BOUNDS ON PERFORMANCE OF OPTIMUM QUANTIZERS

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Peter Elias

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Bounds on Performance of Optimum Quantizers

PETER ELIAS, FELLOW, IEEE

Abstract—A quantizer Q divides the range [0, 1] of a random variable x into K *quantizing intervals* the ith such interval having length Δx_i. We define the *quantization error* for a particular value of x (unusually) as the length of the quantizing interval in which x finds itself, and measure quantizer performance (unusually) by the rth mean value of the quantizing interval lengths $M_r(Q) = \overline{\Delta x^r}^{1/r}$, averaging with respect to the distribution function F of the random variable x. Q_1 is defined to be an *optimum* quantizer if $M_r(Q_1) \leq M_r(Q)$ for all Q. The unusual definitions restrict the results to bounded random variables, but lead to general and precise results.

We define a class Q^* of quasi-optimum quantizers; Q_2 is in Q^* if the different intervals Δx_i make equal contributions to the mean rth power of the interval size so that Pr $\{\Delta x_i\}$ Δx_i^r is constant for all i. Theorems 1, 2, 3, and 4 prove that $Q_2 \in Q^*$ exists and is unique for given F, K, and r: that $1 \geq KM_r(Q_2) \geq KM_r(Q_1) \geq I_r$, where $I_r = \{\int_0^1 f(x)^p dx\}^{1/q}$, f is the density of the absolutely continuous part of the distribution function F of x, $p = 1/(1+r)$, and $q = r/(1+r)$: that $\lim KM_r(Q_2) = I_r$ as $K \to \infty$; and that if $KM_r(Q) = I_r$ for finite K, then $Q \in Q^*$.

I. Introduction

A. Summary

After reviewing the history and literature of the problem, in Section II we define a quantizer Q that divides the range [0, 1] of a random variable x into a set of K quantizing intervals of which the ith has size Δx_i. Using the unusual definition that the *quantizing error* for a particular value x is the *size of the quantizing interval* within which x finds itself, we measure the performance of a quantizer by the rth mean interval size $M_r(Q)$, the rth root of average of the rth powers Δx_i^r of the interval sizes, averaged with respect to the distribution F of the random variable x:

$$M_r(Q) = \overline{\Delta x^r}^{1/r}.$$

Given F, K, and r, we call Q_1 optimum if $M_r(Q_1) \leq M_r(Q)$ for all Q. It is hard to compute the performance of Q_1 directly, but we bound it by the performance of Q_2, a quantizer in the class Q^*, defined by the property that each of the K terms $\Pr\{\Delta x_i\}$ Δx_i^r in the sum $M_r(Q)^r$ are equal.

Prior work defines quantization error (or noise) not as the size of the quantizing interval, but as the absolute value of the *difference* between x, the random variable being quantized, and some representative point $h(x)$ lying in its quantizing interval; $h(x)$ is a staircase function that takes K fixed values, one in each quantizing interval. Performance is then measured by $D_r(Q)$, the mean rth power of this difference:

$$D_r(Q) = \overline{|x - h(x)|^r},$$

the average again being with respect to the distribution F of x. Optimization here requires choosing the K values of h as well as the K quantizing intervals so as to minimize $D_r(Q)$.

Suppose now that F is absolutely continuous, with density f. If this density is constant in each quantizing interval then the average of $|x - h(x)|^r$ over the interval Δx will be minimized when $h(x) = \Delta x/2$. Then

$$\frac{1}{\Delta x}\int_0^{\Delta x}\left|x - \frac{\Delta x}{2}\right|^r = \frac{1}{1+r}\left(\frac{\Delta x}{2}\right)^r = \frac{\Delta x^r}{2^r(1+r)}.$$

Thus for an f that does not change too fast,

$$D_r(Q) = \overline{|x - h(x)|^r} \approx \frac{1}{2^r(1+r)}\overline{\Delta x^r} = \frac{1}{2^r(1+r)} M_r(Q)^r, \tag{1}$$

the two measures will be almost monotonically related for fixed r and F, and the optimum quantizing intervals for one will be approximately optimum for the other, an approximation that will become better, for smooth f, as K increases.

Our unusual definitions have one severe limitation and one major virtue. The limitation is that F must have compact support, i.e., that x lies in a finite interval Ω, which we normalize to [0, 1] for convenience. If x has infinite range, as in the Gaussian case, one or two of the quantizing intervals are infinite, and so is their rth mean.

The virtue is that no further restrictions need be placed on F in order to obtain a firm lower bound to $M_r(Q)$ for any given F, K, and r, and to show that for the optimum quantizer Q_1, $M_r(Q_1)$ approaches this bound as $K \to \infty$. We prove the existence and uniqueness of the quantizer $Q_2 \in Q^*$ for given F, K, r (Theorem 2) and the bounds

$$\frac{1}{K} \geq M_r(Q_2) \geq M_r(Q_1) \geq \frac{I_r}{K}$$

where

$$I_r = \left\{\int_0^1 f^p\, dx\right\}^{1/q} \qquad q = \frac{r}{1+r}, \quad p = \frac{1}{1+r}$$

and $f = dF/dx$ is the density of the absolutely continuous part of F (Theorems 1 and 2). $Q_2 \in Q^*$ may be constructed adaptively without advance or complete knowledge of F

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(Theorem 2). While $Q \in Q^*$ is not in general optimum, for each $Q \in Q^*$ there is a distribution F_Q consistent with Q for which it is (Theorem 3). If $Q(K)$ is a K-interval quantizer in Q^* and F and r are fixed, then $KM_r(Q(K)) \rightarrow I_r$ as $K \rightarrow \infty$ (Theorem 4).

Given these results about $M_r(Q)$, it is possible to impose smoothness conditions on F and f and thus to derive results about $D_r(Q)$ from (1). For arbitrary F, results of the precision and generality of those available for $M_r(Q)$ seem unlikely to hold for $D_r(Q)$. A reader interested in $D_r(Q)$ may want to rewrite our results using $M_r^r/2^r(1 + r)$ as a performance measure to facilitate comparison. We have not done so below because the denominator complicates the expressions, while the use of the rth mean power rather than the rth mean prevents consideration of $r = 0$, where the rth mean becomes the geometric mean and the mean rth power becomes uninteresting.

While a firm bound has value for applications, the fact that for arbitrary F the bound is also a limit for large K is only of mathematical interest unless one can say something about rate of approach. This is explored in Section III.

There is no general answer; for arbitrary F, convergence can be arbitrarily slow. However, a kind of a priori knowledge that is often available about F, namely that it has a density f that is monotone, or unimodal, can be used to derive bounds on rate of approach. It is in fact as easy to be more general, and to define a class C_J of distributions, proving results on rate of convergence of $KM_r(Q)$ to I_r for $Q \in Q^*$ and $F \in C_J$.

A distribution F is in C_J if the graph of F may be divided into no fewer than J convex pieces, alternately convex $\cap$ and convex $\cup$. Thus an F with monotone density is in C_1; an F with unimodal density and the arcsin distribution are in C_2. Fig. 1 gives other examples.

For $F \in C_J$, convergence is governed by $k = K/J$, the average number of quantizing intervals per convex domain of F. Theorem 5 gives a number of results, of which the most general is

$$I_r \leq KM_r(Q) \leq I_r^{(k-1)/k} \exp\left\{\frac{1}{q}\,\frac{1 + \ln k}{k - 1}\right\}.$$

If $f \leq f_{\max} < \infty$, a tighter result for small q is

$$I_r \leq KM_r(Q) \leq I_r \exp\left\{\frac{q + \ln\left[k(f_{\max} - 1) + 1\right]}{k - q}\right\}$$

while if also $f \geq f_{\min} > 0$, the tightest result is

$$I_r \leq KM_r(Q) \leq I_r \exp\left\{\frac{p \ln\left[f_{\max}/f_{\min}\right]}{k}\right\}.$$

Theorem 5 also permits an experimenter who has found one quantizer Q in Q^* for some K, but has no further knowledge of F beyond the integer J, to bound I_r above and below in terms of $M_r(Q)$ and other measured quantities.

In Section IV we give examples of bounds for $K =$ 2—a very nonasymptotic case—and an entropy bound for equiprobable quantization. We also state results analogous to Theorems 1 and 4 for the multidimensional quantizer.

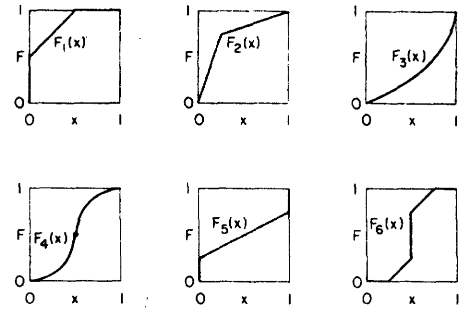

Fig. 1. Distribution functions $F_1(x)$, $F_2(x)$, and $F_3(x)$ are in C_1. $F_4(x)$, $F_5(x)$, and $F_6(x)$ are in C_2. The dot in the center at $(\frac{1}{2}, \frac{1}{2})$ is the unique boundary point separating the two domains of convexity of $F_4(x)$. Any point in the slanted section of $F_5(x)$ is such a boundary, as is any point in the vertical section of $F_6(x)$, for example, the point at $(\frac{1}{2}, \frac{1}{2})$ will do in both cases.

B. History and Literature

All prior work on quantization seems to use the error measure $|x - h(x)|$ described above. In the work dealing with optimum choice of quantization intervals and of values of $h(x)$, performance is measured by the mean square of this error $D_2(Q)$ [1]–[3], the mean rth power $D_r(Q)$ [4]–[6], or the average of more general functions of $|x - h(x)|$ [6]–[8]. Using these measures permits these authors to include cases of unbounded random variables, which many do.

Since $M_r(Q)$ is new as a measure of quantizer performance, the results below are all new in detail. And since the strict lower bounding properties of I_r do not hold for the difference measure, Theorem 1 is also new in principle. So are all results for $r = 0$, the class C_J of distributions, and the rate of approach results of Theorem 5. However for $r \neq 0$ the definition of the class Q^* and Theorem 4 on its asymptotic optimality have approximate precursors.

In a fundamental and widely overlooked paper in 1951, Panter and Dite [1] define as approximately optimum the quantizer for which each of the K quantizing intervals makes an equal contribution to the sum $\overline{|x - h(x)|^r}$—analogous to Q^*—and give the approximation for large K and smooth f

$$\overline{|x - h(x)|^r} \approx \frac{1}{2^r(1 + r)K^r}\left[\int_0^1 f(x)^{1/1+r}\,dx\right]^{1+r} \tag{2}$$

for the particular case $r = 2$, $2^r(1 + r) = 12$ (we have normalized the range of integration). They credit P. R. Aigrain with the first deduction of this result.

Smith [2] in 1957 summarizes [1] but omits mention of the optimum quantizer and the role of I_r, presumably as not relevant to his different objectives. Lozovoy [3], referencing Smith but not Panter and Dite, treats their problem and misses (2) entirely; Roe [5], referencing neither, rediscovers their quantizer but not (2) for general

$r > 0$; Zador [4] referencing Panter and Dite and unpublished work by Lloyd [32] proves Theorem 4 for the difference error under some restrictions on f, for general $r > 0$, with an unevaluated constant replacing $2^r(1 + r)$, but generalized to vector $\boldsymbol{x}$ (multidimensional quantization) and permitting x to be unbounded. Max [7], Bruce [8], and Bluestein [9], looking for algorithms for finding optimum quantizers (for signal plus noise in [9]) miss the reference and the result; Algazi [6], finding simpler suboptimal algorithms also misses the reference but cites the quantizer from Roe and rediscovers (2).

As Smith [2] notes, Sheppard [10] was the first to give the effect of a uniform quantizer (i.e., grouping statistical data in uniform intervals) on variance in 1898. Sheppard's correction, missed by Clavier *et al.* [11] in the first paper on PCM distortion, is rederived by Bennett [12] and by Oliver *et al.* [13]. Smith gives later statistical references on uniform quantization; Cox [14], in 1957, writes on optimum spacing of $K - 1$ levels to minimize the mean-square error in grouping normally distributed data into K intervals, and designs an optimum K-interval quantizer in the mean-square difference sense for $K = 2$ to $K = 6$ for the Gaussian distribution by numerical calculation, not referenced by and not referencing the communications literature and missing (2).

Bertram [15], in 1958, considers uniform quantization in automatic control systems and says "as far as the author has been able to determine there is nothing in the control system literature concerning this problem." He cites Bennett [12] and Widrow [16] on uniform quantization, and the numerical analysis literature on roundoff error in the numerical solution of differential equations. Tou [17], in 1963, devotes a chapter to design of optimal quantizers in control systems, includes nonuniform spacing of levels in his formulation, is concerned with effects of quantization on the dynamics of the systems as well as the static considerations dealt with here, references only Bertram on quantization per se and misses (2).

This paper, and [1]–[9] and [17] deal with minimization of a measure of the average distortion introduced by a quantizer by varying the location of $K - 1$ quantizing levels, given a fixed probability distribution. This is a zero-delay zero-memory encoding operation. Kolmogorov (see Lorentz [18] and Vitushkin [19] for English presentations and further references) and Shannon [20] have dealt with related problems that include delay and memory. Both divide a space (of continuous functions in Kolmogorov's case, of continuous or discrete random processes in Shannon's) into many small domains and find the trading relations between the logarithm of the number of domains (source rate for Shannon, ϵ entropy for Kolmogorov) and the error made in mapping any random point in the domain into one fixed representative point. Kolmogorov and his school deal with the minimax and covering or packing problems of approximating every function in some class to within ϵ on some distortion measure—Shannon with the minimization of source rate for a given *average* level of distortion.

As applied to one-dimensional quantization, Kolmogorov's problem leads directly to the uniform quantizer. Shannon's leads to the problem of minimizing average distortion for a given entropy of the set of K output symbols. In our notation below, minimizing distortion for given

$$H(Y) = -\sum_{i=1}^{K} \Delta y_i \ln \Delta y_i.$$

This is a sensible problem when the variable delay and equipment complexity required for efficient encoding of the quantizer output is permissable.

Pinkston [21] for a class of cases and Goblick and Holsinger [22] for the Gaussian case examine how closely one-dimensional quantization approaches Shannon's rate-distortion function. Gish and Pierce [23] (whose preprint received in the fall of 1967 introduced me to the Panter and Dite reference after my first presentation of some of these results) show that uniform, one-dimensional quantization is asymptotically (large K, small error) optimum in minimizing distortion for given entropy for a large class of measures, including mean rth-power difference measures but not the geometric mean. Wood [33] independently reaches the same conclusion as Gish and Pierce in the mean-square one-dimensional case, citing Roe but not Panter and Dite or Algazi and missing (2). The multidimensional minimal-entropy quantization problem is considered by Schutzenberger [24], who gives inequalities with unknown coefficients on trading relations between entropy and mean rth-power difference measures, and by Zador [4], who gives an asymptotic (large K, small error) result with unknown coefficient for this case too. The rate-distortion function of nonwhite Gaussian noise, given by McDonald and Schultheiss [25] and Goblick [26], has been compared with results obtained by varying the spacing of samples in time as well as the spacing of quantization levels, by Goblick and Holsinger [22] and Kellog [27].

The only point of contact of the present work with the minimal-entropy quantization problem occurs at $r = 0$, when the equiprobable quantizer is in Q^* and its output has entropy $\ln K$, monotonically related to K, so that for a geometric mean-error criterion the two problems have more or less the same solution. The equiprobable quantizer has the virtue of requiring no variable delay or encoding, for a stationary source without memory, while the equal-interval quantizer, even for such a source, requires recoding to represent its input in an average number of binary digits nearly equal to its entropy.

II. Bounds and Asymptotes

A. Definitions

Quantizers: A *quantizer*, or *analog-digital encoder*, maps a random variable x into a discrete set of output symbols $\{S_i\}$, $1 \leq i \leq K$. Let x be a real number in the closed interval [0, 1] with probability distribution function $y = F(x)$. We define the quantizer $Q = \{x_i, y_i\}$ as a set of

$K + 1$ distinct points in the unit square, with

$$x_{i-1} \le x_i, \quad y_{i-1} \le y_i, \quad 1 \le i \le K \qquad x_0 = y_0 = 0, \quad x_K = y_K = 1 \tag{3}$$

and say that the distribution F is *compatible* with Q if the graph of $y = F(x)$ passes through the $K + 1$ points of Q. The x_i are the *quantizing levels* of Q; the y_i are its probability levels; the $\Delta x_i = x_i - x_{i-1}$ are the lengths of its *quantizing intervals*; and $\Delta y_i = y_i - y_{i-1}$ is the probability that x falls in the ith quantizing interval and is encoded into S_i.

Note that the quantizer Q is also determined by the $2K$ nonnegative numbers $\{\Delta x_i\}$, $\{\Delta y_i\}$ subject to

$$\sum_{i=1}^{K} \Delta x_i = \sum_{i=1}^{K} \Delta y_i = 1, \qquad \Delta x_i + \Delta y_i > 0, \quad 1 \le i \le K. \tag{4}$$

If F is known and none of the $\{x_i, y_i\}$ lie on steps or flats of the graph of F, then the quantizer is completely determined by F and either the $\{x_i\}$ or the $\{y_i\}$. However, if the graph of F has steps or flats, the $\{x_i\}$ or the $\{y_i\}$ alone may not determine Q, and the pairs $\{x_i, y_i\}$ may be needed.

This model implies that if $F(x)$ has a step at $x = x_i$, a quantizing level, then when the random variable x takes on the value x_i, an independent random selection is made to determine whether to emit S_i or S_{i+1}, with probabilities, respectively, proportional to $y_i - F(x_i^-)$ and $F(x_i^+) - y_i$. We use it rather than the usual model, in which the quantizer is just the set $\{x_i\}$ and only absolutely continuous F are quantized, because we want to discuss different F compatible with the same Q, and because we want to be able, for example, to use the equiprobable quantizer for $K = 2$ given by $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, and $(1, 1)$ on the distribution $F_6(x)$ in Fig. 1.

Measures of Performance: We measure the performance of a quantizer Q by the rth mean of the Δx_i with weights Δy_i,

$$M_r(Q) = M_r(\{\Delta x_i, \Delta y_i\}) = \left\{\sum_{i=1}^{K} \Delta y_i (\Delta x_i)^r\right\}^{1/r} \tag{5}$$

including the limiting cases $r \to 0$ (geometric mean of the Δx_i) and $r \to \infty$ (maximum absolute value of the Δx_i) (see Hardy *et al.* [28], ch. 1). Note that $M_r(Q)$ can be computed from the quantizer itself, with no knowledge of F beyond that given by the $\{\Delta x_i\}$ and $\{\Delta y_i\}$ values.

For given F, K, and r we also ask how small the rth mean quantizing interval may be made by adjusting the Δx_i and Δy_i. We define a quantizer Q_1 whose rth mean quantizing interval is given by

$$M_r(Q_1) = \min_{\{\Delta x_i, \Delta y_i\}} M_r(\{\Delta x_i, \Delta y_i\}) \tag{6}$$

subject to the constraints of (4), as an *optimum* quantizer for F, K, and r.

Characterization of F: We define $F(x) = \mu([0, x])$ in terms of a probability measure μ defined on the interval $\Omega = [0, 1]$. Let λ be Lebesgue measure on Ω. Then μ has a Lebesgue decomposition, (e.g., Munroe [31], Theorem 41.6):

$$\mu = \mu_0 + \mu_1$$

where μ_0 is singular and μ_1 is absolutely continuous with respect to λ. Let Ω_0 be the set on which μ_0 does not vanish, and Ω_1 the set on which μ_1 does not vanish. Then we have

$$\begin{aligned} &\lambda(\Omega_0) = 0; \qquad \lambda(\Omega) = 1; \\ &\mu(\Omega) = \mu_0(\Omega_0) + \mu_1(\Omega_1) = 1 \\ &f = \frac{d\mu}{d\lambda} = \frac{dF}{dx} \quad \text{almost everywhere in } \Omega \\ &\int_S f\,dx = \mu_1(S) \le \mu(S). \end{aligned} \tag{7}$$

B. Bounds on $M_r(opt)$

For any F and r, choosing all $\Delta x_i = 1/K$ gives an rth mean quantizing error $M_r(1/K, \Delta y_i) = 1/K$. The optimum quantizer Q_1 can do no worse. Thus

$$1 \ge KM_r(Q_1), \tag{8}$$

with equality for all r and K when $F(x) = x$, the uniform distribution.

We should expect behavior like (8), since doubling the number of quantizing intervals permits halving the size of each, and thus of their rth mean. However, it is usually possible to do better than the constant 1 on the left in (8) by making Δx_i small when Δy_i is large and vice versa. We next derive a lower bound to $KM_r(\text{opt})$, which limits how much better one can do.

Let p and q be defined by

$$p = \frac{1}{1+r}, \qquad q = \frac{r}{1+r}. \tag{9}$$

For any quantizer Q, from the definition (5) we have

$$\begin{aligned} KM_r(Q) &= K\left[\sum_{i=1}^{K} \Delta y_i\, \Delta x_i^r\right]^{1/r} \\ &= K^{1/q}\left[\sum_{i=1}^{K} (1/K)(\Delta y_i^{1/r}\, \Delta x_i)^r\right]^{1/r} \\ &\ge K^{1/q}\left[\sum_{i=1}^{K} (1/K)(\Delta y_i^{1/r}\, \Delta x_i)^{r/(1+r)}\right]^{(1+r)/r} \\ &= \left[\sum_{i=1}^{K} \Delta y_i^p\, \Delta x_i^q\right]^{1/q} \\ &= \left[\sum_{i=1}^{K} \bar{f}_i^p\, \Delta x_i\right]^{1/q} \end{aligned} \tag{10}$$

using the inequality for rth means ([28], Theorem 16) and, if Δx_i does not vanish, defining the average densities

$$\bar{f}_i = \Delta y_i / \Delta x_i. \tag{11}$$

For an i for which Δx_i vanishes, we define the term in the last line of (10) to be zero, as in the preceding line.

Considering a typical term in the last line of (10),

$$\begin{aligned} \bar{f}_i^p \,\Delta x_i &= \left[\frac{y_i - y_{i-1}}{\Delta x_i}\right]^p \Delta x_i \\ &\geq \left[\frac{1}{\Delta x_i}\int_{x_{i-1}}^{x_i} f(x)\,dx\right]^p \Delta x_i \\ &\geq \left[\frac{1}{\Delta x_i}\int_{x_{i-1}}^{x_i} f(x)^p\,dx\right] \Delta x_i \\ &= \int_{x_{i-1}}^{x_i} f(x)^p\,dx = \int_{x_{i-1}}^{x_i} f(x)^{-q}(f(x)\,dx) \\ &= \int_{y_{i-1}}^{y_i} g(y)^q\,dy \qquad g = 1/f = dx/dy. \end{aligned} \tag{12}$$

The first inequality follows from $\mu \geq \mu_1$ in (7) and the second follows from the convexity $\cap$ of the pth power for $p \leq 1$; the pth power of the average of f over $[x_{i-1}, x_i]$ is greater than the average of its pth power.

Summing (12) on i and substituting in (10) gives the following theorem.

Theorem 1

Given an F as in (7) and a nonnegative r, let Q_1 be an optimum quantizer consistent with F. Then $M_r(Q_1)$, is bounded:

$$1 \geq KM_r(Q_1) \geq I_r = \left[\int_0^1 f(x)^p\,dx\right]^{1/q} \tag{13}$$

where p and q are given by (9). In the limit $r = 0$ $(q = 0)$, (13) bounds the geometric mean $M_0(Q_1)$ of the Δx_i, while in the limit $r = \infty$ $(q = 1)$, (13) bounds the maximum value $M_\infty(Q_1)$ of the Δx_i:

$$\begin{aligned} KM_0(Q_1) &\geq \exp\left\{\int_0^1 \ln g(y)\,dy\right\} \\ &= \begin{cases} \exp\{H(f)\} & \mu_0(\Omega) = 0 \\ 0 & \mu_0(\Omega) > 0 \end{cases} \end{aligned} \tag{14}$$

$$KM_\infty(Q_1) \geq \int_{\Omega_1} 1\,dx = \int_0^1 g(y)\,dy = \lambda(\Omega_1)$$

where μ_0 is the singular measure of (7); Ω_1 is the set of x on which $f(x)$ does not vanish; and $H(f)$ is the usual entropy functional:

$$H(f) = -\int_{\Omega_1} f(x)\ln f(x)\,dx.$$

Proof: All but the limiting cases have been proved. The fact that M_0 exists and is the geometric mean and that M_∞ exists and is the maximum interval size is shown in [28], Theorems 3 and 5. Thus only the limits of I_r need evaluation.

From the definition of Ω given in (7),

$$\lim_{r\to\infty}\left[\int_0^1 f(x)^p\,dx\right] = \lim_{r\to\infty}\left[\int_{\Omega_1} f(x)^p\,dx\right]^{1/q} = \int_{\Omega_1} 1\,dx = \lambda(\Omega_1) \tag{15}$$

As $r \to 0$, we have

$$\begin{aligned} &\lim_{r\to 0}\left[\int_0^1 f(x)^p\,dx\right]^{1/q} \\ &\quad= \lim_{r\to 0}\left[\int_{\Omega_1} f(x)^p\,dx\right]^{1/q} = \lim_{r\to 0}\left[\int_{\Omega_1} f(x)^{-q}f(x)\,dx\right]^{1/q} \\ &\quad= \lim_{q\to 0}\left\{M_q(1/f, f)_{\Omega_1}\left[\int_{\Omega_1} f(x)\,dx\right]^{1/q}\right\} \end{aligned} \tag{16}$$

or alternately

$$\lim_{r\to 0}\left[\int_0^1 g(y)^q\,dy\right]^{1/q} = \lim_{q\to 0} M_q(g, 1)$$

where we have used the notation

$$M_a(u, v)_E = \left[\int_E u(x)^a v(x)\,dx \Big/ \int_E v(x)\,dx\right]^{1/a},$$

i.e., $M_a(u, v)_E$ is the ath mean of the function u with weights v on the set E. By [28], Theorem 187, the limit of M_a as $a \to 0$ is the corresponding geometric mean. The geometric mean $M_0(1/f, f)\Omega_1$ is the exponential of the usual entropy expression and appears on the right in the top line of (14). The limit of the other factor on the right in (16) is $\mu_1(\Omega_1)^{1/q} = \mu_1(\Omega)^{1/q}$, and its limit as $q \to 0$ is 0 unless $\mu_1(\Omega_1) = 1$, when the limit is 1. The geometric mean $M_0(g, 1)$ is the center term in the top line of (14). Q.E.D.

Note incidentally that if F is defined on $[A, B]$ rather than $[0, 1]$ the only change is to replace the upper bound 1 on the left in (13) by $B-A$, and the limits 0, 1 on the integral I_r^q by A, B. The bounds in (14) are unchanged.

C. The Class Q^* of Quantizers

We next define a particular class Q^* of quantizers that are asymptotically optimum as $K \to \infty$ and have other useful properties.

First, given any quantizer $\{\Delta x_i, \Delta y_i\}$ and any $r \geq 0$ (and thus p, q by (9)) we define σ_i:

$$\sigma_i = \begin{cases} \Delta y_i, & r = 0 \\ \Delta y_i^p\,\Delta x_i^q = \bar{f}_i^p\,\Delta x_i, & r \neq 0 \neq 1/r \\ \Delta x_i, & r = \infty. \end{cases} \tag{17}$$

By the normalization (4) and Holder's inequality ([28], Theorem 11),

$$\begin{aligned} \sum_{i=1}^K \sigma_i &= \sum_{i=1}^K \Delta y_i^p\,\Delta x_i^q \\ &\leq \left[\sum_{i=1}^K \Delta y_i\right]^p\left[\sum_{i=1}^K \Delta x_i\right]^q = 1\cdot 1 = 1. \end{aligned} \tag{18}$$

A K-interval quantizer Q is defined to be in the class Q^* for a particular value of r if σ_i is defined and is the same for all K quantizing intervals:

$$\sigma_i = \sigma, \qquad 1 \leq i \leq K; \qquad \sum_{i=1}^K \sigma_i = K\sigma \leq 1 \tag{19}$$

by (18). For $Q \in Q^*$, the average rth mean quantizing interval $M_r(Q)$ of (5) is related to σ by

$$KM_r(Q) = K\left[\sum_{i=1}^{K} \sigma_i^{1+r}\right]^{1/r} = (K\sigma)^{1/q} = \left[\sum_{i=1}^{K} \sigma_i\right]^{1/q} \leq 1. \tag{20}$$

By (20), $Q \in Q^*$ can be inserted into the inequality chain (13), to give

$$1 \geq KM_r(Q) \geq KM_r(Q_1) \geq I_r. \tag{21}$$

To make (21) meaningful we next show the existence of quantizers in Q^*.

Theorem 2

Given F as in (7), let r be positive and finite ($p \neq 0 \neq q$) and let K be any positive integer. Then the average quantizing interval $M_r(Q)$ for any K-interval quantizer $Q \in Q^*$ compatible with F is uniquely determined by F, K, r, and the requirement (18). If $M_r(Q) > 0$, then the quantizer $Q = \{x_i, y_i\} \in Q^*$ is itself also uniquely determined. If $M_r(Q) = 0$, then F is a pure step function with a finite number of steps, $\mu(\Omega_0) = 1$ and $f = 0$ in (7), and $Q \in Q^*$ exists but may not be unique.

For $r = 0$, the equiprobable quantizer $\Delta y_i = 1/K$ is in Q^* and is unique; for $r = \infty$, the equal interval quantizer $\Delta x_i = 1/K$ is in Q^* and is unique.

Proof: For $r = 0$ and $r = \infty$, the theorem follows directly from (17) and (19), defining σ_i and Q^*.

For $p \neq 0 \neq q$, we consider two cases. Given F, suppose first that every K-interval quantizer $Q(K)$ consistent with F has $\sigma_i = 0$ for at least *one* value of i, $1 \leq i \leq K$. Then it is not possible to choose more than K points on the graph of F, including (0, 0) and (1, 1), without having two of them share a coordinate value, i.e., lie on the same step or flat of F. Thus F is a pure staircase function with a total number of steps and flats $\leq K$. Let $Q_2(K)$ contain (0, 0), (1, 1), one intermediate point at each corner of the staircase, and any leftover points anywhere else. Then Q_2 has $\sigma_i = 0$ for *all* i, so by (19) $Q_2 \in Q^*$, and $M_r(Q_2) = 0$ by (20). $Q_2(K)$ is unique if and only if the number of steps and flats in F is just K.

In the second case, there exists a quantizer $Q(K)$ consistent with F for which all $\sigma_i > 0$, $1 \leq i \leq K$. We parametrize the graph of $y = F(x)$ by defining a new variable s:

$$s = \frac{x+y}{2} = \frac{x + F(x)}{2} = \frac{y + G(y)}{2}. \tag{22}$$

As s increases from 0 to 1 the graph of $F(x)$ goes from (0, 0) to (1, 1), and $x(s)$ and $y(s)$ are monotone nondecreasing functions of s. Since a set of $K + 1$ numbers $\{s_i\}$, $0 \leq i \leq K$, with $0 = s_0 < s_1 < \cdots < s_K = 1$, determine a quantizer $Q = \{x(s_i), y(s_i)\} = \{x_i, y_i\}$ uniquely, we can speak of "the quantizer $\{s_i\}$" as well as "the quantizer $\{x_i, y_i\}$." By the definition of s, $\sigma_i > 0$ is a continuous strictly decreasing function of s_{i-1}. It is therefore possible to invert, in the domain $\sigma_i > 0$, $s_i \leq 1$, to obtain each s_i as a function of the $\{\sigma_i\}$:

$$s_i = h_i(\sigma_1, \sigma_2, \cdots, \sigma_i) \qquad 1 \leq i \leq K - 1 \tag{23}$$

where h_i is continuous and strictly monotone in each σ_{i-j}, and the domain $\sigma_i > 0$, $s_i \leq 1$ is nonempty by the assumed existence of $Q(K)$. Since σ_K is a monotone decreasing function of s_{K-1}, the equation

$$\sigma_K(h_{K-1}(\sigma, \sigma, \cdots, \sigma)) = \sigma$$

has a left side strictly decreasing in σ, a right side strictly increasing, and thus a unique solution for $\sigma > 0$. Then the unique $Q_2(K) \in Q^*$ is determined by (23) at $\sigma_i = \sigma$, and by (19) has $M_r(Q_2) > 0$. Q.E.D.

Comment: The procedure of Theorem 2 for finding quantizers in Q^* is tedious analytically. Practically, however, it can be implemented as an adaptive feedback system without advance or complete knowledge of F, by simply adjusting the x_i and measuring the resulting relative frequency estimates of the Δy_i until condition (19) is satisfied.

D. Properties of Quantizers in Q^*

Next come two theorems that show the unique role played by Q^*, and the fact that the lower bound of Theorem 1 is best possible in two senses. For each $Q \in Q^*$ and no other Q, the bound is attained by a unique distribution F_Q, and for any F it is approached as $K \to \infty$ by quantizers in Q^* consistent with F and thus a fortiori by optimum quantizers.

Theorem 3

Given a K-interval quantizer $Q = \{x_i, y_i\}$ and finite positive r (i.e., $p \neq 0 \neq q$), if and only if $Q \in Q^*$ it is possible to find a unique distribution function $F_Q(x)$ with density $f_Q(x) = dF_Q/dx$ defined a.e. such that Q attains the lower bound of (21), i.e.,

$$KM_r(Q) = I_r = \left[\int_0^1 f_Q(x)^p \, dx\right]^{1/q}, \tag{24}$$

and is thus the optimum quantizer for F_Q, K, and r. The result still holds if $p = 0$ and none of the Δy_i of the given quantizer vanish, or if $q = 0$ and none of the given Δx_i vanish. If $q = 0$ and one or more x_i vanish a quantizer in Q^* will satisfy (24) but so will other quantizers not in Q^*. If $p = 0$ and one or more Δy_i vanish no quantizer will satisfy (24).

Proof: The graph of the distribution function F_Q of the theorem is constructed by connecting each pair of adjacent points (x_i, y_i) and (x_{i+1}, y_{i+1}) by a straight line segment. For $p \neq 0 \neq q$, if any of the Δx_i or Δy_i vanish, then (sufficiency) either all $\sigma_i = 0$, $f_Q(x) = 0$ a.e. and the quantizer Q is in Q^* with $M_r(Q) = 0 = I_r$, or (necessity) some vanish and some do not, Q is not in Q^*, and equality is not attained in the rth mean inequality in (10) and thus is also not attained in (13) and (24). This completes the proof for $p \neq 0 \neq q$ unless none of the Δx_i or Δy_i vanish. In that case the F_Q defined above is given by

$$F_Q(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}} y_i + \frac{x_i - x}{x_i - x_{i-1}} y_{i-1}$$

$$x_{i-1} \leq x \leq x_i \quad (25)$$

$$f_Q(x) = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x_i} \qquad x_{i-1} < x < x_i.$$

To show sufficiency, note that for $Q \in Q^*$, and only then, equality holds in the rth mean inequality and thus in (10). Equality also holds for $F = F_Q$ in (12), and thus in (13) and (24), since there are no steps in $F_Q(x)$ for $x \in [x_{i-1}, x_i]$ so that

$$\Delta y_i = y_i - y_{i-1} = \int_{x_{i-1}}^{x_i} f_Q(x)\,dx = F_Q(x_i) - F_Q(x_{i-1})$$

and f_Q is constant in each quantizing interval, so that

$$\left\{\frac{1}{\Delta x_i}\int_{x_{i-1}}^{x_i} f(x)\,dx\right\}^p = \frac{1}{\Delta x_i}\int_{x_{i-1}}^{x_i} f_Q(x)^p\,dx.$$

To show necessity, note that if $\{x_i, y_i\}$ is not in Q^*, while the distribution (25) will still give equality in (12), the unequal values of σ_i implied by not satisfying (19) for all i will lead to strict inequality in the rth moment inequality and thus in (10) and (24).

This completes the proof for $p \neq 0 \neq q$. For $p = 0$ and all $\Delta y_i > 0$ the result still holds. The unique $Q \in Q^*$ has all $\Delta x_i = 1/K$ by (23), $KM_\infty(Q) = 1$ and this is also the limiting value of I_r as $r \to \infty$. A quantizer not in Q^* must have $\Delta x_{\max} > 1/K$, and since $\Delta x_{\max} = M_\infty$, the two sides of (24) are not equal in that case. For $q = 0$ and all $\Delta x_i > 0$ the result also holds. $KM_0(Q)$ is K times the geometric mean of the Δx_i with weights Δy_i, and it is easy to show that for the quantizer in Q^*, for which all $\Delta y_i = 1/K$, and only for it, this quantity is equal to the exponential of the entropy of the density function f_Q defined in (25), which has been shown in Theorem 1 to be the limit of I_r for $q = 0$.

This completes all the "if and only if" cases. If $q = 0$ and some $\Delta x_i = 0$, a quantizer in Q^* will satisfy (24) by making both sides vanish, but so will any other quantizer that has any $\Delta x_i = 0$. If $p = 0$ and some $\Delta y_i = 0$ so that F_Q or any other compatible distribution must have $\lambda(\Omega_1) < 1$ neither a quantizer in Q^* nor any other can satisfy (24), for the right side is $\lambda(\Omega_1)$ by Theorem 1 while the left is at least $K/(K-1)$ times $\lambda(\Omega_1)$ because at least one of the K labels must be saved for the interval in which $\Delta y_i = 0$. Q.E.D.

Theorem 4

Given a distribution function F and any nonnegative finite r, let $Q = Q(K) \in Q^*$ be any sequence of K-interval quantizers in Q^* compatible with F. Then

$$\lim_{K\to\infty} KM_r(Q(K)) = I_r \quad (26)$$

where the right side is to be interpreted as the limit in (14) of Theorem 1 for $r = 0$.

Proof: If $M_r(Q_1)$ vanishes for some finite K_0 and some $Q_1 = Q_1(K_0) \in Q^*$, then (26) is satisfied by the vanishing of both sides. For I_r must vanish by (13). And any quantizer $Q_2(K)$, $K > K_0$, obtained from $Q_1(K_0)$ by keeping the $K_0 + 1$ original levels and adding $K - K_0$ more will have all $\sigma_i = 0$ so that $M_r(Q_2) = 0$, which by Theorem 2 means that any $Q \in Q^*$ with K intervals will have $M_r(Q) = M_r(Q_2) = 0$.

What remains is to prove (26) when $M_r(Q) > 0$ for all K. Then by Theorem 2, $Q \in Q^*$ is unique for each K in all cases, and by the construction of Theorem 3 it determines a unique distribution F_Q with density f_Q for which it is optimum. By Theorem 3, for each K, $KM_r(Q) = I_r(f_Q) \geq I_r(f)$.

First, assume $q > 0$. Then, as we will prove below, the set $\Omega_2 = \Omega_2(K, \epsilon)$ given by

$$\Omega_2 = \{x: f_Q^p(x) \geq f^p(x) + \epsilon\} \quad (27)$$

has Lebesgue measure $\lambda(\Omega_2)$ that approaches 0 as $K \to \infty$:

$$\lim_{K\to\infty} \lambda(\Omega_2(K)) = 0. \quad (28)$$

Then letting $\Omega = [0, 1]$, we have (assuming (28))

$$\begin{aligned} I_r(f_Q)^{1/q} &= \int_0^1 f_Q^p\,dx \leq \int_{\Omega-\Omega_2} (f^p + \epsilon)\,dx + \int_{\Omega_2} f_Q^p\,dx \\ &\leq \int_\Omega f^p\,dx + \epsilon\int_\Omega dx + \lambda(\Omega_2)\int_{\Omega_2} \frac{f_Q^p}{\lambda(\Omega_2)}\,dx \\ &\leq \int_0^1 f^p\,dx + \epsilon + \lambda(\Omega_2)\left\{\frac{\int_{\Omega_2} f_Q\,dx}{\lambda(\Omega_2)}\right\}^p \quad (29) \\ &\leq \int_0^1 f^p\,dx + \epsilon + \lambda^q(\Omega_2)\cdot 1, \end{aligned}$$

$$\lim_{K\to\infty} I_r(f_Q)^{1/q} \leq \int_0^1 f^p\,dx + \epsilon = I_r(f)^{1/q} + \epsilon.$$

where we have used (27) in the first line, the convexity $\cap$ of the pth power in the third, and (28) in the fifth. Taking qth powers proves the theorem for $q \neq 0$, given (28).

To prove (28), we note from (27) that for $x \in \Omega_2$, $f_Q^p(x) \geq \epsilon$, since $f^p(x) \geq 0$. Then from (17), (19), and (25), for $x \in \Omega_2$,

$$1/K \geq \sigma_i = \Delta y_i^p\,\Delta x_i^q = \left(\frac{\Delta y_i}{\Delta x_i}\right)^p \Delta x_i = f_Q^p(x)\,\Delta x_i \geq \epsilon\,\Delta x_i,$$

so the size Δx_i of the quantizing interval in which $x \in \Omega_2$ lies is bounded above:

$$\Delta x_i \leq 1/K\epsilon. \quad (30)$$

As K increases, a fixed x will be in $\Omega_2(K)$ for some values of K but not for others. To take limits, we define

$$g_K(x) = \begin{cases} f_{Q(K)}(x), & x \in \Omega_2(K) \\ f(x), & x \notin \Omega_2(K). \end{cases} \tag{31}$$

Then taking pointwise limits,

$$\lim_{K\to\infty} g_K(x) = f(x) \qquad \text{a.e. in } \Omega. \tag{32}$$

For if $x \in \Omega_2(K)$ for only finitely many K, (32) follows trivially from (31). If $x \in \Omega_2(K)$ for infinitely many K, then on this subsequence of K values $g_K(x) = f_{Q(K)}(x)$ is a difference quotient $\Delta y_i/\Delta x_i$, with $\Delta x_i \to 0$ by (30), and approaches the derivative $dF(x)/dx = f(x)$ a.e. in Ω, (see e.g., Munroe [31], Theorem 41.3) while it approaches the same value on the complementary subsequence by (31), proving (32). Since the pth power, $p > 0$, is continuous, $g_K^p \to f^p$ a.e. and thus (see e.g., Munroe [31], Theorem 31.3) in Lebesgue measure in Ω, and thus in Ω_2. But in Ω_2, $g_K^p = f_{Q(K)}^p$ by (31). Thus $f_{Q(K)}^p \to f^p$ in measure in Ω_2, and (28) holds.

For $q = 0$, if F has a step then $I_0(f_{Q(K)})$ vanishes for sufficiently large K, since the equiprobable quantizer will have a complete quantizing interval lying in the step, $\Delta y_i = 1/K$, so the geometric mean of the $\{\Delta x_i\}$ will vanish. In this case the theorem is already proved above. Thus if $I_0(f_{Q(K)}) > 0$ for all K, F is continuous (though not necessarily absolutely continuous). Then for each quantizing interval

$$\int_{x_{i-1}}^{x_i} f_Q\,dx = \Delta y_i = \mu([x_{i-1}, x_i])$$
$$\geq \mu_1([x_{i-1}, x_i]) = \int_{x_{i-1}}^{x_i} f\,dx. \tag{33}$$

We define $\Omega_3 = \Omega_3(K, \epsilon)$ by

$$\Omega_3 = \{x: f_Q(x) < \epsilon\}. \tag{34}$$

Since f_Q is piecewise constant, Ω_3 is a union of quantizing intervals to each of which (33) applies, so

$$\int_{\Omega_3} f\,dx = \mu_1(\Omega_3) \leq \mu(\Omega_3)$$
$$= \int_{\Omega_3} f_Q\,dx < \int_{\Omega_3} \epsilon\,dx = \epsilon\lambda(\Omega_3) \leq \epsilon. \tag{35}$$

On $\Omega - \Omega_3$, $f_Q \geq \epsilon$. Then the quantizing intervals in $\Omega - \Omega_3$ are bounded above as in (30), we can define $g_K(x)$ as in (31) but with $\Omega - \Omega_3$ replacing Ω_2, (32) holds, and since $-z \ln z$ is continuous in z, $-g_K \ln g_K \to -f \ln f$ pointwise a.e. and thus in Lebesgue measure, in Ω and thus in $\Omega - \Omega_3$. Since in $\Omega - \Omega_3$ we have $f_{Q(K)} = g_K$, $-f_Q \ln f_Q \to -f \ln f$ in measure in $\Omega - \Omega_3$. Thus there is a set $\Omega_4 = \Omega_4(K, \epsilon)$ such that, for sufficiently large K,

$$-f_Q \ln f_Q < -f \ln f + \epsilon \quad \text{on} \quad \Omega - \Omega_3 - \Omega_4, \qquad \lambda(\Omega_4) < \epsilon. \tag{36}$$

Then we have, for large K,

$$\int_0^1 - f_Q \ln f_Q\,dx$$
$$\leq \int_{\Omega_3} - f_Q\,dx + \int_{\Omega_4} - f_Q \ln f_Q\,dx$$
$$+ \int_{\Omega-\Omega_3-\Omega_4} (-f \ln f + \epsilon)\,dx \tag{37}$$
$$\leq \epsilon \ln(1/\epsilon) + \epsilon/e + \epsilon + \int_{\Omega-\Omega_3-\Omega_4} - f \ln f\,dx$$
$$\leq 3\,\epsilon \ln(1/\epsilon) + \int_\Omega - f \ln f\,dx + \int_{\Omega_3\cup\Omega_4} f \ln f\,dx,$$

where we use (36) in the first line, and in the second the fact that $-z \ln z$ is monotone increasing in $z \leq \epsilon < 1/e$ to bound the integral over Ω_3, and the fact that $-z \ln z \leq 1/e$ and (36) to bound the integral over Ω_4.

Now let $\Omega_5 = \{x : f(x) \geq 1\}$. Then $f \ln f < 0$ outside Ω_5, so

$$\int_{\Omega_3} f \ln f\,dx = \int_{\Omega_3-\Omega_5} f \ln f\,dx$$
$$+ \int_{\Omega_3\cap\Omega_5} f \ln f\,dx \leq \int_{\Omega_3\cap\Omega_5} f \ln f\,dx. \tag{38}$$

And from (35) and the definition of Ω_5,

$$\epsilon \geq \mu_1(\Omega_3) = \int_{\Omega_3} f\,dx \geq \int_{\Omega_3\cap\Omega_5} f\,dx$$
$$\geq \int_{\Omega_3\cap\Omega_5} dx = \lambda(\Omega_3 \cap \Omega_5). \tag{39}$$

Thus from (37) and (38),

$$\lim_{K\to\infty} \int_0^1 - f_Q \ln f_Q\,dx$$
$$\leq \int_{\Omega-(\Omega_3\cap\Omega_5)\cup\Omega_4} - f \ln f\,dx + 3\,\epsilon \ln(1/\epsilon)$$
$$\leq \sup_{E:\lambda(E)<2\epsilon} \left\{\int_{\Omega-E} - f \ln f\,dx\right\} + 3\,\epsilon \ln(1/\epsilon). \tag{40}$$

Since, by (36) and (39), $\lambda((\Omega_3 \cap \Omega_5) \cup \Omega_4) < 2\epsilon$. As $\epsilon \to 0$ the right side of (40) approaches the value $H(f)$, finite or $-\infty$, and the exponential of (40) gives

$$\lim_{K\to\infty} I_0(f_{Q(K)}) \leq I_0(f).$$

Q.E.D.

Comment: For $r = \infty$, (26) does not hold, since by (17), (19), $KM_\infty(Q) \equiv 1$ for $Q \in Q^*$, while by Theorem 1 the lower bound $I_\infty(f) = \lambda(\Omega_1) < 1$ if f vanishes on a set of positive measure. This is not due to the nonoptimality of $Q \in Q^*$ for arbitrary F, since assigning a positive probability to each rational number in [0, 1] gives $I_\infty(f) = 0$, while the optimum quantizer Q_1 has $KM_\infty(Q_1) = 1$, since every quantizing interval in [0, 1] has positive probability and thus must not be larger than any other interval to minimize $M_\infty(Q_1)$. The quantity $I_\infty(f)$ does have asymptotic significance, however. We state without proof two results. First, for sufficiently large K there exist quantizers Q such that all but a set Ω_δ of x of probability $\mu(\Omega_\delta) < \delta$ lie in quantizing intervals of size $< (1/K) \cdot (I_\infty + \epsilon)$. Second, if F has only steps and an absolutely continuous part, and thus no continuous singular part, then (26) holds at $r = \infty$ for a quantizer Q obtained from an equiprobable $Q \in Q^*$ by merging adjacent quantizing intervals that lie on a single flat of F. This gives some quantizing intervals with $\Delta x_i > 1/K$ but $\Delta y_i = 0$, so that they do not increase $M_\infty(Q)$, which is the essential supremum of the $\{\Delta x_i\}$.

III. Rate of Approach

A. Monotone Densities and Convex Distributions

Theorem 4 guarantees convergence but says nothing about rate of approach. In fact it is not possible to do so without restricting F. Given $\epsilon > 0$, it is possible for any K_0 to construct an F that has $KM_r(Q) > 1 - \epsilon$ for all $Q(K)$ with $K < K_0$, but has $KM_r(Q(K)) \rightarrow 0$ for $K \rightarrow \infty$ and $Q \in Q^*$. A staircase with sufficiently small steps and flats will obviously do.

However if F has a density f that is monotone, or more generally if F is a convex ($\cup$ or $\cap$) curve in the unit rectangle, bounds can be obtained on the rate at which $KM_r(Q) \rightarrow I_r$ for $Q \in Q^*$, as a function of K. These results can be extended to an F that has J domains of convexity rather than 1 (see Fig. 1).

We define a distribution F to be in class C_J if there exists a set of $J + 1$ points (ξ_i, η_i) on the graph of F, with $(\xi_0, \eta_0) = (0, 0)$, $(\xi_J, \eta_J) = (1, 1)$, the graph of $y = F(x)$ is a convex curve between the points (ξ_i, η_i) and (ξ_{i+1}, η_{i+1}) (regions of convex $\cup$ F alternating with regions of convex $\cap$ F), and there is no set of less than $J + 1$ points on the graph of F having this property.

The division points (ξ_i, η_i) are not unique when the graph of F has straight-line portions. Any step or flat must pass through one of the (ξ_i, η_i), possible steps alternating with possible flats.

Given that $F \in C_J$, a designer knowing F knows J, can compute I_r by (13), and knows

$$f_{\max} = \max_{x \in [0,1]} f(x), \qquad f_{\min} = \min_{x \in [0,1]} f(x). \tag{41}$$

An experimenter who has constructed a quantizer in Q^* by the iterative procedure of Theorem 2, but does not have complete knowledge of F, cannot compute I_r. He knows $M_r(Q)$ and

$$\bar{f}_{\max} = \max_{1 \le i \le K} \bar{f}_i, \qquad \bar{f}_{\min} = \min_{1 \le i \le K} \bar{f}_i \tag{42}$$

and we assume that he also knows J a priori. He may also know one or both of $f_{\max}$ and $f_{\min}$, which are not measureable and thus must be known a priori if at all.

Theorem 5 permits the designer to bound $M_r(Q)$ as a function of K and the parameters he knows, without designing a quantizer. It also permits the experimenter to bound I_r, and thus to predict how well a quantizer might do for some other K, without further experiment or knowledge of F.

Theorem 5

Let $F \in C_J$, $Q \in Q^*$ with K intervals, $J' = \min(J, K)$, $k = K/J'$ (not necessarily integer), p and q as in (9), $f_{\max}$, $f_{\min}$, $\bar{f}_{\max}$, $\bar{f}_{\min}$ as in (41), (42). Then

$$KM_r(Q) \ge I_r \ge \begin{cases} KM_r(Q) \exp\left\{-\dfrac{p \ln [f_{\max}/f_{\min}]}{k}\right\} \\ KM_r(Q) \exp\left\{-\dfrac{p \ln [f_{\max}/\bar{f}_{\min}] + q}{k - q}\right\} \\ KM_r(Q) \exp\left\{-\dfrac{p \ln [\bar{f}_{\max}/f_{\min}] + p^2/q}{k - p}\right\} \\ KM_r(Q) \exp\left\{-\dfrac{p \ln [\bar{f}_{\max}/\bar{f}_{\min}] + 1/q}{k - 1}\right\} \end{cases} \tag{43}$$

and

$$I_r \le KM_r(Q) \le \begin{cases} I_r \exp\left\{\dfrac{p \ln [f_{\max}/f_{\min}]}{k}\right\} \\ I_r \exp\left\{\dfrac{\ln [k(f_{\max} - 1) + 1] + q}{k - q}\right\} \\ I_r \exp\left\{\left(\dfrac{p}{q}\right)\dfrac{\ln [k(f_{\min}^{-1} - 1) + 1] + p}{k - p}\right\} \\ I_r^{k-1/k} \exp\left\{\left(\dfrac{1}{q}\right)\dfrac{1 + \ln k}{k - 1}\right\}, \end{cases} \tag{44}$$

where $k \ge 2$ in the last line of (44).

Comment: We prove Theorem 5 by means of a number of lemmas. Before doing so, we note that at $r = q = 0$ only the first two lines of each of (43) and (44) are nontrivial, and that they require $f_{\max}/f_{\min} < \infty$, or at least $f_{\max} < \infty$. This is not a weakness in the result. If f is not bounded, $I_0(f)$ may vanish for any value of $\bar{f}_{\max}/\bar{f}_{\min} > 1$, for there is no way of ruling out a step in F or an f that becomes infinite so fast that $H(f) = -\infty$.

For $q \neq 0$, however, while $f_{\max}$ and $f_{\min}^{-1}$ may become infinite for $F \in C_J$, $\bar{f}_{\max}$ and $\bar{f}_{\min}^{-1}$ cannot for a $Q \in Q^*$ unless $KM_r(Q)$ vanishes. Thus the last line of (43) provides a firm bound for $F \in C_J$ in terms of a $Q \in Q^*$, with J as the only a priori information. And the last line of (44) provides the designer who knows I_r with an equally general bound when F has steps and flats.

B. Lower Bounds on T_i

Lemma 1 bounds below the integral of $f(x)^p$ over the ith quantizing interval by means of the convexity $\cap$ of the pth power ($p \le 1$). The bounds are functions of ratios of any two of the three quantities.

$$f_{\max i} = \max_{x \in [x_{i-1}, x_i]} f(x)$$
$$f_{\min i} = \min_{x \in [x_{i-1}, x_i]} f(x) \qquad (45)$$
$$\bar{f}_i = \frac{\Delta y_i}{\Delta x_i}.$$

and hold for any F and any Q.

We start by obtaining lower bounds to the integral on the right in (12). We define T_i:

$$T_i = \frac{1}{\sigma_i}\int_{x_{i-1}}^{x_i} f(x)^p\,dx = \frac{1}{\sigma_i}\int_{y_{i-1}}^{y_i} g(y)^q\,dy \qquad (46)$$

and note from (13) that

$$I_r^q = \int_0^1 f(x)^p\,dx = \sum_{i=1}^K \int_{x_{i-1}}^{x_i} f(x)^p\,dx = \sum_{i=1}^K \sigma_i T_i \qquad (47)$$

and that for quantizers Q in Q^*, from (20) and (21),

$$KM_r(Q) \geq I_r\left(\sigma \sum_{i=1}^K\right)^{1/q} = (K\sigma)^{1/q}\left(\frac{1}{K}\sum_{i=1}^K T_i\right)^{1/q}$$

or

$$KM_r(Q) \geq I_r \geq KM_r(Q)\bar{T}^{1/q},$$
$$K\sigma = [KM_r(Q)]^q \geq I_r^q \geq [KM_r(Q)]^q\bar{T} = K\sigma\bar{T}, \qquad (48)$$
$$\bar{T} = \frac{1}{K}\sum_{i=1}^K T_i.$$

Thus for $Q \in Q^*$ lower bounds on I_r can be derived from lower bounds on the T_i, or their average $\bar{T}$. We now derive such bounds.

Lemma 1

T_i as defined by (46) is bounded below by

$$T_i \geq e^{-qu_i} = (\bar{f}_i/f_{\max i})^q \qquad (49a)$$
$$T_i \geq e^{-pv_i} = (f_{\min i}/\bar{f}_i)^p \qquad (49b)$$
$$T_i \geq e^{-pqz_i} = (f_{\min i}/f_{\max i})^{pq} \qquad (49c)$$

where we have defined $z_i = \ln[f_{\max i}/f_{\min i}]$, $u_i = \ln[f_{\max i}/\bar{f}_i]$, $v_i = \ln[\bar{f}_i/f_{\min i}]$ for later use; $f_{\max i}$, $f_{\min i}$, and $\bar{f}_i$ are defined by (45).

Proof: Given $\bar{f}_i$ and $f_{\max i}$, the $f(x)$ that minimizes T_i for fixed Δx_i (and thus fixed σ_i), by the convexity of the $p = 1/(1+r)$ power, is at $f_{\max i}$ for a set of measure $\alpha\,\Delta x_i$, and at its lower bound 0 for a set of measure $(1-\alpha)\Delta x_i$, with α determined by

$$\alpha\,\Delta x_i\, f_{\max i} + (1-\alpha)\,\Delta x_i\cdot 0 = \bar{f}_i\,\Delta x_i,$$

and thus

$$\alpha = \bar{f}_i/f_{\max i}.$$

Then from (46) and the definition (17) of σ_i,

$$T_i \geq \frac{1}{\sigma_i}\{\alpha\,\Delta x_i f_{\max i}^p + (1-\alpha)\,\Delta x_i\cdot 0^p\}$$
$$\geq \frac{1}{\sigma_i}\frac{\bar{f}_i\,\Delta x_i f_{\max i}^p}{f_{\max i}} = (\bar{f}_i/f_{\max i})^q,$$

which proves (49a). Similarly, given $\bar{f}$ and $f_{\min i}$,

$$T_i \geq (f_{\min i}/\bar{f}_i)^p,$$

proving (49b). Next, if both $f_{\max i}$ and $f_{\min i}$ are known, from (49a) and (49b).

$$T_i = T_i^p T_i^q \geq (\bar{f}_i/f_{\max i})^{pq}(f_{\min i}/\bar{f}_i)^{pq} = (f_{\min i}/f_{\max i})^{pq}$$

which proves (49c). Q.E.D.

C. Lower Bounds on $\bar{T}$ for Convex F

For F convex $\cup$, f exists a.e. in [0, 1] and is increasing. We complete its definition there by setting it equal to the right-hand derivative, which exists throughout [0, 1] (Hardy *et al.* [28], Theorem 111). Then f may be infinite only at $x = 1$, and may vanish only on an interval containing $x = 0$, and

$$f_{\max i} = f(x_i), \qquad f_{\max} = \max_i f_{\max i} = f(1),$$
$$\bar{f}_{\max} = \max_i \bar{f}_i = \bar{f}_K$$
$$f_{\min i} = f(x_{i-1}), \qquad f_{\min} = \min_i f_{\min i} = f(0), \qquad (50)$$
$$\bar{f}_{\min} = \min_i \bar{f}_i = \bar{f}_1.$$

We define

$$\sum_{i=1}^K z_i = \sum_{i=1}^K \ln\frac{f(x_i)}{f(x_{i-1})} = \ln\frac{f(1)}{f(0)} = \ln\frac{f_{\max}}{f_{\min}} = z \qquad (51a)$$

$$u_1 + \sum_{i=2}^K z_i = \ln\frac{f(x_1)}{\bar{f}_1} + \ln\frac{f(1)}{f(x_1)} = \ln\frac{f_{\max}}{\bar{f}_{\min}} = u \qquad (51b)$$

$$\sum_{i=1}^{K-1} z_i + v_K = \ln\frac{f(x_{K-1})}{f(0)} + \ln\frac{\bar{f}_K}{f(x_{K-1})} = \ln\frac{\bar{f}_{\max}}{f_{\min}} = v \qquad (51c)$$

$$u_1 + \sum_{i=2}^{K-1} z_i + v_K = \ln\frac{\bar{f}_{\max}}{\bar{f}_{\min}} = w \qquad (51d)$$

and note that by inverting the order of numbering the intervals, the right-most expressions also apply if F is convex $\cap$, i.e., for any $F \in C_1$.

Then from (48) and Lemma 1 we have the simple bounds

$$\bar{T} \geq \frac{1}{K}\sum_{i=1}^K e^{-pqz_i} \geq e^{-pqz/K} = \left(\frac{f_{\min}}{f_{\max}}\right)^{pq/K} \qquad (52a)$$

$$\bar{T} \geq \frac{1}{K}\left[e^{-qu_1} + \sum_{i=2}^K e^{-pqz_i}\right] \geq e^{-qu/K} = \left(\frac{\bar{f}_{\min}}{f_{\max}}\right)^{q/K} \qquad (52b)$$

$$\bar{T} \geq \frac{1}{K}\left[\sum_{i=1}^{K-1} e^{pqz_i} + e^{-pv_K}\right] \geq e^{-pv/K} = \left(\frac{f_{\min}}{\bar{f}_{\max}}\right)^{p/K} \qquad (52c)$$

$$\bar{T} \geq \frac{1}{K}\left[e^{-qu_1} + \sum_{i=2}^{K-1} e^{-pqz_i} + e^{-pv_K}\right] \geq e^{-[\min(p,q)w/K]}$$
$$= \left(\frac{\bar{f}_{\min}}{\bar{f}_{\max}}\right)^{[\min(p,q)/K]}, \qquad (52d)$$

where e^{-pqz_i} is bounded below by e^{-qz_i}, e^{-pz_i}, and $e^{-\min(p,q)z_i}$, and the convexity $\cup$ of the negative exponential is used to replace averages by functions of average argument.

The exponents in (52b), (52c), and (52d) can be improved. We note from $\ln p \leq p - 1$ that

$$pe^q = e^{q+\ln p} \leq e^{q+p-1} = e^0 = 1$$

and thus that, by convexity $\cup$ of the exponential and (52b),

$$K\bar{T} \geq e^{-qu_1} + \sum_{i=2}^{K} e^{-pqz_i} \geq pe^{-q(u_1-1)} + \sum_{i=2}^{K} e^{-pqz_i}$$
$$\geq (p + K - 1)e^{-pq(u-1)/(p+K-1)}, \quad (53a)$$

so that

$$\bar{T} \geq \left(1 - \frac{q}{K}\right)e^{-pq(u-1)/(K-q)}$$
$$\geq e^{-[pq(u-1)+q]/(K-q)} = e^{-q(pu+q)/(K-q)} \quad (53b)$$

where the last inequality follows from

$$1 - \frac{\alpha}{k} = \frac{k-\alpha}{k} = \frac{1}{1 + \dfrac{\alpha}{k-\alpha}} \geq e^{-\alpha/(k-\alpha)}.$$

Similarly, from $qe^p \leq 1$ and (51c) and (51d)

$$\bar{T} \geq \left(1 - \frac{p}{K}\right)e^{-pq(v-1)/(K-p)}$$
$$\geq e^{-[pq(v-1)+p]/(K-p)} = e^{-p(qv+p)/(K-p)} \quad (53c)$$
$$\bar{T} \geq \left(1 - \frac{1}{K}\right)e^{-pq(w-2)/(K-1)}$$
$$\geq e^{-[pq(w-2)+1]/(K-1)} \geq e^{-(pqw+1)/(K-1)}. \quad (53d)$$

The bounds (52) and (53) only permit the estimation of behavior as K increases if f_{max}/f_{min} is known and finite. Otherwise design of a quantizer $Q \in Q^*$ for each K is required, to evaluate $\bar{f}_{max}$, $\bar{f}_{min}$ or both; these change as K increases and the averaging is done over smaller intervals. The following lemmas permit eliminating $\bar{f}_{max}$ or $\bar{f}_{min}$ or both.

Lemma 2

For F convex $\cup$, Q a K-interval quantizer in Q^* compatible with F and any $r \geq 0$,

$$\left(\frac{f_{max}}{\bar{f}_{min}}\right)^p = e^{pu} \leq K(f_{max} - 1) + 1 \quad (54a)$$

$$\left(\frac{\bar{f}_{max}}{f_{min}}\right)^q = e^{qv} \leq K(f_{min}^{-1} - 1) + 1 \cdot \quad (54b)$$

Proof: Since $Q \in Q^*$, if $\sigma = 0$ then by Theorem 2, F is a staircase, $f_{max} = f_{min}^{-1} = \infty$ and the lemma is proved. If $\sigma \neq 0$, from (4), (17), and (19)

$$1 = \sum_{i=1}^{K} \Delta x_i = \sum_{i=1}^{K} \sigma_i \bar{f}_i^{-p} = \sigma \sum_{i=1}^{K} \bar{f}_i^{-p}$$
$$1 = \sum_{i=1}^{K} \Delta y_i = \sum_{i=1}^{K} \sigma_i \bar{f}_i^{q} = \sigma \sum_{i=1}^{K} \bar{f}_i^{q}$$

so

$$\sum_{i=1}^{K} \bar{f}_i^{-p} = \sum_{i=1}^{K} \bar{f}_i^{q} = \frac{1}{\sigma}. \quad (55)$$

Multiplying (55) by f_{max}^p and using $f_{max} \geq \bar{f}_i$ for all i on all but the first term of the first sum and all terms of the second sum and noting from (50) that $\bar{f}_1 = \bar{f}_{min}$ gives

$$\left(\frac{f_{max}}{\bar{f}_{min}}\right)^p + (K - 1) \leq Kf_{max}$$

proving (54a). Equation (54b) follows by a dual derivation. Q.E.D.

Lemma 2 permits the elimination of $\bar{f}_{max}$ and $\bar{f}_{min}$ from the bounds of (52) and (53) as long as either of f_{max} and f_{min}^{-1} are finite. If both are infinite, Lemma 3 is needed.

Lemma 3

Let F be convex $\cup$, $Q \in Q^*$, $K \geq 2$, $r \geq 0$. Then

$$(\bar{f}_{max}/\bar{f}_{min})^{pq} = e^{pqw} \leq \frac{K}{I_r^q}. \quad (56)$$

Proof: We apply Hölder's inequality to the two sums in (55) with exponent q for the first and p for the second ([28], Theorem 11), but interchange the first and last terms in the second set, so that $\bar{f}_1^q$ faces $\bar{f}_K^{-p}$ and $\bar{f}_K^q$ faces $\bar{f}_1^{-p}$ while the other terms match in subscript. We get

$$\left(\frac{\bar{f}_{max}}{\bar{f}_{min}}\right)^{pq} + \left(\frac{\bar{f}_{min}}{\bar{f}_{max}}\right)^{pq} + (K - 2) \leq \left(\frac{1}{\sigma}\right)^p\left(\frac{1}{\sigma}\right)^q = \frac{1}{\sigma}$$

or

$$e^{pqw} = \left(\frac{\bar{f}_{max}}{\bar{f}_{min}}\right)^{pq} \leq \frac{1}{\sigma} - K + 2 \leq \frac{1}{\sigma} \leq \frac{K}{I_r^q}$$

since $K \geq 2$. In the last step we use (48) to replace σ, which requires finding the $Q(K)$ in Q^*, by I_r^q, which can be computed from f.

Proof of Theorem 5: From (48), taking the $1/q$ power we have

$$KM_R(Q) \geq I_r \geq KM_r(Q)(\bar{T})^{1/q}. \quad (57)$$

Lower-bounding T successively by the right-most terms in each of (52a), (53b), (53c), and (53d) gives

$$KM_r(Q) \geq I_r \geq \begin{cases} KM_r(Q) \exp\left\{-\dfrac{pz}{K}\right\} \\ KM_r(Q) \exp\left\{-\dfrac{pu+q}{K-q}\right\} \\ KM_r(Q) \exp\left\{-\left(\dfrac{p}{q}\right)\dfrac{qv+p}{K-p}\right\} \\ KM_r(Q) \exp\left\{-\left(\dfrac{1}{q}\right)\dfrac{pqw+1}{K-1}\right\} \end{cases} \quad (58)$$

which becomes (43), the first half of Theorem 5, for $k = K$, $J = 1$, by substitution of the definitions (51) of z, u, v, and w. Substituting in (58) for u and v from (54a) and (54b) and for w from (57) gives (44), the second half of the theorem, also for $k = K$, $J = 1$.

For $J > 1$, Lemmas 2 and 3 hold with K replaced by k on the right in (54) and (56). The proofs can be extended to deal with the J largest and J smallest of the $\{\bar{f}_i\}$. It is possible to show that the quantizing intervals can be divided into $\leq J$ subsets of adjacent intervals, in each of which the monotone properties used in (51) hold, even through the quantizing points $\{x_i, y_i\}$ and the points separating domains of convexity $\{\xi_i, \eta_i\}$ may not coincide. Then z, u, v, and w are upper bounds to the sums over each such subset of the terms to their left, and minimization of the sum of the exponential bounds to the T_i subject to the constraints (51) gives the results in (53a), (53b), and (53c) and the right-most result in (53d), with K replaced by k throughout, completing the proof of Theorem 5 for arbitrary J.

IV. Special Cases and Extensions

A. Case $K = 2$, $r = 1$

The case $K = 2$, $J = 1$ (and thus $k = K$) is the smallest quantizer about which an experimenter can say anything interesting. For $r = 1$, $p = q = \frac{1}{2}$, a detailed analysis that will not be reproduced here bounds I_1 above and below in terms of $KM_1(Q)$ and vice versa. We have, for $Q \in Q^*$,

$$\sqrt{I_1}\,(2 - \sqrt{I_1}) \geq 2M_1(Q) \geq I_1 \geq [1 - \sqrt{1 - 2M_1(Q)}]^2 . \qquad (59)$$

All of the bounds in (59) are attained. The distribution $F(x)$, which is a square with the upper-left corner cut off illustrated as $F_1(x)$ in Fig. 1 and given by $(0 < \alpha \leq \frac{1}{2})$

$$\begin{array}{ll} F(0^-) = 0, & f(0^-) \text{ undefined} \\ F(x) = 1 - 2\alpha + x,\ f(x) = 1 & 0 < x \leq 2\alpha \\ F(x) = 1,\ f(x) = 0 & 2\alpha < x \leq 1 , \end{array}$$

has $Q \in Q^*$ given by (0, 0), $(\alpha, 1 - \alpha)$, and (1, 1). For $r = 1$ this gives

$$M_1(Q) = \Delta y_1 \Delta x_1 + \Delta y_2 \Delta x_2 = (1 - \alpha)\alpha + \alpha(1 - \alpha) = 2\alpha(1 - \alpha)$$

$$I_1 = \left\{\int_0^{2\alpha} f^{1/2} dx\right\}^2 = (2\alpha)^2$$

which satisfy the left-most and right-most inequalities in (59). The central inequality is satisfied by $F_\bullet$ of Theorem 3 illustrated by $F_2(x)$ in Fig 1 A line segment from (0, 0) to $(\alpha, 1 - \alpha)$ and another from $(\alpha, 1 - \alpha)$ to (1, 1). This gives $M_1(Q) = 2\alpha(1 - \alpha)$ as before, since the quantizer is unchanged, and

$$I_1 = \{\Delta x_1 f_1^{1/2} + \Delta x_2 f_2^{1/2}\}^2 = \left\{\alpha\sqrt{\frac{1 - \alpha}{\alpha}} + (1 - \alpha)\sqrt{\frac{\alpha}{1 - \alpha}}\right\}^2 = 4\alpha(1 - \alpha) .$$

B. Case $r = 0$—An Entropy Bound

At $q = 0$, $M_r(Q)$ for $Q \in Q^*$ becomes the geometric mean of the $\{\Delta x_i\}$ taken with equal weights, since at $q = 0$, $\sigma_i = \Delta_i = 1/K$. Only the first two lines of (43) are useful: they give, for $F \in C_J$,

$$\overline{e^{\ln \Delta x}} = M_0(Q) \geq \frac{e^{H(f)}}{K} \geq \begin{cases} M_0(Q) \exp\left\{-\dfrac{\ln [f_{\max}/f_{\min}]}{k}\right\} \\ M_0(Q) \exp\left\{-\dfrac{\ln [k(f_{\max} - 1) + 1]}{k}\right\} \end{cases}$$

giving a $1/k$ or $\ln k/k$ approach depending on whether $f_{\min} > 0$ or not; $f_{\max} < \infty$ is necessary in either case.

Taking logarithms gives a two-sided bound on the entropy (always ≤ 0) of a distribution $F \in C_J$ on [0, 1],

$$H(f) = -\int_0^1 f(x) \ln f(x) dx = \int_0^1 \ln g(y) dy$$

which is of some independent interest.

$$0 \geq H(f) - \ln K_1 - \overline{\ln \Delta x} \geq \begin{cases} -\dfrac{\ln [f_{\max}/f_{\min}]}{k} \\ -\dfrac{\ln [k(f_{\max} - 1) + 1]}{k} \end{cases} \qquad (60)$$

for a K_1-interval equiprobable quantizer ($y_i = 1/K_1$) with $k = \max(K_1/J, 1)$. The more usual bound is one-sided.

$$0 \geq H(f) - \overline{\ln \Delta y} + \ln K_2 \qquad (61)$$

for a K_2-interval uniform quantizer ($\Delta x_i = 1/K_2$) where $\overline{\ln \Delta y}$ is the entropy of the discrete distribution $\{\Delta y_i\}$.

C. Multidimensional Quantization

In the multidimensional case we give only some definitions and results. A more complete treatment will appear elsewhere.

Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_N)$ be a vector-valued random variable with probability measure μ defined on the N-dimensional unit cube $\Omega = [0, 1]^N$. Let Ω be the union of K disjoint quantization regions R_i, $1 \leq i \leq K$. Let λ be Lebesgue measure on Ω, and let $\lambda(R_i) = \Delta V_i$ be the volume of R_i and $\mu(R_i) = \Delta P_i$ be the probability that $\boldsymbol{x}$ will fall in R_i. The K-region quantizer is defined as the set $\{R_i, \Delta P_i\}$, $1 \leq i \leq K$, with

$$\sum_{i=1}^{K} \Delta V_i = \sum_{i=1}^{K} \Delta P_i = 1 .$$

To define a measure of performance for Q, we first define the quantization error in the nth coordinate when $\boldsymbol{x}$ is in R_i as the width $\Delta_i(x_n)$ of R_i in the direction of x_n.

$$\Delta_i(x_n) = \sup_{\boldsymbol{x} \in R_i} \{x_n\} - \inf_{\boldsymbol{x} \in R_i} \{x_n\}$$

We define the (rth mean) quantization error of $\boldsymbol{x}$ in R_i as ϵ_i, where $0 \leq r \leq \infty$, the limiting cases having the usual interpretation, and

$$\epsilon_i^r = \frac{1}{N}\sum_{n=1}^{N}\Delta_i(x_n)^r$$

and measure the performance of Q by the rth mean of the ϵ_i:

$$M_r(Q) = \overline{\epsilon^r}^{1/r} = \left\{\sum_{i=1}^{K}\Delta P_i\epsilon_i^r\right\}^{1/r} = \left\{\sum_{i=1}^{K}\Delta P_i\frac{1}{N}\sum_{n=1}^{N}\Delta_i(x_n)^r\right\}^{1/r}. \quad (62)$$

The justification of this definition of error and measure of performance, as in the one-dimensional case, lies in the simple, precise, and general results to which they lead. For smooth distributions, these results may again be used to make approximate or asymptotic statements about the behavior of other measures. Again the major restriction is bounded $|x|$.

Let μ be an arbitrary probability measure on Ω. Then we have an analog to Theorem 1, for $0 \le r \le \infty$, any K-region quantizer Q consistent with μ has

$$K^{1/N}M_r(Q) \ge \left\{\int_\Omega f^{N/N+r}d\lambda\right\}^{N+r/Nr}. \quad (63)$$

And there is an analog to Theorem 4; for $0 \le r \le \infty$, there is an increasing sequence of positive integers K_m and a sequence of quantizers $Q(K_m)$ such that

$$\lim_{m\to\infty} K^{1/N}M_r(Q(K_m)) = \left\{\int_\Omega f^{N/N+r}d\lambda\right\}^{N+r/Nr}. \quad (64)$$

In both these results, $f = d\mu/d\lambda$ is the density of the absolutely continuous part of μ, and the usual interpretations hold at $r = 0$, $r = \infty$. Equation (64) is close to Zador's results [4], which require absolutely continuous μ with bounded f and include an unknown constant, use a mean rth-absolute-difference performance measure, and apply to unbounded $|x|$ as well.

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13. ABSTRACT

A quantizer Q divides the range [0, 1] of a random variable x into K quantizing intervals the i^{th} such interval having length Δx_i. We define the quantization error for a particular value of x (unusually) as the length of the quantizing interval in which x finds itself, and measure quantizer performance (unusually) by the r^{th} mean value of the quantizing interval lengths $M_r(Q) = \overline{\Delta x^r}^{1/r}$, averaging with respect to the distribution function F of the random variable x. Q_1 is defined to be an optimum quantizer if $M_r(Q_1) \leq M_r(Q)$ for all Q. The unusual definitions restrict the results to bounded random variables, but lead to general and precise results.

We define a class Q^* of quasi-optimum quantizers; Q_2 is in Q^* if the different intervals Δx_i make equal contributions to the mean r^{th} power of the interval size so that $\Pr\{\Delta x_i\} \Delta x_i^r$ is constant for all i. Theorems 1, 2, 3, and 4 prove that $Q_2 \in Q^*$ exists and is unique for given F, K, and r: that $1 \geq KM_r(Q_2) \geq KM_r(Q_1) \geq I_r$, where $I_r = \{\int_0^1 f(x)^p\, dx\}^{1/q}$, f is the density of the absolutely continuous part of the distribution function F of x, $p = 1/(1+r)$, and $q = r/(1+r)$: that $\lim KM_r(Q_2) = I_r$ as $K \to \infty$; and that if $KM_r(Q) = I_r$ for finite K, then $Q \in Q^*$.

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