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BOUNDS FOR THE PROBABILITY OF A UNION, WITH APPLICATIONS[1]

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1. Inequalities. Consider a probability space $(\Omega, \mathfrak{F}, P)$ and the sequence of events $A_i \varepsilon \mathfrak{F}$, $i = 1, 2, \cdots, n$. Let $I_i(\omega)$ be the indicator random variable of the event A_i, $i = 1, 2, \cdots, n$, then $\max_{i=1,\cdots,n} I_i(\omega)$ is the indicator random variable of the event $\bigcup_{i=1}^n A_i$.

In the present paper we assume that we only know $P_i = P(A_i)$ and $P_{i,j} = P(A_i \cap A_j)$ for all $i, j = 1, 2, \cdots, n$, and we find bounds for the $P(\bigcup_{i=1}^n A_i)$.

It is not difficult to prove that the inequality

$$(1)\quad \sum_{i\varepsilon J} I_i(\omega) - \sum_{i<j,i,j\varepsilon J} I_i(\omega)I_j(\omega) \leqq \max_{i\varepsilon J} I_i(\omega) \leqq (1 - I_k(\omega)) \sum_{i\varepsilon J} I_i(\omega) + I_k(\omega)$$

holds for all points $\omega \varepsilon \Omega$, all subsets J of the set $\{1, 2, \cdots, n\}$ and all $k \varepsilon J$.

Taking expectations in (1) we obtain

$$(2)\quad \sum_{i\varepsilon J} P_i - \sum_{i<j,i,j\varepsilon J} P_{i,j} \leqq P(\bigcup_{i\varepsilon J} A_i) \leqq \sum_{i\varepsilon J} P_i - \sum_{i\varepsilon J, i\neq k} P_{k,i}.$$

Hence, the best lower bound for the $P(\bigcup_{i=1}^n A_i)$, among the lower bounds (2), is

$$(3)\quad \max_J \left(\sum_{i\varepsilon J} P_i - \sum_{i<j,i,j\varepsilon J} P_{ij}\right).$$

Now if we take disjoint subsets $J_1, \cdots, J_m$ such that $\bigcup_{i=1}^m J_i = \{1, \cdots, n\}$ and denote by $B_r = \bigcup_{i\varepsilon J_r} A_i$, then $\bigcup_{r=1}^m B_r = \bigcup_{i=1}^n A_i$ and using (2) obtain $P(B_r) \leqq \min (1, \sum_{i\varepsilon J_r} P_i - \max_{k\varepsilon J_r} \sum_{i\varepsilon J_r, i\neq k} P_{k,i}) = T_r$ and $P(\bigcup_{i=1}^n A_i) = P(\bigcup_{r=1}^m B_r) \leqq \sum_{r=1}^m T_r$. In particular for $m = 1$ we obtain the upper bound

$$(4)\quad \min \left[\sum_{i=1}^n P_i - \max_{k=1,2,\cdots,n} \sum_{i=1, i\neq k}^n P_{k,i}, 1\right].$$

Actually, it is easy to construct examples where the bounds (3) and (4) are attained.

Below we derive another lower bound which is not as sharp but more elegant than (3). For this, let us use the following notation:

$$P' = (P_1, \cdots, P_n), \qquad Q = \{P_{i,j}\}, \qquad I'(\omega) = (I_1(\omega), \cdots, I_n(\omega)),$$

i.e., P, $I(\omega)$ are $n \times 1$ vectors and Q is $n \times n$ matrix. Let us also denote by Q^- the generalized inverse of Q, i.e., satisfying $QQ^-Q = Q$. Such an inverse always exists, (see [4], page 24). The lower bound (5), we derive below, for non-singular Q was obtained by Gallot [3].

LEMMA 1.1. *The vector P is in the range of Q.*

PROOF. If Q is non-singular, the lemma is obvious. If Q is singular, then for

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every vector a in the null space of Q, i.e., $Qa = 0$, we have

$$E(a'I)^2 = a'Qa = 0.$$

Hence, $a'I = 0$ with probability one and, taking expectations, we obtain $a'P = 0$, i.e., P is orthogonal to the null space of Q and thus lies in the range of Q. Q.E.D.

THEOREM 1.1.

$$P(\bigcup_{i=1}^n A_i) \geqq P'Q^-P. \tag{5}$$

PROOF. For any vector $a' = (a_1, \cdots, a_n)$, the following inequality

$$(a'I(\omega))^2 - 2(a'I(\omega)) + \max_{i=1,\cdots,n} I_i(\omega) \geqq 0 \tag{6}$$

holds for all points $\omega \varepsilon \Omega$; hence, taking expectations, we obtain

$$P(\bigcup_{i=1}^n A_i) \geqq 2a'P - a'Qa. \tag{7}$$

The vector a which maximizes $2a'P - a'Qa$ satisfies the relation $Qa = P$. Therefore,

$$P(\bigcup_{i=1}^n A_i) \geqq a'Qa \qquad \text{subject to} \qquad Qa = P,$$

but

$$a'Qa = a'QQ^-Qa = P'Q^-P. \qquad \text{Q.E.D.}$$

2. Related work. Recently, Dawson and Sankoff [2] have proved a result equivalent to

$$P(\bigcup_{i=1}^n A_i) \geqq 2(B + C)/(2 + \rho) - 2C/(1 + \rho) \tag{8}$$

where

$$B = \textstyle\sum_{i=1}^n P_i, \qquad C = \sum_{i=1}^n \sum_{i<j}^n P_{ij}, \quad \text{and} \quad \rho = [2C/B].$$

They show that (8) is stronger than

$$P(\bigcup_{i=1}^n A_i) \geqq B^2/(2C + B) \tag{9}$$

which was derived by Chung and Erdös [1]. Nevertheless, inequality (8) is not better than (5) as will be clear from the examples below.

In the following nt merical examples, inequalities (3), (4), (5), (8), and (9) are compared.

(1) For $n = 3$ and

$$P_1 = 0.5, \qquad P_2 = 0.4, \qquad P_3 = 0.3,$$
$$P_{12} = 0, \qquad P_{13} = 0.25, \qquad P_{23} = 0.05,$$

the lower bounds (3), (5), (8), and (9) are 0.9, 0.9, 0.9, and 0.8, respectively. The upper bound (4) is 0.9.

(2) For $n = 3$ and

$$P_1 = 0.50, \qquad P_2 = 0.40, \qquad P_3 = 0.60,$$

$$P_{12} = 0.20, \qquad P_{13} = 0.30, \qquad P_{23} = 0.10,$$

the lower bounds (3), (5), (8), and (9) are 0.9, 0.9, 0.858 and 0.833 respectively. The upper bound (4) is 1.

(3) For $n = 3$ and

$$P_1 = \tfrac{1}{3}, \qquad P_2 = \tfrac{2}{3}, \qquad P_3 = \tfrac{1}{3},$$

$$P_{12} = \tfrac{1}{6}, \qquad P_{13} = \tfrac{1}{3}, \qquad P_{23} = \tfrac{1}{6},$$

the lower bounds (3), (5), (8), and (9) are 0.83, 0.76, 0.66, and 0.66, respectively. The upper bound (4) equals 0.83.

Notice that inequality (8) is equivalent to (5) in the first example, better in the second, and worse in the last example in which Q is singular. For $n = 2$, (8) gives the exact result i.e., equality holds, whereas equality holds in (5) only if Q is singular or $P_1 > P_2 = P_{12}$.

3. Applications. Assume now that we have an infinite sequence of events $A_i \varepsilon \mathfrak{F}$, $i = 1, 2, \cdots$, and denote by $D'_{r,n} = (P_r, \cdots, P_{r+n})$, $Q_{r,n} = \{P_{i,j}\}$, $i, j = r, r+1, \cdots, r+n$, and the quadratic form $D'_{r,n}Q^{-}_{r,n}D_{r,n} = c_{r,n}$.

LEMMA 3.1. *The* $\lim_{r\to\infty} \lim_{n\to\infty} c_{r,n} = c$ *exists and lies between zero and one.*

PROOF. By the definition of $c_{r,n}$ and relation (7), we obtain

$$1 \geqq c_{r,n+1} = \max_a [2a'D_{r,n+1} - a'Q_{r,n+1}a]$$
$$\geqq \max_b [2b'D_{r,n+1} - b'Q_{r,n+1}b] = c_{r,n} \geqq 0$$

where $a' = (a_0, \cdots, a_{n+1})$ and $b' = (b_0, \cdots, b_n, 0)$.

Thus, for any r the sequence $c_{r,n}$, $n = 1, 2, \cdots$, is increasing and bounded by 1; hence,

$$\lim_{n\to\infty} c_{r,n} = c_r .$$

Similarly, we prove that $0 \leqq c_{r,n} \leqq c_{r-1,n+1} \leqq 1$ so that $1 \geqq c_{r-1} \geqq c_r \geqq 0$ and the sequence c_r is decreasing and bounded below by 1; hence, the result. Q.E.D.

THEOREM 3.1.

$$\liminf_{r\to\infty} \liminf_{n\to\infty} \Big(\sum_{i=r}^{n} P_i - \max_{k=r,\cdots,n} \sum_{i=r, i\neq k}^{n} P_{k,i}\Big)$$
$$\geqq P\Big(\bigcap_{r=1}^{\infty} \bigcup_{n=r}^{\infty} A_n\Big) \geqq \lim_{r\to\infty} \lim_{n\to\infty} c_{r,n} .$$

PROOF.

$$P\Big(\bigcap_{r=1}^{\infty} \bigcup_{n=r}^{\infty} A_n\Big) = \lim_{r\to\infty} P\Big(\bigcup_{n=r}^{\infty} A_n\Big).$$

But, by virtue of (4) and (5) we have

$$\liminf_{n\to\infty} \Big(\sum_{i=r}^{n} P_i - \max_{k=r,\cdots,n} \sum_{i=r, i\neq k}^{n} P_{k,i}\Big)$$
$$\geqq P\Big(\bigcup_{n=r}^{\infty} A_n\Big) \geqq \lim_{n\to\infty} c_{r,n} = c_r$$

and hence the result. Q.E.D.

COROLLARY 3.1. *The lower bound of Theorem* 3.1 *is better than the lower bound of Erdös-Rényi* ([5], p. 326), i.e.,

$$\text{(10)} \qquad \lim_{r\to\infty} \lim_{n\to\infty} c_{r,n} \geqq \limsup_{n\to\infty} [(\textstyle\sum_{i=1}^{n} P_i)^2/\sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij}].$$

PROOF. First, observe that

$$c_{r,n} = D'_{r,n} Q^{-}_{r,n} D_{r,n}$$
$$= \max_a (2a'D_{r,n} - a'Q_{r,n}a) \geqq (\textstyle\sum_{i=r}^{r+n} P_i)^2/\sum_{i=r}^{r+n} \sum_{j=r}^{r+n} P_{ij}$$

because the last expression is found by setting the $n \times 1$ vector $a' = c(1, 1, \cdots, 1)$ with

$$c = \textstyle\sum_{i=r}^{n+r} P_i / \sum_{i=r}^{n+r} \sum_{j=r}^{n+r} P_{ij} .$$

This shows that (5) is stronger than (9) which in consequence proves our claim. Q.E.D.

The following example, although not very interesting, shows that strict inequality might hold in (10) and hence the Borel-Cantelli type lemma obtainable from Theorem 3.1 is stronger than the one obtained by Erdös and Rényi. Let the events A_i, $i = 1, 2, \cdots$, such that,

$$P_{2i-1} = p, \quad P_{2i} = q, \quad \text{with} \quad 0 < q < p \leqq 1, \quad \text{and}$$
$$P_{i,j} = p, \quad \text{if } i \text{ and } j \text{ are even,}$$
$$P_{i,j} = q, \quad \text{if } i \text{ and } j \text{ are not both even, for all } i, j, = 1, \cdots,$$

then we easily see that $c_r = p$ for all $r = 1, \cdots$, so that $\lim_{r\to\infty} c_r = p$, whereas $\lim_{n\to\infty} (\sum_{i=1}^{n} P_i)^2/(\sum_{i=1}^{n} \sum_{j=1}^{n} P_{i,j}) = (p + q)^2/(p + 3q)$ which is less than p.

As the last application, consider the sequence of random variables X_n, $n = 1, \cdots$, and let A_n be the event $|X_n| > e$ for some $e > 0$, then

$$P(\sup_{n\geqq r} |X_n| > e) = P(\textstyle\bigcup_{n=r}^{\infty} A_n)$$

and

$$\liminf_{r\to\infty} \liminf_{n\to\infty} (\textstyle\sum_{i=r}^{n} P_i - \max_{r\leqq k\leqq n} \sum_{i=r, i\neq k}^{n} P_{k,i})$$
$$\geqq \lim_{r\to\infty} P(\sup_{n\geqq r} |X_n| > e) = P(\textstyle\bigcap_{r=1}^{\infty} \bigcup_{n=r}^{\infty} A_n) \geqq \lim_{r\to\infty} \lim_{n\to\infty} c_{r,n} .$$

For example, if $P(X_n = 0) = 1 - p/n$, $P(X_n = 1) = p/n$ and $P(X_n = 1, X_k = 1) = p \min (1/n - 1/n^2, 1/k - 1/k^2)$ for $n \neq k$ where $0 \leqq p \leqq 6/\pi^2$ then from (4)

$$P(\textstyle\bigcup_{n=r}^{i} A_n) \leqq \sum_{n=r}^{i} P_n - \sum_{n=r+1}^{i} P_{r,n}$$
$$= (\textstyle\sum_{n=r}^{i} (1/n) - \sum_{n=r+1}^{i} (1/n - 1/n^2))p = 1/r + \sum_{n=r+1}^{i} (1/n^2).$$

Hence $P(\bigcup_{n=r}^{\infty} A_n) \leqq 1/r + \sum_{n=r+1}^{\infty} (1/n^2)$, and $\lim_{r\to\infty} P(\bigcup_{n=r}^{\infty} A_n) = 0$, i.e., $X_n \to_{\text{a.s.}} 0$ as $n \to \infty$.

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