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Bounds for the general sum-connectivity index of composite graphs

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Abstract

The general sum-connectivity index is a molecular descriptor defined as $\chi_\alpha(X) = \sum_{xy \in E(X)} (d_X(x) + d_X(y))^\alpha$, where $d_X(x)$ denotes the degree of a vertex $x \in X$, and α is a real number. Let X be a graph; then let $R(X)$ be the graph obtained from X by adding a new vertex x_e corresponding to each edge of X and joining x_e to the end vertices of the corresponding edge $e \in E(X)$. In this paper we obtain the lower and upper bounds for the general sum-connectivity index of four types of graph operations involving R -graph. Additionally, we determine the bounds for the general sum-connectivity index of line graph $L(X)$ and rooted product of graphs.

MSC: 05C07

Keywords: R -graphs; corona product; line graph; rooted product

1 Introduction

Topological indices are useful tools for theoretical chemistry. A structural formula of a chemical compound is represented by a molecular graph. The atoms of the compounds and chemical bonds represent the vertices and edges of the molecular graphs, respectively. Topological indices related to their use in quantitative structure-activity (QSAR) and structure-property (QSPR) relationships are very interesting. In the QSAR/QSPR study, physico-chemical properties and topological indices such as the Wiener index, the Szeged index, the Randić index, the Zagreb indices and the ABC index are used to predict the bioactivity of the chemical compounds.

A single number that characterizes some properties corresponding to a molecular graph represents a topological index. There are many classes of topological indices, some of them are distance-based topological indices, degree-based topological indices and counting related polynomials and indices of graphs. All topological indices are useful in different fields, but degree-based topological indices play an important role in chemical graph theory and particularly in theoretical chemistry.

In this paper, we consider simple, connected and finite graphs. Let X be a graph with vertex set $V(X)$ and edge set $E(X)$. For $x \in V(X)$, $N_X(x)$ denotes the set of neighbors of x . The degree of a vertex $x \in V(X)$ is the number of vertices adjacent to x and represented by $d_X(x) = |N_X(x)|$. The numbers of vertices and number of edges in the graph X are represented by n_X and m_X , respectively. The maximum and minimum vertex degree of X are denoted by Δ_X and δ_X , respectively.

In the chemical and mathematical literature, several dozens of vertex-based graph invariants have been considered, in hundreds of published papers. For details see the books [1–3], and the surveys [4, 5].

The two oldest degree based molecular descriptors, called Zagreb indices [6], are defined as

$$M_1(X) = \sum_{x \in V(X)} (d_X(x))^2, \quad M_2(X) = \sum_{xy \in E(X)} d_X(x)d_X(y).$$

The general Randić index of X was proposed by Bollobás and Erdős [7], denoted by $R_\alpha(X)$ and defined as follows:

$$R_\alpha(X) = \sum_{xy \in E(X)} (d(x)d(y))^\alpha,$$

where α is a real number. Then $R_{-\frac{1}{2}}$ is the classical Randić index proposed by Randić [8] in 1975. Recently, a closely related topological index to the Randić index, called the sum-connectivity index [9], denoted by $\chi(X)$, is as follows:

$$\chi(X) = \sum_{xy \in E(X)} (d_X(x) + d_X(y))^{-\frac{1}{2}}.$$

Zhou and Trinajstić [10] introduced the general sum-connectivity index, denoted by $\chi_\alpha(X)$ and defined as follows:

$$\chi_\alpha(X) = \sum_{xy \in E(X)} (d_X(x) + d_X(y))^\alpha, \quad (1.1)$$

where α is a real number. Then $\chi_{-\frac{1}{2}}$ is the sum-connectivity index. Su and Xu [11] introduced a new topological index, the general sum-connectivity co-index, denoted by $\overline{\chi}_\alpha$ and is defined as follows:

$$\overline{\chi}_\alpha(X) = \sum_{xy \notin E(X)} (d_X(x) + d_X(y))^\alpha,$$

where α is a real number. Researchers introduced many graph operations such as the cartesian product, join of graphs, line graphs, the corona product, the edge corona product, the subdivision-vertex join, the subdivision edge join, the neighborhood corona, the subdivision vertex neighborhood corona and the subdivision edge neighborhood corona. Much work has been done related to these graph operations. Lan and Zhou [12] defined four new graph operations based on R -graphs and determined the adjacency (respectively, Laplacian and singles Laplacian) spectra of these graph operations. Godsil and McKay [13] introduced a new graph operation (rooted product) and then found its spectrum. Recently, Azari *et al.* [14] determined the Zagreb indices of chemical graphs that are constructed by a rooted product. Khalifeh *et al.* [15] gave the exact formulas for the Zagreb indices of several graph operations. For a detailed study of the topological indices of graph operations, we refer to [16–27].

2 Methods

In this paper, we obtain the lower and upper bounds for the general sum-connectivity index of four types of graph operations involving the R -graph. Additionally, we determine the bounds for the general sum-connectivity index of line graph $L(X)$ and the rooted product of graphs by using graph-theoretic tools and mathematical inequalities.

3 Results and discussion

In this section, we derive some bounds on the general sum-connectivity index of several graph operations such as R -graphs, line graphs and the rooted product. Let X and Y be two simple connected graphs whose vertex sets are disjoint. For each $x \in V(X)$ and $y \in V(Y)$, we have

$$\begin{aligned} \Delta_X &\geq d_X(x), & \delta_X &\leq d_X(x), \\ \Delta_Y &\geq d_Y(y), & \delta_Y &\leq d_Y(y). \end{aligned} \tag{3.1}$$

The equality holds if and only if X and Y are regular graphs.

3.1 R -graphs

Let X be a graph; then $R(X)$ is the graph obtained from X by adding a new vertex x_e corresponding to each edge of X and joining x_e to the end vertices of the corresponding edge $e \in E(X)$. Let $V(R(X)) = V(X) \cup I(X)$, where $I(X) = V(R(X)) \setminus V(X)$ is the set of newly added vertices.

The corona product of two graphs X and Y , denoted by $X \bullet Y$, is a graph obtained by taking one copy of graph X and n_X copies of graph Y and joining the vertex X , that is, on the i th position in X to every vertex in i th copy of Y . The order and size of $X \bullet Y$ are $n_X(1 + n_Y)$ and $m_X + n_X m_Y + n_X n_Y$, respectively. The degree of a vertex $x \in V(X \bullet Y)$ is given by

$$d_{X \bullet Y}(x) = \begin{cases} d_X(x) + n_Y & \text{if } x \in V(X), \\ d_Y(x) + 1 & \text{if } x \in V(Y). \end{cases} \tag{3.2}$$

Let X and Y be two connected and vertex-disjoint graphs. The R -vertex corona product of $R(X)$ and Y , denoted by $R(X) \odot Y$, is a graph obtained from one copy of vertex-disjoint graph $R(X)$ and n_X copies of Y and joining a vertex of $V(X)$, that is, on the i th position in $R(X)$ to every vertex in the i th copy of Y . The graph $R(X) \odot Y$ has a number of $n_X + m_X + n_X n_Y$ vertices and $3m_X + n_X m_Y + n_X n_Y$ edges. The degree of a vertex $x \in V(R(X) \odot Y)$ is given by

$$d_{R(X) \odot Y}(x) = \begin{cases} 2d_X(x) + n_Y & \text{if } x \in V(X), \\ 2 & \text{if } x \in I(X), \\ d_Y(x) + 1 & \text{if } x \in V(Y). \end{cases} \tag{3.3}$$

The R -edge corona product of $R(X)$ and Y , denoted by $R(X) \ominus Y$, is a graph obtained from one copy of vertex-disjoint graph $R(X)$ and m_X copies of Y and joining a vertex of $I(X)$, that is, on i th position in $R(X)$ to every vertex in the i th copy of Y . The graph $R(X) \ominus Y$

has $n_X + m_X + m_X n_Y$ number of vertices and $3m_X + m_X m_Y + m_X n_Y$ number of edges. The degree of a vertex $x \in V(R(X) \ominus Y)$ is given by

$$d_{R(X) \ominus Y}(x) = \begin{cases} 2d_X(x) & \text{if } x \in V(X), \\ 2 + n_Y & \text{if } x \in I(X), \\ d_Y(x) + 1 & \text{if } x \in V(Y). \end{cases} \tag{3.4}$$

The R -vertex neighborhood corona product of $R(X)$ and Y , denoted by $R(X) \square Y$, is a graph obtained from one copy of vertex-disjoint graph $R(X)$ and n_X copies of Y and joining the neighbors of a vertex of X in $R(X)$, that is, on the i th position in $R(X)$ to every vertex in the i th copy of Y . The graph $R(X) \square Y$ has $n_X + m_X + n_X n_Y$ vertices and $3m_X + n_X m_Y + 4m_X n_Y$ edges. The degrees of vertices of $R(X) \square Y$ are given by

$$\begin{aligned} d_{R(X) \square Y}(x) &= d_X(x)(2 + n_Y) \quad \text{if } x \in V(X), \\ d_{R(X) \square Y}(x) &= 2(n_Y + 1) \quad \text{if } x \in I(X), \\ d_{R(X) \square Y}(y) &= d_Y(y) + 2d_X(x) \quad \text{if } y \in V(Y), x \in V(X). \end{aligned} \tag{3.5}$$

In the last expression, $y \in V(Y)$ is the vertex in i th copy of Y corresponding to i th vertex $x \in V(X)$ in $R(X)$.

The R -edge neighborhood corona product of $R(X)$ and Y , denoted by $R(X) \boxplus Y$, is a graph obtained from one copy of vertex-disjoint graph $R(X)$ and m_X copies of Y and joining the neighbors of a vertex of $I(X)$ in $R(X)$, that is, on the i th position in $R(X)$ to every vertex in the i th copy of Y . The graph $R(X) \boxplus Y$ has $n_X + m_X + m_X n_Y$ vertices and $3m_X + m_X m_Y + 2m_X n_Y$ edges. The degree of a vertex $x \in V(R(X) \boxplus Y)$ is given by

$$d_{R(X) \boxplus Y}(x) = \begin{cases} d_X(x)(2 + n_Y) & \text{if } x \in V(X), \\ 2 & \text{if } x \in I(X), \\ d_Y(x) + 2 & \text{if } x \in V(Y). \end{cases} \tag{3.6}$$

In the following theorem, we compute the bounds on the general sum-connectivity index of R -vertex corona product of $R(X)$ and Y .

Theorem 3.1 *Let $\alpha < 0$. Then the bounds for the general sum-connectivity index of $R(X) \odot Y$ are given by*

$$\begin{aligned} \chi_\alpha(R(X) \odot Y) &\geq 2^\alpha m_X (2 \Delta_X + n_Y)^\alpha + 2m_X (2 \Delta_X + n_Y + 2)^\alpha + 2^\alpha n_X m_Y (\Delta_Y + 1)^\alpha \\ &\quad + n_X n_Y (2 \Delta_X + \Delta_Y + n_Y + 1)^\alpha, \\ \chi_\alpha(R(X) \odot Y) &\leq 2^\alpha m_X (2 \delta_X + n_Y)^\alpha + 2m_X (2 \delta_X + n_Y + 2)^\alpha + 2^\alpha n_X m_Y (\delta_Y + 1)^\alpha \\ &\quad + n_X n_Y (2 \delta_X + \delta_Y + n_Y + 1)^\alpha. \end{aligned}$$

The equality holds if and only if X and Y are regular graphs.

Proof Using (3.1) and (3.3) in equation (1.1), we get

$$\begin{aligned}
 \chi_\alpha(R(X) \odot Y) &= \sum_{xy \in E(R(X))} (d_{R(X)}(x) + d_{R(X)}(y))^\alpha + n_X \sum_{xy \in E(Y)} (d_Y(x) + d_Y(y))^\alpha \\
 &\quad + \sum_{x \in V(R(X))} \sum_{y \in V(Y)} (d_{R(X)}(x) + d_Y(y))^\alpha \\
 &= \sum_{\substack{xy \in E(R(X)), \\ x, y \in V(X)}} (2d_X(x) + 2d_Y(y) + 2n_Y)^\alpha + \sum_{\substack{xy \in E(R(X)), \\ x \in V(X), y \in I(X)}} (2d_X(x) + n_Y + 2)^\alpha \\
 &\quad + n_X \sum_{xy \in E(Y)} (d_Y(x) + d_Y(y) + 2)^\alpha + \sum_{\substack{x \in V(R(X)) \\ x \in V(X)}} \sum_{y \in V(Y)} (2d_X(x) + n_Y + d_Y(y) + 1)^\alpha \\
 &\geq 2^\alpha m_X (2\Delta_X + n_Y)^\alpha + 2m_X (2\Delta_X + n_Y + 2)^\alpha + 2^\alpha n_X m_Y (\Delta_Y + 1)^\alpha \\
 &\quad + n_X n_Y (2\Delta_X + \Delta_Y + n_Y + 1)^\alpha. \tag{3.7}
 \end{aligned}$$

One can analogously compute the following:

$$\begin{aligned}
 \chi_\alpha(R(X) \odot Y) &\leq 2^\alpha m_X (2\delta_X + n_Y)^\alpha + 2m_X (2\delta_X + n_Y + 2)^\alpha + 2^\alpha n_X m_Y (\delta_Y + 1)^\alpha \\
 &\quad + n_X n_Y (2\delta_X + \delta_Y + n_Y + 1)^\alpha. \tag{3.8}
 \end{aligned}$$

The equality in (3.7) and (3.8) obviously holds if and only if X and Y are regular graphs. This completes the proof. □

We compute the bounds on the general sum-connectivity index for the R -edge corona product of $R(X)$ and Y in the following theorem.

Theorem 3.2 *Let $\alpha < 0$. Then the bounds for the general sum-connectivity index of $R(X) \ominus Y$ are given by*

$$\begin{aligned}
 \chi_\alpha(R(X) \ominus Y) &\geq 4^\alpha m_X \Delta_X^\alpha + 2m_X (2\Delta_X + n_Y + 2)^\alpha + 2^\alpha m_X m_Y (\Delta_Y + 1)^\alpha \\
 &\quad + m_X n_Y (\Delta_Y + n_Y + 3)^\alpha, \\
 \chi_\alpha(R(X) \ominus Y) &\leq 4^\alpha m_X \delta_X^\alpha + 2m_X (2\delta_X + n_Y + 2)^\alpha + 2^\alpha m_X m_Y (\delta_Y + 1)^\alpha \\
 &\quad + m_X n_Y (\delta_Y + n_Y + 3)^\alpha.
 \end{aligned}$$

The equality holds if and only if X and Y are regular graphs.

Proof Using (3.1) and (3.4) in equation (1.1), we get

$$\begin{aligned}
 \chi_\alpha(R(X) \ominus Y) &= \sum_{xy \in E(R(X))} (d_{R(X)}(x) + d_{R(X)}(y))^\alpha + m_X \sum_{xy \in E(Y)} (d_Y(x) + d_Y(y))^\alpha \\
 &\quad + \sum_{x \in V(R(X))} \sum_{y \in V(Y)} (d_{R(X)}(x) + d_Y(y))^\alpha
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{xy \in E(R(X)), \\ x, y \in V(X)}} (2d_X(x) + 2d_Y(y))^\alpha + \sum_{\substack{xy \in E(R(X)), \\ x \in V(X), y \in I(X)}} (2d_X(x) + n_Y + 2)^\alpha \\
 &\quad + m_X \sum_{xy \in E(Y)} (d_Y(x) + d_Y(y) + 2)^\alpha + \sum_{\substack{x \in V(R(X)), y \in V(Y) \\ x \in I(X)}} (n_Y + 2 + d_Y(y) + 1)^\alpha \\
 &\geq 4^\alpha m_X \Delta_X^\alpha + 2m_X(2\Delta_X + n_Y + 2)^\alpha + 2^\alpha m_X m_Y (\Delta_Y + 1)^\alpha \\
 &\quad + m_X n_Y (\Delta_Y + n_Y + 3)^\alpha. \tag{3.9}
 \end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
 \chi_\alpha(R(X) \ominus Y) &\leq 4^\alpha m_X \delta_X^\alpha + 2m_X(2\delta_X + n_Y + 2)^\alpha + 2^\alpha m_X m_Y (\delta_Y + 1)^\alpha \\
 &\quad + m_X n_Y (\delta_Y + n_Y + 3)^\alpha. \tag{3.10}
 \end{aligned}$$

The equality in (3.9) and (3.10) obviously holds if and only if X and Y are regular graphs. This completes the proof. \square

In the following theorem, we calculate bounds on the general sum-connectivity index for the R -vertex neighborhood corona product of $R(X)$ and Y .

Theorem 3.3 *Let $\alpha < 0$. Then the bounds for the general sum-connectivity index of $R(X) \square Y$ are given by*

$$\begin{aligned}
 \chi_\alpha(R(X) \square Y) &\geq 2^\alpha m_X (n_Y + 2)^\alpha \Delta_X^\alpha + 2m_X (n_Y (\Delta_X + 2) + 2(\Delta_Y + 1))^\alpha \\
 &\quad + 2^\alpha n_X m_Y (\Delta_Y + 2\Delta_X)^\alpha \\
 &\quad + n_Y \sum_{\substack{x \in V(X), \\ w_i \in N_X(x), w_i \in V(X)}} (\Delta_X (n_Y + 2) + \Delta_Y + 2\Delta_X)^\alpha \\
 &\quad + n_Y \sum_{\substack{x \in V(X), \\ w_i \in N_X(x), w_i \in I(X)}} (2(n_Y + 1) + \Delta_Y + 2\Delta_X)^\alpha, \\
 \chi_\alpha(R(X) \square Y) &\leq 2^\alpha m_X (n_Y + 2)^\alpha \delta_X^\alpha + 2m_X (n_Y (\delta_X + 2) + 2(\delta_Y + 1))^\alpha \\
 &\quad + 2^\alpha n_X m_Y (\delta_Y + 2\delta_X)^\alpha \\
 &\quad + n_Y \sum_{\substack{x \in V(X), \\ w_i \in N_X(x), w_i \in V(X)}} (\delta_X (n_Y + 2) + \delta_Y + 2\delta_X)^\alpha \\
 &\quad + n_Y \sum_{\substack{x \in V(X), \\ w_i \in N_X(x), w_i \in I(X)}} (2(n_Y + 1) + \delta_Y + 2\delta_X)^\alpha.
 \end{aligned}$$

The equality holds if and only if X and Y are regular graphs.

Proof Using (3.1) and (3.5) in equation (1.1), we get

$$\begin{aligned}
 \chi_\alpha(R(X) \square Y) &= \sum_{xy \in E(R(X))} (d_{R(X)}(x) + d_{R(X)}(y))^\alpha + n_X \sum_{xy \in E(Y)} (d_Y(x) + d_Y(y))^\alpha \\
 &\quad + \sum_{x \in V(R(X))} \sum_{y \in V(Y)} (d_{R(X)}(x) + d_Y(y))^\alpha
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{xy \in E(R(X)), \\ x, y \in V(X)}} (d_X(x)(n_Y + 2) + d_Y(y)(n_Y + 2))^\alpha \\
 &+ \sum_{\substack{xy \in E(R(X)), \\ x \in V(X), y \in I(X)}} (d_X(x)(n_Y + 2) + 2(n_Y + 1))^\alpha \\
 &+ n_X \sum_{xy \in E(Y)} (d_Y(x) + 2d_X(w_i) + d_Y(y) + 2d_X(w_i))^\alpha \\
 &+ \sum_{\substack{x \in V(X) \\ w_i \in N_X(x), w_i \in V(X)}} \sum_{y \in V(Y)} (d_X(w_i)(n_Y + 2) + d_Y(y) + 2d_X(x))^\alpha \\
 &+ \sum_{\substack{x \in V(X), \\ w_i \in N_X(x), w_i \in I(X)}} \sum_{y \in V(Y)} (2(n_Y + 1) + d_Y(y) + 2d_X(x))^\alpha \\
 &\geq 2^\alpha m_X(n_Y + 2)^\alpha \Delta_X^\alpha + 2m_X(n_Y(\Delta_X + 2) + 2(\Delta_Y + 1))^\alpha \\
 &+ 2^\alpha n_X m_Y(\Delta_Y + 2\Delta_X)^\alpha \\
 &+ n_Y \sum_{\substack{x \in V(X), \\ w_i \in N_X(x), w_i \in V(X)}} (\Delta_X(n_Y + 2) + \Delta_Y + 2\Delta_X)^\alpha \\
 &+ n_Y \sum_{\substack{x \in V(X), \\ w_i \in N_X(x), w_i \in I(X)}} (2(n_Y + 1) + \Delta_Y + 2\Delta_X)^\alpha. \tag{3.11}
 \end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
 \chi_\alpha(R(X) \square Y) &\leq 2^\alpha m_X(n_Y + 2)^\alpha \delta_G^\alpha \\
 &+ 2m_X(n_Y(\delta_X + 2) + 2(\delta_Y + 1))^\alpha + 2^\alpha n_X m_Y(\delta_Y + 2\delta_X)^\alpha \\
 &+ n_Y \sum_{\substack{x \in V(X), \\ w_i \in N_X(x), w_i \in V(X)}} (\delta_X(n_Y + 2) + \delta_Y + 2\delta_X)^\alpha \\
 &+ n_Y \sum_{\substack{x \in V(X), \\ w_i \in N_X(x), w_i \in I(X)}} (2(n_Y + 1) + \delta_Y + 2\delta_X)^\alpha. \tag{3.12}
 \end{aligned}$$

The equality in (3.11) and (3.12) obviously holds if and only if X and Y are regular graphs. This completes the proof. \square

In the following theorem, we compute lower and upper bounds on the general sum-connectivity index for R -edge neighborhood corona product of $R(X)$ and Y .

Theorem 3.4 *Let $\alpha < 0$. Then the bounds for the general sum-connectivity index of $R(X) \boxtimes Y$ are given by*

$$\begin{aligned}
 \chi_\alpha(R(X) \boxtimes Y) &\geq 2^\alpha m_X(n_Y + 2)^\alpha \Delta_X^\alpha + 2m_X(n_Y \Delta_X + 2(\Delta_X + 1))^\alpha + 2^\alpha n_X m_Y(\Delta_Y + 2)^\alpha \\
 &+ n_Y \sum_{\substack{x \in I(X), \\ w_i \in N_X(x), w_i \in V(X)}} (\Delta_X(n_Y + 2) + \Delta_Y + 2)^\alpha,
 \end{aligned}$$

$$\begin{aligned} \chi_\alpha(R(X) \boxminus Y) &\leq 2^\alpha m_X(n_Y + 2)^\alpha \delta_X^\alpha + 2m_X(n_Y \delta_X + 2(\delta_X + 1))^\alpha + 2^\alpha n_X m_Y (\delta_Y + 2)^\alpha \\ &\quad + n_Y \sum_{\substack{x \in I(X), \\ w_i \in N_X(x), w_i \in V(X)}} (\delta_X(n_Y + 2) + \delta_Y + 2)^\alpha. \end{aligned}$$

The equality holds if and only if X and Y are regular graphs.

Proof Using (3.1) and (3.6) in equation (1.1), we get

$$\begin{aligned} \chi_\alpha(R(X) \boxminus Y) &= \sum_{xy \in E(R(X))} (d_{R(X)}(x) + d_{R(X)}(y))^\alpha + m_X \sum_{xy \in E(Y)} (d_Y(x) + d_Y(y))^\alpha \\ &\quad + \sum_{x \in V(R(X))} \sum_{y \in V(Y)} (d_{R(X)}(x) + d_Y(y))^\alpha \\ &= \sum_{\substack{xy \in E(R(X)), \\ x, y \in V(X)}} (d_X(x)(n_Y + 2) + d_X(y)(n_Y + 2))^\alpha \\ &\quad + \sum_{\substack{xy \in E(R(X)), \\ x \in V(X), y \in I(X)}} (d_X(x)(n_Y + 2) + 2)^\alpha + n_X \sum_{xy \in E(Y)} (d_Y(x) + 2 + d_Y(y) + 2)^\alpha \\ &\quad + \sum_{\substack{x \in I(X), \\ w_i \in N_X(x), w_i \in V(X)}} \sum_{y \in V(Y)} (d_X(w_i)(n_Y + 2) + d_Y(x) + 2)^\alpha \\ &\geq 2^\alpha m_X(n_Y + 2)^\alpha \Delta_X^\alpha + 2m_X(n_Y \Delta_X + 2(\Delta_X + 1))^\alpha + 2^\alpha n_X m_Y (\Delta_Y + 2)^\alpha \\ &\quad + n_Y \sum_{\substack{x \in I(X), \\ w_i \in N_X(x), w_i \in V(X)}} (\Delta_X(n_Y + 2) + \Delta_Y + 2)^\alpha. \end{aligned} \tag{3.13}$$

Analogously, one can compute the upper bound,

$$\begin{aligned} \chi_\alpha(R(X) \boxminus Y) &\leq 2^\alpha m_X(n_Y + 2)^\alpha \delta_X^\alpha + 2m_X(n_Y \delta_X + 2(\delta_X + 1))^\alpha + 2^\alpha n_X m_Y (\delta_Y + 2)^\alpha \\ &\quad + n_Y \sum_{\substack{x \in I(X), \\ w_i \in N_X(x), w_i \in V(X)}} (\delta_X(n_Y + 2) + \delta_Y + 2)^\alpha. \end{aligned} \tag{3.14}$$

The equality in (3.13) and (3.14) obviously holds if and only if X and Y are regular graphs. This completes the proof. □

3.2 Line graph

The line graph of X , denoted by $L(X)$, is a graph with vertex set $V(L(X)) = E(X)$ and any two vertices e_1 and e_2 have an arc in $L(X)$ if and only if they share a common endpoint in X . The graph $L(X)$ has m_X vertices and $\frac{1}{2}M_1(X) - m_X$ edges. The degree of a vertex $x \in L(X)$ is given by

$$d_{L(X)}(x) = d_X(w_i) + d_X(w_j) - 2 \quad \text{if } x = w_i w_j, \forall w_i, w_j \in V(X). \tag{3.15}$$

We compute lower and upper bounds on the general sum-connectivity index of $L(X)$ in the following theorem.

Theorem 3.5 *Let $\alpha < 0$. Then the bounds for the general sum-connectivity index of $L(X)$ are given by*

$$4^\alpha (\Delta_X - 1)^\alpha \left(\frac{1}{2} M_1(X) - m_X \right) \leq \chi_\alpha(L(X)) \leq 4^\alpha (\delta_X - 1)^\alpha \left(\frac{1}{2} M_1(X) - m_X \right).$$

The equality holds if and only if X is a regular graph.

Proof Using (3.1) and (3.15) in equation (1.1), we get

$$\begin{aligned} \chi_\alpha(L(X)) &= \sum_{xy \in E(L(X))} (d_{L(X)}(x) + d_{L(X)}(y))^\alpha \\ &= \sum_{x=w_i, w_j \in E(X), y=w_j, w_k} (d_X(w_i) + d_X(w_j) - 2 + d_X(w_j) + d_X(w_k) - 2) \\ &\geq 4^\alpha (\Delta_X - 1)^\alpha \left(\frac{1}{2} M_1(X) - m_X \right). \end{aligned} \tag{3.16}$$

One can analogously compute the following:

$$\chi_\alpha(L(X)) \leq 4^\alpha (\delta_X - 1)^\alpha \left(\frac{1}{2} M_1(X) - m_X \right). \tag{3.17}$$

The equality in (3.16) and (3.17) holds if and only if X is a regular graph. □

3.3 Rooted product

A rooted graph is graph in which one vertex is labeled as a special vertex and that vertex is called root vertex of graph. The rooted graph is also known as a pointed graph and a flow graph. Let Y be a labeled graph with n_Y and X be a sequence of n_Y rooted graphs X_1, X_2, \dots, X_{n_Y} . The rooted product of X and Y , denoted by $Y(X)$, is a graph that obtained from one copy of Y and n_Y copies of X and identifying the rooted vertex of X_i ($1 \leq i \leq n_Y$) with i th vertex of Y . The number of vertices and edges in $Y(X)$ are $n_X = n_{X_1} + n_{X_2} + \dots + n_{X_{n_Y}}$ and $m_X + m_Y$. The degree of a vertex $x \in V(Y(X))$ (where w_i is a rooted vertex of X_i) is given by

$$d_{Y(X)}(x) = \begin{cases} d_Y(x) + d_{X_i}(w_i) & \text{if } x \in V(Y), \\ d_Y(x) + d_{X_i}(w_i) & \text{if } x \in V(X), x = w_i, \\ d_{X_i}(x) & \text{if } x \in V(X), x \neq w_i. \end{cases} \tag{3.18}$$

In the following theorem, the bounds on the general sum-connectivity index of rooted product are computed.

Theorem 3.6 *Let $\alpha < 0$. Then the bounds for the general sum-connectivity index of $Y(X)$ are given by*

$$\begin{aligned} \chi_\alpha(Y(X)) &\geq \sum_{ij \in E(Y)} (2 \Delta_Y + \omega_i + \omega_j)^\alpha + \sum_{i=1}^{n_Y} \chi_\alpha(X_i) \\ &\quad + \sum_{i=1}^{n_Y} |N_{X_i}(w_i)| [(2 \Delta_{X_i} + \Delta_Y)^\alpha - 2^\alpha \Delta_{X_i}^\alpha], \end{aligned}$$

$$\chi_\alpha(Y(X)) \leq \sum_{ij \in E(Y)} (2\delta_Y + \omega_i + \omega_j)^\alpha + \sum_{i=1}^{n_Y} \chi_\alpha(X_i) + \sum_{i=1}^{n_Y} |N_{X_i}(w_i)| [(2\delta_{X_i} + \delta_Y)^\alpha - 2^\alpha \delta_{X_i}^\alpha].$$

The equality holds if and only if Y and X_i are regular graphs.

Proof Let the degree of w_i in X_i be denoted by ω_i and the number of neighbors of w_i in X_i be denoted by $|N_{X_i}(w_i)|$. Using (3.1) and (3.18) in equation (1.1), we get

$$\begin{aligned} \chi_\alpha(Y(X)) &= \sum_{ij \in E(Y)} (d_Y(i) + \omega_i + d_Y(j) + \omega_j)^\alpha \\ &\quad + \sum_{i=1}^{n_Y} \left[\sum_{\substack{xy \in E(X_i), \\ x, y \neq w_i}} (d_{X_i}(x) + d_{X_i}(y))^\alpha + \sum_{\substack{xy \in E(X_i), \\ x \in V(X_i), y = w_i}} (d_{X_i}(x) + d_Y(i) + \omega_i)^\alpha \right] \\ &= \sum_{ij \in E(Y)} (d_Y(i) + \omega_i + d_Y(j) + \omega_j)^\alpha \\ &\quad + \sum_{i=1}^{n_Y} \left[\chi_\alpha(X_i) - \sum_{\substack{xy \in E(X_i), \\ x \in V(X_i), y = w_i}} (d_{X_i}(x) + \omega_i)^\alpha + \sum_{\substack{xy \in E(X_i), \\ x \in V(X_i), y = w_i}} (d_{X_i}(x) + d_Y(i) + \omega_i)^\alpha \right] \\ &\geq \sum_{ij \in E(Y)} (2\Delta_Y + \omega_i + \omega_j)^\alpha + \sum_{i=1}^{n_Y} \chi_\alpha(X_i) + \sum_{i=1}^{n_Y} |N_{X_i}(w_i)| [(2\Delta_{X_i} + \Delta_Y)^\alpha - 2^\alpha \Delta_{X_i}^\alpha]. \end{aligned}$$

Similarly, we can show that

$$\chi_\alpha(Y(X)) \leq \sum_{ij \in E(Y)} (2\delta_Y + \omega_i + \omega_j)^\alpha + \sum_{i=1}^{n_Y} \chi_\alpha(X_i) + \sum_{i=1}^{n_Y} |N_{X_i}(w_i)| [(2\delta_{X_i} + \delta_Y)^\alpha - 2^\alpha \delta_{X_i}^\alpha].$$

The equality in (3.13) and (3.14) obviously holds if and only if Y and X_i are regular graphs. This completes the proof. □

In the special case, when all $X_1, X_2, X_3, \dots, X_{n_Y}$ are isomorphic to a graph G , then the rooted product of Y and G is denoted by $Y\{G\}$. This rooted product is called a cluster of Y and G . The following corollary is an easy consequence of Theorem 3.6.

Corollary 3.1 *Let $\alpha < 0$. Then the bounds for the general sum-connectivity index of cluster $Y\{G\}$ are given by*

$$\begin{aligned} \chi_\alpha(Y\{G\}) &\geq 2^\alpha m_Y (\Delta_Y + \omega)^\alpha + n_Y \chi_\alpha(G) + n_Y |N_G(w)| [(2\Delta_G + \Delta_Y)^\alpha - 2^\alpha \Delta_G^\alpha], \\ \chi_\alpha(Y\{G\}) &\leq 2^\alpha m_Y (\delta_Y + \omega)^\alpha + n_Y \chi_\alpha(G) + n_Y |N_G(w)| [(2\delta_G + \delta_Y)^\alpha - 2^\alpha \delta_G^\alpha], \end{aligned}$$

where $\omega = d_G(w)$ and $|E(G)| = m_Y$.

4 Conclusion

In this article, we obtained the lower and upper bounds for the general sum-connectivity index of four types of graph operations involving R -graph. Additionally, we have deter-

mined the bounds for the general sum-connectivity index of line graph $L(X)$ and the rooted product of graphs.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The idea to obtain the lower and upper bounds for the general sum-connectivity index of four types of graph operations involving the R -graph was proposed by SA, while the idea to obtain bounds for the general sum-connectivity index of the line graph $L(X)$ and the rooted product of graphs was proposed by MI and ZR. After several discussions, SA and MI obtained some sharp lower bounds. MI and ZR checked these results and suggested to improve them. The first draft was prepared by SA, which was verified and improved by MI and ZR. The final version was prepared by SA and MI. All authors read and approved the final manuscript.

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