T) := \inf\{m \in \mathbb{Z} \big| T = R \cdot T_{\leq m} := \sum_{\substack{n \geq 0 \\ k \leq m}} R_n T_k\}.$$

As R is homogeneous we also may write

$$d(T) := \inf\{m \in \mathbb{Z} \big| T_{\geq m} = R \cdot T_m := \Sigma_{n \geq 0} R_n T_m\}.$$

2.3. Definition and Remark. A) Let $M = \oplus_{n \in \mathbb{Z}} M_n$ be a graded R-module. For $i \in \mathbb{N}_0$ we denote by $H^i_{R_+}(M)$ the i-th *local cohomology module of* M *with respect to the irrelevant ideal* R_+ of R. We identify $H^0_{R_+}(M)$ with the R_+*-torsion submodule of* M, thus $H^0_{R_+}(M) = \Gamma_{R_+}(M) := \bigcup_{n \in \mathbb{N}} (0 \underset{M}{:} (R_+)^n)$.

B) (cf [**5**, Chap 15]) Let M and i be as above. Then $H^i_{R_+}(M)$ carries a natural grading as an R-module. For all $n \in \mathbb{Z}$ we use $H^i_{R_+}(M)_n$ to denote the n-th graded component of $H^i_{R_+}(M)$.

Assume now in addition, that M is finitely generated. Then, the R_0-module $H^i_{R_+}(M)_n$ is finitely generated for all $n \in \mathbb{Z}$ and vanishes if $n >> 0$. Moreover $H^i_{R_+}(M) = 0$ for all i which exceed the minimal number of generators of the ideal R_+. So, for each $k \in \mathbb{N}_0$ we may define the *Castelnuovo-Mumford regularity of* M *at and above level* k *by*

$$\text{reg}^k(M) := \sup\{\text{end}(H^i_{R_+}(M)) + i \big| i \geq k\}$$

and obtain $\text{reg}^k(M) \in \mathbb{Z} \cup \{-\infty\}$.

C) Let M be a finitely generated graded R-module. Then, the *Castelnuovo-Mumford regularity of* M is defined by

$$\text{reg}(M) := \text{reg}^0(M) = \sup\ \{\text{end}(H^i_{R_+}(M)) + i \big| i \in \mathbb{N}_0\}.$$

Keep in mind that we always have (cf [**5**, 15.3.1])

$$d(M) \leq \operatorname{reg}(M).$$

We now recall a few basic facts on graded modules and their local cohomology, which shall be used repeatedly in our arguments.

2.4. *Remark.* (*Replacement arguments*) A) Let R'_0 be a noetherian R_0-algebra and furnish $R'_0 \otimes_{R_0} R$ with its natural grading, given by $(R'_0 \otimes_{R_0} R)_n := R'_0 \otimes_{R_0} R_n$ for all $n \in \mathbb{N}_0$. Then $R'_0 \otimes_{R_0} R$ becomes a homogeneous noetherian ring with $(R'_0 \otimes_R R)_+ = R_+ \cdot (R'_0 \otimes_{R_0} R) = R'_0 \otimes_{R_0} R_+$.

If $M = \oplus_{n\in\mathbb{Z}} M_n$ is a graded R-module, we furnish $R'_0 \otimes_{R_0} M$ with its natural grading as an $R'_0 \otimes_{R_0} R$-module, given by $(R'_0 \otimes_{R_0} M)_n := R'_0 \otimes_{R_0} M_n$ for all $n \in \mathbb{Z}$.

If M is finitely generated over R, then $R'_0 \otimes_{R_0} M$ is a finitely generated graded $R'_0 \otimes_{R_0} R$-module.

B) Let R'_0 and M be as in part A), but assume in addition that R'_0 is flat over R_0. Then the graded flat base-change property of local cohomology (cf [**5**, 15.2.3]) yields an isomorphism of graded R_0-modules

$$H^i_{(R'_0 \otimes_{R_0} R)_+}(R'_0 \otimes_{R_0} M) \cong R'_0 \otimes_{R_0} H^i_{R_+}(M)$$

for each $i \in \mathbb{N}_0$. In particular $\operatorname{reg}^k(R'_0 \otimes_{R_0} M) \leq \operatorname{reg}^k(M)$ for all $k \in \mathbb{N}_0$.

C) Let R'_0 be a noetherian faithfully flat R_0-algebra. Then, for each graded R-module T we have

$$\operatorname{beg}(R'_0 \otimes_{R_0} T) = \operatorname{beg}(T), \quad \operatorname{end}(R'_0 \otimes_{R_0} T) = \operatorname{end}(T)$$

and $d(R'_0 \otimes_{R_0} T) = d(T)$.

Also if $\mathfrak{a} \subseteq R$ is a graded ideal and $S \subseteq T$ is a graded submodule, then

$$\left(R'_0 \otimes_{R_0} S :_{R'_0 \otimes_{R_0} T} R'_0 \otimes_{R_0} \mathfrak{a}\right) = R'_0 \otimes_{R_0} (S \underset{T}{:} \mathfrak{a}).$$

Finally, if M is a finitely generated graded R-module, statement B) implies that $\operatorname{reg}^k(R'_0 \otimes_{R_0} M) = \operatorname{reg}^k(M)$ for each $k \in \mathbb{N}_0$.

So, in order to prove a statement on beginnings, ends, generating degrees, Castelnuovo-Mumford regularities, annihilators of finitely generated graded modules we may perform a faithfully flat base change $\bullet \mapsto R'_0 \otimes_{R_0} \bullet$.

D) Assume that $(R_0, \mathfrak{m}_0)$ is local and $(R'_0, \mathfrak{m}'_0)$ is a local flat R_0-algebra with $\mathfrak{m}'_0 = \mathfrak{m}_0 R'_0$. Then, $\text{length}_{R'_0}(R'_0 \otimes_{R_0} V) = \text{length}_{R_0}(V)$ for each R_0-module V.

So, in order to prove a statement on R_0-lengths of graded R-modules, we may again perform the base change $\bullet \mapsto R'_0 \otimes_{R_0} \bullet$.

3. Ends of Torsion Modules.

We keep the notations and hypotheses of the previous section. A crucial point needed to get our regularity bound is an appropriate estimate for the end of an R_+-torsion module T which occurs as a homomorphic image of a finitely generated graded R-module U. The aim of this section is to establish such an estimate (cf 3.8).

3.1 *Reminder and Remark.* (cf [**5**, Chap 18].) A) Let T be a finitely generated graded R-module. An element $f \in R^h_+$ is said to be *filter-regular* (or *almost-regular*) *with respect to* T if it is a non-zero divisor with respect to $T/H^0_{R_+}(T)$.

In this situation, we call the graded submodule $(0 \underset{T}{:} f)$ of T the *filter-kernel of* T *with respect* to f.

B) Let f and T be as in part A). Then, the following statements are equivalent:

(i) f is filter-regular with respect to T;

(ii) $f \notin \bigcup[\text{Ass}_R(T) \cap \text{Proj}(R)]$;

(iii) $\frac{f}{1} \in \text{NZD}_{R_\mathfrak{p}}(T_\mathfrak{p})$ for all $\mathfrak{p} \in \text{Proj}(R)$;

(iv) $(0 \underset{T}{:} f) \subseteq H^0_{R_+}(T)$;

(v) $\text{end}(0 \underset{T}{:} f) < \infty$.

C) Let f and T be as above. Let $W \subseteq T$ be a graded submodule which is R_+-torsion. It follows immediately by part A) that f is filter-regular with respect to T if and only if it is with respect to T/W.

D) Let f and T be as above. Let $\mathfrak{b} \subseteq R$ be a graded ideal such that $\mathfrak{b}T = 0$. Then, f is filter-regular with respect to T if and only if its image $f + \mathfrak{b} \in R/\mathfrak{b}$ is.

E) Let $d \in \mathbb{N}$ and let $f \in R_d$ be filter-regular with respect to the finitely generated graded R-module T. Then f^m is filter-regular with respect to T for all $m \in \mathbb{N}$, (cf B) (ii)). Also in this situation it follows easily (cf B) (iv)) that $(0 \underset{T}{:} f)_n = H^0_{R_+}(T)_n$ for all $n \geq \mathrm{end}(H^0_{R_+}(T)) - d + 1$.

F) Let T and $f \in R_d$ be as in part E). Let R'_0 be a flat noetherian R_0-algebra. Then, the element $1_{R'_0} \otimes f \in (R'_0 \otimes_{R_0} R)_d = R'_0 \otimes_{R_0} R_d$ is filter regular with respect to the graded $R'_0 \otimes_{R_0} R$-module $R'_0 \otimes_{R_0} T$.

3.2. *Reminder and Remark.* A) Let T be a finitely generated graded R-module. A sequence of elements $f_1, \cdots, f_r \in R^h_+$ is called a *filter-regular (or almost regular) sequence with respect to* T if f_i is filter-regular with respect to $T/\sum_{j=1}^{i-1} f_j T$ for all $i \leq r$.

B) Let T be as in part A), let $f_1, \cdots, f_r \in R^h_+$ and let $W \subseteq T$ be a graded submodule which is R_+-torsion. Then, by 3.1 C) it is immediate that $f_1, \cdots, f_r$ form a filter regular sequence with respect to T if and only if they do with respect to T/W.

C) Let T and $f_1, \cdots, f_r$ be as in part B). Let $\mathfrak{b} \subseteq R$ be a graded ideal such that $\mathfrak{b}T = 0$. It follows by 3.1 D) that the elements $f_1, \cdots, f_r$ form a filter-regular sequence with respect to T if and only if their images $f_1 + \mathfrak{b}, \cdots, f_r + \mathfrak{b} \in R/\mathfrak{b}$ do.

D) Finally if $f_1, \cdots, f_r \in R^h_+$ form a filter-regular sequence with respect to the finitely generated and graded R-module T, so do $f_1^{m_1}, \cdots, f_r^{m_r}$ for any choice of exponents $m_1, \cdots, m_r \in \mathbb{N}$ (cf 3.1 E)).

3.3. Lemma. *Let T be a finitely generated graded R-module, let $r \in \mathbb{N}$, let $d_1, \cdots, d_r \in \mathbb{N}$ and let $f_1, \cdots, f_r$ be a filter-regular sequence with respect to T such that $f_j \in R_{d_j}$ for $j = 1, \cdots, r$. Then, for all $k \in \mathbb{N}_0$ and all $i \in \{0, \cdots, r\}$:*

a) $\mathrm{end}(H^k_{R_+}(T/\sum_{j=1}^{i} f_j T)) \leq \max_{j=0}^{i}\{\mathrm{end}(H^{k+j}_{R_+}(T)) + j\} - i + \sum_{j=1}^{i} d_j$;

b) $\mathrm{end}(H^{k+i}_{R_+}(T)) + \sum_{j=1}^{i} d_j \leq \mathrm{end}(H^k_{R_+}(T/\sum_{j=1}^{i} f_jT))$;

c) $\mathrm{reg}^k(T/\sum_{j=1}^{i} f_jT) \leq \mathrm{reg}^k(T) - i + \sum_{j=1}^{i} d_j$;

d) $\mathrm{reg}^{k+i}(T) \leq \mathrm{reg}^k(T/\sum_{j=1}^{i} f_jT) + i - \sum_{j=1}^{d} d_j$.

Proof. It suffices to prove statements a) and b). For $i = 0$, both statements are clear. So, let $i > 0$. Let $\ell \in \mathbb{N}_0$. As f_1 is filter-regular with respect to T we have $0 \underset{T}{:} f_1 \subseteq \Gamma_{R_+}(T)$ and hence get an epimorphism of graded R-modules

$$H^\ell_{R_+}(T) \twoheadrightarrow H^\ell_{R_+}(T/(0 \underset{T}{:} f_1))$$

and an isomorphism of graded R-modules

$$H^{\ell+1}_{R_+}(T) \cong H^{\ell+1}_{R_+}(T/(0 \underset{T}{:} f_1)).$$

Applying cohomology to the short exact sequence of graded R-modules

$$0 \to (T/(0 \underset{T}{:} f_1))(-d_1) \to T \to T/f_1T \to 0$$

we thus get

$$\begin{aligned}\mathrm{end}(H^\ell_{R_+}(T/f_1T)) &\leq \max\{\mathrm{end}(H^\ell_{R_+}(T)), \mathrm{end}(H^{\ell+1}_{R_+}(T)) + d_1\} \\ &\leq \max\{\mathrm{end}(H^\ell_{R_+}(T)), \mathrm{end}(H^{\ell+1}_{R_+}(T)) + 1\} + d_1 - 1\end{aligned}$$

and

$$\mathrm{end}(H^{\ell+1}_{R_+}(T)) + d_1 \leq \mathrm{end}(H^\ell_{R_+}(T/f_1T)).$$

Applying the first estimate with $T/\sum_{j=1}^{i-1} f_jT$ instead of T and f_i instead of f_1, we get

$$\begin{aligned}\mathrm{end}(H^\ell_{R_+}(T/\Sigma_{j=1}^{i} f_jT)) \leq \max\{&\mathrm{end}(H^\ell_{R_+}(T/\Sigma_{j=1}^{i-1} f_jT)), \\ &\mathrm{end}(H^{\ell+1}_{R_+}(T/\Sigma_{j=1}^{i-1} f_jT)) + 1\} + d_i - 1\end{aligned}$$

By induction on i it follows that

$$\mathrm{end}(H^\ell_{R_+}(T/\Sigma_{j=1}^{i} f_jT)) \leq \max_{j=0}^{i}\{\mathrm{end}(H^{\ell+j}_{R_+}(T)) + j\} - i + \Sigma_{j=1}^{i} d_j$$

and

$$\operatorname{end}(H^{\ell+i}_{R_+}(T)) + \Sigma^i_{j=1} d_j \leq \operatorname{end}(H^{\ell}_{R_+}(T/\Sigma^i_{j=1} f_j T))$$

for all $i \in \{0, \cdots, r\}$. □

3.4. *Reminder and Remark.* A) Let T be a finitely generated graded R-module and let $f_1, \cdots, f_r \in R^h_+$ form a filter-regular sequence with respect to T. We call this sequence *saturated* if $T/\sum_{j=1}^r f_j T$ is an R_+-torsion module.

B) Let T and $f_1, \cdots, f_r \in R^h_+$ as in part A). Then, the filter-regular sequence $f_1, \cdots, f_r$ is saturated if and only if $R_+ \subseteq \sqrt{0 \underset{R}{:} (T/\sum_{j=1}^r f_j T)}$ or – equivalently – if and only if

$$\sqrt{(0 \underset{R}{:} T) + R_+} = \sqrt{(0 \underset{R}{:} T) + \Sigma^r_{j=1} f_j R}.$$

C) Let T and $f_1, \cdots, f_r$ be as above and let $W \subseteq T$ be a graded submodule which is R_+-torsion. It easily follows from 3.2 B) that $f_1, \cdots, f_r$ form a saturated filter-regular sequence with respect to T if and only if they do with respect to T/W.

D) Let T and $f_1, \cdots, f_r$ be as above. Let $\mathfrak{b} \subseteq R$ be a graded ideal such that $\mathfrak{b}T = 0$. It follows easily from 3.2 C) that the elements $f_1, \cdots, f_r$ form a saturated filter-regular sequence with respect to T if and only if their images $f_1 + \mathfrak{b}, \cdots, f_r + \mathfrak{b} \in R/\mathfrak{b}$ do.

3.5. *Reminder and Remark.* A) Let T be a finitely generated graded R-module. Then, the *cohomological dimension of T (with respect to R_+)* is defined as

$$\operatorname{cd}_R(T) = \operatorname{cd}(T) := \sup\{i \in \mathbb{Z} \big| H^i_{R_+}(T) \neq 0\}.$$

B) Let T be as in part A). Keep in mind the following facts:

a) $\operatorname{cd}(T) < \infty$;

b) $\operatorname{cd}(T) \leq 0 \Longleftrightarrow R_+ \subseteq \sqrt{0 \underset{R}{:} T} \Longleftrightarrow T$ is R_+-torsion;

c) $\operatorname{cd}(T) = -\infty \Longleftrightarrow \operatorname{cd}(T) < 0 \Longleftrightarrow T = 0$.

C) Let T be as above and let $\mathfrak{b} \subseteq R$ be a graded ideal such that $\mathfrak{b}T = 0$. It follows easily from the base-ring independence of local cohomology that $\operatorname{cd}_R(T) = \operatorname{cd}_{R/\mathfrak{b}}(T)$.

D) Let T be as above and assume that $(R_0, \mathfrak{m}_0)$ is local. Then $\operatorname{cd}(T) = \dim(T/\mathfrak{m}_0 T)$ (cf [**4**, 2.3 a)]).

E) Let R_0' be a flat noetherian R_0-algebra. Then the flat base change property of local cohomology yields $\operatorname{cd}_{R_0' \otimes_{R_0} R}(R_0' \otimes_{R_0} T) \leq \operatorname{cd}_R(T)$, with equality if R_0' is faithfully flat over R.

F) Let T be as above and let $f_1, \cdots, f_r \in R_+^h$ form a saturated filter-regular sequence with respect to T, so that $\sqrt{(0 \underset{R}{:} T) + R_+} = \sqrt{(0 \underset{R}{:} T) + \sum_{j=1}^r f_j R}$. Then $H^i_{R_+}(T) \cong H^i_{(f_1, \cdots, f_r)}(T)$ for all $i \in \mathbb{N}_0$ (cf [**5**, 2.1.9]), thus $r \geq \operatorname{cd}(T)$.

3.6. Lemma. *Assume that R_0 has infinite residue fields, let $\mathfrak{b} \subseteq R$ be a graded ideal such that $R_+ \subseteq \sqrt{\mathfrak{b}}$, let $d \geq \max\{d(\mathfrak{b}), 1\}$ and let $\mathcal{Q} \subseteq \operatorname{Proj}(R)$ be a finite set. Then $\mathfrak{b}_d \nsubseteq \bigcup_{\mathfrak{q} \in \mathcal{Q}} \mathfrak{q}$.*

Proof. We may assume that $\mathcal{Q} \neq \varnothing$. For each $\mathfrak{m}_0 \in \operatorname{Max}(R_0)$ we set

$$\mathcal{Q}(\mathfrak{m}_0) := \{\mathfrak{q} \in \mathcal{Q} \big| \mathfrak{q} \cap R_0 \subseteq \mathfrak{m}_0\}.$$

Then, there is a finite set $\mathbb{M} \subseteq \operatorname{Max}(R_0)$ such that $\mathcal{Q}(\mathfrak{m}_0) \neq \varnothing$ for each $\mathfrak{m}_0 \in \mathbb{M}$ and such that $\mathcal{Q} = \bigcup_{\mathfrak{m}_0 \in \mathbb{M}} \mathcal{Q}(\mathfrak{m}_0)$.

Let $\mathfrak{m}_0 \in \mathbb{M}$. As $R_+ \subseteq \sqrt{\mathfrak{b} \cap R_+} = \sqrt{\mathfrak{b}_d \cdot R}$ and as $\mathcal{Q}(\mathfrak{m}_0) \cap \operatorname{Var}(R_+) = \varnothing$ it follows that $\mathfrak{b}_d \nsubseteq \mathfrak{q}_d$ for all $\mathfrak{q} \in \mathcal{Q}(\mathfrak{m}_0)$. So, by Nakayama $\mathfrak{q}_d \cap \mathfrak{b}_d + \mathfrak{m}_0 \mathfrak{b}_d \subsetneqq \mathfrak{b}_d$ for each $\mathfrak{q} \in \mathcal{Q}(\mathfrak{m}_0)$. As $\mathcal{Q}(\mathfrak{m}_0)$ is finite and $R_0/\mathfrak{m}_0$ is infinite, there is some $v_{\mathfrak{m}_0} \in \mathfrak{b}_d \setminus \bigcup_{\mathfrak{q} \in \mathcal{Q}(\mathfrak{m}_0)} (\mathfrak{q}_d \cap \mathfrak{b}_d + \mathfrak{m}_0 \mathfrak{b}_d)$. Now, for each $\mathfrak{m}_0 \in \mathbb{M}$ we find some element $a_{\mathfrak{m}_0} \in \left(\bigcap_{\mathfrak{n}_0 \in \mathbb{M} \setminus \{\mathfrak{m}_0\}} \mathfrak{n}_0\right) \setminus \mathfrak{m}_0$. With $v := \sum_{\mathfrak{m}_0 \in \mathbb{M}} a_{\mathfrak{m}_0} v_{\mathfrak{m}_0}$ it follows that

$$v \in \mathfrak{b}_d \setminus \bigcup_{\mathfrak{m}_0 \in \mathbb{M}} \bigcup_{\mathfrak{q} \in \mathcal{Q}(\mathfrak{m}_0)} (\mathfrak{q}_d \cap \mathfrak{b}_d + \mathfrak{m}_0 \mathfrak{b}_0);$$

hence $v \in \mathfrak{b}_d \setminus \bigcup_{\mathfrak{q} \in \mathcal{Q}(\mathfrak{m}_0)} \mathfrak{q}$. ◻

3.7. Lemma. *Let $(R_0, \mathfrak{m}_0)$ be local such that $R_0/\mathfrak{m}_0$ is infinite. Let $\mathcal{S} \subseteq \mathrm{Proj}(R)$ be a finite set. Let U be a finitely generated graded R-module and let $\mathfrak{a} \subseteq R$ be a graded ideal such that $R_+ \subseteq \sqrt{\mathfrak{a} + (0 \underset{R}{:} U)}$. Assume that $c := \mathrm{cd}(U) > 0$ and let $d_1, \cdots, d_c$ be positive integers $\geq d(\mathfrak{a})$. Then, there is a saturated filter-regular sequence $f_1, \cdots, f_c$ with respect to U such that $f_i \in \mathfrak{a}_{d_i} \setminus \bigcup_{\mathfrak{s} \in \mathcal{S}} \mathfrak{s}$ for all $i = 1, \cdots, c$.*

Proof. By 3.4 D) and 3.5 C) we may replace R by $R/(0 \underset{R}{:} U)$ and hence assume that $(0 \underset{T}{:} U) = 0$. By 3.5 D) we have $\dim(U/\mathfrak{m}_0 U) = c > 0$, so that the set $\min \mathrm{Ass}_R(U/\mathfrak{m}_0 U)$ of minimal associated primes of $U/\mathfrak{m}_0 U$ belongs to $\mathrm{Proj}(R)$. Therefore, we may apply 3.6 with $\mathfrak{b} := \mathfrak{a}, T := U$ and $\mathcal{Q} := \mathcal{S} \cup \min \mathrm{Ass}_R(U/\mathfrak{m}_0 U) \cup (\mathrm{Ass}_R(U) \cap \mathrm{Proj}(R))$ and thus find some $f_1 \in \mathfrak{a}_{d_1} \setminus \bigcup_{\mathfrak{s} \in \mathcal{S}} \mathfrak{s}$ which is filter-regular with respect to U and such that $\dim((U/\mathfrak{m}_0)/f_1(U/\mathfrak{m}_0)) = c - 1$, whence $\mathrm{cd}(U/f_1 U) = \dim((U/f_1 U)/\mathfrak{m}_0(U/f_1 U)) = c - 1$.

Repeating this argument we may construct a filter-regular sequence $f_1, \cdots, f_c$ with respect to U with $f_i \in \mathfrak{a}_{d_i} \setminus \bigcup_{\mathfrak{s} \in \mathcal{S}} \mathfrak{s}$ for $i = 1, \cdots, c$ and such that $\mathrm{cd}(U/\sum_{j=1}^{c} f_j U) = 0$. According to 3.5 B) b) it follows in particular that $U/\sum_{j=1}^{c} f_j U$ is R_+-torsion, so that the filter-regular sequence $f_1, \cdots, f_c$ is saturated. □

3.8. Proposition. *Let $U \twoheadrightarrow T$ be an epimorphism of finitely generated graded R-modules such that T is R_+-torsion. Let $\mathfrak{a} \subseteq R$ be a graded ideal such that $\mathfrak{a}T = 0$ and $R_+ \subseteq \sqrt{\mathfrak{a} + (0 \underset{R}{:} U)}$. Then*

$$\mathrm{end}(T) \leq \mathrm{reg}(U) + \max\{0, d(\mathfrak{a}) - 1\}\mathrm{cd}(U).$$

Proof. Let $d := d(\mathfrak{a}), c := \mathrm{cd}(U)$. It suffices to show that

$$\mathrm{end}(T_{\mathfrak{p}_0}) \leq \mathrm{reg}(U) + \max\{0, d - 1\}c \text{ for all } \mathfrak{p}_0 \in \mathrm{Spec}(R_0).$$

So, let $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$. Then clearly $d(\mathfrak{a}_{\mathfrak{p}_0}) \leq d, \mathfrak{a}_{\mathfrak{p}_0} T_{\mathfrak{p}_0} = 0$ and $(0 \underset{R}{:} U)_{\mathfrak{p}_0} \subseteq (0 \underset{R_{\mathfrak{p}_0}}{:} U_{\mathfrak{p}_0})$. In particular $(R_{\mathfrak{p}_0})_+ \subseteq \sqrt{\mathfrak{a}_{\mathfrak{p}_0} + (0 \underset{R_{\mathfrak{p}_0}}{:} U_{\mathfrak{p}_0})}$.

Moreover $\operatorname{reg}(U_{\mathfrak{p}_0}) \leq \operatorname{reg}(U)$, (cf 2.4 B)) and $\operatorname{cd}_{R\mathfrak{p}_0}(U_{\mathfrak{p}_0}) \leq c$ (cf 3.5 E)). This allows us to replace $R, \mathfrak{a}, U$ and T respectively by $R_{\mathfrak{p}_0}, \mathfrak{a}_{\mathfrak{p}_0}, U_{\mathfrak{p}_0}$ and $T_{\mathfrak{p}_0}$, and hence to assume that $(R_0, \mathfrak{m}_0)$ is local.

Next, let x be an indeterminate and consider the noetherian faithfully flat local R_0-algebra $R'_0 := R_0[x]_{\mathfrak{m}_0 R_0[x]}$. Then, by 2.4 C) and 3.5 E) we may replace $R, \mathfrak{a}, U, T$ respectively by $R'_0 \otimes_{R_0} R, R'_0 \otimes_{R_0} \mathfrak{a}, R'_0 \otimes_{R_0} U$ and $R'_0 \otimes_{R_0} T$ and hence assume that $R_0/\mathfrak{m}_0$ is infinite.

Assume first that $c \leq 0$. Then U is R_+-torsion (cf 3.5 B) b)), and our claim is obvious. So, let $c > 0$. Assume that $d = 0$, so that $\mathfrak{a} = \mathfrak{a}_0 R$. If $\mathfrak{a}_0 = R_0$ we have $T = 0$, and our claim is obvious. So, let $\mathfrak{a}_0 \subseteq \mathfrak{m}_0$. Then, there is some $r \in \mathbb{N}$ with $(R_+)^r \subseteq \mathfrak{m}_0 R + (0 \underset{R}{:} U)$. Therefore $R_n = \mathfrak{m}_0 R_n + (0 \underset{R}{:} U)_n$ for all $n \geq r$. Hence, by Nakayama we get $R_n = (0 \underset{R}{:} U)_n$ for all $n \geq r$. So U is R_+-torsion, which contradicts the assumption that $c > 0$. Therefore we have $d > 0$.

Applying 3.7 we find elements $f_1, \cdots, f_c \in \mathfrak{a}_d$ which form a saturated filter-regular sequence with respect to U. So, by 3.3 c) we get

$$\begin{aligned}\operatorname{end}(U/\Sigma_{j=1}^{c} f_j U) = \operatorname{reg}^0(U/\Sigma_{j=1}^{c} f_j U) &\leq \operatorname{reg}^0(U) - c + \operatorname{cd} \\ &= \operatorname{reg}(U) + (d-1)c.\end{aligned}$$

In view of the induced epimorphism $U/\sum_{j=1}^{c} f_j U \twoheadrightarrow T$ this proves our claim. □

4. Lengths of Filter-Kernels. We keep our previous hypotheses and notations. In this section we provide the basic technical tool needed to prove our main result: An appropriate bound for the length of the filter-kernels of a finitely generated graded R-module N which is a homomorphic image of a graded Cohen-Macaulay module U (cf 4.6).

4.1. *Reminder and Remark.* A) Let $\dim(R_0) = 0$ and let T be a finitely generated graded R-module. Then $\operatorname{length}_{R_0}(T_n) < \infty$ for all $n \in \mathbb{Z}$ and $\operatorname{cd}(T) = \dim(T)$.

B) Let R_0 and T be as in part A) and let $f \in R_+^h$ be filter-regular with respect to T. If $\dim(T) > 0$, f avoids all minimal members of $\operatorname{Ass}_R(T)$, so that $\dim(T/fT) = \dim(T) - 1$. Therefore, if $f_1, \cdots, f_r \in R_+^h$ is a filter-regular sequence with respect to T, it follows by induction that

$$\dim(T/\Sigma_{j=1}^{i} f_j T) = \max\{0, \dim(T) - i\} \text{ for all } i \leq r.$$

In particular, the filter-regular sequence $f_1, \cdots, f_r$ is saturated if and only if $r \geq \dim(T)$.

C) Let R_0 and T be as above. Assume that $T \neq 0$. We denote the *Hilbert-Serre multiplicity* of T by $e(T)$. So there is a polynomial $P_T(x) \in \mathbb{Q}[x]$ of degree $\dim(T)$ such that $\sum_{m \leq n} \text{length}_{R_0}(T_m) = P_T(n)$ for all $n >> 0$ and such that $e(T)/\dim(T)!$ is the leading coefficient of P_T. If $\dim(T) = 0$, then $e(T) = \text{length}_{R_0}(T)$.

D) Let R_0 and T be as above, let $d \in \mathbb{N}$ and let $f \in R_d$ be filter regular with respect to T. Then, for all $n >> 0$ we have an exact sequence $0 \to T_{n-d} \xrightarrow{f} T_n \to (T/fT)_n \to 0$ and hence

$$c := P_{T/fT}(x) - (P_T(x) - P_T(x-d))$$

is constant. If f is regular with respect to T, then the above sequence is exact for all $n \in \mathbb{Z}$, so that $c = 0$.

Therefore, if $\dim(T) > 1$ or if f is regular with respect to T, we have $e(T/fT) = de(T)$.

E) Let R_0 and T be as above but assume in addition that $(R_0, \mathfrak{m}_0)$ is local and T is Cohen-Macaulay of dimension $r > 0$. Let $d_1, \cdots, d_r \in \mathbb{N}$ and let $f_1, \cdots f_r$ be a filter-regular sequence with respect to T such that $f_i \in R_{d_i}$ for $i = 1, \cdots, r$. Then, by statement B) the elements $f_1, \cdots, f_r$ form a system of parameters and hence a regular sequence with respect to T. So, by the last observation of part C) and by a repeated application of part D) we get

$$\text{length}_{R_0}(T/\Sigma_{j=1}^r f_j T) = d_1 d_2 \cdots d_r e(T).$$

4.2. Proposition. *Assume that* $(R_0, \mathfrak{m}_0)$ *is local with* $\dim(R_0) = 0$ *and let* $U \twoheadrightarrow T$ *be an epimorphism of finitely generated graded* R*-modules such that* U *is Cohen-Macaulay and* T *is* R_+*-torsion. Let* $\mathfrak{a} \subseteq R$ *be a graded ideal such that* $\mathfrak{a}T = 0$ *and* $R_+ \subseteq \sqrt{\mathfrak{a} + (0 \underset{R}{:} U)}$. *Then*

$$\text{length}_{R_0}(T) \leq \max\{1, d(\mathfrak{a})\}^{\dim(U)} \cdot e(U).$$

Proof. Let $c := \dim(U)$. If $c = 0$ we have $\text{length}_{R_0}(T) \leq \text{length}_{R_0}(U) = e(U)$, (cf 4.1 C)). So, let $c > 0$. Set $d := \max\{1, d(\mathfrak{a})\}$.

As in the proof of 3.8 we may assume that $R_0/\mathfrak{m}_0$ is infinite (cf 2.4 D)). According to 3.7 there are elements $f_1, \cdots, f_c \in \mathfrak{a}_d$ which form a saturated filter-regular sequence with respect to U.

By 4.1 E) it follows that $\mathrm{length}_{R_0}(U/\sum_{j=1}^{c} f_j U) = d^c e(U)$, and the induced epimorphism $U/\sum_{j=1}^{c} f_j U \twoheadrightarrow T$ gives our claim. □

4.3 *Definition and Remark.* A) Let T be a finitely generated R_+-torsion module. We define the *span* of T by

$$\mathrm{span}(T) := \begin{cases} 0, & \text{if } T = 0, \\ \mathrm{end}(T) - \mathrm{beg}(T) + 1, & \text{if } T \neq 0. \end{cases}$$

B) Let $f \in R_d$ with $d \in \mathbb{N}$ and let T be a finitely generated graded R_+-torsion module. Then, clearly

$$f^n T = 0 \quad \text{for all} \quad n \geq \left\lceil \frac{\mathrm{span}(T)}{d} \right\rceil,$$

where we use the notation

$$\lceil a \rceil := \min\{n \in \mathbb{Z} \big| n \geq a\}, \quad (a \in \mathbb{R}).$$

4.4. Lemma. *Assume that* $\dim(R_0) = 0$*. Let* $d \in \mathbb{N}$*, let* W *be a finitely generated graded* R*-module and let* $f \in R_d$ *be filter-regular with respect to* W*. Then*

a) $\mathrm{length}_{R_0}(0 \underset{W}{:} f) \leq \mathrm{length}_{R_0}(H^0_{R_+}(W/fW))$;

b) $\mathrm{length}_{R_0}(H^0_{R_+}(W)) \leq \left\lceil \frac{\mathrm{span}(H^0_{R_+}(W))}{d} \right\rceil \mathrm{length}_{R_0}(H^0_{R_+}(W/fW))$.

Proof. "a)": As f is filter-regular with respect to W we have $(H^0_{R_+}(W) \underset{W}{:} f) = H^0_{R_+}(W)$ and hence $(H^0_{R_+}(W) + fW)/fW \cong H^0_{R_+}(W)/fH^0_{R_+}(W)$. So, there is a monomorphism of graded R-modules $H^0_{R_+}(W)/fH^0_{R_+}(W) \rightarrowtail H^0_{R_+}(W/fW)$; hence

$$(*) \quad \mathrm{length}_{R_0}(H^0_{R_+}(W)/fH^0_{R_+}(W)) \leq \mathrm{length}_{R_0}(H^0_{R_+}(W/fW)).$$

Moreover, the exact sequence of graded R-modules

$$0 \to (0 \underset{W}{:} f) \to H^0_{R_+}(W) \xrightarrow{f\cdot} (fH^0_{R_+}(W))(d) \to 0$$

shows that

$$\begin{aligned}\mathrm{length}_{R_0}(0 \underset{W}{:} f) &= \mathrm{length}_{R_0}(H^0_{R_+}(W)) - \mathrm{length}_{R_0}(fH^0_{R_+}(W)) \\ &= \mathrm{length}_{R_0}(H^0_{R_+}(W)/fH^0_{R_+}(W))\end{aligned}$$

and this proves our claim.

"b)": Let $m := \left\lceil \frac{\mathrm{span}(H^0_{R_+}(W))}{d} \right\rceil$ so that $f^m H^0_{R_0}(W) = 0$ (cf 4.3 B)), hence

$$\mathrm{length}_{R_0}(H^0_{R_+}(W)) = \Sigma_{n=0}^{m-1} \mathrm{length}_{R_0}(f^n H^0_{R_+}(W)/f^{n+1}H^0_{R_+}(W)).$$

In view of the epimorphisms of graded R-modules

$$H^0_{R_+}(W)/fH^0_{R_+}(W) \overset{f^n}{\twoheadrightarrow} (f^n H^0_{R_+}(W)/f^{n+1}H^0_{R_+}(W))(dn)$$

we thus get $\mathrm{length}_{R_0}(H^0_{R_+}(W)) \leq m\ \mathrm{length}_{R_0}(H^0_{R_+}(W)/fH^0_{R_+}\ (W))$. Now, we conclude by the inequality $(*)$. □

4.5. Lemma. *Assume that* $\dim(R_0) = 0$. *Let* N *be a finitely generated graded* R*-module of dimension* $s > 0$. *Let* $f_1, f_2, \cdots, f_s \in R_1$ *form a filter-regular sequence with respect to* N. *Then:*

a) $\mathrm{length}_{R_0}(H^0_{R_+}(N))$

$$\leq \mathrm{length}_{R_0}(N/\sum_{j=1}^{s} f_j N)\Pi_{i=0}^{s-1}\mathrm{span}(H^0_{R_+}(N/\sum_{j=1}^{i} f_j N)).$$

b) $\mathrm{length}_{R_0}(0 \underset{N}{:} f_1)$

$$\leq \mathrm{length}_{R_0}(N/\sum_{j=1}^{s} f_j N)\Pi_{i=1}^{s-1}\mathrm{span}(H^0_{R_+}(N/\sum_{j=1}^{i} f_j N)).$$

Proof. "a)": Let $s = 1$. Then, N/f_1N is an R_+-torsion module (cf 4.1 B)) and hence 4.4 b) yields $\mathrm{length}_{R_0}(H^0_{R_+}(N)) \leq \mathrm{span}(H^0_{R_+}(N)) \cdot \mathrm{length}_{R_0}(N/f_1N)$.

So, let $s > 1$. Then by induction and setting $\overline{N} := N/f_1N$ we have (cf 4.1 B))

$$\begin{aligned}&\mathrm{length}_{R_0}(H^0_{R_+}(\overline{N}))\\ &\quad\leq \mathrm{length}_{R_0}(\overline{N}/\Sigma_{j=1}^{s-1} f_{j+1}\overline{N})\Pi_{i=0}^{s-2}\mathrm{span}(H^0_{R_+}(\overline{N}/\Sigma_{j=1}^{i} f_{j+1}\overline{N}))\\ &\quad= \mathrm{length}_{R_0}(N/\Sigma_{j=1}^{s} f_jN)\Pi_{i=1}^{s-1}\mathrm{span}(H^0_{R_+}(N/\Sigma_{j=1}^{i} f_jN)).\end{aligned}$$

By 4.5 b) we also have

$$\mathrm{length}_{R_0}(H^0_{R_+}(N)) \leq \mathrm{span}(H^0_{R_+}(N)) \cdot \mathrm{length}_{R_0}(H^0_{R_+}(\overline{N})).$$

Altogether, this proves our claim.

"b)": Let $s = 1$. Observing that N/f_1N is an R_+-torsion module (cf 4.1 B)) we conclude by 4.4 a) that

$$\mathrm{length}_{R_0}(0 \underset{N}{:} f_1) \leq \mathrm{length}_{R_0}(N/f_1N).$$

So, let $s > 1$. Writing $\overline{N} := N/f_1N$ we obtain from 4.4 a) that $\mathrm{length}_{R_0}(0 \underset{N}{:} f_1) \leq \mathrm{length}_{R_0}(H^0_{R_+}(\overline{N}))$. Applying statement a) to $\overline{N}$ and the sequence $f_2, \cdots, f_s$ we get

$$\begin{aligned}&\mathrm{length}_{R_0}(H^0_{R_+}(\overline{N}))\\ &\quad\leq \mathrm{length}_{R_0}(\overline{N}/\Sigma_{j=2}^{s} f_j\overline{N}))\Pi_{i=1}^{s-1}\mathrm{span}(H^0_{R_+}(\overline{N}/\Sigma_{j=2}^{i} f_j\overline{N}))\\ &\quad= \mathrm{length}_{R_0}(N/\Sigma_{j=1}^{s} f_jN)\Pi_{i=1}^{s-1}\mathrm{span}(H^0_{R_+}(N/\Sigma_{j=1}^{i} f_jN)),\end{aligned}$$

and our claim follows. □

4.6. Proposition. *Assume that* $(R_0, \mathfrak{m}_0)$ *is local with* $\dim(R_0) = 0$ *and let* $U \twoheadrightarrow N$ *be an epimorphism of finitely generated graded* R*-modules such that* U *is Cohen-Macaulay and* $\dim(N) =: s > 0$*. Let* $\mathfrak{b} \subseteq R$ *be a graded ideal such that* $\mathfrak{b}N = 0$ *and* $(0 \underset{R}{:} N) \subseteq \sqrt{\mathfrak{b}}$*. Let* $f_1, \cdots, f_s \in R_1$ *be a filter-regular sequence with respect to* N *and with respect to* U*. Let* $t \geq \max\{1, d(\mathfrak{b})\}$*. Then*

$$\mathrm{length}_{R_0}(0 \underset{N}{:} f_1) \leq e(U)t^{\dim(U)-s}\Pi_{i=1}^{s-1}\left(\mathrm{reg}(N/\Sigma_{j=1}^{i} f_jN) - \mathrm{beg}(N) + 1\right).$$

Proof. For each $i \in \{1, \cdots, s-1\}$ we have

$$\begin{aligned}\operatorname{span}(H^0_{R_+}(N/\Sigma^i_{j=1} f_j N)) &\leq \operatorname{end}(H^0_{R_+}(N/\Sigma^i_{j=1} f_j N)) - \operatorname{beg}(N) + 1 \\ &\leq \operatorname{reg}(N/\Sigma^i_{j=1} f_j N) - \operatorname{beg}(N) + 1.\end{aligned}$$

By 4.1 E), the graded R-module $\overline{U} := U/\sum_{j=1}^s f_j U$ is CM and satisfies $\dim(\overline{U}) = \dim(U) - s$ and $e(\overline{U}) = e(U)$. Moreover $f_1, \ldots, f_s$ form a saturated filter-regular sequence with respect to N (cf 4.1 B)) so that $R_+ \subseteq \sqrt{(0 \underset{R}{:} N) + \sum_{j=1}^s f_j R} = \sqrt{\mathfrak{b} + \sum_{j=1}^s f_j R} \subseteq \sqrt{\mathfrak{b} + (0 \underset{R}{:} \overline{U})}$. Clearly $\mathfrak{b}(N/\sum_{j=1}^s f_j N) = 0$. By 4.2, applied to the epimorphism $\overline{U} \twoheadrightarrow T := N/\sum_{j=1}^s f_j N$ and with $\mathfrak{a} := \mathfrak{b} + \sum_{j=1}^s f_j R$ (cf 2.4 E)) we thus get

$$\operatorname{length}_{R_0}(N/\Sigma^s_{j=1} f_j N) \leq e(U) t^{\dim(U)-s}.$$

Now, we conclude by 4.5. □

5. Regularity of Submodules. In this section we shall establish our main result and apply it to draw a few more conclusions (cf 5.4 - 5.8). We keep the previous notations and hypotheses.

5.1. Lemma. *Assume that R_0 has infinite residue fields. Let V be a finitely generated graded R-module, let $M \subseteq V$ be a graded submodule and let $f \in R_1$ be filter-regular with respect to V/M and to V. Let $m \in \mathbb{Z}$ be such that*

$$m \geq \max\{d(M), \operatorname{reg}(V) + 1, \operatorname{reg}(M + fV)\}$$

and assume that $(M \underset{V}{:} f)_m = M_m$. Then $\operatorname{reg}(M) \leq m$.

Proof. We have $m > \operatorname{reg}(V), m \geq \max\{\operatorname{reg}(V) + 1, d(M)\} \geq \max\{d(V) + 1, d(M)\} \geq d(M + fV)$ and $m \geq \operatorname{reg}(M + fV)$. If we apply the generalized Bayer-Stillman criterion [**3**, 4.7] to the modules $M + fV$ and V, we thus find elements $f_2, \cdots, f_r \in R_1$ which are filter-regular with respect to V and such that with $f_1 := f$ the equations

$$((M + f_1 V + \Sigma^{i-1}_{j=2} f_j V) \underset{V}{:} f_i)_m = (M + f_1 V + \Sigma^{i-1}_{j=2} f_j V)_m$$

for $i = 2, 3, \cdots, r$ and

$$(M + f_1 V + \Sigma^r_{j=2} f_j V)_m = V_m$$

hold. As $(M \underset{V}{:} f_1)_m = M_m$ we thus get

$$((M + \Sigma_{j=1}^{i-1} f_j V) : f_i)_m = (M + \Sigma_{j=1}^{i-1} f_j V)_m \quad \text{for } i = 2, 3, \cdots, r.$$

In addition

$$(M + \Sigma_{j=1}^{r} f_j V)_m = V_m.$$

As f_1 is filter-regular with respect to V we now may apply the criterion [**3**, 4.7] in the opposite direction to the modules M and V and get $\operatorname{reg}(M) \leq m$. □

5.2. Lemma. *Assume that* $\dim(R_0) = 0$ *and that* R_0 *has infinite residue fields. Let* V *be a finitely generated graded* R*-module, let* $M \subseteq V$ *be a graded submodule and let* $f \in R_1$ *be filter-regular with respect to* V/M *and to* V*. Then*

$$\operatorname{reg}(M) \leq \max\{d(M), \operatorname{reg}(V)+1, \operatorname{reg}(M+fV)\} + \operatorname{length}_{R_0}((M \underset{V}{:} f)/M).$$

Proof. Let $D := \max\{d(M), \operatorname{reg}(V) + 1, \operatorname{reg}(M + fV)\}$. Then, there is an integer $m \in [D, D + \operatorname{length}_{R_0}((M \underset{V}{:} f)/M)]$ such that $(M \underset{V}{:} f)_m/M_m = ((M \underset{V}{:} f)/M)_m = 0$. Now, we conclude by 5.1. □

5.3. Theorem. *Assume that* $(R_0, \mathfrak{m}_0)$ *is local with* $\dim(R_0) = 0$*, let* $U \twoheadrightarrow V$ *be an epimorphism of finitely generated graded* R*-modules such that* U *is a Cohen-Macaulay module. Let* $M \subsetneqq V$ *be a graded submodule, let* $\mathfrak{b} \subseteq R$ *be a graded ideal such that* $\mathfrak{b}V \subseteq M$ *and* $(M \underset{R}{:} V) \subseteq \sqrt{\mathfrak{b}}$*. Let* $d \geq d(M)$ *and let* $t \geq \max\{1, d(\mathfrak{b})\}$*. Finally set* $s := \dim(V/M)$ *and* $c := \dim(U) - s$*.*

a) *If* $s = 0$*, then:* $\operatorname{reg}(M) \leq \operatorname{reg}(V) + (t-1)c + 1$.

b) *If* $s > 0$*, then:*

$$\operatorname{reg}(M) \leq [\max\{d, \operatorname{reg}(V) + (t-1)c + 1\} + e(U)t^c - \operatorname{beg}(U)]^{2^{s-1}} + \operatorname{beg}(U).$$

c) *If* $r := \dim(U) > 1$ *and* $s < r$*, then:*

$$\operatorname{reg}(M) \leq [\max\{d, reg(V) + t\} + e(U)t - \operatorname{beg}(U)]^{2^{r-2}} + \operatorname{beg}(U).$$

Proof. "a)": As $s = 0$, we have $R_+ \subseteq \sqrt{M \underset{R}{:} V} \subseteq \sqrt{\mathfrak{b}} \subseteq \sqrt{\mathfrak{b} + (0 \underset{R}{:} V)}$. If we apply 3.8 to the epimorphism $V \to V/M$ with $\mathfrak{a} := \mathfrak{b}$ we get $\operatorname{reg}(V/M) = \operatorname{end}(V/M) \leq \operatorname{reg}(V) + (t-1)\dim(V) = \operatorname{reg}(V) + (t-1)c$. Now, the short exact sequence $0 \to M \to V \to V/M \to 0$ yields

$$\operatorname{reg}(M) \leq \max\{\operatorname{reg}(V), \operatorname{reg}(V/M) + 1\} \leq \operatorname{reg}(V) + (t-1)c + 1.$$

"b)": As above we may assume that $R_0/\mathfrak{m}_0$ is infinite.

After an appropriate shift of U, V and M we may assume that $\operatorname{beg}(U) = 0$, so that $\operatorname{reg}(V)$ and d are non-negative. We set

$$A := \max\{d, \operatorname{reg}(V) + (t-1)c + 1\} + e(U)t^c.$$

Applying 3.7 with $\mathfrak{a} := R_+$ to the R-module $V/M \oplus V \oplus U$ we find a sequence $f_1, \cdots, f_s \in R_1$ which is filter-regular with respect to V/M, to V and to U.

Now, let $i \in \{1, \cdots, s\}$ and observe that $\dim(U) \geq \dim(V) \geq \dim(V/M) = s > 0$. We set

$$U^{(i)} := U/\Sigma_{j=1}^{i} f_j U, \ V^{(i)} := V/\Sigma_{j=1}^{i} f_j V \quad \text{and}$$
$$M^{(i)} := (M + \Sigma_{j=1}^{i} f_j V)/\Sigma_{j=1}^{i} f_j V \subseteq V^{(i)}.$$

Then, according to 4.1 B), C) the module $U^{(i)}$ is again Cohen-Macaulay and satisfies $\dim(U^{(i)}) = \dim(U) - i$ and $e(U^{(i)}) = e(U)$.

Moreover, $\dim(V^{(i)}/M^{(i)}) = \dim((V/M)/\sum_{j=1}^{i} f_j(V/M)) = s - i$, so that

$$(*_1) \qquad \dim(U^{(i)}) - \dim(V^{(i)}/M^{(i)}) = c.$$

In addition we have

$$(*_2) \qquad d(M^{(i)}) \leq d(M)$$

and 3.3 c) yields

$$(*_3) \qquad \operatorname{reg}(V^{(i)}) \leq \operatorname{reg}(V).$$

Assume first that $s = 1$. Then $\dim(V/(M+f_1V)) = \dim(V^{(1)}/M^{(1)}) = 0$, so that $V/(M+f_1V)$ is R_+-torsion. In particular

$$\begin{aligned} R_+ &\subseteq \sqrt{0 \underset{R}{:} (V/(M+f_1V))} \subseteq \sqrt{f_1R + (0 \underset{R}{:} V/M)} \\ &= \sqrt{f_1R + (M \underset{R}{:} V)} = \sqrt{f_1R + \mathfrak{b}} \subseteq \sqrt{\mathfrak{b} + (0 \underset{R}{:} V^{(1)})}. \end{aligned}$$

If we apply 3.8 to the epimorphism

$$V^{(1)} \twoheadrightarrow V^{(1)}/M^{(1)} \cong V/(M+f_1V)$$

with $\mathfrak{a} := \mathfrak{b}$ we thus get

$$\begin{aligned} \operatorname{reg}(V/(M+f_1V)) &= \operatorname{end}(V/(M+f_1V)) \\ &\leq \operatorname{reg}(V^{(1)}) + (t-1)\dim(V^{(1)}) \\ &\leq \operatorname{reg}(V) + (t-1)(\dim(V)-1) \\ &\leq \operatorname{reg}(V) + (t-1)(\dim(U)-1) \\ &= \operatorname{reg}(V) + (t-1)c. \end{aligned}$$

So, the short exact sequence $0 \to (M+f_1V) \to V \to V/(M+f_1V) \to 0$ yields

$$\operatorname{reg}(M+f_1V) \leq \max\{\operatorname{reg}(V), \operatorname{reg}(V)+(t-1)c+1\} = \operatorname{reg}(V)+(t-1)c+1.$$

Moreover, if we apply 4.6 to the epimorphism $U \twoheadrightarrow V/M$ we get the inequality $\operatorname{length}_{R_0}((M \underset{V}{:} f_1)/M) = \operatorname{length}_{R_0}(0 \underset{V/M}{:} f_1) \leq e(U)t^c$. By 5.2 we obtain $\operatorname{reg}(M) \leq A$.

So, let $s > 1$ and let $i \in \{1, \cdots, s-1\}$. Then clearly $(\mathfrak{b} + \sum_{j=1}^i f_jR)V^{(i)} \subseteq M^{(i)}$ and $(M^{(i)} \underset{R}{:} V^{(i)}) = (0 \underset{R}{:} (V^{(i)}/M^{(i)})) = 0 \underset{R}{:} (V/(M + \sum_{j=1}^i f_jV)) \subseteq \sqrt{(0 \underset{R}{:} V/M) + \sum_{j=1}^i f_jR} = \sqrt{\mathfrak{b} + \sum_{j=1}^i f_jR}$. Clearly we also have $\max(1, d(\mathfrak{b})) = \max(1, d(\mathfrak{b} + \sum_{j=1}^i f_jR))$. If we apply induction to the epimorphism $U^{(i)} \to V^{(i)}$, the submodule $M^{(i)} \subseteq V^{(i)}$ and the ideal $\mathfrak{b} + \sum_{j=1}^i f_jR$ (cf 2.4 E)) and keep in mind $(*_1), (*_2)$ and $(*_3)$, we get $\operatorname{reg}(M^{(i)}) \leq A^{2^{s-i-1}}$. The short exact sequence

$$0 \to M^{(i)} \to V^{(i)} \to (V/M)/\Sigma_{j=1}^i f_j(V/M) \to 0$$

and the inequalities $\operatorname{reg}(V^{(i)}) \leq \operatorname{reg}(V) < A \leq A^{2^{s-i-1}}$ yield

$$(*)\ \operatorname{reg}(V/(M+\Sigma_{j=1}^{i} f_j V)) = \operatorname{reg}((V/M)/\Sigma_{j=1}^{i} f_j(V/M)) \leq A^{2^{s-i-1}} - 1.$$

Applying this estimate for $i = 1$ we get $\operatorname{reg}(V/(M + f_1 V)) \leq A^{2^{s-2}} - 1$. Now, the short exact sequence $0 \to (M + f_1 V) \to V \to V/(M + f_1 V) \to 0$ and the inequality $\operatorname{reg}(V) < A$ imply

$$(**) \qquad \operatorname{reg}(M + f_1 V) \leq A^{2^{s-2}}.$$

If we apply 4.6 with $N := V/M$ and keep in mind that $\operatorname{beg}(N) \geq \operatorname{beg}(U) = 0$, we obtain from $(*)$

$$\begin{aligned}\operatorname{length}_{R_0}((M \underset{V}{:} f_1)/M) &= \operatorname{length}_{R_0}(0 \underset{V/M}{:} f_1) \leq e(U)t^c \cdot \Pi_{i=1}^{s-1} A^{2^{s-i-1}} \\ &= e(U)t^c A^{2^{s-1}-1}.\end{aligned}$$

As $d, \operatorname{reg}(V) + 1$ and $1 + e(U)t^c \leq A$ we now get from 5.2 and $(**)$

$$\begin{aligned}\operatorname{reg}(M) &\leq A^{2^{s-2}} + e(U)t^c A^{2^{s-1}-1} = A^{2^{s-2}}(1 + e(U)t^c A^{2^{s-2}-1}) \\ &\leq A^{2^{s-2}}(1 + e(U)t^c)A^{2^{s-2}-1} \leq A^{2^{s-2}} A A^{2^{s-2}-1} = A^{2^{s-1}}.\end{aligned}$$

"c)": As above, we may assume that $\operatorname{beg}(U) = 0$. Consequently $\varrho := \operatorname{reg}(V) \geq d(V) \geq \operatorname{beg}(V) \geq 0$. Keep in mind that t and $e := e(U)$ are both > 0. We set $B := \max\{d, \varrho + t\} + et$ and aim to show that $\operatorname{reg}(M) \leq B^{2^{r-2}}$.

Assume first, that $s = 0$. Then, by statement a) we have $\operatorname{reg}(M) \leq \varrho + (t-1)r + 1$. If $r = 2$ we thus get $\operatorname{reg}(M) < \varrho + 2t \leq \varrho + t + et \leq B$. If $r > 2$, we may write $\operatorname{reg}(M) < (\varrho + t) + (r-1)t \leq ((\varrho + t) + et)^{2^{r-2}} \leq B^{2^{r-2}}$.

Now, let $s > 0$, so that $1 \leq c = r - s \leq r - 1$. We set

$$A(c) := \max\{d, \varrho + (t-1)c + 1\} + et^c.$$

According to statement b) it suffices to show that $[A(c)]^{2^{r-c-1}} \leq B^{2^{r-2}}$. If $c = 1$, this is immediate. So, let $2 \leq c \leq r - 1$. Then, the inequalities

$$\begin{aligned}[A(c)]^{2^{r-c-1}} &\leq [t \cdot \max\{d, \varrho + (t-1)(c-1) + 1\} + et^c]^{2^{r-c-1}} \\ &\leq [(A(c-1))^2]^{2^{r-c-1}} = [A(c-1)]^{2^{r-(c-1)-1}}\end{aligned}$$

allow one to conclude by induction. □

5.4. Corollary. *Assume that* $(R_0, \mathfrak{m}_0)$ *is local with* $\dim(R_0) = 0$ *and that* R *is a Cohen-Macaulay ring. Let* V *be a graded* R*-module which is generated by* $\mu(< \infty)$ *homogeneous elements. Let* $M \subsetneqq V$ *be a graded submodule, let* $\mathfrak{b} \subseteq R$ *be a graded ideal such that* $\mathfrak{b}V \subseteq M$ *and* $(M \underset{R}{:} V) \subseteq \sqrt{\mathfrak{b}}$*, let* $d \geq d(M)$ *and* $t \geq \max\{1, d(\mathfrak{b})\}$*. Finally set* $s := \dim(V/M)$ *and* $c := \dim(R) - s$*.*

a) *If* $s = 0$*, then:* $\operatorname{reg}(M) \leq \operatorname{reg}(V) + (t-1)c + 1$*.*

b) *If* $s > 0$*, then:*

$$\operatorname{reg}(M) \leq [\max\{d, \mathit{reg}(V)+(t-1)c+1\}+\mu e(R)t^c-\operatorname{beg}(V)]^{2^{s-1}}+\operatorname{beg}(V).$$

c) *If* $r := \dim(R) > 1$ *and* $s < r$*, then:*

$$\operatorname{reg}(M) \leq [\max\{d, \mathit{reg}(V) + t\} + \mu e(R)t - \operatorname{beg}(V)]^{2^{r-2}} + \operatorname{beg}(V).$$

Proof. There are integers $a_1 \leq a_2 \leq \cdots \leq a_\mu$ such that $\operatorname{beg}(V) = a_1$ and such that there is an epimorphism of graded R-modules $U := \oplus_{i=1}^{\mu} R(-a_i) \twoheadrightarrow V$. Applying 5.3 to this epimorphism, we get our claim. □

5.5. Corollary. *Assume that* $(R_0, \mathfrak{m}_0)$ *is local with* $\dim(R_0) = 0$ *and that* R *is a Cohen-Macaulay ring of dimension* $r > 0$*. Let* F *be a graded free* R*-module of finite rank* μ*. Let* $M \subsetneqq F$ *be a graded submodule, set* $d := d(M), t := \max\{1, d(M \underset{R}{:} F)\}$ *and* $c := \mathit{height}(M \underset{R}{:} F)$*.*

a) *If* $c = r$*, then:* $\operatorname{reg}(M) \leq d(F) + \operatorname{reg}(R) + (t-1)r + 1$*.*

b) *If* $c < r$*, then:* $\operatorname{reg}(M) \leq$

$$[\max\{d, d(F)+\operatorname{reg}(R)+(t-1)c+1\}+\mu e(R)t^c-\operatorname{beg}(F)]^{2^{\dim(R)-c-1}}+\operatorname{beg}(F).$$

c) *If* $r > 1$ *and* $c > 0$*, then:*

$$\operatorname{reg}(M) \leq [\max\{d, d(F)+\operatorname{reg}(R)+t\}+\mu e(R)t-\operatorname{beg}(F)]^{2^{r-2}}+\operatorname{beg}(F).$$

Proof. Apply 5.4 with $V := F$ and observe that $\operatorname{reg}(F) = d(F) + \operatorname{reg}(R)$. □

5.6. Corollary. *Assume that* $(R_0, \mathfrak{m}_0)$ *is local with* $\dim(R_0) = 0$ *and that* R *is a Cohen-Macaulay ring of dimension* $r > 0$. *Let* $\mathfrak{a} \subsetneqq R$ *be a graded ideal, let* $d := \max\{1, d(\mathfrak{a})\}$ *and* $c := \mathrm{height}(\mathfrak{a})$.

a) *If* $c = r$, *then* $\mathrm{reg}(\mathfrak{a}) \leq \mathrm{reg}(R) + (d-1)r + 1$.

b) *If* $c < r$, *then* $\mathrm{reg}(\mathfrak{a}) \leq [\max\{d, \mathrm{reg}(R) + (d-1)c + 1\} + e(R)d^c]^{2^{r-c-1}}$.

c) *If* $0 < c < r$, *then* $\mathrm{reg}(\mathfrak{a}) \leq [\mathrm{reg}(R) + d(\mathfrak{a})(1 + e(R))]^{2^{r-2}}$.

Proof. Apply 5.5 with $F := R$ and $M := \mathfrak{a}$ and observe that $d(\mathfrak{a}) > 0$ if $c > 0$. □

5.7. Corollary. *Let* $r > 1$, *let* $(R_0, \mathfrak{m}_0)$ *be local with* $\dim(R_0) = 0$ *and let* $\mathfrak{a}$ *be a proper graded ideal of the polynomial ring* $R_0[x_1, \cdots, x_r]$.

a) *If* $\mathrm{height}(\mathfrak{a}) = 0$, *then* $\mathrm{reg}(\mathfrak{a}) \leq [\max\{d(\mathfrak{a}), 1\} + \mathrm{length}(R_0)]^{2^{r-1}}$.

b) *If* $\mathrm{height}(\mathfrak{a}) > 0$, *then* $\mathrm{reg}(\mathfrak{a}) \leq [d(\mathfrak{a})(1 + \mathrm{length}(R_0))]^{2^{r-2}}$.

Proof. Apply 5.6 with $R := R_0[x_1, \cdots, x_r]$ and observe that $\mathrm{reg}(R) = 0$ and $e(R) = \mathrm{length}(R_0)$. □

5.8. Corollary. (cf [**7**]) *Let* $r > 1$, *let* K *be a field and let* $\mathfrak{a}$ *be a proper graded ideal of the polynomial ring* $K[x_1, \cdots, x_r]$.

Then $\mathrm{reg}(\mathfrak{a}) < (2d(\mathfrak{a}))^{2^{r-2}}$.

Proof: Apply 5.7 and observe that $\mathfrak{a} \neq 0$ *implies height*$(\mathfrak{a}) > 0$. □

6. Free Presentations and Regularity. In this section we give a bound on the regularity of a finitely generated graded module M in terms of a free presentation of M (cf 6.3 - 6.5). We briefly discuss our results in the context of the "problem of the finitely many steps".

We keep the previous notations and hypotheses. As a further application of Theorem 5.3 we have

6.1. Proposition. *Assume that* $(R_0, \mathfrak{m}_0)$ *is local with* $\dim(R_0) = 0$ *and that* R *is a Cohen-Macaulay ring of dimension* $r > 0$. *Let* V *be a graded* R*-module which is generated by* $\mu(< \infty)$ *homogeneoous elements. Let* $\alpha := \min\{\mathrm{beg}(V), \mathrm{reg}(V) - \mathrm{reg}(R)\}$. *Let* $M \subseteq V$ *be a graded submodule. Then*

$$\mathrm{reg}(M) \leq [\max\{d(M), \mathrm{reg}(V) + 1\} + (\mu + 1)e(R) - \alpha]^{2^{r-1}} + \alpha.$$

Proof. Consider M as a graded submodule of $W := V \oplus R(-\alpha)$ and observe that W is generated by $\mu + 1$ homogeneous elements, that $\mathrm{beg}(W) = \alpha, \mathrm{reg}(W) = \mathrm{reg}(V)$, $(M \underset{R}{:} W) = 0$ and $\dim(W/M) = r$. Then, apply 5.4 b) with $\mathfrak{b} = 0$. □

6.2. Corollary. *Let* R, V, r, α *and* μ *be as in 6.1 and let* $f : W \to V$ *be a homomorphism of graded* R*-modules such that* W *is finitely generated. Then:*

$$\mathrm{reg}(\mathrm{Im}(f)) \leq [\max\{d(W), \mathrm{reg}(V) + 1\} + (\mu + 1)e(R) - \alpha]^{2^{r-1}} + \alpha.$$

Proof. Observe that $d(\mathrm{Im}(f)) \leq d(W)$ and apply 6.1. □

6.3. Theorem. *Let* $(R_0, \mathfrak{m}_0)$ *be local with* $\dim(R_0) = 0$ *and assume that* R *is a Cohen-Macaulay ring of dimension* $r > 0$. *Let*

$$\oplus_{j=1}^{\nu} R(-b_j) \xrightarrow{h} \oplus_{i=1}^{\mu} R(-a_i) \to M \to 0$$

be an exact sequence of graded R*-modules, with integers* $b_1 \leq b_2 \leq \cdots \leq b_\nu$ *and* $a_1 \leq a_2 \leq \cdots \leq a_\mu$. *Let* $\mu^* := \sup\{i \in \{1, \cdots, \mu\} \big| a_i \leq b_\nu\}$. *Then*

$$\mathrm{reg}(M) \leq \max\{a_\mu + \mathrm{reg}(R), [b_\nu + \mathrm{reg}(R) + 1 + (\mu^* + 1)e(R) - a_1]^{2^{r-1}} + a_1 - 1\}.$$

Proof. If $a_1 > b_\nu$, the map h vanishes and $\mu^* = -\infty$. Hence

$$\mathrm{reg}(M) = \mathrm{reg}(\oplus_{i=1}^{\mu} R(-a_i)) = a_\mu + \mathrm{reg}(R).$$

So, let $a_1 \leq b_\nu$. Then μ^* is a positive integer. We set

$$W := \Sigma_{j=1}^{\nu} R(-b_j),\ V := \Sigma_{i=1}^{\mu^*} R(-a_i), F := \Sigma_{i=1}^{\mu} R(-a_i).$$

Clearly the map h factors through the submodule V of F. So, if we apply 6.2 to the induced homomorphism $h : W \to V$ and observe that $d(W) = b_\nu$, $\mathrm{reg}(V) = a_{\mu^*} + \mathrm{reg}(R) \leq b_\nu + \mathrm{reg}(R)$ and $\alpha = \min\{\mathrm{beg}(V), \mathrm{reg}(V) - \mathrm{reg}(R)\} = \min\{a_1, a_{\mu^*} + \mathrm{reg}(R) - \mathrm{reg}(R)\} = a_1$, we get

$$\mathrm{reg}(\mathrm{Im}(h)) \leq [b_\nu + \mathrm{reg}(R) + 1 + (\mu^* + 1)e(R) - a_1]^{2^{r-1}} + a_1.$$

As $\mathrm{reg}(F) = a_\mu + \mathrm{reg}(R)$ we now may conclude by the exact sequence of graded R-modules

$$0 \to \mathrm{Im}(h) \to F \to M \to 0. \qquad \square$$

6.4. Corollary. *Let* $(R_0, \mathfrak{m}_0)$ *be local with* $\dim(R_0) = 0$*, let* $r > 0$ *and let* $R_0[\underline{x}] = R_0[x_1, \cdots, x_r]$ *be a polynomial ring. Let*

$$\oplus_{j=1}^{\nu} R_0[\underline{x}](-b_j) \to \oplus_{i=1}^{\mu} R_0[\underline{x}](-a_i) \to M \to 0$$

be an exact sequence of graded $R_0[\underline{x}]$*-modules, with integers* $b_1 \leq b_2 \leq \cdots \leq b_\nu, a_1 \leq b_\nu$ *and* $a_1 \leq a_2 \leq \cdots \leq a_\mu$*. Let* $\mu^* := \sup\{i \in \{1, \cdots, \mu\} \big| a_i \leq b_\nu\}$*. Then*

$$\mathrm{reg}(M) \leq \max\{a_\mu, [b_\nu + 1 + (\mu^* + 1)\mathrm{length}(R_0) - a_1]^{2^{r-1}} + a_1 - 1\}.$$

Proof. Observe that $\mathrm{reg}(R_0[\underline{x}]) = 0$ and that $R_0[\underline{x}]$ is a Cohen-Macaulay ring of dimension r with $e(R_0[\underline{x}]) = \mathrm{length}(R_0)$. Then apply 6.3. $\square$

6.5. Corollary. *Let* $r > 0$ *and let* K *be a field and let* $K[\underline{x}] = K[x_1, \cdots, x_r]$ *be a polynomial ring. Let*

$$\oplus_{j=1}^{\nu} K[\underline{x}](-b_j) \to \oplus_{i=1}^{\mu} K[\underline{x}](-a_i) \to M \to 0$$

be an exact sequence of graded $K[\underline{x}]$-modules, with integers $b_1 \leq b_2 \leq \cdots \leq b_\nu, a_1 \leq b_\nu$ and $a_1 \leq a_2 \leq \cdots \leq a_\mu$. Let $\mu^* := \sup\{i \in \{1, \cdots, \mu\} \big| a_i \leq b_\nu\}$. Then

$$\operatorname{reg}(M) \leq \max\{a_\mu, [b_\nu + \mu^* + 2 - a_1]^{2^{r-1}} + a_1 - 1\}.$$

Proof. Apply 6.4. ◻

6.6. *Remark.* A) Let $r > 1$ and let $K[\underline{x}] = K[x_1, \cdots, x_r]$ be a polynomial ring over the field K, let M be a finitely generated graded $K[\underline{x}]$-module with a minimal graded free presentation

$$(*) \qquad \oplus_{j=1}^{\nu} K[\underline{x}](-b_j) \xrightarrow{h} \oplus_{i=1}^{\mu} K[\underline{x}](-a_i) \to M \to 0$$

with integers $b_1 \leq b_2 \leq \cdots \leq b_\nu$ and $a_1 \leq a_2 \leq \cdots \leq a_\mu$. The classical "*problem of the finitely many steps*" is the question whether "*the discrete data $(b_1, b_2, \cdots, b_\nu; a_1, \cdots, a_\mu)$ of the presentation $(*)$ already bound the computational complexity of the complete minimal free resolution of M*".

To make this more precise, let

$$(**) \quad 0 \to \oplus_{j=1}^{\varrho_p} K[\underline{x}](-\beta_{pj}) \to \cdots \to \oplus_{j=1}^{\varrho_i} K[\underline{x}](-\beta_{ij}) \to \cdots \to M \to 0$$

be a minimal graded free resolution of M with $p := \operatorname{proj}\dim(M) (\leq r)$ and integers $\beta_{i1} \leq \cdots \leq \beta_{i\varrho_i}$ for $i = 0, \cdots, p$. Then $\varrho_0 = \mu, \varrho_1 = \nu, \beta_{0j} = a_j$ for $j = 1, \cdots, \mu, \beta_{1j} = b_j$ for $j = 1, \cdots, \nu$ and $\beta_{01} < \beta_{11} < \cdots < \beta_{p1}$. Moreover

$$\operatorname{reg}(M) = \max\{\beta_{i\varrho_i} - i \big| i = 0, \cdots, p\}$$

and

$$\varrho_i \leq \varrho_{i-1} \binom{r + \beta_{i\varrho_i} - \beta_{(i-1)1}}{r} \qquad \text{for } i = 1, \cdots, p.$$

Therefore, $\operatorname{reg}(M)$ and a_1 give an upper bound on the numbers $\beta_{i\varrho_i}$ and ϱ_i, and hence on the size of the systems of K-linear equations which successively express the condition that the sequence $(**)$ is exact. Thus $\operatorname{reg}(M)$ (together with the beginning a_1 of M) bounds the computational complexity of the minimal free resolution $(**)$ of M.

Consequently, the problem of the finitely many steps may be asked in the form: "*Is there an upper bound on* $\operatorname{reg}(M)$ *in terms of the degrees* $b_1, \cdots, b_\nu, a_1, \cdots, a_\mu$?"

Basically, this question was already answered affirmatively by K. Henzelt - E. Noether [**12**] and G. Hermann [**13**] – as a consequence of a bound on the generating degree of the kernel of a matrix with entries in $K[\underline{x}]$.

An iteration of this bound (corrected according to [**14**], for example) yields the following "*squarely-exponential*" estimate (cf [**11**, 4.31]):

$$(***) \qquad \operatorname{reg}(M) \leq \max\{a_\mu, [2\mu(b_\nu - a_1 + 1)]^{2^{r^2-2}} + a_1 - 1\},$$

which is weaker than the "linear-exponential" bound in 6.5.

B) Keep the notations and hypotheses of part A) and let μ^* be as in 6.5. Applying the extended Bayer-Mumford bound [**3**, Corollary 5.8] to $\operatorname{Im}(h) \subseteq \oplus_{i=1}^{\mu^*} K[\underline{x}](-a_i) \subseteq K[\underline{x}] \oplus \mu^*(-a_1)$ and observing the short exact sequence $0 \to \operatorname{Im}(h) \to \oplus_{i=1}^{\mu} K[\underline{x}](-a_i) \to M \to 0$, one obtains:

$$\operatorname{reg}(M) \leq \max\{a_\mu, (\mu^*)^{\lfloor (r-1)!(e-1)\rfloor} (2(b_\nu - a_1))^{(r-1)!} + a_1 - 1\}.$$

This is sharper than the bound $(***)$ but still weaker than the estimate of 6.5.

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Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Schwitzerland
Email address: brodmann@math.uzh.ch

Zurich Global Corporate, Corporate, Captive Related Services, Austrasse 46, CH-8054 Zürich, Switzerland
Email address: thomas.goetsch@zurich.com