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Sana Akram, Muhammad Javaid*, and Muhammad Jamal

Bounds on F-index of tricyclic graphs with fixed pendant vertices

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Abstract: The F-index $F(G)$ of a graph G is obtained by the sum of cubes of the degrees of all the vertices in G. It is defined in the same paper of 1972 where the first and second Zagreb indices are introduced to study the structure-dependency of total π-electron energy. Recently, Furtula and Gutman [J. Math. Chem. **53** (2015), no. 4, 1184–1190] reinvestigated F-index and proved its various properties. A connected graph with order n and size m, such that $m = n + 2$, is called a tricyclic graph. In this paper, we characterize the extremal graphs and prove the ordering among the different subfamilies of graphs with respect to F-index in Ω_n^α, where Ω_n^α is a complete class of tricyclic graphs with three, four, six and seven cycles, such that each graph has $\alpha \geq 1$ pendant vertices and $n \geq 16 + \alpha$ order. Mainly, we prove the bounds (lower and upper) of $F(G)$, i.e

$$8n + 12\alpha + 76 \leq F(G) \leq 8(n-1) - 7\alpha + (\alpha+6)^3 \quad \text{for each} \quad G \in \Omega_n^\alpha.$$

Keywords: extremal graphs, tricyclic graphs, F-index

MSC 2010: 05C12, 05C50, 05C35

1 Introduction and preliminaries

A representative number of a molecular graph that expresses the various features of the involved organic molecules, usually known as a topological index (TI). It plays an important role to study the certain changes in the molecular structures which may be physical or chemical. Moreover, Cheminformatics studies quantitative structural activity and property relationships that are used to examine the bioactivities and chemical reactivities of the chemical compounds in a molecular graph on the bases of obtained computational results for the different topological indices (TI's), see [1]. Most importantly, all the TI's are invariants under the parameter of graphs-isomorphism. For a connected graph, there are many TI's in literature. These are classified into three main classes degree-based TI's, distance-based TI's and polynomial-based TI's. The TI's depending upon degrees are more familiar than the others, see [2].

Wiener (1947) defined the first distance based TI, when he was working on paraffin, see [3]. Later on, it was called by Wiener index and much more work has been done on it. Recently, Furtula and Gutman (2015) [4] reinvestigated a degree-based TI and named it forgotten index (F-index). They also proposed its basic properties in the same paper and reported that it can enhance the physico-chemical capability of the molecules. The F-index and its co-index of the different graphs are studied by De et al. [5], Milovanovic et

***Corresponding Author: Muhammad Javaid:** Department of Mathematics, School of Science, University of Management and Technology, Lahore 54770, Pakistan; E-mail: javaidmath@gmail.com
Sana Akram: Department of Mathematics, School of Science, University of Management and Technology, Lahore 54770, Pakistan; E-mail: sanaakram.umt@gmail.com
Muhammad Jamal: Department of Mathematics and Statistics, PMAS-Arid Agriculture University, Rawalpindi, Pakistan; E-mail: jamal@uaar.edu.pk

al. [6] and Basavanagoud et al. [7]. Khaksari and Ghorbani [8] studied the certain product of graphs with the same index. The extremal graphs with respect to F-index among the unicyclic and bicyclic graphs are studied in [9, 10]. For more studies, we refer to [11] and [12-25].

In this paper, we prove the existence of extremal graphs with respect to F-index in the class of tricyclic graphs with three, four, six and seven cycles under the condition of certain pendant vertices. We also investigate the ordering and compute the bounds (lower and upper) of the F-index in the same class of graphs.

Throughout the paper, $G(V(G), E(G))$ for vertex-set $V(G)$ and edge-set $E(G)$ is considered as simple (no loops and parallel edges), finite and undirected graph. For $r \in V(G)$, $d(r)$ shows its degree (number of incident edges on r). For more theoretic terminologies, we refer [26]. Now, some important TI's are defined as follows:

Definition 1.1. For a (molecular) graph G, the first and second Zagreb indices are

$$M_1(G) = \sum_{rs\in E(G)} [d(r) + d(s)] \quad \text{and} \quad M_2(G) = \sum_{rs\in E(G)} [d(r) \times d(s)].$$

Definition 1.2. For a (molecular) graph G, the general Randić index ($R_\alpha(G)$) is

$$R_\alpha(G) = \sum_{rs\in E(\Gamma)} [d(r) \times d(s)]^\alpha.$$

For $\alpha = -\frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\alpha = 1$, we obtain Randić, reciprocal Randić and second Zagreb indices respectively.

Definition 1.3. For a (molecular) graph G the forgotten index (F-index) is defined as follow:

$$F(G) = \sum_{s\in V(G)} [d(s)]^3.$$

For more studies, we refer to [4, 11, 27-29]. Following lemma is frequently used in the main results.

Lemma 1.1. [9] For $1 \le i \le n$ and $1 \le j \le 2$, assume that $< d_1^1, d_2^1, d_3^1, \dots, d_n^1 >$ and $< d_1^2, d_2^2, d_3^2, \dots, d_n^2 >$ are degree sequences with the condition of $d_i^1 = d_i^2$, where d_i^j is degree of the vertex $v_i^j \in V(G_j)$ and $n = |V(G_1)| = |V(G_2)|$. Then, $F(G_1) = F(G_2)$.

2 Computational results of F-index

A connected graph with order n and size m such that $m = n - 1 + c$ is called a c-cyclic graph. In particular, if $c = 0$, $c = 1$, $c = 2$ or $c = 3$ then it is a tree, unicyclic, bicyclic or tricyclic graph respectively. A tricyclic graph contains at least three and at most seven cycles except of exactly five cycles. There are seven possibilities for a tricyclic graph with three cycles as shown in Figure 1. Moreover, the possibilities for the tricyclic graphs with four, six and seven cycles are four, three and one respectively, see Figure 2. Now, we define some more tricyclic graphs with respect to the attachment of $k \ge 1$ pendent vertices to the l vertices of the graphs which are defined in Figure 1. To choose l vertices, we have the following choices:

(i) cycle-vertex of degree 2,
(ii) tree-vertex of degree 2,
(iii) cycle-vertex of degree greater or equal to 2,
(iv) cycle-vertex and tree-vertex of degree exactly 2,
(v) cycle-vertex of degree greater or equal to 2 and tree-vertex of degree exactly 2.

More precisely, we define that l vertices are either of degree exactly 2 or, greater or equal to 2. By joining $k \ge 1$ pendant vertices to l vertices of degree 2, and the vertices of degree greater or equal to 2 of the graph G_1 in Figure 1, the tricyclic graphs $A_{l,k,1}^{m,r} = A_1$ and $A_{l,k,2}^{m,r} = A_2$ are obtained respectively. In G_1, vertices of degree 3 are four and of degree 2 are $m_1 + m_2 + m_3 + r$ such that $m = m_1 + m_2 + m_3$ are cycle-vertex and r are tree-vertex. Table 1 shows the vertex-partition with respect to degrees of vertices of graphs A_1 and A_2.

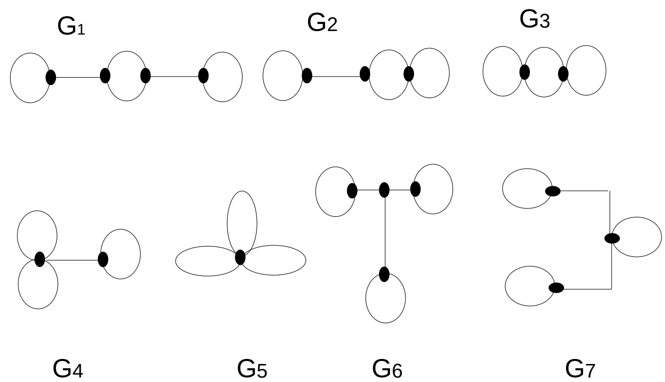

Figure 1: Tricyclic graphs with three cycles.

Table 1: Vertex-partitions of the tricyclic graphs A_1 and A_2.

$d(v)$, for $v \in V(A_1)$	1	2	3	$k+2$
$\|d(v)\|$	lk	$m+r-l$	4	l
$d(v)$, for $v \in V(A_2)$	1	2	$k+2$	$k+3$
$\|d(v)\|$	lk	$m+r+4-l$	$l-4$	4

In G_2 (Figure 1), the vertices of degrees 4, 3, and 2 are 1, 2 and $m_1+m_2+m_3+r+1$ such that $m = m_1+m_2+m_3$ are cycle-vertex and $r+1$ are tree-vertex. By joining $k \geq 1$ pendant vertices to l vertices of degree 2, and the vertices of degree greater or equal to 2 of G_2 in Figure 1, the tricyclic graphs $B^{m,r}_{l,k,1} = B_1$ and $B^{m,r}_{l,k,2} = B_2$, are obtained, respectively. The Table 2 presents the vertex-partitions of graphs B_1 and B_2.

The graph G_3 (Figure 1) has 2 and $m_1+m_2+m_3+r+2$ vertices of degrees 4 and 2, respectively such that $m = m_1+m_2+m_3$ and $r+2$ are cycle-vertex. The tricyclic graphs $C^{m,r}_{l,k,1} = C_1$ and $C^{m,r}_{l,k,2} = C_2$ are obtained by joining $k \geq 1$ pendent vertices to l vertices of degree 2, and degree greater or equal to 2 of the graph G_3 in Figure 1, respectively. The Table 3 presents the vertex-partitions with respect to the degrees of vertices of the graphs C_1 and C_2.

Similarly, we obtain the tricyclic graphs $D^{m,r}_{l,k,1} = D_1$, $D^{m,r}_{l,k,2} = D_2$, $E^{m,r}_{l,k,1} = E_1$ and $E^{m,r}_{l,k,2} = E_2$, by joining $k \geq 1$ pendent vertices to l vertices of degree 2, and greater or equal to 2 of the graphs G_4 and G_5 in Figure 1, respectively. In Figure 1, G_4 has m cycle-vertex and $r+2$ tree-vertex of degrees 2 and G_5 has m cycle-vertex and $r+3$ tree-vertex of degrees 2. The vertex-partitions of these derived tricyclic graphs are shown in Table 4 and Table 5.

Moreover for $i \in \{1,2\}$, we note that (i) $|V(A_i)| = |V(B_i)| = |V(C_i)| = |V(D_i)| = |V(E_i)| = m_1+m_2+m_3+lk+r+4 = m+lk+r+4$ and (ii) the graphs G_6 and G_7 have the same degree sequences as of G_1 and G_2 respectively. For more explanation, B_1, B_2, E_1 and E_2 are given in Figure 2 with certain value of the parameters l, m, k and r.

Now, A_1^1 from A_1 are obtained by deleting k pendant vertices from a vertex of degree $k+2$ and joining these vertices to another vertex of degree $k+2$. Similarly, A_1^2 is derived from A_1^1 by deleting $2k$ pendant vertices

Table 2: Vertex-partitions of the tricyclic graphs B_1 and B_2.

$d(v)$, for $v \in V(B_1)$	1	2	3	4	$k+2$
$\|d(v)\|$	lk	$m+r+1-l$	2	1	l
$d(v)$, for $v \in V(B_2)$	1	2	$k+2$	$k+3$	$k+4$
$\|d(v)\|$	lk	$m+r+4-l$	$l-3$	2	1

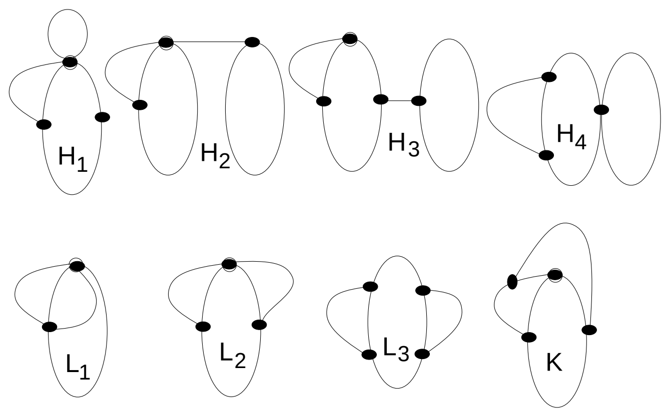

Figure 2: Tricyclic graphs with four cycles (H_1, H_2 H_3 and H_4), six cycles (L_1, L_2 and L_3) and seven cycles (K).

Table 3: Vertex-partitions of the tricyclic graphs C_1 and C_2.

$d(v)$, for $v \in V(C_1)$	1	2	4	$k+2$
$\lvert d(v)\rvert$	lk	$m+r+2-l$	2	l
$d(v)$, for $v \in V(C_2)$	1	2	$k+2$	$k+4$
$\lvert d(v)\rvert$	lk	$m+r+4-l$	$l-2$	2

Table 4: Vertex-partitions of the tricyclic graphs D_1 and D_2.

$d(v)$, for $v \in V(D_1)$	1	2	3	5	$k+2$
$\lvert d(v)\rvert$	lk	$m+r+2-l$	1	1	l
$d(v)$, for $v \in V(D_2)$	1	2	$k+2$	$k+3$	$k+5$
$\lvert d(v)\rvert$	lk	$m+r+4-l$	$l-2$	1	1

Table 5: Vertex-partitions of the tricyclic graphs E_1 and E_2.

$d(v)$, for $v \in V(E_1)$	1	2	6	$k+2$
$\lvert d(v)\rvert$	lk	$m+r+3-l$	1	l
$d(v)$, for $v \in V(E_2)$	1	2	$k+2$	$k+6$
$\lvert d(v)\rvert$	lk	$m+r+4-l$	$l-1$	1

Table 6

Base Graphs (BG)	H_1	H_2	H_3
Joining k vertices to l vertices of degree = 2	$R^{m,r}_{l,k,1} = R_1$	$S^{m,r}_{l,k,1} = S_1$	$T^{m,r}_{l,k,1} = T_1$
Joining k vertices to l vertices of degree ≥ 2	$R^{m,r}_{l,k,2} = R_2$	$S^{m,r}_{l,k,2} = S_2$	$T^{m,r}_{l,k,2} = T_2$
Classes of tricyclic graphs generated from BG	ξ_1	ξ_2	ξ_3

Table 7

Base Graphs (BG)	L_1	L_2	L_3
Joining k vertices to l vertices of degree = 2	$X^{m,r}_{l,k,1} = X_1$	$Y^{m,r}_{l,k,1} = Y_1$	$Z^{m,r}_{l,k,1} = Z_1$
Joining k vertices to l vertices of degree ≥ 2	$X^{m,r}_{l,k,2} = X_2$	$Y^{m,r}_{l,k,2} = Y_2$	$Z^{m,r}_{l,k,2} = Z_2$
Classes of tricyclic graphs generated from BG	ζ_1	ζ_2	ζ_3

from the vertex of degree $2k + 2$ and joining these vertices to the vertex of degree $k + 2$. After $l - 1$ iteration, we obtain A_1^{l-1} from A_1^{l-2} by deleting $(l - 1)k$ pendent vertices from a vertex of degree $(l - 1)k + 2$ and joining these vertices to the last vertex of degree $k+2$, where $2 \le l \le m+r$. Using the same transformation, we obtain A_2^i from A_2^{i-1} for $1 \le i \le l-5$ by the deletion of ik pendent vertices from a vertex of degree $ik + 2$ and joining these vertices to the vertex of degree $k+2$. Moreover, we obtain A_2^i from A_2^{i-1} for $l-4 \le i \le l-1$ by the deletion of ik pendent vertices from a vertex of degree $ik + 3$ and joining these vertices to the last vertex of degree $k + 3$, where $A_2^0 = A_2$. Similarly, for $1 \le i \le l - 1$, we obtain B_2^i, C_2^i, D_2^i and E_2^i from B_2^{i-1}, C_2^{i-1}, D_2^{i-1}, and E_2^{i-1} respectively.

Assume that $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ $\mathcal{U}_4$ and $\mathcal{U}_5$, are classes of the tricyclic graphs obtained from G_1, G_2, G_3 G_4 and G_5 (shown in Figure 1) respectively such that the order of each graph is $m+lk+r+4$ with lk pendent vertices. Let $\mathcal{U}_{\mathbf{n}}^{\mathbf{lk}}$ be a class of all the tricyclic graphs with three cycles such that each graph has order n and pendant vertices lk, where $k \ge 1$, $n \ge 16$ and $1 \le l \le n$. Similarly, tricyclic graphs with four and six cycles obtained from the base graphs presented in Figure 2 are given in Table 6 and Table 7. Moreover, $\xi_{\mathbf{n}}^{\mathbf{lk}}$ and $\zeta_{\mathbf{n}}^{\mathbf{lk}}$ are classes of all the tricyclic graphs with four and six cycles respectively that include each graph of order n and pendant vertices lk. Finally, we obtain the tricyclic graphs with seven cycles ($K^{m,r}_{l,k,1} = K_1$) and ($K^{m,r}_{l,k,2} = K_2$) from the base graph K (see, Figure 2) and $\mu_{\mathbf{n}}^{\mathbf{lk}}$ be a class of all the tricyclic graphs with seven cycles such that each graph has order n and pendant vertices lk. Now, by the deletion and addition of pendant vertices, we have R_j^i, S_j^i, T_j^i, X_j^i, Y_j^i, Z_j^i and K_j^i from R_j^{i-1}, S_j^{i-1}, T_j^{i-1}, X_j^{i-1}, Y_j^{i-1}, Z_j^{i-1} and K_j^{i-1} respectively, where $1 \le i \le l-1$ and $1 \le j \le 2$.

Now, we present some important lemmas which are frequently used in the next section of main results.

Lemma 2.1. For $u, v, a, b \ge 1$, the functions (i) $f_1(u) = -au(u + b)$, (ii) $f_2(u) = -au^2(u + b)$, (iii) $f_3(u, v) = -3u^3(v^2 + 3v + 2) - 12u^2(v + 1)$, and (iv) $f_4(u, v) = -auv(uv + b) - c$ are strictly decreasing functions.

Proof: Since, (i) $\frac{df_1(u)}{du} = -a(2u + b) < 0$, (ii) $\frac{df_2(u)}{du} = -au(3u + 2b) < 0$, (iii) $\frac{\partial f_3(u,v)}{\partial u} = -9u^2(v^2 + 3v + 2) - 24u(v + 1) < 0$ and $\frac{\partial f_3(u,v)}{\partial v} = -3u^3(2v + 3) - 12u^2 < 0$, (iv) $\frac{\partial f_4(u,v)}{\partial u} = -av(2uv + b) < 0$ and $\frac{\partial f_4(u,v)}{\partial v} = -au(2uv+b) < 0$ for $u, v, a, b \ge 1$. Therefore, $f_1(u)$, $f_2(u)$, $f_3(u, v)$ and $f_4(u, v)$ are strictly decreasing functions.

By the use of Definition 1.3 and the generalization of Table 1-Table 5 for the ith iteration of the deletion of ik pendant vertices from the vertex of degree $ik+2$ and joining them to a vertex of degree $k+2$, we obtain the F-index of the tricyclic graphs A_1^i, B_1^i, C_1^i, D_1^i and E_1^i with three cycles and lk pendant vertices for $0 \le i \le l-1$ in the following lemma.

Lemma 2.2. For $m = m_1 + m_2 + m_3 \ge 9$, $k \ge 1$, $r \ge 2$, $2 \le l \le m + r$, $0 \le i \le l - 1$ and $m_j \ge 3$ with $j = 1, 2, 3$, the F-index of the tricyclic graphs defined above are

(i) $F(A_1^i) = lk+(l-i-1)(k+2)^3+8(m+r-l+i)+[(i+1)k+2]^3+108,$

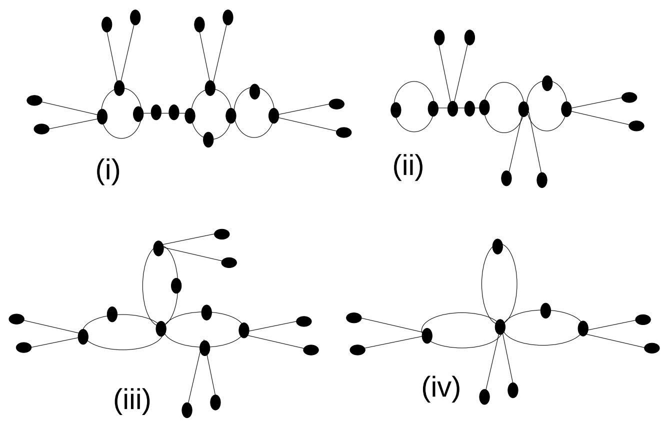

Figure 3: (i)$B^{6,2}_{4,2,1}$ (ii)$B^{3,2}_{3,2,2}$ (iii)$E^{7,0}_{4,2,1}$ and (iv)$E^{4,0}_{3,2,2}$.

(*ii*) $F(B^i_1) = lk+(l-i-1)(k+2)^3+8(m+r+1-l+i)+[(i+1)k+2]^3+118,$

(*iii*) $F(C^i_1) = lk+(l-i-1)(k+2)^3+8(m+r-l+2+i)+[(i+1)k+2)]^3+128,$

(*iv*) $F(D^i_1) = lk+(l-i-1)(k+2)^3+8(m+r-l+2+i)+[(i+1)k+2)]^3+152,$

(*v*) $F(E^i_1) = lk+(l-i-1)(k+2)^3+8(m+r-l+3+i)+[(i+1)k+2)]^3+216.$

Again using Definition 1.3 and Tables 1-5 (3rd and 4th rows), we obtain the F-index of A^i_2, B^i_2, C^i_2, D^i_2 and E^i_2 for $0 \le i \le l-1$ in the following lemma.

Lemma 2.3. For $m = m_1 + m_2 + m_3 \ge 9$, $k \ge 1$, $r \ge 2$, $2 \le l \le m + r + 4$, $0 \le i \le l - 1$ and $m_j \ge 3$ with $j = 1, 2, 3$, the F-index of the tricyclic graphs A^i_2, B^i_2, C^i_2, D^i_2 and E^i_2 are

$$(i)\quad F(A^i_2) = \begin{cases} lk + (l-5-i)(k+2)^3 + 4(k+3)^3 + 8(m+r+4-l+i) + [(i+1)k+3]^3; & \text{for } 1 \le i \le l-5,\\ lk + (l-1-i)(k+3)^3 + 8(m+r) + 27(i-l+4) + [(i+1)k+3]^3; & \text{for } l-4 \le i \le l-1,\end{cases}$$

$$(ii)\quad F(B^i_2) = \begin{cases} lk + (l-4-i)(k+2)^3 + 2(k+3)^3 + (k+4)^3 + 8(m+r+4-l+i) + [(i+1)k+3]^3; \\ \qquad \text{for } 1 \le i \le l-4,\\ lk + 8(m+r+1) + (k+3)^3 + (k+4)^3 + (lk-2k+3)^3; \text{for } i = l-3,\\ lk + 8(m+r+1) + (k+4)^3 + (lk-k+3)^3 + 27; \text{for } i = l-2,\\ lk + (lk+4)^3 + 8(m+r+1) + 54; \text{for } i = l-1,\end{cases}$$

$$(iii)\quad F(C^i_2) = \begin{cases} lk + (l-3-i)(k+2)^3 + 2(k+4)^3 + 8(m+r+4-l+i) + [(i+1)k+3]^3; \\ \qquad \text{for } 1 \le i \le l-3,\\ lk + (k+4)^3 + 8(m+r+2) + (lk-k+4)^3; \quad \text{for } i = l-2,\\ lk + (lk+4)^3 + 8(m+r+2) + 64; \quad \text{for } i = l-1,\end{cases}$$

$$(iv)\quad F(D^i_2) = \begin{cases} lk + (l-3-i)(k+2)^3 + (k+3)^3 + (k+5)^3 + 8(m+r+4-l+i) + [(i+1)k+3]^3; \\ \qquad \text{for } 1 \le i \le l-3,\\ lk + (k+5)^3 + (lk-k+3)^3 + 8(m+r+2); \quad \text{for } i = l-2,\\ lk + (lk+5)^3 + 8(m+r+2) + 27; \quad \text{for } i = l-1,\end{cases}$$

$$(v)\quad F(E^i_2) = \begin{cases} lk + (l-2-i)(k+2)^3 + (k+6)^3 + 8(m+r+4-l+i) + [(i+1)k+3]^3; & \text{for } 1 \le i \le l-2,\\ lk + (lk+6)^3 + 8(m+r+3); \text{for } i = l-1.\end{cases}$$

Lemma 2.4. For $m = m_1 + m_2 + m_3 \ge 9$, $k \ge 1$, $r \ge 2$, $2 \le l \le m + r$, $0 \le i \le l - 1$, $1 \le j \le 2$ and $m_p \ge 3$ with $p = 1, 2, 3$, we have

(a) $F(T_j^i) = F(A_j^i)$, $F(S_j^i) = F(B_j^i)$ and $F(R_j^i) = F(D_j^i)$
(b) $F(X_j^i) = F(C_j^i)$, $F(Y_j^i) = F(B_j^i)$ and $F(Z_j^i) = F(A_j^i)$
(c) $F(L_j^i) = F(A_j^i)$.

Proof. (a) Since the degree sequences of the base graphs of the tricyclic graph with four cycles T_j^i (see, H_3 in Figure 2) and tricyclic graph with three cycles A_j^i (see, G_1 in Figure 1) are equal. Therefore, the degree sequences of the graphs T_j^i and A_j^i having $k \geq 1$ pendant vertices attached with l vertices of degree (i) exactly 2 for $j = 1$ and (ii) greater or equal 2 for $j = 2$ are equal. Consequently, by Lemma 1.1, $F(T_j^i) = F(A_j^i)$. Similarly, the degree sequences of the tricyclic graphs with four cycles S_j^i and R_j^i are equal to the degree sequences of the tricyclic graphs with three cycles B_j^i and D_j^i respectively. Thus, by Lemma 1.1, we have $F(S_j^i) = F(B_j^i)$ and $F(R_j^i) = F(D_j^i)$. (b) Proof is same as of part (a). (c) Proof is same as of part (a).

3 Extremal graphs with respect to F-index

The results of extremal graphs in the complete class of tricyclic graphs with fixed pendant vertices are obtained in this section.

Lemma 3.1. For $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq m + r$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(A_1) \leq F(A_2)$, $F(B_1) \leq F(B_2)$, $F(C_1) \leq F(C_2)$, $F(D_1) \leq F(D_2)$ and $F(E_1) \leq F(E_2)$.

Proof. Consider:

Case 1: Using Lemma 2.2(i) and Lemma 2.3(i) for $i = 0$, we have $F(A_1) - F(A_2) = F(A_{l,k,l}^{m,r}) - F(A_{l,k,2}^{m,r}) = -12k(k+5)$. By Lemma 2.1(i), it follows that the tricyclic graph A_1 has F-index less than of A_2.

Case 2: Using Lemma 2.2(ii) and Lemma 2.3(ii) for $i = 0$, we have $F(B_1) - F(B_2) = F(B_{l,k,l}^{m,r}) - F(B_{l,k,2}^{m,r}) = -6k(2k + 11)$. By Lemma 2.1(i), it follows that F-index of the tricyclic graph B_1 is less than the F-index of the tricyclic graph B_2.

Case 3: Using Lemma 2.2(iii) and Lemma 2.3(iii) for $i = 0$, we have $F(C_1) - F(C_2) = F(C_{l,k,l}^{m,r}) - F(C_{l,k,2}^{m,r}) = -12k(k + 6)$. By Lemma 2.1(i), it follows that F-index of the tricyclic graph C_1 is less than the F-index of the tricyclic graph C_2.

Case 4: Using Lemma 2.2(iv) and Lemma 2.3(iv) for $i = 0$, we have $F(D_1) - F(D_2) = F(D_{l,k,l}^{m,r}) - F(D_{l,k,2}^{m,r}) = -12k(k + 6)$. By Lemma 2.1(i), it follows that F-index of the tricyclic graph D_1 is less than the F-index of the tricyclic graph D_2.

Case 5: Using Lemma 2.2(v) and Lemma 2.3(v) for $i = 0$, we have $F(E_1) - F(E_2) = F(E_{l,k,l}^{m,r}) - F(E_{l,k,2}^{m,r}) = -12k(k + 8)$. By Lemma 2.1(i), it follows that F-index of the tricyclic graph E_1 is less than the F-index of E_2.

Consequently, from all the cases, we have $F(A_1) \leq F(A_2)$, $F(B_1) \leq F(B_2)$, $F(C_1) \leq F(C_2)$, $F(D_1) \leq F(D_2)$ and $F(E_1) \leq F(E_2)$.

Lemma 3.2. For $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq m + r$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(A_1) \leq F(B_1) \leq F(C_1) \leq F(D_1) \leq F(E_1)$.

Proof. Consider:

Case 1: By Lemma 2.2((i) and (ii)) for $i = 0$, we have $F(A_1) - F(B_1) = F(A_{l,k,l}^{m,r}) - F(B_{l,k,1}^{m,r}) = -18 < 0$. It follows that $F(A_1) < F(B_1)$.

Case 2: Using Lemma 2.2((ii) and (iii)) for $i = 0$, we have $F(B_1) - F(C_1) = F(B_{l,k,l}^{m,r}) - F(C_{l,k,1}^{m,r}) = -36 < 0$. Which implies that $F(B_1) < F(C_1)$.

Case 3: By Lemma 2.2((iii) and (iv)) for $i = 0$, we have $F(C_1) - F(D_1) = F(C^{m,r}_{l,k,l}) - F(D^{m,r}_{l,k,1}) = -24 < 0$. It follows that $F(C_1) < F(D_1)$.

Case 4: By Lemma 2.2((iv) and (v)) for $i = 0$, we have $F(D_1) - F(E_1) = F(D^{m,r}_{l,k,l}) - F(E^{m,r}_{l,k,1}) = -72 < 0$. It implies that $F(D_1) < F(E_1)$.

From all the cases, we conclude that $F(A_1) \le F(B_1) \le F(C_1) \le F(D_1) \le F(E_1)$.

Theorem 3.3. If $m = m_1 + m_2 + m_3 \ge 9$, $k \ge 1$, $r \ge 2$, $2 \le l \le m + r$ and $m_j \ge 3$ with $j = 1, 2, 3$. Then, $F(A_1) \le F(G)$, $F(B_1) \le F(G)$, $F(C_1) \le F(G)$, $F(D_1) \le F(G)$ and $F(E_1) \le F(G)$ for each $G \in \mathcal{U}_\mathbf{1}$, $G \in \mathcal{U}_\mathbf{2}$, $G \in \mathcal{U}_\mathbf{3}$, $G \in \mathcal{U}_\mathbf{4}$ and $G \in \mathcal{U}_\mathbf{5}$ respectively. Moreover, equalities hold if $G \cong A_1$, $G \cong B_1$, $G \cong C_1$, $G \cong D_1$ and $G \cong E_1$ respectively.

Proof. We consider the following cases:

Case 1: If $G = A^1_1$ then by Lemma 2.2(i) (for $i = 0$ and $i = 1$) $F(A_1) - F(A^1_1) = -6k^2(k+2)$. Moreover using Lemma 2.1(ii), we have $F(A_1) < F(A^1_1)$. Similarly, by Lemma 2.2((ii)-(iv)) and Lemma 2.1(ii), we have $F(B_1) - F(B^1_1) = F(C_1) - F(C^1_1) = F(D_1) - F(D^1_1) = F(E_1) - F(E^1_1) = -6k^2(k+2) < 0$ which implies that $F(B_1) \le F(B^1_1)$, $F(C_1) \le F(C^1_1)$, $F(D_1) \le F(D^1_1)$, $F(E_1) \le F(E^1_1)$. Consequently, if $G = A^1_1$, $G = B^1_1$, $G = C^1_1$, $G = D^1_1$ and $G = E^1_1$ then $F(A_1) \le F(G)$, $F(B_1) \le F(G)$ $F(C_1) \le F(G)$, $F(D_1) \le F(G)$ and $F(E_1) \le F(G)$ for each $G \in \mathcal{U}_\mathbf{1}$, $G \in \mathcal{U}_\mathbf{2}$, $G \in \mathcal{U}_\mathbf{3}$, $G \in \mathcal{U}_\mathbf{4}$ and $G \in \mathcal{U}_\mathbf{5}$ respectively.

Case 2: If $G = A^i_1$ for $1 \le i \le l - 2$, by Lemma 2.2(i)

$$F(A^i_1) - F(A^{i+1}_1) = -8 + (k + 2)^3 + (ik + k + 2)^3 - (ik + 2k + 2)^3 = -3k^3(i^2 + 3i + 2) - 12k^2(i + 1).$$

By Lemma 2.1(iii), $F(A^i_1) < F(A^{i+1}_1)$. Using $i = 1, 2, 3, \dots, l - 2$, we have $F(A^1_1) < F(A^2_1), F(A^2_1) < F(A^3_1), \dots, F(A^{l-2}_1) < F(A^{l-1}_1)$. By Combining these inequalities

$$F(A^1_1) < F(A^2_1) < F(A^3_1) < \dots < F(A^{l-1}_1).$$

Using Case 1 and above inequality, $F(A_1) < F(A^i_1)$ for $1 \le i \le l - 1$ which implies that $F(A_1) < F(G)$. Similarly, by Lemma 2.2((ii)-(v)) $F(B^i_1) - F(B^{i+1}_1) = F(C^i_1) - F(C^{i+1}_1) = F(D^i_1) - F(D^{i+1}_1) = F(E^i_1) - F(E^{i+1}_1) = -8 + (k + 2)^3 + [k(i + 1) + 2]^3 - [k(i + 2) + 2)]^3 = -3k^3(i^2 + 3i + 2) - 12k^2(i + 1)$. By Lemma 2.1(ii), $F(B^i_1) < F(B^{i+1}_1)$, $F(C^i_1) < F(C^{i+1}_1)$, $F(D^i_1) < F(D^{i+1}_1)$ and $F(E^i_1) < F(E^{i+1}_1)$, where $1 \le i \le l - 2$. Using Case 1 and above inequalities, we have $F(B_1) < F(B^i_1)$, $F(C_1) < F(C^i_1)$, $F(D_1) < F(D^i_1)$ and $F(E_1) < F(E^i_1)$, for $1 \le i \le l - 1$. Consequently, if $G = A^i_1$, $G = B^i_1$, $G = C^i_1$, $G = D^i_1$ and $G = E^i_1$ for $1 \le i \le l - 1$ then $F(A_1) \le F(G)$, $F(B_1) \le F(G)$ $F(C_1) \le F(G)$, $F(D_1) \le F(G)$ and $F(E_1) \le F(G)$ for each $G \in \mathcal{U}_\mathbf{1}$, $G \in \mathcal{U}_\mathbf{2}$, $G \in \mathcal{U}_\mathbf{3}$, $G \in \mathcal{U}_\mathbf{4}$ and $G \in \mathcal{U}_\mathbf{5}$ respectively.

Case 3: If $G = A^i_2$, $G = B^i_2$, $G = C^i_2$, $G = D^i_2$ and $G = E^i_2$, using the same way as of Case 2, we can prove that $F(A_2) \le F(G)$, $F(B_2) \le F(G)$ $F(C_2) \le F(G)$, $F(D_2) \le F(G)$ and $F(E_2) \le F(G)$ respectively, where $1 \le i \le l - 1$. Moreover, by Lemma 3.1 $F(A_1) \le F(A_2)$, $F(B_1) \le F(B_2)$, $F(C_1) \le F(C_2)$, $F(D_1) \le F(D_2)$ and $F(E_1) \le F(E_2)$. Consequently, $F(A_1) \le F(G)$, $F(B_1) \le F(G)$ $F(C_1) \le F(G)$, $F(D_1) \le F(G)$ and $F(E_1) \le F(G)$ for each $G \in \mathcal{U}_\mathbf{1}$, $G \in \mathcal{U}_\mathbf{2}$, $G \in \mathcal{U}_\mathbf{3}$, $G \in \mathcal{U}_\mathbf{4}$ and $G \in \mathcal{U}_\mathbf{5}$ respectively.

Case 4: If $G \in \mathcal{U}_\mathbf{1} - \{A^i_1, A^i_2\}$, $G \in \mathcal{U}_\mathbf{2} - \{B^i_1, B^i_2\}$, $G \in \mathcal{U}_\mathbf{3} - \{C^i_1, C^i_2\}$, $G \in \mathcal{U}_\mathbf{4} - \{D^i_1, D^i_2\}$ and $G \in \mathcal{U}_\mathbf{5} - \{E^i_1, E^i_2\}$. After using transformation of the deletion of pendant vertices and joining them with degree greater or equal to two, we obtain A^i_1 or A^i_2, B^i_1 or B^i_2, C^i_1 or C^i_2, D^i_1 or D^i_2, and E^i_1 or E^i_2 respectively. Thus, we follow the Case 2 or Case 3.

Thus, from all the cases, we have $F(A_1) \le F(G)$ for each $G \in \mathcal{U}_\mathbf{1}$, $F(B_1) \le F(G)$ for each $G \in \mathcal{U}_\mathbf{2}$, $F(C_1) \le F(G)$ for each $G \in \mathcal{U}_\mathbf{3}$, $F(D_1) \le F(G)$ for each $G \in \mathcal{U}_\mathbf{4}$ and $F(E_1) \le F(G)$ for each $G \in \mathcal{U}_\mathbf{5}$, where equalities hold if $G \cong A_1$, $G \cong B_1$, $G \cong C_1$, $G \cong D_1$ and $G \cong E_1$ respectively.

Theorem 3.4. If $k \ge 1$, $n \ge 16 + lk$ and $1 \le l \le n - lk$. Then $F(A_1) \le F(G)$ for each $G \in \mathcal{U}^\mathbf{lk}_\mathbf{n}$, where $\mathcal{U}^\mathbf{lk}_\mathbf{n}$ is a class of all the tricyclic graphs with three cycles such that each graph has order n and pendant vertices lk. Moreover, equality holds if $G \cong A_1$.

Proof. We consider the following cases:

Case 1: If $G \in \mathcal{U}_{\mathbf{1}}$, by Theorem 3.3 $F(A_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{1}}$.

Case 2: If $G \in \mathcal{U}_{\mathbf{2}}$, by Theorem 3.3 $F(B_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{2}}$. Also, by Lemma 3.6 $F(A_1) \leq F(B_1)$ which implies that $F(A_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{2}}$.

Case 3: Assume that $G \in \mathcal{U}_{\mathbf{3}}$. Using Theorem 3.3, we have $F(C_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{3}}$. Now, by Lemma 3.2 $F(A_1) \leq F(C_1)$. Consequently, $F(A_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{3}}$.

Case 4: If $G \in \mathcal{U}_{\mathbf{4}}$ then by Theorem 3.3 $F(D_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{4}}$. Moreover, by Lemma 3.2 $F(A_1) \leq F(D_1)$ which implies that $F(A_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{4}}$.

Case 5: If $G \in \mathcal{U}_{\mathbf{5}}$, by Theorem 3.3 $F(C_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{5}}$ and by Lemma 3.2 $F(A_1) \leq F(E_1)$. Consequently, $F(A_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{5}}$.

So, from all the cases, we conclude that $F(A_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{n}}^{\mathbf{lk}}$ and equality holds if $G \cong A_1$.

Lemma 3.5. For $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq n - lk$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(A_1^{l-1}) \leq F(A_2^{l-1})$, $F(B_1^{l-1}) \leq F(B_2^{l-1})$, $F(C_1^{l-1}) \leq F(C_2^{l-1})$, $F(D_1^{l-1}) \leq F(D_2^{l-1})$ and $F(E_1^{l-1}) \leq F(E_2^{l-1})$.

Proof. Consider:

Case 1: By Lemma 2.2(i) and Lemma 2.3(i) for $i = l - 1$, we have $F(A_1^{l-1}) - F(A_2^{l-1}) = -3lk(lk + 5)$. By Lemma 2.1(iv), $F(A_1^{l-1}) < F(A_2^{l-1})$.

Case 2: Using Lemma 2.2(ii) and Lemma 2.3(ii) for $i = l - 1$, we have $F(B_1^{l-1}) - F(B_2^{l-1}) = -6lk(lk + 6)$. Using Lemma 2.1(iv), we have $F(B_1^{l-1}) \leq F(B_2^{l-1})$.

Case 3: By Lemma 2.2(iii) and Lemma 2.3(iii) for $i = l-1$, we have $F(C_1^{l-1}) - F(C_2^{l-1}) = -6lk(lk+6)$. By Lemma 2.1(iv), $F(C_1^{l-1}) < F(C_2^{l-1})$.

Case 4: By Lemma 2.2(iv) and Lemma 2.3(iv) for $i = l-1$, we have $F(D_1^{l-1}) - F(D_2^{l-1}) = -9lk(lk+7)$. By Lemma 2.1(iv), $F(D_1^{l-1}) < F(D_2^{l-1})$.

Case 5: By Lemma 2.2(v) and Lemma 2.3(v) for $i = l - 1$, we have $F(E_1^{l-1}) - F(E_2^{l-1}) = -12lk(lk + 8)$. Using Lemma 2.1(iv), $F(E_1^{l-1}) < F(E_2^{l-1})$.

From all the cases, we conclude that $F(A_1^{l-1}) \leq F(A_2^{l-1})$, $F(B_1^{l-1}) \leq F(B_2^{l-1})$, $F(C_1^{l-1}) \leq F(C_2^{l-1})$, $F(D_1^{l-1}) \leq F(D_2^{l-1})$, $F(E_1^{l-1}) \leq F(E_2^{l-1})$.

Lemma 3.6. For $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq m + r + 4$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(A_2^{l-1}) \leq F(B_2^{l-1}) \leq F(C_2^{l-1}) \leq F(D_2^{l-1}) \leq F(E_2^{l-1})$.

Proof. Consider:

Case 1: By Lemma 2.3((i) and (ii)) for $i = l - 1$, we have $F(A_2^{l-1}) - F(B_2^{l-1}) = -3l^2k^2 - 21lk - 28$. Using Lemma 3.1(iv), we have $F(A_2^{l-1}) < F(B_2^{l-1})$.

Case 2: Using Lemma 2.3((ii) and (iii)) for $i = l - 1$, we have $F(B_2^{l-1}) - F(C_2^{l-1}) = -18 < 0$. By Lemma 3.1(iv), we have $F(B_2^{l-1}) \leq F(C_2^{l-1})$.

Case 3: By Lemma 2.3((iiii) and (iv)) for $i = l-1$, we have $F(C_2^{l-1}) - F(D_2^{l-1}) = -3l^2k^2 - 27lk - 24$. Using Lemma 3.1(iv), we have $F(C_2^{l-1}) < F(D_2^{l-1})$.

Case 4: By Lemma 2.3((iv) and (v)) for $i = l - 1$, we have $F(D_2^{l-1}) - F(E_2^{l-1}) = -3l^2k^2 - 33lk - 72$. By Lemma 3.1, we have $F(D_2^{l-1}) < F(E_2^{l-1})$.

From all the cases, we conclude that $F(A_2^{l-1}) \leq F(B_2^{l-1}) \leq F(C_2^{l-1}) \leq F(D_2^{l-1}) \leq F(E_2^{l-1})$.

Theorem 3.7. If $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq m + r + 4$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(G) \leq F(A_2^{l-1})$, $F(G) \leq F(B_2^{l-1})$, $F(G) \leq F(C_2^{l-1})$, $F(G) \leq F(D_2^{l-1})$ and $F(G) \leq F(E_2^{l-1})$ for each $G \in \mathcal{U}_{\mathbf{1}}$, $G \in \mathcal{U}_{\mathbf{2}}$,

$G \in \mathcal{U}_3$, $G \in \mathcal{U}_4$ and $G \in \mathcal{U}_5$ respectively. Moreover, equalities hold if $G \cong A_2^{l-1}$, $G \cong B_2^{l-1}$, $G \cong C_2^{l-1}$, $G \cong D_2^{l-1}$ and $G \cong E_2^{l-1}$ respectively.

Proof. Proof is same as of Theorem 3.3 with the help of Lemma 2.3, Lemma 3.5 and Lemma 3.6.

Theorem 3.8. If $k \geq 1$, $n \geq 16 + lk$ and $1 \leq l \leq n - lk$. Then $F(G) \leq F(E_2^{l-1})$ for each $G \in \mathcal{U}_{\mathbf{n}}^{\mathbf{lk}}$, where $\mathcal{U}_{\mathbf{n}}^{\mathbf{lk}}$ is a class of all the tricyclic graphs with three cycles such that each graph has order n and pendant vertices lk. Moreover, equality holds if $G \cong E_2^{l-1}$.

Proof. Proof follows by Theorem 3.4 with the help of Theorem 3.3, Lemma 3.5 & Lemma 3.6.

By the similar arguments as of the tricyclic graphs with three cycles, we obtain the following result for the tricyclic graphs with four, six and seven cycles.

Theorem 3.9. If $k \geq 1$, $n \geq 16 + lk$ and $1 \leq l \leq n - lk$. Then

(a) $F(T_1) \leq F(G) \leq F(R_2^{l-1})$ for each $G \in \xi_{\mathbf{n}}^{\mathbf{lk}}$, where lower and upper bounds holds for $G \cong T_1$ and $G \cong R_2^{l-1}$ respectively.
(b) $F(Z_1) \leq F(G) \leq F(X_2^{l-1})$ for each $G \in \zeta_{\mathbf{n}}^{\mathbf{lk}}$, where lower and upper bounds holds for $G \cong Z_1$ and $G \cong X_2^{l-1}$ respectively.
(c) $F(L_1) \leq F(G) \leq F(L_2^{l-1})$ for each $G \in \mu_{\mathbf{n}}^{\mathbf{lk}}$, where lower and upper bounds holds for $G \cong L_1$ and $G \cong L_2^{l-1}$ respectively.

Proof. (a) By Lemma 2.4, $F(T_1) = F(A_1)$, $F(T_2) = F(A_2)$, $F(S_1) = F(B_1)$, $F(S_2) = F(B_2)$, $F(R_1) = F(D_1)$ and $F(R_2) = F(D_2)$. Now, by Lemma 3.1 $F(A_1) \leq F(A_2)$, $F(B_1) \leq F(B_2)$ and $F(D_1) \leq F(D_2)$. Consequently, $F(T_1) \leq F(T_2)$, $F(S_1) \leq F(S_2)$ and $F(R_1) \leq F(R_2)$. Moreover, by Lemma 3.2 $F(A_1) \leq F(B_1) \leq F(D_1)$ which implies that $F(T_1) \leq F(S_1) \leq F(R_1)$. Finally, by Theorem 3.3 and Theorem 3.4, we have $F(A_1) \leq F(G)$ for each $G \in \mathcal{U}_{\mathbf{n}}^{\mathbf{lk}}$, where $\mathcal{U}_{\mathbf{n}}^{\mathbf{lk}}$ is a class of all the tricyclic graphs with three cycles such that each graph has order n and pendant vertices lk. Consequently, $F(T_1) \leq F(G)$ for each $G \in \xi_{\mathbf{n}}^{\mathbf{lk}}$, where lower bound holds for $G \cong T_1$. Similarly, by Theorem 3.5, Lemma 3.6, Theorem 3.7 and Theorem 3.8, we have $F(G) \leq F(D_2^{l-1}) \leq F(E_2^{l-1})$ for each $G \in \mathcal{U}_{\mathbf{n}}^{\mathbf{lk}}$, where $\mathcal{U}_{\mathbf{n}}^{\mathbf{lk}}$ is a class of all the tricyclic graphs with three cycles such that each graph has order n and pendant vertices lk which implies that $F(G) \leq F(R_2^{l-1})$ for each $G \in \xi_{\mathbf{n}}^{\mathbf{lk}}$, where $\xi_{\mathbf{n}}^{\mathbf{lk}}$ is a class of all the tricyclic graphs with four cycles such that each graph has order n and pendant vertices lk. Thus, $F(T_1) \leq F(G) \leq F(R_2^{l-1})$ for each $G \in \xi_{\mathbf{n}}^{\mathbf{lk}}$, where lower and upper bounds holds for $G \cong T_1$ and $G \cong R_2^{l-1}$ respectively. (b) Proof is same as of part (a). (c) Proof is same as of part (a).

Theorem 3.10. If $k \geq 1$, $n \geq 16 + \alpha$ and $1 \leq l \leq n - \alpha$. Then $F(A_1) \leq F(G) \leq F(E_2^{l-1})$ for each $G \in \Omega_n^\alpha$, where $\Omega_n^\alpha = \{\mathcal{U}_{\mathbf{n}}^\alpha, \xi_{\mathbf{n}}^\alpha, \zeta_{\mathbf{n}}^\alpha, \mu_{\mathbf{n}}^\alpha\}$, α is number of pendant vertices and bounds (lower and upper) holds for $G \cong T_1$ and $G \cong R_2^{l-1}$ (respectively).

Proof. Using Lemma 2.4, Theorem 3.4, Theorem 3.8 and Theorem 3.9.

4 Lower and upper bounds

The ordering and investigate of bounds (lower and upper) of the F-index in the complete class of tricyclic graphs of three, four, six or seven cycles with fixed pendant vertices is given in this section.

Theorem 4.1. If $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq m + r + 4$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, (a) $F(\mathcal{U}_1) < \mathbf{F}(\mathcal{U}_2) < \mathbf{F}(\mathcal{U}_3) < \mathbf{F}(\mathcal{U}_4) < \mathbf{F}(\mathcal{U}_5)$, (b)$F(\xi_3) < \mathbf{F}(\xi_2) < \mathbf{F}(\xi_1)$ and (c)$F(\zeta_3) < \mathbf{F}(\zeta_2) < \mathbf{F}(\zeta_1)$.

Proof.(a) Firstly, we prove that $F(\mathcal{U}_1) < \mathbf{F}(\mathcal{U}_2)$. For the purpose, we show that for each $G \in \mathcal{U}_1$ there exists $G^* \in \mathcal{U}_2$ such that $F(G) < F(G^*)$, where $n = m + r + 4 + lk$ is order of both G and G^* with lk pendant vertices in each. We assume that $G = A_1^i$ and $G^* = B_1^i$ for $1 \leq i \leq l - 1$. By Lemma 2.2, $F(G) - F(G^*) = -10 < 0$ which implies that $F(G) < F(G^*)$. Similarly, by Lemma 2.3 it can be proved that if $G = A_2^i$ for $1 \leq i \leq l - 1$ then there exists $G^* = B_2^i$ such that $F(G) < F(G^*)$. In addition, if $G \in \mathcal{U}_1 - \{\mathbf{A_1^i}, \mathbf{A_2^i}\}$, using transformation of delation and joining the pendant vertices. Then, $G = A_1^i$ or $G = A_2^i$. Thus, we get $G^\star \in \mathcal{U}_2$ such that $F(G) < F(G^\star)$. So, we conclude that $F(\mathcal{U}_1) < \mathbf{F}(\mathcal{U}_2)$. Similarly, we can prove that $F(\mathcal{U}_2) < \mathbf{F}(\mathcal{U}_3)$, $F(\mathcal{U}_3) < \mathbf{F}(\mathcal{U}_4)$, and $F(\mathcal{U}_4) < \mathbf{F}(\mathcal{U}_5)$. Consequently, we have $F(\mathcal{U}_1) < \mathbf{F}(\mathcal{U}_2) < \mathbf{F}(\mathcal{U}_3) < \mathbf{F}(\mathcal{U}_4) < \mathbf{F}(\mathcal{U}_5)$. Proves of (b) and (c) are same as of (a).

Theorem 4.2. If $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq m + r + 4$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(\mathcal{U}_1) = \mathbf{F}(\xi_3) = \mathbf{F}(\zeta_3) < \mathbf{F}(\mathcal{U}_2) = \mathbf{F}(\xi_2) = \mathbf{F}(\zeta_2) < \mathbf{F}(\mathcal{U}_3) = \mathbf{F}(\zeta_1) < \mathbf{F}(\mathcal{U}_4) = \mathbf{F}(\xi_1) < \mathbf{F}(\mathcal{U}_5) < \mathbf{F}(\mathcal{U}_6)$.
Proof. Proof is obvious using Lemma 2.4, Theorem 4.1 (a), (b) and (c).

Theorem 4.3. Let $G \in \Omega_n^\alpha$ be a tricyclic graph of order $n \geq 16 + \alpha$ with three, four, six or seven cycles and $\alpha \geq 1$ pendant vertices. Then, $8n + 12\alpha + 76 \leq F(G) \leq 8(n-1) - 7\alpha + (\alpha + 6)^3$, where the lower and upper bounds are achieved if and only if $G \cong A_1$ with $k = 1$ and $G \cong E_2^{l-1}$ respectively.

Proof. Assuming $l = \alpha$ and $k = 1$ in Lemma 2.2 (i) for $i = 0$ and Lemma 2.3(v) for $i = l - 1$, we have $F(A_1) = 8n + 12\alpha + 76$ and $F(E_2^{l-1}) = 8(n-1) - 7\alpha + (\alpha + 6)^3$. Moreover, by Theorem 3.4 and Theorem 3.8 $F(A_1) < F(G)$ and $F(G) \leq F(E_2^{l-1})$ for each G being a tricyclic graph of order $n \geq 16$ with three cycles and $\alpha \geq 1$ pendant vertices. Thus, we have $8n + 12\alpha + 76 \leq F(G) \leq 8(n-1) - 7\alpha + (\alpha + 6)^3$, where lower and upper bounds are achieved if and only if $G \cong A_1$ with $k = 1$ and $G \cong E_2^{l-1}$ respectively.

5 Conclusion

In this paper, we studied the complete class of tricyclic graphs consisting on three, four, six and seven cycles for certain number of pendant vertices with respect to F-index. We proved the existence of the extremal graphs and construct the ordering of graphs with respect to F-index. Mainly, we computed the bounds (lower and upper) of F-index for the same family of graphs.

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