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# Bounds for the Ratio of Two Gamma Functions: from Gautschi's and Kershaw's Inequalities to Complete Monotonicity

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**Abstract** In the expository and survey paper, along one of main lines of bounding the ratio of two gamma functions, the author looks back and analyses some inequalities, the complete monotonicity of several functions involving ratios of two gamma or $q$-gamma functions, the logarithmically complete monotonicity of a function involving the ratio of two gamma functions, some new bounds for the ratio of two gamma functions and divided differences of polygamma functions, and related monotonicity results.

***Keywords:*** *bound, ratio of two gamma functions, completely monotonic function, logarithmically completely monotonic function, divided difference, gamma function, q-gamma function, psi function, polygamma function, inequality*

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## 1. Introduction

For the sake of proceeding smoothly, we briefly introduce some necessary concepts and notation.

### 1.1. The Gamma and $q$-gamma Functions

It is well-known that the classical Euler gamma function may be defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt \tag{1.1}$$

for $x>0$. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ are called the polygamma functions. It is common knowledge that special functions $\Gamma(x)$, $\psi(x)$ and $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ are fundamental and important and have much extensive applications in mathematical sciences.

The $q$-analogue of $\Gamma$ is defined [[6], pp. 493-496] for $x>0$ by

$$\Gamma_q(x)=(1-q)^{1-x}\prod_{i=0}^\infty\frac{1-q^{i+1}}{1-q^{i+x}},\quad 0<q<1, \tag{1.2}$$

$$\Gamma_q(x)=(q-1)^{1-x}q^{\binom{x}{2}}\prod_{i=0}^\infty\frac{1-q^{-(i+1)}}{1-q^{-(i+x)}},\quad q>1. \tag{1.3}$$

The $q$-gamma function $\Gamma_q(z)$ has the following basic properties:

$$\lim_{q\to1^+}\Gamma_q(z)=\lim_{q\to1^-}\Gamma_q(z)=\Gamma(z) \tag{1.4}$$

and

$$\Gamma_q(x)=q^{\binom{x-1}{2}}\Gamma_{1/q}(x). \tag{1.5}$$

The $q$-analogue of the psi or digamma function $\psi$ is defined by

$$\begin{aligned}\psi_q(x)&=\frac{\Gamma_q(x)}{\Gamma_q(x)}\\&=-\ln(1-q)+\ln q\sum_{k=0}^\infty\frac{q^{k+x}}{1-q^{k+x}}\\&=-\ln(1-q)-\int_0^\infty\frac{e^{-xt}}{1-e^{-t}}\,d\gamma_q(t)\end{aligned}$$

for $0<q<1$, where $d\gamma_q(t)$ is a discrete measure with positive masses $-\ln q$ at the positive points $-k\ln q$ for $k\in\mathbb{N}$, more accurately,

$$\gamma_q(t)=-\ln q\sum_{k=1}^\infty\delta(t+k\ln q),\quad 0<q<1. \tag{1.6}$$

See [[33], p. 311] and its corrected version [34].

### 1.2. The Generalized Logarithmic Mean

The generalized logarithmic mean $L_p(a,b)$ of order $p\in\mathbb{R}$ for positive numbers $a$ and $b$ with $a\ne b$ may be defined [[13], p. 385] by

$$L_p(a,b)=\begin{cases}\left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p\ne -1,0;\\ \dfrac{b-a}{\ln b-\ln a}, & p=-1;\\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p=0.\end{cases} \tag{1.7}$$

It is well-known that

$$L_{-2}(a,b)=\sqrt{ab}=G(a,b), \tag{1.8}$$

$$L_{-1}(a,b)=L(a,b),\quad L_0(a,b)=I(a,b), \tag{1.9}$$

and

$$L_1(a,b)=\frac{a+b}{2}=A(a,b) \tag{1.10}$$

are called respectively the geometric mean, the logarithmic mean, the identric or exponential mean, and the arithmetic mean. It is also known [[13], pp. 386-387, Theorem 3] that the generalized logarithmic mean $L_p(a,b)$ of order $p$ is increasing in $p$ for $a\ne b$. Therefore, inequalities

$$G(a,b)<L(a,b)<I(a,b)<A(a,b) \tag{1.11}$$

are valid for $a>0$ and $b>0$ with $a\ne b$. See also [70,71,72,115]. Moreover, the generalized logarithmic mean $L_p(a,b)$ is a special case of $E(r,s;x,y)$, that is, $L_p(a,b)=E(1,p+1;a,b)$.

In passing, we remark that the complete monotonicity of the logarithmic mean was established in [69,84].

### 1.3. Logarithmically Completely Monotonic Functions

A function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and

$$(-1)^n f^{(n)}(x)\ge 0 \tag{1.12}$$

for $x\in I$ and $n\ge 0$.

**Theorem 1.1.** [[118], p. 161] *A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0<x<\infty$ is that*

$$f(x)=\int_0^\infty e^{-xt}\,d\alpha(t), \tag{1.13}$$

*where $\alpha(t)$ is nondecreasing and the integral converges for $0<x<\infty$.*

**Theorem 1.2.** *[[11], p. 83] If $f(x)$ is completely monotonic on $I$, $g(x)\in I$, and $g'(x)$ is completely monotonic on $(0,\infty)$, then $f(g(x))$ is completely monotonic on $(0,\infty)$.*

A positive function $f(x)$ is said to be logarithmically completely monotonic on an interval $I\subseteq R$ if it has derivatives of all orders on $I$ and its logarithm $\ln f(x)$ satisfies

$$(-1)^k[\ln f(x)]^{(k)}\ge 0$$

for $k\in\mathbb{N}$ on $I$.

The notion "logarithmically completely monotonic function" was first put forward in [7] without an explicit definition. This terminology was explicitly recovered in [88] whose revised and expanded version was formally published as [83,90].

It has been proved once and again in [10,23,66,67,83,87,88,89,103] that a logarithmically completely monotonic function on an interval $I$ must also be completely monotonic on $I$. C. Berg points out in [10] that these functions are the same as those studied by Horn [32] under the name infinitely divisible completely monotonic functions. For more information, please refer to [10,92,93] and related references therein.

### 1.4. Outline of this Paper

In this expository and survey paper, along one of main lines of bounding the ratio of two gamma functions, we look back and analyse Gautschi's double inequality and Kershaw's second double inequality, the complete monotonicity of several functions involving ratios of two gamma or $q$-gamma functions by Alzer, Bustoz-Ismail, Elezović-Giordano-Pečarić and Ismail-Muldoon, the logarithmically complete monotonicity of a function involving the ratio of two gamma functions, some new bounds for the ratio of two gamma functions and the divided differences of polygamma functions, and related monotonicity results by Batir, Elezović-Pečarić, Qi and others.

## 2. Gautschi's and Kershaw's Double Inequalities

In this section, we begin with the papers [22,35] to introduce a kind of inequalities for bounding the ratio of two gamma functions.

### 2.1. Gautschi's Double Inequalities

The first result of the paper [22] was the double inequality

$$\frac{(x^p+2)^{1/p}-x}{2}<e^{x^p}\int_x^\infty e^{-t^p}\,dt \le c_p\left[\left(x^p+\frac{1}{c_p}\right)^{1/p}-x\right] \tag{2.1}$$

for $x\ge 0$ and $p>1$, where

$$c_p=\left[\Gamma\left(1+\frac{1}{p}\right)\right]^{p/(p-1)} \tag{2.2}$$

or $c_p = 1$. By an easy transformation, the inequality (2.1) was written in terms of the complementary gamma function

$$\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1}\, dt \tag{2.3}$$

as

$$\begin{aligned} \frac{p[(x+2)^{1/p} - x^{1/p}]}{2} &< e^x \Gamma\left(\frac{1}{p}, x\right) \\ &\le pc_p \left[\left(x + \frac{1}{c_p}\right)^{1/p} - x^{1/p}\right] \end{aligned} \tag{2.4}$$

for $x \ge 0$ and $p > 1$. In particular, if letting $p \to \infty$, the double inequality

$$\frac{1}{2}\ln\left(1+\frac{2}{x}\right) \le e^x E_1(x) \le \ln\left(1+\frac{1}{x}\right) \tag{2.5}$$

for the exponential integral $E_1(x) = \Gamma(0,x)$ for $x > 0$ was derived from (2.4), in which the bounds exhibit the logarithmic singularity of $E_1(x)$ at $x = 0$. As a direct consequence of the inequality (2.4) for $p = \frac{1}{s}$ and $x = 0$, the following simple inequality for the gamma function was deduced:

$$2^{s-1} \le \Gamma(1+s) \le 1, \quad 0 \le s \le 1. \tag{2.6}$$

The second result of the paper [22] was a sharper and more general inequality

$$e^{(s-1)\psi(n+1)} \le \frac{\Gamma(n+s)}{\Gamma(n+1)} \le n^{s-1} \tag{2.7}$$

for $0 \le s \le 1$ and $n \in \mathbb{N}$ than (2.6). It was obtained by proving that the function

$$f(s) = \frac{1}{1-s}\ln\frac{\Gamma(n+s)}{\Gamma(n+1)} \tag{2.8}$$

is monotonically decreasing for $0 \le s < 1$ and that

$$\lim_{s\to 1^-} f(s) = -\lim_{s\to 1^-} \psi(n+s) = -\psi(n+1).$$

**Remark 2.1.** *For more information on refining the inequality (2.1), please refer to [38,96,110] and related references therein.*

**Remark 2.2.** *The left-hand side inequality in (2.7) can be rearranged as*

$$\frac{\Gamma(n+s)}{\Gamma(n+1)}\exp\big((1-s)\psi(n+1)\big) \ge 1 \tag{2.9}$$

*or*

$$\left[\frac{\Gamma(n+s)}{\Gamma(n+1)}\right]^{1/(s-1)} e^{-\psi(n+1)} \le 1 \tag{2.10}$$

*for $n \in \mathbb{N}$ and $0 \le s \le 1$. Since the limit*

$$\lim_{n\to\infty}\left\{\left[\frac{\Gamma(n+s)}{\Gamma(n+1)}\right]^{1/(s-1)} e^{-\psi(n+1)}\right\} = 1 \tag{2.11}$$

*can be verified by using Stirling's formula in [1, p. 257, 6.1.38]: For $x > 0$, there exists $0 < \theta < 1$ such that*

$$\Gamma(x+1) - \sqrt{2\pi}x^{x+1/2}\exp\left(-x + \frac{\theta}{12x}\right), \tag{2.12}$$

*it is natural to guess that the function*

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+1)}\right]^{1/(s-1)} e^{-\psi(x+1)} \tag{2.13}$$

*for $0 \le s < 1$ is possibly increasing with respect to $x$ on $(-s,\infty)$. This guess was verified and generalized in [[52], Theorem 1], [[53], Theorem 1], [[85], Theorem 1], [[86], Theorem 1] and others. See also Section 4.*

**Remark 2.3.** *For information on the study of the right-hand side inequality in (2.7), please refer to [61,62,65,105,106] and a great amount of related references therein.*

## 2.2. Kershaw's Second Double Inequality and Its Proof

In 1983, over twenty years later after the paper [22], among other things, D. Kershaw was motivated by the left-hand side inequality (2.7) in [22] and presented in [35] the following double inequality for $0 < s < 1$ and $x > 0$:

$$\begin{aligned} \exp\left[(1-s)\psi\left(x+\sqrt{s}\right)\right] &< \frac{\Gamma(x+1)}{\Gamma(x+s)} \\ &< \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]. \end{aligned} \tag{2.14}$$

It is called in the literature Kershaw's second double inequality.

*Kershaw's proof for (2.14).* Define the function $f_\alpha$ by

$$f_\alpha(x) = \frac{\Gamma(x+1)}{\Gamma(x+s)}\exp\big((s-1)\psi(x+\alpha)\big) \tag{2.15}$$

for $x > 0$ and $0 < s < 1$, where the parameter $\alpha$ is to be determined.

It is not difficult to show, with the aid of Stirling's formula, that

$$\lim_{x\to\infty} f_\alpha(x) = 1. \tag{2.16}$$

Now let

$$F(x) = \frac{f_\alpha(x)}{f_\alpha(x+1)} = \frac{x+s}{x+1}\exp\frac{1-s}{x+\alpha}. \tag{2.17}$$

Then

$$\frac{F'(x)}{F(x)} = (1-s)\frac{(\alpha^2 - s) + (2\alpha - s - 1)x}{(x+1)(x+s)(x+\alpha)^2}.$$

It is easy to show that

1. if $\alpha = s^{1/2}$, then $F'(x) < 0$ for $x > 0$;
2. if $\alpha = \frac{s+1}{2}$, then $F'(x) > 0$ for $x > 0$.

Consequently if $\alpha = s^{1/2}$ then $F$ strictly decreases, and since $F(x) \to 1$ as $x \to \infty$ it follows that $F(x) > 1$ for $x > 0$. This implies that $f_\alpha(x) > f_\alpha(x+1)$ or $x > 0$, and so $f_\alpha(x) > f_\alpha(x+n)$. Take the limit as $n \to \infty$ to give the result that $f_\alpha(x) > 1$, which can be rewritten as the left-hand side inequality in (2.14). The corresponding upper bound can be verified by a similar argument when $\alpha = \frac{s+1}{2}$, the only difference being that in this case $f_\alpha$ strictly increases to unity.

**Remark 2.4.** *The idea contained in the above stated proof of (2.14) was also utilized by other mathematicians. For detailed information, please refer to related contents and references in [61,62].*

**Remark 2.5.** *The inequality (2.14) can be rearranged as*

$$\frac{\Gamma(x+s)}{\Gamma(x+1)}\exp\left((1-s)\psi\left(x+\sqrt{s}\right)\right) < 1 < \frac{\Gamma(x+s)}{\Gamma(x+1)}\exp\left((1-s)\psi\left(x+\frac{s+1}{2}\right)\right) \tag{2.18}$$

*or*

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+1)}\right]^{1/(s-1)}\exp\left(-\psi\left(x+\sqrt{s}\right)\right) > 1 > \left[\frac{\Gamma(x+s)}{\Gamma(x+1)}\right]^{1/(s-1)}\exp\left(-\psi\left(x+\frac{s+1}{2}\right)\right). \tag{2.19}$$

*By Stirling's formula (2.12), we can prove that*

$$\lim_{x\to\infty}\left\{\left[\frac{\Gamma(x+s)}{\Gamma(x+1)}\right]^{1/(s-1)}\exp\left(-\psi\left(x+\sqrt{s}\right)\right)\right\} = 1$$

*and*

$$\lim_{x\to\infty}\left[\frac{\Gamma(x+s)}{\Gamma(x+1)}\right]^{1/(s-1)}\exp\left(-\psi\left(x+\frac{s+1}{2}\right)\right) = 1.$$

*These clues make us to conjecture that the functions in the every end of inequalities (2.18) and (2.19) are perhaps monotonic with respect to x on $(0,\infty)$.*

## 3. Several Complete Monotonicity Results

The complete monotonicity of the functions in the every end of inequalities (2.18) were first demonstrated in [12], and then several related functions were also proved in [5,19,41] to be (logarithmically) completely monotonic.

### 3.1. Bustoz-Ismail's Complete Monotonicity Results

In 1986, motivated by the double inequality (2.14) and other related inequalities, J. Bustoz and M. E. H. Ismail revealed in [12, Theorem 7 and Theorem 8] that

1. the function [Trial mode]

$$\frac{\Gamma(x+s)}{\Gamma(x+1)}\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \tag{3.1}$$

for $0 \le s \le 1$ is completely monotonic on $(0,\infty)$; When $0 < s < 1$, the function (3.1) satisfies $(-1)^n f^{(n)}(x) > 0$ for $x > 0$;

2. the function

$$\frac{\Gamma(x+1)}{\Gamma(x+s)}\exp\left[(s-1)\psi\left(x+s^{1/2}\right)\right] \tag{3.2}$$

for $0 < s < 1$ is strictly decreasing on $(0,\infty)$.

**Remark 3.1.** *The proof of the complete monotonicity of the function (3.1) in [[12], Theorem 7] relies on the inequality*

$$(y+a)^{-n} - (y+b)^{-n} > (b-a)n\left(y+\frac{a+b}{2}\right)^{-n-1} \tag{3.3}$$

*for $n > 0, y > 0$, and $0 < a < b$, the series representation*

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right) \tag{3.4}$$

*in [[21], p. 15], and the above Theorem 1.2 applied to $f(x) = e^{-x}$.*

**Remark 3.2.** *The inequality (3.3) verified in [[12], Lemma 3.1] can be rewritten as*

$$\left[\frac{1}{-n}\frac{(y+a)^{-n} - (y+b)^{-n}}{(y+a)-(y+b)}\right]^{1/[(-n)-1]} < \frac{(y+a)+(y+b)}{2}, n > 0 \tag{3.5}$$

*for $y > 0$ and $0 < a < b$, which is equivalent to*

$$E(1,n;y+a,y+b) < E(1,2;y+a,y+b), \tag{3.6}$$

*where $E(r,s;x,y)$ stands for extended mean values and is defined for two positive numbers x and y and two real numbers r and s by*

$$E(r,s;x,y) = \left[\frac{r}{s}\frac{y^s - x^s}{y^r - x^r}\right]^{\frac{1}{s-r}}, rs(r-s)(x-y) \ne 0;$$

$$E(r,0;x,y) = \left[\frac{1}{r}\frac{y^r - x^r}{\ln y - \ln x}\right]^{1/r}, r(x-y) \ne 0;$$

$$E(r,r;x,y) = \frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}, r(x-y) \ne 0;$$

$$E(0,0;x,y) = \sqrt{xy}, \quad x \ne y;$$

$$E(r,s;x,x) = x, \quad x \ne y.$$

*Actually, the inequality (3.6) is an immediate consequence of monotonicity of $E(r,s;x,y)$, see [39]. For more information, please refer to [13,17,24,29,51,57,58,72,76, 79,80,91,104,107,112,113,114,119] and related references therein.*

**Remark 3.3.** *The proof of the decreasing monotonicity of the function (3.2) just used the formula (3.4) and and the above Theorem 1.2 applied to* $f(x)=e^{-x}$.

**Remark 3.4.** *Indeed, J. Bustoz and M. E. H. Ismail had proved in [[12], Theorem 7] that the function (3.1) is logarithmically completely monotonic on* $(0,\infty)$ *for* $0\le s\le 1$. *However, because the inequality (1.12) strictly holds for a completely monotonic function* ***[Trial mode]*** *on* $(0,\infty)$ *unless* $f(x)$ *is constant (see [[18], p. 98], [[92], p. 82] and [117]), distinguishing between the cases* $0\le s\le 1$ *and* $0<s<1$ *is not necessary.*

## 3.2. Alzer's and Related Complete Monotonicity Results

Stimulated by the complete monotonicity obtained in [12], including those mentioned above, H. Alzer obtained in [[5], Theorem 1]] that the function

$$\frac{\Gamma(x+s)}{\Gamma(x+1)}\frac{(x+1)^{x+1/2}}{(x+s)^{x+s-1/2}}\times\exp\left[s-1+\frac{\psi'(x+1+\alpha)-\psi'(x+s+\alpha)}{12}\right] \tag{3.7}$$

for $\alpha>0$ and $s\in(0,1)$ is completely monotonic on $(0,\infty)$ if and only if $\alpha\ge\frac{1}{2}$, so is the reciprocal of (3.7) for $\alpha\ge 0$ and $s\in(0,1)$ if and only if $\alpha=0$.

As consequences of the monotonicity of the function (3.7), the following inequalities are deduced in [[5], Corollary 2 and Corollary 3]:

1. The inequalities

$$\begin{aligned}&\exp\left[s-1+\frac{\psi'(x+1+\beta)-\psi'(x+s+\beta)}{12}\right]\\ &\le\frac{(x+s)^{x+s-1/2}}{(x+1)^{x+1/2}}\frac{\Gamma(x+1)}{\Gamma(x+s)}\\ &\le\exp\left[s-1+\frac{\psi'(x+1+\alpha)-\psi'(x+s+\alpha)}{12}\right]\end{aligned} \tag{3.8}$$

for $\alpha>\beta\ge 0$ are valid for all $s\in(0,1)$ and $x\in(0,\infty)$ if and only if $\beta=0$ and $\alpha\ge\frac{1}{2}$.

2. If

$$a_n=\frac{3}{2}\left\{1+\ln\left(\frac{2[\Gamma((n+1)/2)]^2}{[\Gamma(n/2)]^2}\frac{n^{n-1}}{(n+1)^n}\right)\right\},$$

then

$$a_n<(-1)^{n+1}\left[\frac{\pi^2}{12}-\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k^2}\right]<a_{n+1} \tag{3.9}$$

for $n\in\mathbb{N}$.

**Remark 3.5.** *The inequality (3.9) follows from the formula*

$$\frac{1}{4}\left[\psi'\left(\frac{n}{2}+1\right)-\psi'\left(\frac{n+1}{2}\right)\right]=\sum_{k=1}^{\infty}\frac{(-1)^k}{(n+k)^2}=(-1)^n\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}-\frac{\pi^2}{2}\right]$$

*and the inequality (3.8) applied to* $s=\frac{1}{2}$, $\alpha=\frac{1}{2}$ *and* $\beta=0$.

**Remark 3.6.** *The proof of the complete monotonicity of the function (3.7) in [5] is based on Theorem 1.2 applied to* $f(x)=e^{-x}$, *the formulas*

$$\frac{1}{x}=\int_0^\infty e^{-xt}dt,\quad \ln\frac{y}{x}=\int_0^\infty\frac{e^{-xt}-e^{-yt}}{t}dt \tag{3.10}$$

*and*

$$\psi(x)=-\gamma+\int_0^\infty\frac{e^{-t}-e^{-xt}}{1-e^{-t}}dt \tag{3.11}$$

*for* $x,y>0$, *and discussing the positivity of the functions*

$$\frac{12-t^2e^{-\alpha t}}{12(1-e^{-t})}-\frac{1}{2}-\frac{1}{t}\quad\text{and}\quad\frac{1}{2}+\frac{1}{t}-\frac{12-t^2}{12(1-e^{-t})} \tag{3.12}$$

*for* $x\in(0,\infty)$ *and* $\alpha\ge\frac{1}{2}$. *Therefore, H. Alzer essentially gave in [[5], Theorem 1] necessary and sufficient conditions for the function (3.7) to be logarithmically completely monotonic on* $(0,\infty)$.

**Remark 3.7.** *In [[41], Theorem 3], a slight extension of [[5], Theorem 1] was presented: The function*

$$\frac{\Gamma(x+s)}{\Gamma(x+t)}\frac{(x+t)^{x+t-1/2}}{(x+s)^{x+s-1/2}}\times\exp\left[s-t+\frac{\psi'(x+t+\alpha)-\psi'(x+s+\alpha)}{12}\right] \tag{3.13}$$

*for* $0<s<t$ *and* $x\in(0,\infty)$ *is logarithmically completely monotonic if and only if* $\alpha\ge\frac{1}{2}$, *so is the reciprocal of (3.13) if and only if* $\alpha=0$.

*The decreasing monotonicity of (3.13) and its reciprocal imply that the double inequality*

$$\begin{aligned}&\exp\left[t-s+\frac{\psi'(x+s+\beta)-\psi'(x+t+\beta)}{12}\right]\\ &\le\frac{(x+t)^{x+t-1/2}}{(x+s)^{x+s-1/2}}\frac{\Gamma(x+s)}{\Gamma(x+t)}\\ &\le\exp\left[t-s+\frac{\psi'(x+s+\alpha)-\psi'(x+t+\alpha)}{12}\right]\end{aligned} \tag{3.14}$$

*for* $\alpha>\beta\ge 0$ *are valid for* $0<s<t$ *and* $x\in(0,\infty)$ *if and only if* $\beta=0$ *and* $\alpha\ge\frac{1}{2}$.

*It is obvious that the inequality (3.14) is a slight extension of the double inequality (3.8) obtained in [[5], Corollary 2].*

**Remark 3.8.** *Specially we notice that [[33], Theorem 3.4] has been corrected in [[34], Theorem 3.4] as follows: Let* $0<q<1$, $0<s<1$, *and*

$$g_\alpha(x)=(1-q)^x\left(q^{-x}-1\right)\Gamma_q(x)\exp\left[\frac{F\left(q^x\right)}{\ln q}-\frac{\psi_q'(x+\alpha)}{12}\right],$$

*where*

$$F(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}=-\int_0^x\frac{\ln(1-t)}{t}dt.$$

*Then the function* $\left[\ln g_\alpha(x)\right]'$ *is completely monotonic on* $(0,\infty)$ *for* $\alpha\ge\frac{1}{2}$ *and the function* $-\left[\ln g_\alpha(x)\right]'$ *is completely monotonic on* $(0,\infty)$ *for* $\alpha\le 0$.

*As a consequence of [[34], Theorem 3.4], the following result was deduced in [[34], Corollary 3.5]: Let* $0<q<1$, $0<s<1$, *and*

$$f_\alpha(x)=\frac{g_\alpha(x+s)}{g_\alpha(x+1)}=\frac{(1-q)^{s-1}\left(1-q^{x+s}\right)^{1/2}\Gamma_q(x+s)}{\left(1-q^{x+1}\right)^{1/2}\Gamma_q(x+1)}\exp\left[\frac{F\left(q^{x+s}\right)-F\left(q^{x+1}\right)}{\ln q}+\frac{\psi_q'(x+1+\alpha)-\psi_q'(x+s+\alpha)}{12}\right]. \tag{3.15}$$

*Then the function* $\left[\ln f_\alpha(x)\right]'$ *is completely monotonic on* $(0,\infty)$ *for* $\alpha\ge\frac{1}{2}$, *the function* $-\left[\ln f_\alpha(x)\right]'$ *is complete monotonic on* $(0,\infty)$ *for* $\alpha\le 0$, *and neither is completely monotonic on* $(0,\infty)$ *for* $1<\alpha<\frac{1}{2}$.

*Taking the limit* $q\to 1^-$ *in (3.15) yields [[34], Corollary 3.6], a recovery, in a slightly extended form, of [[5], Theorem 1] mentioned above.*

*The preprint [34] is a corrected version of the conference paper [33].*

*It is clear that [[41], Theorem 3] can be derived by taking the limit*

$$\lim_{q\to 1^-}\frac{g_\alpha(x+s)}{g_\alpha(x+t)} \tag{3.16}$$

*for* $0<s<t$.

## 3.3. Ismail-Muldoon's Complete Monotonicity Results

Inspired by inequalities (2.7) and (2.14), Ismail and Muldoon proved in [[33], Theorem 3.2] the following conclusions: For $0<a<b$ and $0<q<1$, let

$$h(x)=\ln\left\{\frac{\Gamma_q(x+a)}{\Gamma_q(x+b)}\exp\left[(b-a)\psi_q(x+c)\right]\right\}. \tag{3.17}$$

If $c\ge\frac{a+b}{2}$, then $-h'(x)$ is completely monotonic on $(-a,\infty)$; if $c\le a$, then $h'(x)$ is completely monotonic on $(-c,\infty)$; Neither $h'(x)$ or $-h'(x)$ is completely monotonic for $a<c<\frac{a+b}{2}$. Consequently, the following inequality was deduced in [[33], Theorem 3.3]: If $0<q<1$, the inequality

$$\frac{\Gamma_q(x+1)}{\Gamma_q(x+s)}<\exp\left[(1-s)\psi_q\left(x+\frac{s+1}{2}\right)\right] \tag{3.18}$$

for $0<s<1$ holds for $x>-s$.

Influenced by (3.18), H. Alzer posed in the final of the paper [[4], p. 13] the following open problem: For real numbers $0<q\ne 1$ and $s\in(0,1)$, determine the best possible values $a(q,s)$ and $b(q,s)$ such that the inequalities

$$\exp\left[(1-s)\psi_q(x+a(q,s))\right]<\frac{\Gamma_q(x+1)}{\Gamma_q(x+s)}<\exp\left[(1-s)\psi_q(x+b(q,s))\right]$$

hold for all $x>0$.

**Remark 3.9.** *Since the paper [33] was published in a conference proceedings, it is not easy to acquire it, so the completely monotonic properties of the function* $h(x)$, *obtained in [[33], Theorem 3.2], were neglected in most circumstances.*

## 3.4. Elezović-Giordano-Pečarić's Inequality and Monotonicity Results

Inspired by the double inequality (2.14), the following problem was posed in [[19], p. 247]: What are the best constants $\alpha$ and $\beta$ such that the double inequality

$$\psi(x+\alpha)\le\frac{1}{t-s}\int_s^t\psi(u)du\le\psi(x+\beta) \tag{3.19}$$

holds for $x>-\min\{s,t,\alpha,\beta\}$?

An answer to the above problem was procured in [[19], Theorem 4]: The double inequality

$$\psi\left(x+\psi^{-1}\left(\frac{1}{t-s}\int_s^t\psi(u)du\right)\right)<\frac{1}{t-s}\int_s^t\psi(x+u)du<\psi\left(x+\frac{s+t}{2}\right) \tag{3.20}$$

is valid for every $x\ge 0$ and positive numbers $s$ and $t$.

Moreover, the function

$$\psi\left(x+\frac{s+t}{2}\right)-\frac{1}{t-s}\ln\frac{\Gamma(x+t)}{\Gamma(x+s)} \tag{3.21}$$

for $s,t>0$ and $r=\min\{s,t\}$ was proved in [[19], Theorem 5] to be completely monotonic on $(-r,\infty)$.

**Remark 3.10.** *It is clear that [[19], Theorem 5] stated above extends or generalizes the complete monotonicity of the function (3.1).*

**Remark 3.11.** *By the way, the complete monotonicity in [[19], Theorem 5] was iterated in [[94], Proposition 4] and [[95], Proposition 4] as follows: The function*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(s-t)}\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right] \tag{3.22}$$

*is logarithmically completely monotonic with respect to $x$ on $(-\alpha,\infty)$, where $s$ and $t$ are real numbers and $\alpha=\min\{s,t\}$.*

**Remark 3.12.** *Along the same line as proving the inequality (3.20) in [19], the inequality (3.20) was generalized in [[16], Theorem 2] as*

$$\begin{aligned}(-1)^n\psi^{(n)}&\left(x+\left(\psi^{(n)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t\psi^{(n)}(u)du\right)\right)\\ &<\frac{(-1)^n\left[\psi^{(n-1)}(x+t)-\psi^{(n-1)}(x+s)\right]}{t-s}\\ &<(-1)^n\psi^{(n)}\left(x+\frac{s+t}{2}\right)\end{aligned} \tag{3.23}$$

*for $x>0$, $n\ge0$, and $s,t>0$, where $\left(\psi^{(n)}\right)^{-1}$ denotes the inverse function of $\psi^{(n)}$.*

**Remark 2.13.** *Since the inverse functions of the psi and polygamma functions are involved, it is much difficult to calculate the lower bounds in (3.20) and (3.23).*

**Remark 2.14.** *In [36], by the method used in [35], it was proved that the double inequality*

$$\begin{aligned}\psi\left(x+\sqrt{st}\right)&<\frac{\ln\Gamma(x+t)-\ln\Gamma(x+s)}{t-s}\\ &<\psi\left(x+\frac{s+t}{2}\right)\end{aligned} \tag{3.24}$$

*holds for $s,t>0$. It s clear that the upper bound in (3.24) is a recovery of (3.20) and an immediate consequence of the complete monotonicity of the function (3.21).*

## 4. Two Logarithmically Complete Monotonicity Results

Suggested by the double inequality (2.14), it is natural to put forward the following problem: What are the best constants $\delta_1(s,t)$ and $\delta_2(s,t)$ such that

$$\begin{aligned}\exp\left[\psi\left(x+\delta_1(s,t)\right)\right]&\le\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}\\ &\le\exp\left[\psi\left(x+\delta_2(s,t)\right)\right]\end{aligned} \tag{4.1}$$

is valid for $x>-\min\{s,t,\delta_1(s,t),\delta_2(s,t)\}$ ? where $s$ and $t$ are real numbers.

It is clear that the inequality (4.1) can also be rewritten as

$$\begin{aligned}&\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}\exp\left[\psi(x+\delta_1)\right]\le1\\ &\le\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}\exp\left[\psi(x+\delta_2)\right]\end{aligned} \tag{4.2}$$

which suggests some monotonic properties of the function

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}\exp\left[-\psi\left(x+\delta(s,t)\right)\right], \tag{4.3}$$

since the limit of the function (4.3) as $x\to\infty$ is 1 by using (2.12).

This problem was considered in [52,53,85,86] along two different approaches and the following results of different forms were established.

**Theorem 4.1.** [[52], Theorem 1] and [[53], Theorem 1] *Let $a,b,c$ be real numbers and $\rho=\min\{a,b,c\}$. Define*

$$F_{a,b;c}(x)=\begin{cases}\left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(a-b)}\exp\left[\psi(x+c)\right], a\ne b\\ \exp\left[\psi(x+c)-\psi(x+a)\right], a=b\ne c\end{cases}$$

*for $x\in(-\rho,\infty)$. Furthermore, let $\theta(t)$ be an implicit function defined by equation*

$$e^t-t=e^{\theta(t)}-\theta(t) \tag{4.4}$$

*on $(-\infty,\infty)$. Then $\theta(t)$ is decreasing and $t\theta(t)<0$ for $\theta(t)\ne t$, and*

1. $F_{a,b;c}(x)$ is logarithmically completely monotonic on $(-\rho,\infty)$ if

$$(a,b;c)\in\{c\ge a,c\ge b\}\cup\{c\ge a,c-b\ge\theta(c-a)\}\cup\{c\le a,c-b\ge\theta(c-a)\}\setminus\{a=b=c\};$$

2. $\left[F_{a,b;c}(x)\right]^{-1}$ is logarithmically completely monotonic on $(-\rho,\infty)$ if

$$(a,b;c)\in\{c\le a,c\le b\}\cup\{c\ge a,c-b\le\theta(c-a)\}\cup\{c\le a,0\le c-b\le\theta(c-a)\}\setminus\{a=b=c\}.$$

**Theorem 4.2.** [[85], Theorem 1] and [[86], Theorem 1] *For real numbers $s$ and $t$ with $s\ne t$ and $\theta(s,t)$ a constant depending on $s$ and $t$, define*

$$v_{s,t}(x)=\frac{1}{\exp\left[\psi\left(x+\theta(s,t)\right)\right]}\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}. \tag{4.5}$$

1. *The function $v_{s,t}(x)$ is logarithmically completely monotonic on the interval $(-\theta(s,t),\infty)$ if and only if $\theta(s,t)\le\min\{s,t\}$;*

2. *The function* $\left[v_{s,t}(x)\right]^{-1}$ *is logarithmically completely monotonic on the interval* $\left(-\min\{s,t\},\infty\right)$ *if and only if* $\theta(s,t)\ge\frac{s+t}{2}$.

**Remark 4.1.** *In [52,53], it was deduced by standard argument that*

$$
\begin{aligned}
&(-1)^i\left[\ln F_{a,b;c}(x)\right]^{(i)}\\
&=\int_0^\infty\left[\frac{e^{(c-a)u}-e^{(c-b)u}}{u(b-a)}-1\right]\frac{u^i e^{-(x+c)u}}{1-e^{-u}}du\\
&=\int_0^\infty\left[\frac{\left\{\left[e^{(c-a)u}-(c-a)u\right]-\left[e^{(c-b)u}-(c-b)u\right]\right\}}{\left[(c-a)-(-b)\right]u}\right]\\
&\qquad\times\frac{u^i e^{-(x+c)u}}{1-e^{-u}}du
\end{aligned}
$$

*for* $i\in\mathbb{N}$ *and* $a\ne b$. *Therefore, the sufficient conditions in [[52], Theorem 1] and [[53], Theorem 1] are stated in terms of the implicit function [Trial mode] defined by (4.4).*

**Remark 4.2.** *In [85,86], the logarithmic derivative of* $v_{s,t}(x)$ *was rearranged as*

$$
\ln v_{s,t}(x)=\int_0^\infty\frac{e^{-[x+\theta(s,t)]u}}{1-e^{-u}}\left(1-e^{u[\theta(s,t)+\ln p_{s,t}(u)]}\right)du,
$$

*where*

$$
p_{s,t}(u)=\left(\frac{1}{t-s}\int_s^t e^{-uv}dv\right)^{1/u}. \tag{4.6}
$$

*Since the function* $p_{s,t}(u)$ *is increasing on* $[0,\infty)$ *with*

$$
\lim_{u\to 0}p_{s,t}(u)=e^{-(s+t)/2}
$$

*and*

$$
\lim_{u\to\infty}p_{s,t}(u)=e^{-\min\{s,t\}},
$$

*the necessary and sufficient conditions in [[85], Theorem 1] and [[86], Theorem 1] may be derived immediately by considering Theorem 1.1.*

*However, the necessary conditions in [[85], Theorem 1] and [[86], Theorem 1] were proved by establishing the following inequalities involving the polygamma functions and their inverse functions in [[85], Theorem 1] and [[86], Theorem 1]:*

1. *If* $m>m\ge 0$ *are two integers, then*

$$
\begin{aligned}
&\left(\psi^{(m)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t\psi^{(m)}(v)dv\right)\\
&\le\left(\psi^{(n)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t\psi^{(n)}(v)dv\right),
\end{aligned}\tag{4.7}
$$

*where* $\left(\psi^{(k)}\right)^{-1}$ *stands for the inverse function of* $\psi^{(k)}$ *for* $k\ge 0$;

2. *The inequality*

$$
\psi^{(i)}\left(L(s,t)\right)\le\frac{1}{t-s}\int_s^t\psi^{(i)}(u)du \tag{4.8}
$$

*is valid for* $i$ *being positive odd number or zero and reversed for* $i$ *being positive even number;*

3. *The function*

$$
\left(\psi^{(\ell)}\right)^{-1}\left(\frac{1}{t-s}\int_s^t\psi^{(\ell)}(x+v)dv\right)-x \tag{4.9}
$$

*for* $\ell\ge 0$ *is increasing and concave in* $x>-\min\{s,t\}$ *and has a sharp upper bound* $\frac{s+t}{2}$.

*Note that if taking [Trial mode], [Trial mode], [Trial mode] and [Trial mode] in (4.7), (4.8) and (4.9), then [[20], Lemma 1] and [[20], Theorem 6] may be derived straightforwardly.*

## 5. Recent Bounds and Monotonicity Results

In this section, we collect some recent bounds for the ratio of two gamma functions and gather several monotonicity results of functions involving the ratio of two gamma functions, divided differences of polygamma functions and mean values. Finally, we pose a conjecture.

### 5.1. Elezović-Pečarić's Lower Bound

The inequality (4.8) for $i=0$, that is, [[20], Lemma 1], may be rewritten as

$$
\frac{\ln\Gamma(t)-\ln\Gamma(s)}{t-s}\ge\psi\left(L(s,t)\right) \tag{5.1}
$$

or, equivalently,

$$
\left[\frac{\Gamma(t)}{\Gamma(s)}\right]^{1/(t-s)}\ge e^{\psi(L(s,t))} \tag{5.2}
$$

for positive numbers $s$ and $t$.

**Remark 5.1.** *From the left-hand side inequality in (1.11), it is easy to see that the inequality (5.2) refines the traditionally lower bound* $e^{\psi(G(s,t))}$.

**Remark 5.2.** *In [[9], Theorem 2.4], the following incorrect double inequality was obtained:*

$$
e^{(x-y)\psi(L(x+1,y+1)-1)}\le\frac{\Gamma(x)}{\Gamma(y)}\le e^{(x-y)\psi(A(x,y))}, \tag{5.3}
$$

*where* $x$ *and* $y$ *are positive real numbers. Accurately speaking, the left-hand side inequality in (5.3) should be (5.2). See the first proof of [[55], Theorem 1] or Section 5.5 below.*

### 5.2. Allasia-Giordano-Pečarić's Inequalities

In Section 4 of [3], as straightforward consequences of Hadamard type inequalities obtained in [2], the following double inequalities for bounding $\ln\frac{\Gamma(y)}{\Gamma(x)}$ were listed: For $y>x>0$, $n\in\mathbb{N}$ and $h=\frac{y-x}{n}$, we have

$$\frac{h}{2}[\psi(x)+\psi(y)]+h\sum_{k=1}^{n-1}\psi(x+kh)<\ln\frac{\Gamma(y)}{\Gamma(x)}$$
$$<h\sum_{k=1}^{n-1}\psi\left(x+\left(k+\frac{1}{2}\right)h\right),$$
$$0<h\sum_{k=1}^{n-1}\psi\left(x+\left(k+\frac{1}{2}\right)h\right)-\ln\frac{\Gamma(y)}{\Gamma(x)}$$
$$<\ln\frac{\Gamma(y)}{\Gamma(x)}-\frac{h}{2}[\psi(x)+\psi(y)]-h\sum_{k=1}^{n-1}\psi(x+kh),$$
$$\frac{h}{2}[\psi(x)+\psi(y)]+h\sum_{k=1}^{n-1}\psi(x+kh)$$
$$-\sum_{i=1}^{m-1}\frac{B_{2i}h^{2i}}{(2i)!}\left[\psi^{(2i-1)}(y)-\psi^{(2i-1)}(x)\right]$$
$$<\ln\frac{\Gamma(y)}{\Gamma(x)}<h\sum_{k=1}^{n-1}\psi\left(x+\left(k+\frac{1}{2}\right)h\right)$$
$$-\sum_{i=1}^{m-1}\frac{B_{2i}h^{2i}}{(2i)!}\left[\psi^{(2i-1)}(y)-\psi^{(2i-1)}(x)\right],$$
$$0<h\sum_{k=1}^{n-1}\psi\left(x+\left(k+\frac{1}{2}\right)h\right)$$
$$-\sum_{i=1}^{m-1}\frac{B_{2i}(1/2)h^{2i}}{(2i)!}\left[\psi^{(2i-1)}(y)-\psi^{(2i-1)}(x)\right]-\ln\frac{\Gamma(y)}{\Gamma(x)}$$
$$<\ln\frac{\Gamma(y)}{\Gamma(x)}-\frac{h}{2}[\psi(x)+\psi(y)]-h\sum_{k=1}^{n-1}\psi(x+kh)$$
$$+\sum_{i=1}^{m-1}\frac{B_{2i}h^{2i}}{(2i)!}\left[\psi^{(2i-1)}(y)-\psi^{(2i-1)}(x)\right],$$
$$h\sum_{k=1}^{n-1}\psi\left(x+\left(k+\frac{1}{2}\right)h\right)$$
$$-\sum_{i=1}^{m-2}\frac{B_{2i}(1/2)h^{2i}}{(2i)!}\left[\psi^{(2i-1)}(y)-\psi^{(2i-1)}(x)\right]$$
$$<\ln\frac{\Gamma(y)}{\Gamma(x)}<h\sum_{k=0}^{n-1}\psi\left(x+\left(k+\frac{1}{2}\right)h\right)$$
$$-\sum_{i=1}^{m-1}\frac{B_{2i}(1/2)h^{2i}}{(2i)!}\left[\psi^{(2i-1)}(y)-\psi^{(2i-1)}(x)\right],$$

and

$$\frac{h}{2}[\psi(x)+\psi(y)]+h\sum_{k=1}^{n-1}\psi(x+kh)$$
$$-\sum_{i=1}^{m-1}\frac{B_{2i}h^{2i}}{(2i)!}\left[\psi^{(2i-1)}(y)-\psi^{(2i-1)}(x)\right]$$
$$<\ln\frac{\Gamma(y)}{\Gamma(x)}<\frac{h}{2}[\psi(x)+\psi(y)]+h\sum_{k=1}^{n-1}\psi(x+kh)$$
$$-\sum_{i=1}^{m-2}\frac{B_{2i}h^{2i}}{(2i)!}\left[\psi^{(2i-1)}(y)-\psi^{(2i-1)}(x)\right],$$

where $m$ is an odd and positive integer,

$$B_k\left(\frac{1}{2}\right)=\left(\frac{1}{2^{k-1}}-1\right)B_k, k\ge 0 \tag{5.4}$$

and $B_i$ for $i\ge 0$ are Bernoulli numbers defined by

$$\frac{t}{e^t-1}=\sum_{i=0}^{\infty}B_i\frac{t^i}{i!}=1-\frac{x}{2}+\sum_{j=0}^{\infty}B_{2i}\frac{x^{2j}}{(2j)!},|x|<2\pi.$$

If replacing $m$ by an even and positive integer, then the last four double inequalities are reversed.

## 5.3. Batir's Double Inequality for Polygamma Functions

It is clear that the double inequality (2.14) can be rearranged as

$$\psi\left(x+\sqrt{s}\right)<\frac{\ln\Gamma(x+1)-\ln\Gamma(x+s)}{1-s}<\psi\left(x+\frac{s+1}{2}\right) \tag{5.5}$$

for $0<s<1$ and $x>1$. The middle term in (5.5) can be regarded as a divided difference of the function $\ln\Gamma(t)$ on $(x+s,x+1)$. Stimulated by this, N. Batir extended and generalized in [[8], Theorem 2.7] the double inequality (5.5) as

$$-\left|\psi^{(n+1)}\left(L_{-(n+2)}(x,y)\right)\right|<\frac{\left|\psi^{(n)}(x)\right|-\left|\psi^{(n)}(y)\right|}{x-y}<-\left|\psi^{(n+1)}(A(x,y))\right| \tag{5.6}$$

where $x,y$ are positive numbers and $n\in\mathbb{N}$.

## 5.4. Chen's Double Inequality in Terms of Polygamma Functions

In [[15], Theorem 2], by virtue of the composite Simpson rule

$$\int_a^b f(t)dt=\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{(b-a)^5}{2880}f^{(4)}(\xi),\xi\in(a,b)$$

in [31] and the formula

$$\frac{1}{y-x}\int_x^y f(t)dt=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{y-x}{2}\right)^{2k}f^{(2k)}\left(\frac{x+y}{2}\right)$$

in [42], the following double inequalities and series representations were trivially shown: For $n\in\mathbb{N}$ and positive numbers $x$ and $y$ with $x\ne y$,

$$\frac{1}{3}A(\psi(x),\psi(y))+\frac{2}{3}\psi(A(x,y))-\frac{(y-x)^4}{2880}\psi^{(4)}(\max\{x,y\})<\frac{\ln\Gamma(y)-\ln\Gamma(x)}{y-x}$$
$$<\frac{1}{3}A(\psi(x),\psi(y))+\frac{2}{3}\psi(A(x,y))-\frac{(y-x)^4}{2880}\psi^{(4)}(\min\{x,y\}),$$
$$(-1)^{n-1}\left[\frac{A\left(\psi^{(n)}(x),\psi^{(n)}(y)\right)}{3}+\frac{2\psi^{(n)}(A(x,y))}{3}-\frac{(y-x)^4\psi^{(n+4)}(\min\{x,y\})}{2880}\right]$$
$$<\frac{(-1)^{n-1}\left[\psi^{(n-1)}(y)-\psi^{(n-1)}(x)\right]}{y-x}$$
$$<(-1)^{n-1}\left[\frac{A\left(\psi^{(n)}(x),\psi^{(n)}(y)\right)}{3}+\frac{2\psi^{(n)}(A(x,y))}{3}-\frac{(y-x)^4\psi^{(n+4)}(\max\{x,y\})}{2880}\right],$$

$$\frac{\ln\Gamma(y)-\ln\Gamma(x)}{y-x}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{y-x}{2}\right)^{2k}\psi^{(2k)}\left(\frac{x+y}{2}\right),$$

$$\frac{\psi^{(n-1)}(y)-\psi^{(n-1)}(x)}{y-x}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{y-x}{2}\right)^{2k}\psi^{(2k+n)}\left(\frac{x+y}{2}\right).$$

## 5.5. Recent Monotonicity Results by Qi and His Coauthors

Motivated by the left-hand side inequality in (5.3), although it is not correct, several refinements and generalizations about inequalities (5.2) and (5.6) were established by Qi and his coauthors in recent years.

### 5.5.1.

In [[55], Theorem 1] and [[56], Theorem 1], by virtue of the method used in [[9], Theorem 2.4] and the inequality (4.8) for i = 0, the inequality (5.2) and the right-hand side inequality in (5.3) were recovered.

### 5.5.2.

In [[55], Theorem 2] and [[56], Theorem 2], the decreasing monotonicity of the function (3.2) and the right-hand side inequality in (3.20) were extended and generalized to the logarithmically complete monotonicity, and the inequality (5.2) was generalized to a decreasing monotonicity.

**Theorem 5.1** ([[55], Theorem 2] and [[56], Theorem 2]).

*For $s,t\in\mathbb{R}$ with $s\neq t$, the function*

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)}\frac{1}{e^{\psi(L(s,t;x))}} \tag{5.7}$$

*is decreasing and*

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(t-s)}e^{\psi(A(s,t;x))} \tag{5.8}$$

*is logarithmically completely monotonic on $(-\min\{s,t\},\infty)$, where*

$$\mathrm{L}(s,t;x)=\mathrm{L}(x+s,x+t),\mathrm{A}(s,t;x)=\mathrm{A}(x+s,x+t)$$

### 5.5.3.

In [97,98], the upper bounds in (2.14), (3.20), (5.3), (5.6) and related inequalities in [52,53,85,86] were refined and extended as follows.

**Theorem 5.2** ([97,98]). *The inequalities*

$$\left[\frac{\Gamma(a)}{\Gamma(b)}\right]^{1/(a-b)}\leq e^{\psi(I(a,b))} \tag{5.9}$$

*and*

$$\frac{(-1)^{n-1}\left[\psi^{(n-1)}(a)-\psi^{(n-1)}(b)\right]}{a-b}\leq(-1)^{n}\psi^{(n)}(I(a,b)) \tag{5.10}$$

*for a > 0 and b > 0 hold true.*

**Remark 5.3.** *The basic tools to prove (5.9) and (5.10) are an inequality in [14] and and a complete monotonicity in [101] respectively. They may be recited as follows:*

*1. If g is strictly monotonic, f is strictly increasing, and $f\circ g^{-1}$ is convex (or concave, respectively) on an interval I, then*

$$g^{-1}\left(\frac{1}{t-s}\int_s^t g(u)\,du\right)\leq f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)\,du\right) \tag{5.11}$$

*holds (or reverses, respectively) for $s,t\in I$. See also [[13], p. 274, Lemma 2] and [[20], p. 190, Theorem A].*

*2. The function*

$$x\left|\psi^{(i+1)}(x)\right|-\alpha\left|\psi^{(i)}(x)\right|, i\in\mathbb{N} \tag{5.12}$$

*is completely monotonic on $(0,\infty)$ if and only if $0\leq a\leq i$. See also [99,100].*

**Remark 5.4.** *By the so-called G-A convex approach, the inequality (5.9) was recovered in [120]: For b > a > 0,*

$$[b-L(a,b)]\psi(b)+[L(a,b)-a]\psi(a)<\ln\frac{\Gamma(b)}{\Gamma(a)}<(b-a)\psi(I(a,b)).$$

*See also MR2413632, the review by MathSciNet of the paper [120]. Moreover, by the so-called geometrically convex method, the following double inequality was shown in [[121], Theorem 1.2]: For positive numbers $x$ and $y$,*

$$\frac{x^x}{y^y}\left(\frac{x}{y}\right)^{y[\psi(y)-\ln y]}e^{y-x}\leq\frac{\Gamma(x)}{\Gamma(y)}\leq\frac{x^x}{y^y}\left(\frac{x}{y}\right)^{x[\psi(x)-\ln x]}e^{y-x}$$

### 5.5.4.

In [99,100,101], the function

$$\alpha\left|\psi^{(i)}(x)\right|-x\left|\psi^{(i+1)}(x)\right| \tag{5.13}$$

was proved to be completely monotonic on $(0,\infty)$ if and only if $\alpha\geq i+1$. Utilizing the inequality (5.11) and the completely monotonic properties of the functions (5.12) and (5.13) yields the following double inequality.

**Theorem 5.3** ([[78], Theorem 1] and [[102], Theorem 1]). *For real numbers s > 0 and t > 0 with $s\neq t$ and an integer $i\geq 0$, the inequality*

$$(-1)^i\psi^{(i)}(L_p(s,t))\leq\frac{(-1)^i}{t-s}\int_s^t\psi^{(i)}(u)\,du\leq(-1)^i\psi^{(i)}(L_q(s,t)) \tag{5.14}$$

*holds if $p\leq -i-1$ and $q\geq -i$.*

**Remark 5.5.** *The double inequality (5.14) recovers, extends and refines inequalities (5.2), (5.6), (5.9) and (5.10).*

**Remark 5.6.** *A natural question is whether the above sufficient conditions $p\leq -i-1$ and $q\geq -i$ are also necessary for the inequality (5.14) to be valid.*

**5.5.5.**

As generalizations of the inequalities (5.2), (5.6), the decreasing monotonicity of the function (5.7), and the left-hand side inequality in (5.14), the following monotonic properties were presented.

**Theorem 5.4** ([[78], Theorem 3] and [[102], Theorem 3]). *If* $i \ge 0$ *is an integer,* $s,t \in \mathbb{R}$ *with* $s \ne t$ *, and* $x > -\min\{s,t\}$*, then the function*

$$(-1)^i \psi^{(i)}\left(L_p(s,t;x)\right) - \frac{(-1)^i}{t-s}\int_s^t \psi^{(i)}(x+u)\,du \quad (5.15)$$

*is increasing with respect to* $x$ *for either* $p \le -(i+2)$ *or* $p = -(i+1)$ *and decreasing with respect to* $x$ *for* $p \ge 1$*, where* $L_p(s,t;x) = L_p(x+s, x+t)$.

**Remark 5.7.** *It is not difficult to see that the ideal monotonic results of the function (5.15) should be stated as follows.*

**Conjecture 5.1.** *Let* $i \ge 0$ *be an integer,* $s,t \in \mathbb{R}$ *with* $s \ne t$*, and* $x > -\min\{s,t\}$*. Then the function (5.15) is increasing with respect to* $x$ *if and only if* $p \le -(i+1)$ *and decreasing with respect to* $x$ *if and only if* $p \ge -i$.

**Remark 5.8.** *Corresponding to Conjecture 5.1, the complete monotonicity of the function (5.15) and its negative may also be discussed.*

**Remark 5.9.** *This article is a slightly updated version of [63] and a companion paper of [61,105,106] and their preprints [62,64,65].*

**Remark 5.10.** *Finally, we would like to recommend the articles [25,26,27,28,30,37,40,43-50,54,59,60,68,73,74,75,77,81,82,108,109,111,116] and closely related references therein to the readers for finding new developments and applications of the gamma function, polygamma functions, completely monotonic functions, logarithmically completely monotonic functions, concerned inequalities, asymptotic approximations, and so on.*

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