



Bounds on List Decoding of Rank Metric Codes

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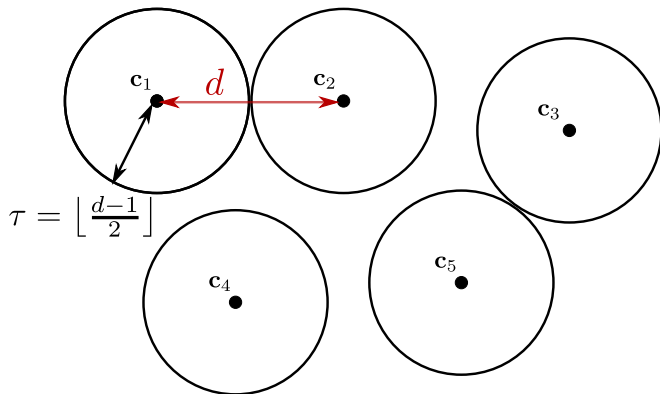
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Motivation — Reed–Solomon vs. Gabidulin Codes

For a code \mathcal{C} of length n , dimension k and minimum distance d , unique decoding is possible up to $\tau = \lfloor \frac{d-1}{2} \rfloor$.



What about decoding algorithms for Gabidulin codes?
Similar to Reed–Solomon codes?

Reed–Solomon vs. Gabidulin Codes — Algorithms

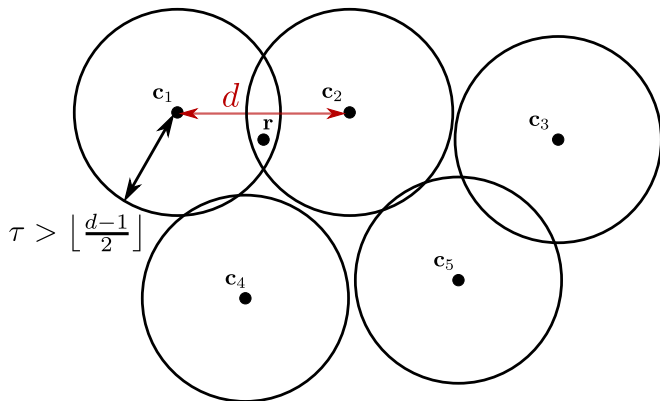
Decoding up to half the minimum distance $\tau = \lfloor \frac{d-1}{2} \rfloor$

	Reed–Solomon Codes	Gabidulin Codes
System of equations	Peterson, ...	Gabidulin
Shift–Register Synthesis	Berlekamp–Massey	Paramonov–Tretjakov, Richter–Plass
Euclidean Algorithm	Sugiyama, ...	Gabidulin
Interpolation	Welch–Berlekamp	Loidreau
⋮	⋮	⋮

Many parallels between Reed–Solomon and Gabidulin codes!

List Decoding

For a code \mathcal{C} of length n , dimension k and minimum distance d , there can be several codewords in a ball of radius $\tau > \lfloor \frac{d-1}{2} \rfloor$.



What about decoding algorithms for Gabidulin codes?
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Reed–Solomon vs. Gabidulin Codes — Algorithms

Decoding **beyond** half the minimum distance $\tau > \lfloor \frac{d-1}{2} \rfloor$

	Reed–Solomon Codes	Gabidulin Codes
Interpolation (List Decoding)	Sudan Guruswami–Sudan (and many accelerations)	?
Syndrome-based (Unique Decoding)	Schmidt–Sidorenko	

Is polynomial–time list decoding possible for Gabidulin codes?

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Rank Metric

- Let \mathcal{B} be a basis of \mathbb{F}_{q^m} over \mathbb{F}_q where q is a power of a prime
- Each vector $\mathbf{x} \in \mathbb{F}_{q^m}^n$ can be mapped on a matrix $\mathbf{X} \in \mathbb{F}_q^{m \times n}$
- **Rank norm:** $\text{rank}(\mathbf{x}) \stackrel{\text{def}}{=} \text{rank of } \mathbf{X} \text{ over } \mathbb{F}_q$

Minimum Rank Distance of a block code \mathcal{C} :

- $d \stackrel{\text{def}}{=} \min\{\text{rank}(\mathbf{c}_1 - \mathbf{c}_2) \mid \mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}, \mathbf{c}_1 \neq \mathbf{c}_2\} \leq n - k + 1$
- Codes with $d = n - k + 1$ are called **Maximum Rank Distance (MRD)** codes

Linearized Polynomial over \mathbb{F}_{q^m}

- $f(x) \stackrel{\text{def}}{=} \sum_{i=0}^{d_f} f_i x^{[i]} = \sum_{i=0}^{d_f} f_i x^{q^i}$ with $f_i \in \mathbb{F}_{q^m}$.
- If $f_{d_f} \neq 0$, define the **q -degree**: $\deg_q f(x) = d_f$.

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Introduced by *Delsarte* (1978), *Gabidulin* (1985), *Roth* (1991)

- A linear **Gabidulin code** $\mathcal{G}(n, k)$ of length $n \leq m$ and dimension k over \mathbb{F}_{q^m} is defined by

$$\mathcal{G}(n, k) \stackrel{\text{def}}{=} \{ \mathbf{c} = (f(\alpha_0) f(\alpha_1) \dots f(\alpha_{n-1})) \mid \deg_q f(x) < k \},$$

where all $f(x)$ are linearized polynomials and $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{F}_{q^m}$ are linearly independent over \mathbb{F}_q .

Minimum Rank Distance of a Gabidulin Code

- $d = \min\{\text{rank}(\mathbf{c}) \mid \mathbf{c} \in \mathcal{G}, \mathbf{c} \neq \mathbf{0}\} = n - k + 1$.

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Problem Statement

Is polynomial-time list decoding possible for rank metric codes (and in particular for Gabidulin codes)?

Problem (Maximum List Size)

Let $\mathcal{C}(n, M, d)$ be a code over \mathbb{F}_{q^m} with $n \leq m$ and minimum rank distance d . Let $\tau < d$. Find a lower and upper bound on the maximum number of codewords ℓ in a ball of rank radius τ . Hence, find a bound on

$$\ell \stackrel{\text{def}}{=} \max_{\mathbf{r} \in \mathbb{F}_{q^m}^n} (|\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{C}(n, M, d)|).$$

Interpretation:

- Lower exponential bound: no polynomial-time list decoding,
- Upper polynomial bound: polynomial-time list decoding might exist.

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Reed-Solomon codes

$$\tau < n - \sqrt{n(n-d)}$$

Johnson bound:

Polynomial list-size

$$\tau \leq \lfloor \frac{d-1}{2} \rfloor$$

Unique Decoding

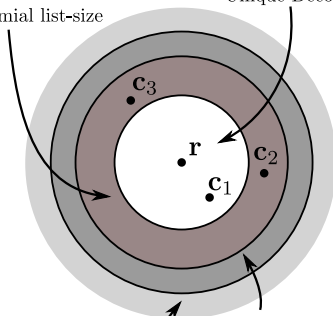
$$\tau > \tau^*$$

Exponential list-size

(Justesen-Hoholdt,

Ben-Sasson-Kopparty-Radhakrishna)

not known



Bounds on the Maximal List-Size

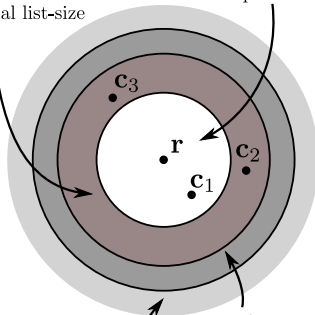
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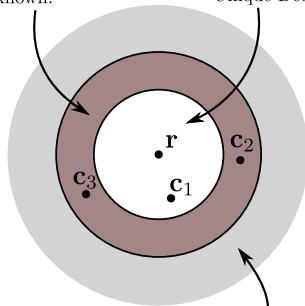
Gabidulin codes

$$\tau < n - \sqrt{n(n-d)}$$

not known!

$$\tau \leq \lfloor \frac{d-1}{2} \rfloor$$

Unique Decoding



$\tau \geq n - \sqrt{n(n-d)}$
Exponential list-size
(this contribution)

Theorem (Lower Bound on the List Size)

Let the Gabidulin code $\mathcal{G}(n, k)$ over \mathbb{F}_{q^m} with $n \leq m$ and $d = n - k + 1$ be given and let $\tau < d$. Then, there exists a word $\mathbf{r} \in \mathbb{F}_{q^m}^n$ such that

$$\ell \geq |\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{G}(n, k)| \geq \frac{\binom{n}{n-\tau}}{(q^m)^{n-\tau-k}} \geq q^m q^{\tau(m+n)-\tau^2-md},$$

and for the special case of $n = m$: $\ell \geq q^n q^{2n\tau-\tau^2-nd}$.

- For $n = m$ this is $\ell \geq q^{n(1-\epsilon)} \cdot q^{2n\tau-\tau^2-nd+n\epsilon}$
- Exponential in n if $\tau \geq n - \sqrt{n(n-d+\epsilon)}$ and $0 \leq \epsilon < 1$ (= Johnson radius).
- Proof similar to the proof of Justesen-Hoholdt for RS codes.

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Proof (i)

- $\mathcal{P}^* \stackrel{\text{def}}{=} \text{set of all monic linearized polynomials of } \deg_q = n - \tau \text{ and a root space over } \mathbb{F}_{q^n} \text{ of dimension } n - \tau > k - 1$
- $|\mathcal{P}^*| = \begin{bmatrix} n \\ n - \tau \end{bmatrix}$
- $\mathcal{P} \stackrel{\text{def}}{=} \text{subset of } \mathcal{P}^* \text{ such that all } q\text{-monomials of } q\text{-degree greater than or equal to } k \text{ have the same coefficients}$
- There are $(q^m)^{n - \tau - k}$ possibilities to choose the highest $n - \tau - (k - 1)$ coefficients
- There exist coefficients such that $|\mathcal{P}| \geq \frac{\begin{bmatrix} n \\ n - \tau \end{bmatrix}}{(q^m)^{n - \tau - k}}$
- For any $f(x), g(x) \in \mathcal{P}$, $\deg_q(f(x) - g(x)) < k$, is evaluation polynomial of a codeword of $\mathcal{G}(n, k)$
- ...

Proof (ii)

- Let $f(x), g(x) \in \mathcal{P}$
- Let $\mathcal{A} = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be a basis of \mathbb{F}_{q^n} over \mathbb{F}_q
- Let $\mathbf{r} = (r_0 \ r_1 \ \dots \ r_{n-1}) = (f(\alpha_0) \ f(\alpha_1) \ \dots \ f(\alpha_{n-1}))$
- Let \mathbf{c} be the evaluation of $f(x) - g(x)$ at \mathcal{A}
- Then, $\mathbf{r} - \mathbf{c}$ is the evaluation of $f(x) - f(x) + g(x) = g(x) \in \mathcal{P}$, whose root space has dimension $n - \tau$ and all roots are in \mathbb{F}_{q^n}
- $\dim \ker(\mathbf{r} - \mathbf{c}) = n - \tau$ and $\text{rk}(\mathbf{r} - \mathbf{c}) = \tau$

Therefore, for **any** $g(x) \in \mathcal{P}$, the evaluation of $f(x) - g(x)$ is a codeword from $\mathcal{G}(n, k)$ and has rank distance τ from \mathbf{r} .

$$\implies \ell \geq |\mathcal{P}| \geq \frac{\binom{n}{n-\tau}}{(q^m)^{n-\tau-k}}.$$



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Theorem (Upper Bound on the List Size)

Let **any** rank metric code $\mathcal{C}(n, M, d)$ over \mathbb{F}_{q^m} with $n \leq m$ and minimum rank distance d be given. Let $\tau < d$. Then, for **any** word $\mathbf{r} \in \mathbb{F}_{q^m}^n$ and hence, for the maximum list size, the following holds

$$\begin{aligned} \ell &= \max_{\mathbf{r} \in \mathbb{F}_{q^m}^n} (|\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{C}(n, M, d)|) \leq \sum_{t=\lfloor \frac{d-1}{2} \rfloor + 1}^{\tau} \frac{\binom{n}{2t+1-d}}{\binom{t}{2t+1-d}} \\ &\leq 4 \sum_{t=\lfloor \frac{d-1}{2} \rfloor + 1}^{\tau} q^{(2t-d+1)(n-t)}. \end{aligned}$$

- Exponential in $n \leq m$ for any $\tau > \lfloor (d-1)/2 \rfloor$
- Does not provide any conclusion if polynomial-time list decoding is possible or not up to the Johnson bound.

Theorem (Lower Bound on the List-Size)

Let $n \leq m$, $\tau \geq \lfloor (d-1)/2 \rfloor + 1$ and $\tau \leq n - \tau$.

Then, there exists a rank metric code $\mathcal{C}(n, M, d_R \geq d)$ over \mathbb{F}_{q^m} of length n and minimum rank distance $d_R \geq d$, and a word $\mathbf{r} \in \mathbb{F}_{q^m}^n$ such that

$$|\mathcal{C}(n, M, d_R \geq d) \cap \mathcal{B}_\tau(\mathbf{r})| \geq q^{(n-\tau)(\tau - \lfloor (d-1)/2 \rfloor)}.$$

- Shows there exists a rank metric code and a received word such that list size is **exponential** in n for $\tau > \lfloor (d-1)/2 \rfloor$.
 \Rightarrow No polynomial-time list decoding for these codes!
- $\mathcal{C}(n, M, d_R \geq d)$ might be non-linear and non-MRD.
- The restriction $\tau \leq n - \tau$ is always fulfilled for $\tau = \lfloor (d-1)/2 \rfloor + 1$ and $k > 1$.
- Proof uses interpretation of $\{\mathbf{r} - \mathbf{c}_1, \mathbf{r} - \mathbf{c}_2, \dots, \mathbf{r} - \mathbf{c}_\ell\}$ as constant-rank code of rank τ and minimum rank distance d .

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We have shown **three bounds on the list size of rank metric codes**:

The lower bound for **Gabidulin codes**

- is based on the evaluation of linearized polynomials,
- shows that polynomial-time list decoding is not possible for $\tau \geq n - \sqrt{n(n-d+\epsilon)}$.

The upper bound for **any** rank metric code

- uses subspace properties,
- is exponential in n .

The lower bound for rank metric codes

- uses the interpretation as constant-rank code,
- **shows that there exists a rank metric code with exponential list size for $\tau \geq \lfloor (d-1)/2 \rfloor + 1$.**

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Merci pour votre attention !