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BOUNDS, INEQUALITIES, AND MONOTONICITY PROPERTIES FOR SOME SPECIALIZED RENEWAL PROCESSES[1]

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Renewal processes with increasing mean residual life and decreasing failure rate interarrival time distributions are investigated. Various two-sided bounds are obtained for $M(t)$, the expected number of renewals in $[0, t]$. It is shown that if the interarrival time distribution has increasing mean residual life with mean μ, then the expected forward recurrence time is increasing in $t \geqslant 0$, as is $M(t) - t/\mu$. If the interarrival time distribution has decreasing failure rate then $M(t)$ is concave, and the forward and backward recurrence time distributions are stochastically increasing in $t \geqslant 0$.

1. Introduction and summary. A random variable X with cdf F is defined to have an increasing mean residual life (IMRL) distribution on $[0, \infty)$ if $\mu_1 = EX < \infty$, $F(0^-) = 0$, $F(0) < 1$, and $E(X - t|X > t)$ is increasing for $t \geqslant 0$. The term increasing (decreasing) is used for monotone nondecreasing (nonincreasing). A random variable X is defined to have a decreasing failure rate (DFR) distribution on $[0, \infty)$ if $F(0^-) = 0$, $F(0) < 1$, and $\Pr(X > s + t|X > t)$ is increasing in $t \geqslant 0$ for each $s > 0$. The DFR property is equivalent to stochastically increasing residual life, which, when $EX < \infty$, implies increasing mean residual life. Thus F DFR and $EX < \infty$ implies F IMRL. It is easy to construct examples for which F is IMRL but not DFR.

In this paper renewal processes with IMRL and DFR interarrival times are studied. A summary of the main results now follows.

If F is DFR then the renewal function, M, is concave and the renewal density, m, which necessarily exists on $(0, \infty)$, is decreasing. The renewal age process, $\{A(t), t \geqslant 0\}$, and the forward recurrence time process, $\{Z(t), t \geqslant 0\}$, are stochastically increasing in t. For any Borel set $A \subset [0, \infty)$, $N(\{A + t\})$, the number of renewals in $A + t = \{x + t, x \in A\}$ is stochastically decreasing in $t \geqslant 0$.

If F is IMRL then the expected forward recurrence time is increasing as is $M(t) - t/\mu_1$. For any Borel set $A \subset [0, \infty)$, $EN(A) \geqslant \mu_1^{-1}l(A)$, where l is Lebesgue measure.

Bounds for the renewal function are obtained for F IMRL, with improvements for F DFR. If F is IMRL with renewal function M, $\mu_i' = EX^i = \int_0^\infty x^i\, dF(x)$, and

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$U(t) = t/\mu_1 + \mu_2/2\mu_1^2$, then $\mu_{k+2} < \infty$ for an integer $k \geqslant 0$ implies

$$U(t) \geqslant M(t) \geqslant U(t) - \min_{0 \leqslant i \leqslant k} c_i t^{-i}. \tag{1}$$

In (1) c_i is an explicitly computed expression involving $\mu_1, \cdots, \mu_{i+2}$. Quantities $0 = v_{-1} \leqslant v_0 \leqslant v_1 \cdots \leqslant v_k = \infty$ are found such that for $t \in [v_{j-1}, v_j]$, $c_j t^{-j} = \min_{0 \leqslant i \leqslant k} c_i t^{-i}$; thus the lower bound in (1) reduces to $U(t) - c_j t^{-j}$ for $t \in [v_{j-1}, v_j]$. It is further shown that if $\psi_F(a_0) = \int_0^\infty e^{a_0 t}\, dF(t) < \infty$ for an $a_0 > 0$, then for $0 < a \leqslant a_0$:

$$U(t) \geqslant M(t) \geqslant U(t) - c(F, a)(e^{at} - 1)^{-1} \tag{2}$$

where $c(F, a) = (\mu_1 a)^{-1} - (\mu_2/2\mu_1^2) - (\psi_F(a) - 1)^{-1}$ does not depend on t.

The results follow from a construction of two dependent renewal processes on the same probability space, one an ordinary renewal process and the other an appropriate delayed renewal process. The delayed renewal process has the property that for all k its kth renewal coincides with the $(N + k)$th renewal of the ordinary renewal process, where N is a random variable. By having a model in which the two processes differ in an easily understandable manner thay can readily be compared. The approach of comparing two processes by constructing a convenient bivariate version is widely used (see, for example, [9], [13], [19], [20], [21] and [22]) although the present construction appears new.

If $X - t | X > t$ is stochastically increasing (increasing in mean) it follows (Theorems 2 and 3) that $Z(t)$, the analogous quantity for the renewal process, is stochastically increasing (increasing in mean). More generally it would be of interest to understand the extent to which monotonicity and aging properties of a distribution are inherited in some fashion by its renewal process. Much work remains to be done on this problem.

The theory of DFR distributions is developed in Barlow [1], Barlow, Marshall and Proschan [6], Barlow and Marshall [2], [3], Barlow and Proschan [4], and Esary, Marshall and Proschan [10]. IMRL distributions are examined in Bryson and Siddiqui [8], Haines and Singpurwalla [12], and Barlow, Marshall and Proschan [6].

Mixtures of DFR distributions are DFR ([5], page 103); in particular, mixtures of exponential distributions are DFR. One would, therefore, expect that data collected by combining approximately exponentially distributed subpopulations would have an empirical distribution which would be DFR in appearance. This phenomena is discussed in Barlow and Proschan ([5], page 103). Keilson [14] has shown that a large class of first passage time distributions for Markov processes are DFR.

2. Representations. Let $X_0 \equiv 0$ and $X_1, X_2, \cdots$ be i.i.d. with distribution F. Define $S_0 = 0$, $S_n = \sum_{i=1}^n X_i$, $n = 1, 2, \cdots$, $N(t) = \max\{i\colon S_i \leqslant t\} = \{\# S_i \leqslant t\} - 1$, and $M(t) = EN(t) + 1 = E\{\# S_i \leqslant t\}$. Further, define $Z(t) = \sum_1^{N(t)+1} X_i - t$, the forward recurrence time at t, and $A(t) = t - \sum_1^{N(t)} X_i$, the renewal age at t. The process $\{N(t), t \geqslant 0\}$ is called an ordinary F renewal process. A class of delayed renewal processes is defined by treating the item at time zero as having a random age T; when $T = t$ the residual life, which is the first renewal epoch, X_0', for the delayed process, has distribution $\bar{F}_t(x) = \bar{F}(t + x)/\bar{F}(t)$, where $\bar{F} = 1 - F$. If H is the cdf of T then $\Pr(X_0' > t) = \int_0^\infty \bar{F}_t(x)\, dH(x)$. Then let $\{X_i', i \geqslant 1\}$ be i.i.d. with distribution F, and independent of X_0'. Define $S_n' = \Sigma_0^n X_i'$, $n \geqslant 0$, $S_{-1} = 0$, $N'(t) = \max\{i: S_i \leqslant t)$ and $M'(t) = EN'(t) + 1$. The process $\{N'(t), t \geqslant 0\}$ is called a delayed F renewal process with initial age distribution H. Note that this definition does not allow an arbitrary distribution for X_0' as the standard definition ([1] page 184). Define $A'(t) = t$ for $0 \leqslant t < S_0'$, $A'(t) = t - S_{N'(t)}'$ for $t \geqslant S_0'$. For a delayed renewal process, Z', A', N', M', X_i', S_n' will denote the analogues of Z, A, N, M, X_i, S_n for an ordinary renewal process. A delayed renewal process with initial age distribution $G(x) = \mu_1^{-1}\int_0^x \bar{F}(y)\, dy$ is called a stationary F renewal process. A stationary F renewal process satisfies $M'(t) = t/\mu_1$ and $Z'(t) \sim G$ for all $t \geqslant 0$ ([11] page 354). Note that the hazard function, h^*, of G satisfies $h^*(t) = 1/E(X - t|X > t)$, where $X \sim F$. Thus F IMRL $\Leftrightarrow$ G DFR.

For a distribution F satisfying $F(0^-) = 0$ and $F(0) < 1$, the definition of DFR given in Section 1 is equivalent to each of the conditions below:

(i) $\bar{F} = 1 - F$ is log convex.

(ii) F is absolutely continuous on $(0, \infty)$ with a density f possessing a version for which the hazard function $h(x) = f(x)/\bar{F}(x)$ is decreasing.

The fact that a DFR distribution on $[0, \infty)$, is absolutely continuous except perhaps for an atom at $\{0\}$ can be proved following an argument of Barlow and Proschan [5] page 77. The other implications are straightforward.

We will either assume F IMRL or DFR on $[0, \infty)$. Two dependent renewal processes will be constructed. Process 1 will be an ordinary F renewal process. Process 2 will be a stationary F renewal process in the IMRL case, and delayed F renewal process of the type described above in the DFR case. The special feature of process 2 is that $S_i' = S_{N+i}$, $i = 0, 1, \cdots$, for a random integer N. Process 1 and 2 differ only in that process 2 has zero renewals in $[0, S_N)$ while process 1 has N renewals in this interval. The construction is based on a simple idea which is obscured by the details of the construction and proof. The hazard function of F is decomposed into two components. The first component causes failure for both processes, the second component only causes failure for process 2. The construction uses the following lemma.

LEMMA 1. *Let X be distributed as F where F will either be assumed* IMRL *on* $[0, \infty)$ *or* DFR *on* $[0, \infty)$. *Set* $\bar{K}(t) = \bar{G}(t)$ *in the* IMRL *case and* $\bar{K}(t) = \int_0^\infty(\bar{F}(t + y)/\bar{F}(y))\, dH(y)$, *where H is an arbitrary probability distribution on* $[0, \infty)$, *in the* DFR *case. Define* $\bar{K}_v(t) = \bar{K}(t + v)/\bar{K}(v)$, $\bar{J}_v(t) = \bar{F}(t)/\bar{K}_v(t)$. *Then* $\bar{J}_v$ *is the survival function of a possibly defective distribution on* $[0, \infty)$.

PROOF. In the IMRL case $\bar{J}_v(t) = (\bar{F}(t)/\bar{G}(t))(\bar{G}(t)/\bar{G}(t + v))\bar{G}(v)$ and since G is DFR both $\bar{F}(t)/\bar{G}(t)$ and $\bar{G}(t)/\bar{G}(t + v)$ are decreasing. Thus $\bar{J}_v$ is decreasing.

In the DFR case $\bar{J}_v(t) = \bar{K}(v)[\int_0^\infty (\bar{F}(t + v + y)/\bar{F}(t))(\bar{F}(y))^{-1}\, dH(y)]^{-1}$. Since F is DFR the denominator is increasing, and then, since the numerator is constant, $\bar{J}_v$ is decreasing in t. Thus, in both cases $\bar{J}_v$ is decreasing. In addition J_v is right continuous, equals 1 for $t < 0$, and is always between 0 and 1. It is thus the survival function of a possibly defective distribution on $[0, \infty)$. □

We proceed with the construction. Again $\bar{K}(t) = \bar{G}(t)$ when F is assumed IMRL, and $\bar{K}(t) = \int_0^\infty (\bar{F}(t + y)/\bar{F}(y))\, dH(y)$ with H an arbitrary distribution on $[0, \infty)$ when F is assumed DFR.

Construct Z_1 and W_1 independent with $Z_1 \sim K$, $W_1 \sim J$ where $\bar{J}(t) = \bar{F}(t)/\bar{K}(t)$. If J is defective let $\Pr(Z_1 = \infty) = \lim_{t\to\infty}\bar{J}(t)$. If $Z_1 \leqslant W_1$ set $X_1 = X_0' = Z_1$ and $X_j = X_{j-1}' = Y_{j-1}, j = 2, 3, \cdots$ where $\{Y_i, i \geqslant 1\}$ is an i.i.d. sequence with distribution F independent of (Z_1, W_1). If $Z_1 > W_1$, set $X_1 = W_1$ and go to stage 2. At stage 2 construct Z_2 and W_2 conditionally independent of each other and of (Z_1, W_1) given W_1, with $Z_2|W_1 = v$ having distribution $\bar{K}_v(t) = \bar{K}(t + v)/\bar{K}(v)$, and $W_2|W_1 = v$ distribution $\bar{J}_v(t) = \bar{F}(t)/\bar{K}_v(t)$. If $Z_2 \leqslant W_2$ then set $X_2 = Z_2$, $X_0' = W_1 + Z_2$ and $X_j = X_{j-2}' = Y_{j-2}, j = 3, 4, \cdots$ where $\{Y_i, i = 1, 2, \cdots\}$ is i.i.d. with distribution F and independent of (Z_1, W_1, Z_2, W_2). If $W_2 < Z_2$ set $X_2 = W_2$ and go to stage 3. We reach stage m if and only if $W_i < Z_i, i = 1, \cdots, m - 1$, in which case $X_i = W_i, i = 1, \cdots, m - 1$. At stage m we construct Z_m and W_m conditionally independent of each other and of $(Z_1, W_1) \cdots (Z_{m-1}, W_{m-1})$ given $\Sigma_{i=1}^{m-1} W_i$, with $(Z_m|\Sigma_1^{m-1} W_i = v) \sim K_v$, $(W_m|\Sigma_1^{m-1} W_i = v) \sim J_v$. If $Z_m \leqslant W_m$ set $X_m = Z_m$, $X_0' = \Sigma_1^{m-1} W_i + Z_m$, $X_j' = X_{j-m}' = Y_{j-m}, j = m + 1, \cdots$ where $\{Y_i, i \geqslant 1\}$ is i.i.d. with distribution F and independent of $(Z_1, W_1, Z_2, W_2, \cdots, Z_{m-1}, W_{m-1})$. If $Z_m > W_m$ then go to stage $m + 1$ and repeat.

THEOREM 1. *Under the above construction*:

(i) $\{X_i, i \geqslant 1\}$ *is an* i.i.d. *sequence with distribution* F.

(ii) $\{X_i', i \geqslant 0\}$ *are independent*, $X_0' \sim K$, $X_j' \sim F$ *for* $j \geqslant 1$.

(iii) $S_i' = S_{N+i}$ *for* $i \geqslant 0$, *where* $N = \min\{i\colon Z_i \leqslant W_i\}$, *and* $\Pr(N < \infty) = 1$.

PROOF. (i) Define $N = \min\{i\colon Z_i \leqslant W_i\}$, $N = \infty$ if $W_i < Z_i$ for all i. Now $(X_i|N \geqslant i, (W_j, Z_j) = (w_j, z_j), j = 1, \cdots, i - 1) \sim \min(Z_v^*, W_v^*)$ where $v = \Sigma_1^{i-1} w_j$, $Z_v^* \sim K_v$, $W_v^* \sim J_v$ and Z_v^* and W_v^* are independent. Since $\bar{K}_v(t)\bar{J}_v(t) = \bar{F}(t)$, $(X_i|N \geqslant i, (W_j, Z_j) = (w_j, z_j), j = 1, \cdots, i - 1) \sim F$. Thus $X_i \sim F$ independent of $X_1, \cdots, X_{i-1}$ for all i; thus $\{X_j, j \geqslant 1\}$ is i.i.d. with distribution F.

(ii) In our construction we generated $W_1, \cdots, W_N$. It will not be convenient to continue constructing W_j's for $j > N$. At stage j construct W_j to be conditionally independent of $W_1, \cdots, W_{j-1}$ given $\Sigma_1^{j-1} W_i$, with $(W_j|\Sigma_1^{j-1} W_i = v) \sim J_v$. Since K is DFR, $\inf_i \Pr(W_i > t) \geqslant \inf_v \Pr(W_v > t) = \bar{F}(t)/\lim_{v\to\infty}\bar{K}_v(t) > 0$. Therefore, $\Pr(\Sigma_1^\infty W_i = \infty) = 1$, so that given t, with probability 1 there exists j so that

$\Sigma_1^{j-1} W_i < t \leqslant \Sigma_1^j W_i$. Then:

$$\Pr(X_0' > t | w_1, w_2, \cdots) = \left(\Pi_{l=1}^{j-1} \bar{K}_{\Sigma_1^{l-1} w_i}(w_l)\right)\left(\bar{K}_{\Sigma_1^{j-1} w_i}(t - \Sigma_1^{j-1} w_i)\right) = \bar{K}(t).$$

Thus $X_0' \sim K$.

(iii) Since $\Sigma_1^\infty W_i = \infty$ with probability 1, $X_0' < \infty$ if and only if $N < \infty$. By (ii), $X_0' \sim K$ thus $\Pr(X_0' < \infty) = 1$ and, therefore, $\Pr(N < \infty) = 1$. By construction $S_i' = S_{N+i}$ for $i \geqslant 0$. □

3. Some properties of IMRL and DFR renewal processes. We will need the following well-known result (Feller [11] page 148).

LEMMA 2. *Let F be a distribution on $[0, \infty)$. If $\mu_1 < \infty$ define $\bar{F}_1(t) = \int_t^\infty \bar{F}(x)\,dx$; if $\mu_2 < \infty$ define $\bar{F}_2(t) = \int_t^\infty \bar{F}_1(x)\,dx$. Then:*

(i) *$\mu_k < \infty \Rightarrow t^k \bar{F}(t) \to 0$ as $t \to \infty$; for $k \geqslant 0$, $\mu_{k+1} < \infty \Rightarrow \int_0^\infty t^k \bar{F}(t)\,dt = \mu_{k+1}/k + 1 < \infty$ and $t^k \bar{F}_1(t) \to 0$ as $t \to \infty$; for $k \geqslant 0$, $\mu_{k+2} < \infty \Rightarrow \int_0^\infty t^k \bar{F}_1(t)\,dt = \mu_{k+2}/(k+1)(k+2) < \infty$ and $t^k \bar{F}_2(t) \to 0$ as $t \to \infty$; for $k \geqslant 0$, $\mu_{k+3} < \infty \Rightarrow \int_0^\infty t^k \bar{F}_2(t)\,dt = \mu_{k+3}/(k+1)(k+2)(k+3) < \infty$.*

(ii) *For $a > 0$, $\psi_F(a) = \int_0^\infty e^{at}\,dF(t) < \infty$ implies $e^{at}\bar{F}(t)$, $e^{at}\bar{F}_1(t)$, and $e^{at}\bar{F}_2(t)$ converge to 0 as $t \to \infty$. Moreover $\int_0^\infty e^{at}\bar{F}(t)\,dt = a^{-1}(\psi_F(a) - 1) < \infty$, $\int_0^\infty e^{at}\bar{F}_1(t)\,dt = a^{-2}(\psi_F(a) - a\mu_1 - 1) < \infty$, and $\int_0^\infty e^{at}\bar{F}_2(t)\,dt = a^{-3}(\psi_F(a) - a^2(\mu_2/2) - a\mu_1 - 1) < \infty$.*

If $\mu_2 = EX^2 < \infty$, define $L(t) = M(t) - t/\mu_1 - \mu_2/2\mu_1^2$. Define $q = \bar{F}(0) = \Pr(X > 0)$. For nonnegative extended real-valued random variables $EX \geqslant EY$ is defined to mean $\infty \geqslant EX \geqslant EY$. For extended real-valued random variables, $X \geqslant_{st} Y$ (X stochastically greater than Y) means $\Pr(X > a) \geqslant \Pr(Y > a)$ for all finite a. In Theorems 2 and 3 below, many of the random variables may be improper (assign positive probability to $\pm\infty$) and the above conventions for $EX \geqslant EY$ and $X \geqslant_{st} Y$ apply.

THEOREM 2. *Let $\{N(t), t \geqslant 0\}$ be an ordinary renewal process with F IMRL, and $\{N'(t), t \geqslant 0\}$ a stationary F renewal process. Then:*

(i) *$M(t) - t/\mu_1$ and $EZ(t)$ (expected forward recurrence time at t) are increasing in $t \geqslant 0$. If $\mu_2 < \infty$ then $L(t) = M(t) - t/\mu_1 - \mu_2/2\mu_1^2 \uparrow 0$ as $t \to \infty$.*

(ii) *If g is a nonnegative measurable function then $\Sigma_0^\infty g(S_i)$ is stochastically larger than $\Sigma_0^\infty g(S_i')$. In particular $N(A)$ is stochastically larger than $N'(A)$ for all Borel sets $A \subset [0, \infty)$, and $EN(A) \geqslant EN'(A) = \mu_1^{-1} l(A)$, where l is Lebesgue measure. Furthermore, $M(t+h) - M(t) \geqslant h/\mu_1$ for all $t \geqslant 0$, $h \geqslant 0$, and converges to h/μ_1 as $t \to \infty$.*

(iii) *If $\mu_2 < \infty$ and g is a bounded measurable function which converges to 0 as $t \to \infty$ then $|\int_0^t g(t-x)\,dM(x) - \mu_1^{-1}\int_0^t g(x)\,dx| \to 0$ as $t \to \infty$.*

(iv) *If $\mu_2 < \infty$ then $0 \geqslant L(t) \geqslant -\mu_1^{-1}\int_t^\infty(\bar{G}(x) - q^{-1}\bar{F}(x))\,dx \geqslant -\mu_1^{-1}\int_t^\infty \bar{G}(x)\,dx$ where $q = \bar{F}(0)$, $L(t) = M(t) - t/\mu_1 - \mu_2/2\mu_1^2$.*

(v) *For $k \geqslant 0$, $\mu_{k+2} < \infty \Rightarrow \lim_{t\to\infty} t^k L(t) = 0$ and $\mu_{k+3} < \infty \Rightarrow 0 \geqslant \int_0^\infty t^k L(t)\,dt$*

$> -\infty$; *for* $a > 0$, $\psi_F(a) = \int_0^\infty e^{at}\, dF(t) < \infty \Rightarrow \lim_{t\to\infty} e^{at}L(t) = 0$ *and* $0 \geqslant \int_0^\infty e^{at}L(t)\, dt > -\infty$.

PROOF. Recall that process 1 is an ordinary F renewal process, process 2 a stationary F renewal process, and $S_i' = S_{N+i}$, $i = 1, 2, \cdots$. Note that by Wald's identity $EN = \mu_2/2\mu_1^2$ for $\mu_1 < \infty$ with both sides equal to ∞ if $\mu_2 = \infty$.

(i) $M(t) - t/\mu_1 = E(N(t) - N'(t))$. Since $N(t) - N'(t)\uparrow N$, $M(t) - t/\mu_1 \uparrow EN = \mu_2/2\mu_1^2$ by the monotone convergence theorem. Thus if $\mu_2 < \infty$ then $0 \geqslant L(t) = M(t) - t/\mu_1 - \mu_2/2\mu_1^2 \uparrow 0$. Since $Z(t) = \Sigma_1^{N(t)+1}X_i - t$, it follows by Wald's identity that $EZ(t) = \mu_1(M(t) - t/\mu_1)$, thus $EZ(t)\uparrow$.

(ii) By construction, $\Sigma_0^\infty g(S_i) \geqslant \Sigma_N^\infty g(S_i) = \Sigma_0^\infty g(S_i')$, thus $\Sigma\, g(S_i)$ is stochastically larger than $\Sigma\, g(S_i')$. Setting $g = I_A$, the indicator function of the Borel set $A \subset [0, \infty)$, gives $N(A)$ stochastically greater than $N'(A)$; taking expectations we get, $EN(A) \geqslant EN'(A) = \mu_1^{-1}l(A)$. Setting $A = [t, t + h\}$ gives $M(t + h) - M(t) \geqslant h/\mu_1$. This last inequality and the elementary renewal theorem $(M(t)/t \to \mu_1^{-1})$ imply that $M(t + h) - M(t) \to h/\mu_1$ as $t \to \infty$. (The convergence of $M(t + h) - M(t)$ also follows from Blackwell's theorem, as an IMRL distribution must be nonlattice.)

(iii) Define $Y(t) = \Sigma_{i=0}^{\min(N-1, N(t))} g(t - S_i)$. Now $\int_0^t g(t - x)\, dM(x) - \mu_1^{-1}\int_0^t g(x)\, dx = E[\Sigma_{i=0}^{N(t)} g(t - S_i) - \Sigma_{i=0}^{N'(t)} g(t - S_i')] = EY(t)$. Now $Y(t) \to 0$ a.s. as $t \to \infty$ and $|Y(t)| \leqslant N(\sup|g|)$ for all t. Since $\mu_2 < \infty$ the result follows from the dominated convergence theorem. An alternative proof is based on the fact that $|\int_0^t g(t - x)\, dM(x) - \mu_1^{-1}\int_0^t g(x)\, dx| \leqslant \sup(|g|)$ multiplied by the total variation of $M(x) - x/\mu_1$ on $[0, \infty)$; the latter quantity is finite when $\mu_2 < \infty$ by part (i) of this theorem.

(iv) By the argument in part (i), $L(t) = \mu_1^{-1}E(Z(t) - Z'(t))$. By construction:

$$\begin{aligned} 0 \geqslant Z(t) - Z'(t) &= 0 && \text{if} \quad t \geqslant X_0' \\ &= Z(t) - (X_0' - t) && \text{if} \quad t < X_0'. \end{aligned}$$

Thus $\mu_1 L(t) = \bar{G}(t)E(Z(t)|X_0' > t) - \int_t^\infty \bar{G}(x)\, dx$. Since $Z(t)|X_0' > t$ is a mixture of distributions of the form $X - v|X > v$ where $X \sim F$, and F is IMRL, it follows that $E(Z(t)|X_0' > t) \geqslant E(X|X > 0) = q^{-1}\mu_1$. Thus $0 \geqslant L(t) \geqslant -\mu_1^{-1}\int_t^\infty(\bar{G}(x) - q^{-1}\bar{F}(x))\, dx \geqslant -\mu_1^{-1}\int_t^\infty \bar{G}(x)\, dx$.

(v) These results follow from (iv) and Lemma 2.

THEOREM 3. *Suppose that* F *is* DFR *on* $[0, \infty)$. *Then*:

(i) $A(t)$ *and* $Z(t)$ *are stochastically increasing in* $t \geqslant 0$.

(ii) *The renewal density* m *on* $(0, \infty)$, *has a version which is decreasing and converges to* μ_1^{-1} *as* $t \to \infty$ (*if* $\mu_1 = \infty$ *then* $\mu_1^{-1} = 0$); $M(t)$ *is concave.*

(iii) *If* g *is a nonnegative measurable function then* $V(t) = \Sigma_{i=1}^\infty g(S_{N(t)+i} - t)$ *is stochastically decreasing in* $t \geqslant 0$. *In particular, for any Borel set* $A \subset [0, \infty)$, $N(\{A + t\})$ *is stochastically decreasing in* $t \geqslant 0$; $N(t + h) - N(t)$ *is stochastically decreasing in* $t \geqslant 0$ *for each* $h \geqslant 0$, *and* $M(t + h) - M(t)\downarrow h/\mu_1$ *as* $t \to \infty$.

(iv) *If g is a measurable function, $R^{\infty} \to \bar{R} = [-\infty, \infty]$, which is decreasing (increasing) then $W(t) = g(\{S_{N(t)+i} - t, i \geq 1\})$ is stochastically decreasing (increasing) in $t \geq 0$. (By g increasing we mean $x_i \leq y_i$, $i \geq 1$, implies $g(\mathbf{x}) \leq g(\mathbf{y})$.)*

PROOF. (i) Give process 2 initial age distribution $F_A(s)$, the age distribution of process 1 at time s. Then $Z'(t) \sim Z(s + t)$, and $A'(t) \sim \min(A(s + t), t)$ so that $A'(t) \leq_{st} A(s + t)$. By construction $Z'(t) \geq Z(t)$, and $A'(t) \geq A(t)$. Thus $Z(t) \leq_{st} Z'(t) \sim Z(s + t)$ and $A(t) \leq_{st} A'(t) \leq_{st} A(s + t)$. Therefore, both $A(t)$ and $Z(t)$ are stochastically increasing in $t \geq 0$.

(ii) Recall that a DFR distribution on $[0, \infty)$ is necessarily absolutely continuous on $(0, \infty)$ and possesses a version of its hazard function, $h(x) = f(x)/\bar{F}(x)$, which is decreasing. Now $m(t) = q^{-1}E(h(A(t)))$, where m is the renewal density, $q = \bar{F}(0)$, and $A(t)$ is the renewal age at time t. Since $A(t)$ is stochastically increasing (part (i)), and h is decreasing, it follows that $h(A(t))$ is stochastically decreasing and therefore $E(h(A(t)))$ is decreasing. Thus $m(t)$ is decreasing and therefore converges as $t \to \infty$. By the elementary renewal theorem ($M(t)/t \to \mu_1^{-1}$) the limit must be μ_1^{-1}. Concavity of M follows from m decreasing.

(iii) Define process 2 as in (i) above. By construction $V(t) \geq V'(t) \sim V(s + t)$. Thus $V(t)$ is stochastically decreasing in $t \geq 0$. For a Borel set $A \subset (0, \infty)$, $g = I_A$ yields $N(\{A + t\})$ stochastically decreasing. If $0 \in A \subset [0, \infty)$ with $B = A - \{0\}$ then $N(\{A + t\}) = N(\{B + t\})$ a.s. for each $t > 0$, while $N(\{A\}) > N(\{B\})$; thus the result also holds for the case $0 \in A$. Setting $A = (0, h]$ gives $N(t + h) - N(t)$ stochastically decreasing, and taking expectations gives $M(t + h) - M(t)\downarrow$; thus $M(t + h) - M(t)$ has a limit as $t \to \infty$, which by the elementary renewal theorem must be h/μ_1.

(iv) Consider an ordinary F renewal process. An observer enters at time s and records renewal epochs in (s, ∞) subtracting s from each one. He then is observing a sequence $\{S_n', n \geq 0\}$ with $S_0' \sim Z_s$ and $S_n' \sim Z_s + S_n$ for $n \geq 1$. Similarly if he enters at time t he observes $\{S_n'', n \geq 0\}$ with $S_0'' \sim Z_t$ and $S_n'' \sim Z_t + S_n$ for $n \geq 1$. From part (i) we know that for $t > s$, Z_t is stochastically larger than Z_s. It follows (Lehman [16], page 73) that we can construct a pair of random variables (S_0', S_0'') with $S_0' \sim Z_s$, $S_0'' \sim Z_t$, and $S_0' \leq S_0''$ a.s. Next take an i.i.d. sequence $\{X_i, i \geq 1\}$ with distribution F, independent of (S_0', S_0''), define $S_n = \Sigma_1^n X_i$, $n \geq 1$, and set $S_n' = S_0' + S_n$, $S_n'' = S_0'' + S_n$, $n \geq 1$. We therefore have versions of the future renewal processes viewed from s and t with $S_n'' - S_n' = S_0'' - S_0' \geq 0$ a.s. Therefore, for g decreasing $g(\{S_n', n \geq 0\}) \geq g(\{S_n'', n \geq 0\})$ a.s., thus $W(s) \geq_{st} W(t)$ and $W(t)$ is stochastically decreasing in $t \geq 0$. □

REMARK. Suppose that $\{Y_i, i \geq 1\}$ is a sequence of nonnegative random variables independent of $\{S_i, i \geq 0\}$. Consider $R(s) = \Sigma_{i=1}^{\infty} Y_i e^{-\lambda(S_{N(s)+i} - s)} = \Sigma_{i=N(s)+1}^{\infty} Y_{i-N(s)} e^{-\lambda(S_i - s)}$. For $\lambda > 0$, $R(s)$ is known as the future discounted reward at time s, and λ is called the discount factor. The process $\{R(s), s \geq 0\}$ is of interest in various stochastic optimization models (Ross [23]). It is easy to show that $\lambda > 0$ and $\sup_i EY_i < \infty$ implies that $ER(t) < \infty$ and thus that $R(t)$ is a proper random variable for all t.

It follows from either part (iii) or part (iv) of Theorem 3, by conditioning on $\{Y_i = y_i, i = 1, \cdots\}$, that if F is DFR, then $R(s)$ is stochastically decreasing in $s \geqslant 0$.

4. Bounds for M(t): IMRL case.

THEOREM 4. (i) *If F is* IMRL *on* $[0, \infty)$, *and* $\mu_{k+2} < \infty$ *for an integer* $k \geqslant 0$ *then*:

$$U(t) \geqslant M(t) \geqslant U(t) - \min_{0 \leqslant i \leqslant k} c_i t^{-i} \tag{3}$$

where $U(t) = t/\mu_1 + \mu_2/2\mu_1^2$, $c_0 = \mu_2/2\mu_1^2 - q^{-1}$ *and*

$$0 \leqslant c_i = -i\int_0^\infty s^{i-1}L(s)\,ds = \int_0^\infty s^i\,d(M(s) - s/\mu_1) < \infty$$

for $i = 1, \cdots, k$.

The term c_i is a function of $\mu_1, \cdots, \mu_{i+2}$, $i = 1, 2, \cdots, k$ *which can be recursively computed from*:

$$c_i = \gamma_i - \mu_1^{-1} i! \Sigma_{s=1}^{i-1}(c_s/s!)\lambda_{i+1-s}, \qquad i = 1, \cdots, k \tag{4}$$

where

$$\gamma_i = \left[\mu_{i+2}/(i+1)(i+2)\mu_1^2\right] - \left[\mu_2\mu_{i+1}/2(i+1)\mu_1^3\right]$$

$$\lambda_i = \mu_i/i!$$

Equation (4) *can be explicitly solved yielding*

$$c_i = \gamma_i + i!\Sigma_{j=1}^{i-1}(\gamma_j/j!)\Sigma_{l=1}^{i-j}(-\mu_1)^{-l}\Sigma_{(i_1, \cdots, i_l) \in A_{i-j,l}}\left(\Pi_1^l \lambda_{i_r+1}\right), \qquad i = 1, \cdots, k \tag{5}$$

where

$$A_{k,l} = \left\{(i_1, \cdots, i_l): i_r \geqslant 1, r = 1, \cdots, l, \Sigma_1^l i_r = k\right\}$$

(ii) *If F is quasiexponential* ($\bar{F}(t) = qe^{-\lambda t}$, $0 < q \leqslant 1$, $\lambda > 0$, $t \geqslant 0$) *then* $c_i = 0$ *for all i. If F is* IMRL *but nonquasiexponential then* $c_i > 0$ *for all i.*

In the nonquasiexponential case define $v_{-1} = 0$, $v_i = c_{i+1}/c_i$, $i = 0, 1, \cdots$ *with* $\infty/\infty = \infty$. *Then v_i is increasing in i and for* $v_{j-1} \leqslant t \leqslant v_j$, $c_j t^{-j} = \inf_i c_i t^{-i}$.

PROOF. (i) Since $M(t) = U(t) + L(t)$ and $L(t) \leqslant 0$ (Theorem 2, (i)) $M(t) \leqslant U(t)$. Since $L(t)$ is increasing (Theorem 2, (i)) $L(t) \geqslant L(0) = M(0) - \mu_2/2\mu_1^2 = q^{-1} - \mu_2/2\mu_1^2 = -c_0$. Thus $M(t) \geqslant U(t) - c_0$. Since $L(t)$ is increasing it is at least as big as its average over $[0, t]$ with respect to any probability measure on $[0, t]$. Thus $L(t) \geqslant it^{-i}\int_0^t s^{i-1}L(s)\,ds$ which since $L \leqslant 0$ exceeds $it^{-i}\int_0^\infty s^{i-1}L(s)\,ds = -c_i t^{-i} > -\infty$ by Theorem 2 part (v). The equivalence between $-i\int_0^\infty s^{i-1}L(s)\,ds$ and $\int_0^\infty s^i\,d(M(s) - s/\mu_1)$ follows by integration by parts and part (v) of Theorem 2.

To compute the c_i's start with the identity $M(t) = 1 + \int_0^t M(t-x)\,dF(x)$, subtract $U(t)$ from both sides, multiply both sides by t^i and use the identity $t^i - (t -$

$x)^i = \Sigma_{r=0}^{i-1}\binom{i}{r}(t - x)^r x^{i-r}$ This yields:

$$t^iL(t) = \int_0^t (t - x)^i L(t - x)\, dF(x) + h(t) \tag{6}$$

where $h(t) = h_1(t) + h_2(t) - h_3(t)$, $h_1(t) = \int_0^t[\Sigma_{r=0}^{i-1}\binom{i}{r}(t - x)^r x^{i-r}]\ L(t - x)\, dF(x)$, $h_2(t) = t^i\mu_1^{-1}\int_t^\infty \bar{F}(x)\, dx$, $h_3(t) = t^i(\mu_2/2\mu_1^2)\bar{F}(t)$. Now (6) is the renewal equation $g = h + g * F = h * M$, with $g(t) = t^iL(t)$. By part (iii) of Theorem 2, if we can show that h is bounded, integrable and that $\lim_{t\to\infty}h(t) = 0$ then we can conclude that $\lim_{t\to\infty}t^iL(t) = \mu_1^{-1}\int_0^\infty h(t)\, dt$. But for $t = 0, \cdots, k$, $\lim_{t\to\infty}t^iL(t) = 0$ by part (v) of Theorem 2. Thus the conclusion will reduce to $\int_0^\infty h(t)\, dt = 0$ and this identity provides us with expressions for c_i.

To show that h is bounded, integrable and convergent to zero we do so separately for each h_i. By Lemma 2, h_2 and h_3 are convergent to zero with:

$$\int_0^\infty h_2(t)\, dt = \mu_{i+2}/\ (i + 1)(i + 2)\mu_1 \tag{7}$$

$$\int_0^\infty h_3(t)\, dt = \mu_{i+1}\mu_2/2(i + 1)\mu_1^2. \tag{8}$$

The boundedness of h_2 and h_3 follows from the boundedness on finite intervals and the convergence to zero as $t \to \infty$.

Defining $L(y) = 0$ for $y < 0$ we write $h_1(t) = \int_0^\infty(\Sigma_{r=0}^{i-1}\binom{i}{r}x^{i-r}((t - x)^rL(t - x))\, dF(x)$. Since $(t - x)^rL(t - x) \to 0$ as $t \to \infty$ by Theorem 2 part (v) the integrand converges pointwise to zero. Since $s^r|L(s)|$ is bounded on finite intervals and converges to 0 as $s \to \infty$, $\sup_s s^r|L(s)| < \infty$, thus the integrand is dominated by the integrable function $\Sigma_{r=0}^{i-1}\binom{i}{r}(\sup_s s^r|L(s)|)x^{i-r}$ and by the dominated convergence theorem $h_1(t) \to 0$. The above argument also shows that $|h_1(t)| \leqslant \Sigma_{r=0}^{i-1}\binom{i}{r}\mu_{i-r}(\sup_s s^r|L(s)|) < \infty$, thus h_1 is bounded; h_1 is integrable by part (v) of Theorem 2 since $\int_0^\infty|h_1(t)|\, dt = \Sigma_{r=0}^{i-1}\binom{i}{r}\mu_{i-r}\int_0^\infty s^r|L(s)|\, ds < \infty$. Moreover:

$$\int_0^\infty h_1(t)\, dt = \Sigma_{r=0}^{i-1}\binom{i}{r}\mu_{i-r}\int_0^\infty s^rL(s)\, ds = -i!\Sigma_{r=1}^{i}(c_r/r!)\lambda_{i+1-r}. \tag{9}$$

The identity $\int_0^\infty h(t)\, dt = 0$ is equivalent, using (7), (8) and (9), to $i!\Sigma_{r=1}^{i}(c_r/r!)\lambda_{i+1-r} = \mu_1\gamma_i$ which reduces to (4).

Define $d_0 = 0$, $d_i = c_i/i!$, $i = 1, \cdots, k$, $\delta_0 = 0$, $\delta_i = \gamma_i/i!$, $i = 1, \cdots, k$, $\beta_0 = 0$, $\beta_i = -\mu_1^{-1}\lambda_{i+1}$, $i = 1, \cdots, k$. Then rewrite (4) as:

$$d_i = \delta_i + \Sigma_{s=0}^{i} d_s\beta_{i-s}. \tag{10}$$

Now (10) is a discrete renewal equation. Its solution is $d_i = \Sigma_{j=0}^{i}\delta_jM_{i-j}$ where $M_i = \Sigma_{s=0}^{\infty}\beta_i^{(s)}$, where $\beta^{(s)}$ is the sth convolution of β. Since $\beta_0 = 0$, $\beta_i^{(s)} = 0$ for $s > i$. We thus obtain

$$d_i = \Sigma_{j=1}^{i}\delta_jM_{i-j} = \delta_i + \Sigma_{j=1}^{i-1}\delta_jM_{i-j} = \delta_i + \Sigma_{j=1}^{i-1}\delta_j\Sigma_{l=1}^{i-j}\Sigma_{(i_1,\cdots,i_l)\in A_{i-j,l}}\left(\Pi_1^l\beta_{i_r}\right),$$

which is equivalent to (5).

(ii) If F is quasiexponential ($\bar{F}(t) = qe^{-\lambda t}$) then $M(t) = q^{-1}\lambda t + q^{-1} = t/\mu_1 + \mu_2/2\mu_1^2(\mu_1 = \lambda q, \mu_2 = 2\lambda^2q)$ thus $L(t) = 0$. If $c_i = -i\int_0^\infty s^{i-1}L(s)\, ds = 0$ for some

i, then since $L \leqslant 0$ and $L\uparrow$ for F IMRL (Theorem 2 (i)), it follows that $L(s) \equiv 0$ or equivalently $M(t) = at + b$ with $b = M(0) \geqslant 1$. But the quasiexponential distribution with $q = b^{-1} \leqslant 1$ and $\lambda = ab^{-1}$ has renewal function $M(t) = at + b$ and the renewal function uniquely determines the distribution. ($\psi_M = (1 - \psi_F)^{-1}$ so the Laplace transform of F determines the Laplace transform of M which determines M). Thus for F IMRL but not quasiexponential, $c_i > 0$ for all i.

Since $L\uparrow$ on $[0, \infty)$ (Theorem 2 (i)) we can interpret L as the distribution function of a measure which assigns negative weight to $\{0\}$ but positive weight to all Borel sets in $(0, \infty)$. Consider the Hilbert space of Borel measurable functions on $(0, \infty)$ satisfying $\int_{0+}^{\infty} f^2\, dL < \infty$ with inner product $(f, g) = \int_{0+}^{\infty} fg\, dL$. Now:

$$c_j^2 = \left(\int_{0+}^{\infty} s^j\, dL(s)\right)^2 = \left(t^{\frac{j+1}{2}}, t^{\frac{j-1}{2}}\right)^2$$
$$\leqslant \|t^{\frac{j+1}{2}}\|^2 \|t^{\frac{j-1}{2}}\|^2 = c_{j+1}c_{j-1}$$

by the Cauchy-Schwartz inequality. Note that $c_i = \infty \Rightarrow c_{i+1} = \infty$ so that the inequality $c_j^2 \leqslant c_{j+1}c_{j-1}$ holds even if not all of c_{j-1}, c_j, c_{j+1} are finite. The inequality is equivalent to $v_j = c_{j+1}/c_j \geqslant c_j/c_{j-1} = v_{j-1}$ (where $\infty/\infty = \infty$). Thus the v_j's are increasing.

Suppose $t \leqslant v_j$. Then since $v_j\uparrow$, $t^m \leqslant \prod_{i=0}^{m-1} v_{j+i} = c_{j+m}/c_j$ for $m = 1, 2, \cdots$. But the inequality $t^m \leqslant c_{j+m}/c_j$ is equivalent to $c_j t^{-j} \leqslant c_{j+m} t^{-(j+m)}$. Similarly if $t \geqslant v_{j-1}$, $c_j t^{-j} \leqslant c_{j-m} t^{-(j-m)}$ for $m = 1, \cdots, j$. Thus for $t \in [v_{j-1}, v_j]$, $c_j t^{-j} = \inf_i c_i t^{-i}$. □

THEOREM 5. *If F is* IMRL *on $[0, \infty)$ and $\psi_F(a_0) = \int_0^{\infty} e^{a_0 t}\, dF(t) < \infty$ for an $a_0 > 0$, then for $0 < a \leqslant a_0$:*

(11)
$$U(t) \geqslant M(t) \geqslant U(t) - (e^{at} - 1)^{-1}\left[(\mu_1 a)^{-1} - (\mu_2/2\mu_1^2) - (\psi_F(a) - 1)^{-1}\right].$$

PROOF. The proof is very similar to that of Theorem 4. Choose $a \in (0, a_0]$. Using $L(t) \leqslant 0$, $L(t)\uparrow$ as in the proof of Theorem 4 we obtain:

$$U(t) \geqslant M(t) \geqslant U(t) + a(e^{at} - 1)^{-1} \int_0^{\infty} e^{as} L(s)\, ds \tag{12}$$

where $0 \geqslant \int_0^{\infty} e^{at} L(t)\, dt > -\infty$ by part (v) of Theorem 2.

To evaluate $\psi_L(a) = \int_0^{\infty} e^{as} L(s)\, ds$ we start with $M(t) = 1 + \int_0^t M(t - x)\, dF(x)$, subtract $U(t)$ from each side and multiply both sides by e^{at}. This gives:

$$e^{at} L(t) = \int_0^t e^{a(t-x)} L(t - x)\, dF(x) + l(t) \tag{13}$$

where

$$l(t) = l_1(t) + l_2(t) - l_3(t),\ l_1(t) = \int_0^t (e^{ax} - 1) e^{a(t-x)} L(t - x)\, dF(x),$$
$$l_2(t) = \mu_1^{-1} e^{at} \int_t^{\infty} \bar{F}(x)\, dx,\ l_3(t) = (\mu_2/2\mu_1^2) e^{at} \bar{F}(t).$$

We verify the conditions of part (iii) of Theorem 2 in a similar manner as in the proof of Theorem 4, making heavy use of Lemma 2 and part (v) of Theorem 2. The conclusion of part (iii) of Theorem 2, in light of $\lim_{t\to\infty} e^{at}L(t) = 0$ (part (v) of Theorem 2) gives us:

(14)

$$(\psi_F(a) - 1)\psi_L(a) = (\mu_2/2\mu_1^2)[(\psi_F(a) - 1)/a] - (\mu_1 a^2)^{-1}(\psi_F(a) - \mu_1 a - 1).$$

Since $\psi_F(a) - 1 \neq 0$ for $a \neq 0$ we can divide both sides of (14) by $\psi_F(a) - 1$ and solve for $\psi_L(a)$. This gives:

$$\psi_L(a) = \mu_2/2\mu_1^2 a - (\mu_1 a^2)^{-1} + [a(\psi_F(a) - 1)]^{-1}. \tag{15}$$

Substituting (15) into (12) gives us (11). □

REMARK. (11) and (15) will hold for $a < 0$ whether or not $\psi_F(a) < \infty$ for an $a > 0$. If $\mu_3 < \infty$ and we let $a \uparrow 0$ in (11) then we obtain $M(t) \geqslant U(t) - c_1 t^{-1}$ where c_1 is given in (4). In general $\mu_{k+3} < \infty$ implies $\psi_L^{(k)}(0^-)$ exists and equals $-(k+1)^{-1}c_{k+1}$. Thus $\psi_L(a)$ can be considered as a generating function for the c_i's. However, unless the particular form of $\psi_F(a)$ leads to a simple expression for $\psi_L(a)$, expressions (4) and (5) of Theorem 4 will be preferable for computing the c_i's.

5. Improved bounds when F is DFR. The bounds given in Theorems 4 and 5 for IMRL distributions can be improved for DFR distributions. Define $\alpha_0 = 1$, $\alpha_i = (i/i+1)^i$, $i \geqslant 1$, $c_i^* = \alpha_i c_i$ where c_i is given in Theorem 3, and $v_i^* = c_{i+1}^*/c_i^* = (\alpha_{i+1}/\alpha_i)v_i$. Also define $g_a(t) = a(e^{at} - 1)^{-1}\int_0^t s e^{as}\, ds = (te^{at}/e^{at} - 1) - a^{-1}$, $h_a(t) = (1 + at)(1 - e^{-at})$, and $\psi(a) = -a\psi_L(a) = [(\mu_1 a)^{-1} - (\mu_2/2\mu_1^2) - (\psi_F(a) - 1)^{-1}]$.

COROLLARY 1. *Assume that F is* DFR *on* $[0, \infty)$. *Then*:

(i) *If $\mu_{k+2} < \infty$ then $U(t) \geqslant M(t) \geqslant U(t) - \min_{0 \leqslant i \leqslant k} c_i^* t^{-i}$*;

(ii) *$v_i^* \uparrow$ and for $v_{j-1}^* \leqslant t \leqslant v_j^*$, $c_j^* t^{-j} = \inf_i c_i^* t^{-i}$; thus for $v_{j-1}^* \leqslant t \leqslant v_j^*$ the bound in* (i) *is given by $U(t) - c_j^* t^{-j}$.*

(iii) *If $\psi_F(a_0) < \infty$ for an $a_0 > 0$ then for $0 < a \leqslant a_0$*:

$$U(t) \geqslant M(t) \geqslant U(t) - (e^{a g_a^{-1}(t)} - 1)^{-1}\psi(a) \geqslant U(t) - (e^{h_a(t)} - 1)^{-1}\psi(a).$$

PROOF. (i) L is concave by Theorem 3 part (ii). Thus $L(jt^{-j}\int_0^t s \cdot s^{j-1}\, ds) = L((j/j+1)t) \geqslant jt^{-j}\int_0^t s^{j-1}L(s)\, ds \geqslant jt^{-j}\int_0^\infty s^{j-1}L(s)\, ds = -c_j t^{-j}$. Thus $L(t) \geqslant -c_j[(j + 1/j)t]^{-j} = -c_j^* t^{-j}$. The argument now proceeds as in Theorem 4.

(ii) A simple differentiation argument shows that $\alpha_{i+1}/\alpha_i \uparrow$. Since $c_{i+1}/c_i \uparrow$ by Theorem 4 and $c_{i+1}^*/c_i^* = (\alpha_{i+1}/\alpha_i)(c_{i+1}/c_i)$, we see that $c_{i+1}^*/c_i^* \uparrow$. The argument now proceeds as in Theorem 4.

(iii) The concavity argument in (i) shows that $L(g_a(t)) \geqslant a(e^{at} - 1)^{-1}\int_0^t e^{as}L(s)\, ds \geqslant -(e^{at} - 1)^{-1}(-a\psi_L(a))$; thus $L(t) \geqslant -(e^{-ag_a^{-1}(t)} - 1)^{-1}(-a\psi_L(a)) = -(e^{ag_a^{-1}(t)} - 1)^{-1}[(\mu_1 a)^{-1} - (\mu_2/2\mu_1^2) - (\psi_F(a) - 1)^{-1}]$. If $s = g_a^{-1}(t)$ then $s = (t + a^{-1})(1 - e^{-as}) \geqslant (t + a^{-1})(1 - e^{-at}) = a^{-1}h_a(t)$, thus $-(e^{ag_a^{-1}(t)} - 1)^{-1} \geqslant -(e^{h_a(t)} - 1)^{-1}$. □

EXAMPLE. Consider $f(x) = (\Gamma(.5))^{-1}x^{-.5}e^{-x}$, $x > 0$; this is the $\Gamma(.5, 1) = \chi_1^2/2$ distribution, which is DFR ([5], page 378). We compute the moment based bounds for $M(t)$, using Theorem 4 and Corollary 1, to illustrate the method, and to see how much the DFR bounds improve upon the IMRL bounds in this case. Using the recursive formula (4) we compute $c_0 = \frac{1}{2}$, $c_1 = c_2 = \frac{1}{8}$, $c_3 = \frac{15}{64}$, $c_4 = \frac{21}{32}$, $c_5 = \frac{315}{128}$, $c_6 = \frac{1485}{128}$. Next $v_{-1} = 0$, $v_0 = \frac{1}{4}$, $v_1 = 1$, $v_2 = \frac{15}{8}$, $v_3 = \frac{14}{5}$, $v_4 = \frac{15}{4}$, $v_5 = \frac{33}{7}$, $v_6 = \frac{91}{16}$. Denoting the lower bound given in Theorem 4 by $B(t) = U(t) - \inf_i c_i t^{-i}$ we obtain:

$$
\begin{aligned}
B(t) &= 2t + 1, & 0 \leqslant t \leqslant \tfrac{1}{4}\\
&= 2t + \tfrac{3}{2} - (8t)^{-1}, & \tfrac{1}{4} \leqslant t \leqslant 1\\
&= 2t + \tfrac{3}{2} - (8t^2)^{-1}, & 1 \leqslant t \leqslant \tfrac{15}{8}\\
&= 2t + \tfrac{3}{2} - \left(\tfrac{15}{64t^3}\right), & \tfrac{15}{8} \leqslant t \leqslant \tfrac{14}{5}\\
&= 2t + \tfrac{3}{2} - \left(\tfrac{21}{32t^4}\right), & \tfrac{14}{5} \leqslant t \leqslant \tfrac{15}{4}\\
&= 2t + \tfrac{3}{2} - \left(\tfrac{315}{128t^t}\right), & \tfrac{15}{4} \leqslant t \leqslant \tfrac{33}{7}\\
&= 2t + \tfrac{3}{2} - \left(\tfrac{1485}{128t^6}\right), & \tfrac{33}{7} \leqslant t \leqslant \tfrac{91}{16},
\end{aligned}
$$

The lower bound, given in Corollary 1, which we denote by $B^*(t) = U(t) - \inf_i c_i^* t^{-i}$ is similarly computed.

The table below gives a few values of t along with the corresponding intervals $[B(t), U(t)]$ and $[B^*(t), U(t)]$ for $M(t)$.

t	$(B(t), U(t))$	$(B^*(t), U(t))$
.1	[1.2, 1.7]	[1.2, 1.7]
.5	[2.25, 2.5]	[2.375, 2.5]
1	[3.375, 3.5]	[3.444, 3.5]
1.5	[4.444, 4.5]	[4.475, 4.5]
2	[5.471, 5.5]	[5,488, 5.5]
3	[7.492, 7.5]	[7.497, 7.5]
4	[9.4976, 9.5]	[9.4990, 9.5]
5	[11.49926, 11.5]	[11.49971, 11.5]

6. Comments and additions.

(1) For IMRL distributions we have the representation (Theorem 1) $X_0' = \sum_1^N X_i$, where $X_i \sim F$, $X_0' \sim G$, $G(x) = \mu_1^{-1}\int_0^x \bar{F}(y)\,dy$, and N is a stopping time. A simple argument shows that $\Pr(N = 1) > 0$. Since G is absolutely continuous, it follows that an IMRL distribution must have an absolutely continuous component. I do not know whether, aside from an atom at $\{0\}$, an IMRL distribution can have a singular component.

(2) If the above mentioned representation is valid for distribution F, with N a stopping time, then a renewal process with interarrival time distribution F will

enjoy many of the properties which were derived for IMRL renewal processes. I do not know whether F IMRL is a necessary condition for such a representation to hold.

(3) By Theorem 2(i), if F is IMRL and $\mu_2 < \infty$, then $L(t) - L(0)$ is the distribution function of a finite positive measure over the Borel sets in $(0, \infty)$. If F is DFR then, in addition, L has a decreasing density (Theorem 3 (ii)). I suspect that L has an improvement with age property for F IMRL, with a strengthened version for F DFR. This is based on the behavior of the moments of L in a few special cases which I have worked out.

(4) If we imitate the construction of Theorem 1 for increasing failure rate (IFR) distributions we obtain the following. The ordinary and delayed renewal processes alternate renewals until a random time at which their renewal epochs coincide; then all future renewal epochs coincide. So far this representation has not yielded much in the way of special properties for DMRL and IFR renewal processes. An example of Berman ([7] page 429) shows that F IFR does not imply $m(t)$ increasing, nor $M(t) - t/\mu_1$ decreasing, nor $EZ(t)$ decreasing. Thus the obvious analogues of the DFR and IMRL results of this paper do not hold for IFR and DMRL distributions.

(5) Identities (4) and (5) may be of independent renewal theory interest. They hold for distributions on $[0, \infty)$ possessing the required moments and having the property that some convolution has an absolutely continuous component. To prove this one would follow the argument of Theorem 4, invoking results of W. L. Smith [25] to justify both $\lim t^k L(t) = 0$, and the applicability of the key renewal argument.

(6) The results $c_i \geq 0$ and $v_i \uparrow$ (Theorem 4) are inequalities among the moments of IMRL distributions which may be of independent interest.

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