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Bounds on the availability function

Butterworth, Richard W.

Monterey, California. Naval Postgraduate School

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BOUNDS ON THE AVAILABILITY FUNCTION

by

Richard W. Butterworth

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NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral A. S. Goodfellow, USN
Superintendent

M. U. Clauser
Provost

ABSTRACT:

This report deals with the problem of computing the availability function for an alternating renewal process with exponential failure times and general repair times. General expressions giving upper and lower bounds on the availability are derived, as well as an estimate of the error. Several of the bounds with greatest practical consequence are worked out and illustrated for a gamma repair distribution with any integer shape parameter. The case of constant repair is compared to the case of gamma repair with a large shape parameter. Finally, some simulation results using a log normal repair distribution are compared to the gamma results, suggesting that a gamma distribution can be used in lieu of a log normal for purposes of computing the availability function.

Prepared by:

I. Introduction and Summary.

The standard model in the reliability literature for a one unit repairable system is the alternating renewal process. This model is appropriate for a system (or subsystem) which begins in an operative condition and operates for some random time, begins repair whenever failure occurs and when the repair is completed, is fully restored to an operating condition where it remains until failure followed by repair, ... etc.

The availability of such a system is defined as the probability that the system is operative. The basic problem treated here is to isolate some cases in which the availability can be computed exactly or at least approximated with reasonable accuracy.

The results obtained are upper and lower bounds on the availability for the case of exponential failure times and general repair times. The bounds can (in principle) be computed to any degree of accuracy. With a minimal amount of calculation, the bounds for the case of repair times having a gamma distribution with any integer shape parameter are worked out and illustrated along with comparisons to the exact availability for shape parameters equal to one through four. The accuracy for all illustrated computations is $=$ (failure rate/repair rate)^3. For example, if the steady state availability is 90%, the error of either bound is at most $(1/9)^3 = .00137$. If steady state availability is 90%, the error is less than .0000085.

The availability when repair times are constant is computed exactly and compared to the case of gamma repair with a large shape parameter. Also, some justification for using gamma distributed repair times when they are actually log-normal distributed is given.

An application of the present work is to correctly account for the availability as a function of time in lieu of the steady state value which must frequently be taken as a constant value in practical situations. It may, in fact, justify the use of steady state values.

II. Conclusions.

The problem of computing the availability of an alternating renewal process with exponential failure and general repair has been treated here. The general expression (3) is used to find lower and upper bounds with arbitrarily small maximum error. Further, the error in the bounds does not depend on the specific repair distribution in question.

Several of the bounds with the greatest practical consequence are worked out for a gamma repair distribution, integer shape parameter, and illustrated along with the exact values of availability. The error was, as theoretically predicted, quite small when the steady state availability was 90% or more. In fact, the upper and lower bounds differ by at most .00137 for the case when steady state availability = 90%, by at most .0000085 for the 98% case.

The availability for constant repair is given along with a comparison to gamma repair, showing that generally the shape parameter must be quite large for the gamma repair to closely approximate constant repair in so far as availability is concerned. The bounds given allow one to judge this approximation for himself.

The log normal repair distribution is briefly discussed. Some simulation results are presented which suggest that the gamma distribution may be used in lieu of log normal for purposes of stochastic modelling of availability.

III. Specific Problem Description.

An alternating renewal process is specified by two sequences $\{X_1, X_2, ...\}$ and $\{Y_1, Y_2, ...\}$, of random variables; the first sequence represents the durations of operation between failures and the second the durations of repair between failures. Thus, the process is in an operative state during $[0, X_1)$, under repair during $[X_1, X_1+Y_1)$, operative during $[X_1+Y_1, X_1+Y_1+X_2)$, ... etc. (Note that we could begin in a repair state if desired.) The operative times $X_1, X_2, ...,$ will be given the exponential distribution, parameter λ, and the repair times $Y_1, Y_2, ...,$ will have an unspecified distribution G, with some finite positive mean. All random times are assumed to be jointly independent. Thus we have,

$$P(X_n \le t) = 1 - e^{-\lambda t} \qquad n = 1,2,\ldots$$

$$P(Y_n \le t) = G(t) \qquad n = 1,2,\ldots \tag{1}$$

$$\{X_1, Y_1, X_2, Y_2, \ldots\} \text{ independent random variables}$$

Within the assumptions of (1), we wish to compute the availability function $A(t)$, the probability of being in an operative state at time t. A precise definition would be

$$A(t) = P\left(\sum_{k=1}^{n}(X_k+Y_k) \le t < \sum_{k=1}^{n}(X_k+Y_k) + X_{n+1} \quad \text{for some} \quad n \ge 0\right),$$

where empty summations are taken as zero in the probability statement.

From our convention of beginning operative at time = 0, we have that $A(0) = 1$. Due to the assumptions in (1), we know that $A(\infty) = \lim_{t\to\infty} A(t) = \frac{\nu}{\nu+\lambda}$, where ν^{-1} = mean repair time under the distribution G [Barlow & Proschan, 1967]. For obvious reasons, we will call λ the failure rate and ν the repair rate (even when G is not an exponential). Note that the steady state value of availability depends only on the first moment of G.

Using script letters to denote Laplace-Stieltjes transforms, so $\mathcal{A}(s) = \int_0^\infty e^{-sx} dA(x)$, it is easily established that [Barlow & Proschan, 1967]

$$A(s) = \frac{1 - \frac{\lambda}{\lambda+s}}{1 - \frac{\lambda}{\lambda+s} G(s)}$$

which simplifies to

$$A(s) = \frac{1}{1 + rG_e(s)} = \sum_{n=0}^{\infty} (-rG_e(s))^n$$

where $r = \lambda/\nu$

$$G_e(t) = \nu \int_0^t (1-G(x))dx \tag{2}$$

Assuming $r < 1$, the series in (2) is valid and gives, on inversion,

$$A(t) = 1 - rG_e(t) + r^2G_e{*}G_e(t) - r^3G_e{*}G_e{*}G_e(t) + \dots \tag{3}$$

where $*$ denote convolution

and we assume $r = \lambda/\nu < 1$.

The distribution G_e appearing in (3) is well known in renewal theory as the equilibrium excess distribution for a renewal process with inter-event distribution G. Unfortunately, no such interpretation applies to the present work.

The expansion given by (3) can be terminated to give either upper or lower bounds on $A(t)$; an estimate of the error shows that arbitrarily high accuracy can be obtained by taking sufficiently many terms. First notice that the terms in (3) are, in absolute value, monotonic decreasing, for any t. In fact, if F is any distribution on $[0,\infty)$ and $0 \leq r \leq 1$, then $r^n F^{n*}(t) \geq r^{n+1} F^{(n+1)*}(t)$ holds for all t (here F^{n*} is the n-fold convolution of F with itself). This follows from the interpretation $F^{(n+1)*}(t) = P(X+Y \leq t) \leq P(X \leq t)P(Y \leq t) = F^{n*}(t)F(t) \leqq F^{n*}(t)$ where X and Y are independent random variables with $P(X \leq t) = F^{n*}(t)$ and $P(Y \leq t) = F(t)$. Now define the following functions;

$$\text{for } n \text{ odd, } L_n(t) = \sum_{k=0}^{n} r^k G_e^{k*}(t)$$

$$\text{for } n \text{ even, } U_n(t) = \sum_{k=0}^{n} r^k G_e^{k*}(t)$$

Using the above inequality and (3), we have that

$$\begin{aligned} &\text{for } n \text{ odd; } L_n(t) \leqq A(t), \quad A(t) - L_n(t) \leqq r^{n+1} \\ &\text{for } n \text{ even; } U_n(t) \geqq A(t), \quad U_n(t) - A(t) \leqq r^{n+1} \end{aligned} \qquad (4)$$

The upper and lower bounds U_n and L_n can be made to have an arbitrarily small error by choosing n sufficiently large, since $r < 1$. The restriction $r < 1$ is equivalent to $A(\infty) > .5$, since $A(\infty) = 1/(1+r)$. It should not be a practical restriction; in fact, for the reasonable case of $A(\infty) = 90\%$, we have $r = 1/9$ and with $A(\infty) = 98\%$, $r = 1/49$. Then, for example, the lower bound $L_1(t) = 1 - \lambda \int_0^t (1-G(x))dx$ has a maximum error of $r^2 = .0123$ or $r^2 = .000416$ for $A(\infty) = 90\%$ or 98% respectively.

The first (non-trivial) upper bound is $U_2(t)$. This bound, as $L_1(t)$, is easily computed for a repair distribution G which is gamma with an integer shape parameter. Numerical examples are fully illustrated in the following section. The maximum error of U_2 is r^3 which is .00138 or .0000085 for $A(\infty) = 90\%$ or 98% respectively.

The bounds U_n and L_n are usually cumbersome to compute for $n \geq 3$ unless the distribution G_e takes on a simple form. One can, however, define a lower bound whose error is at most r^3; it involves no more computation than that involved in U_2, namely $U_2(t) - r^3 = 1 - rG_e(t) + r^2G_e^{2*}(t) - r^3$, for which we have

$$0 \leq A(t) - (U_2(t)-r^3) \leq r^3. \quad (5)$$

By combining the lower bound L_1 and (5), we define a revised version of L_1, namely $L_1'(t) = \text{Max}\ (L_1(t),U_2(t) - r^3)$, for which we have

$$0 \leq A(t) - L_1'(t) \leq r^3. \quad (6)$$

IV. Specific Examples Illustrated.

In this section, numerous specific examples are illustrated by graphs showing the theoretical bounds and, when available, the exact values of availability.

The first case treated is when the repair distribution G belongs to the gamma family with an integer shape parameter. If α, an integer, is the shape parameter and β, a positive number, the scale parameter, we have

$$G(t) = 1 - \sum_{j=1}^{\alpha} \frac{(\beta t)^{j-1}}{(j-1)!} e^{-\beta t}$$

so $$G_e(t) = 1 - \sum_{j=1}^{\alpha} \frac{(\beta t)^{j-1}}{(j-1)!} e^{-\beta t} \frac{(\alpha-j+1)}{\alpha} .$$

Recall that we are always taking the failure distribution to be exponential with parameter λ. Since $r = \frac{\lambda}{\nu} = \frac{\lambda \cdot \alpha}{\beta}$, the lower bound L_1 becomes

$$L_1(t) = 1 - \frac{\lambda\alpha}{\beta} + \frac{\lambda}{\beta} \sum_{j=1}^{\alpha} \frac{(\beta t)^{j-1}}{(j-1)!} e^{-\beta t} (\alpha-j+1).$$

The upper bound U_2 is also a straight-forward computation with

$$U_2(t) = 1 - \frac{\lambda\alpha}{\beta}\left(1 - \frac{\lambda\alpha}{\beta}\right)G_e(t) - \left(\frac{\lambda}{\beta}\right)^2 \sum_{k=1}^{\alpha} \sum_{j=1}^{\alpha} (\beta t)^{j+k+1} \frac{(\alpha-j+1)}{(j+k-1)!} e^{-\beta t}.$$

The revised version $L_1'(t)$ based on $U_2(t)$ is simply

$$\text{Max}\left(L_1(t), U_2(t) - \left(\frac{\lambda\alpha}{\beta}\right)^3\right).$$

Figure 1 shows the bounds, L_1, L_1', U_2, and the exact availability A for $\alpha = 4$, $\beta = 18.0$ and $\lambda = .5$. The exact availability was found by inverting the transform equation (2). The

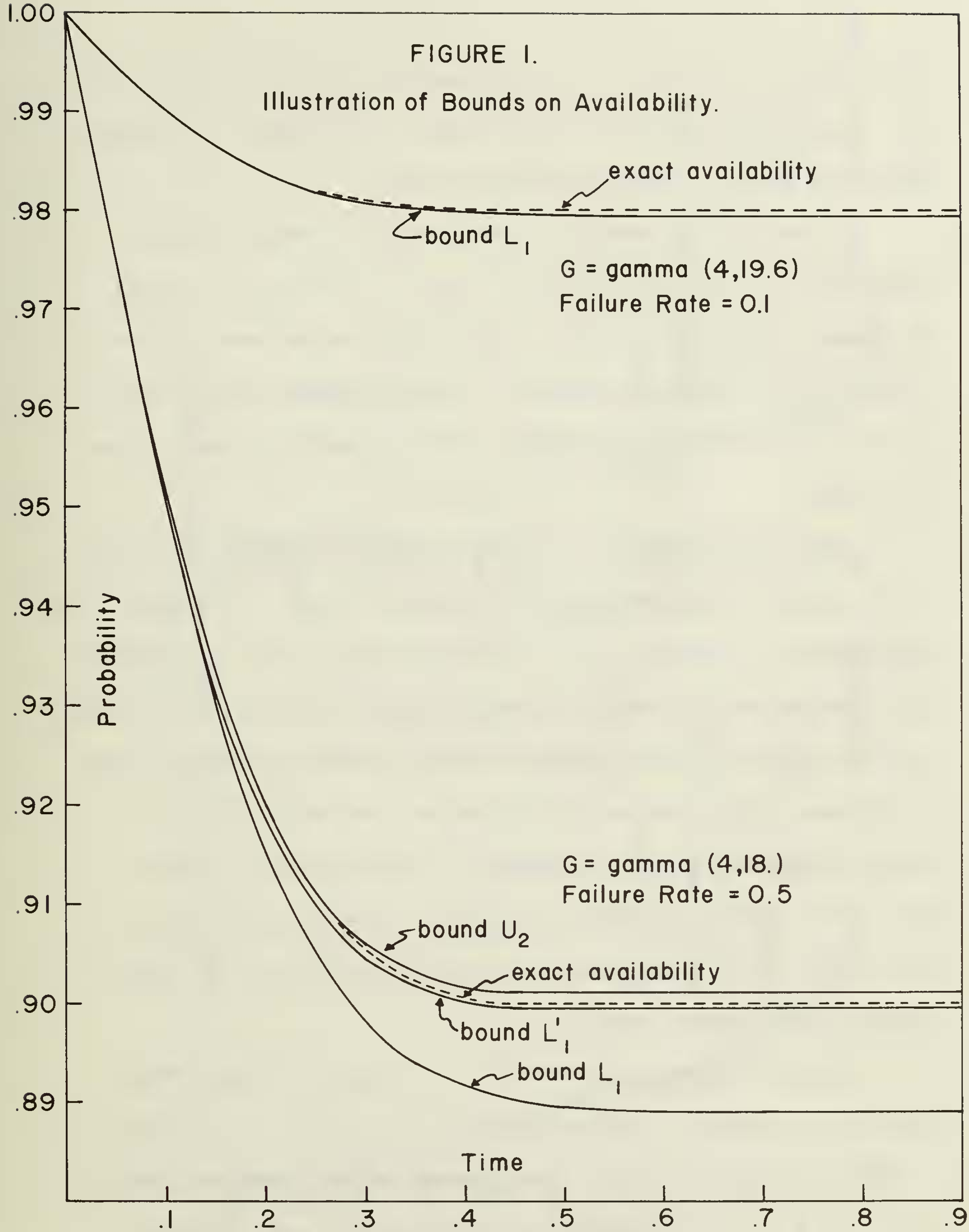


FIGURE I.
Illustration of Bounds on Availability.
exact availability
bound L_1
G = gamma (4,19.6)
Failure Rate = 0.1
G = gamma (4,18.)
Failure Rate = 0.5
bound U_2
exact availability
bound L'_1
bound L_1
Probability
Time
1.00
.99
.98
.97
.96
.95
.94
.93
.92
.91
.90
.89
.1
.2
.3
.4
.5
.6
.7
.8
.9

repair rate ν equals 4.5 so the steady state value $A(\infty) = 90\%$. This graph is typical for α less than, say 10, with β such that the repair rate ν remains fixed at 4.5.

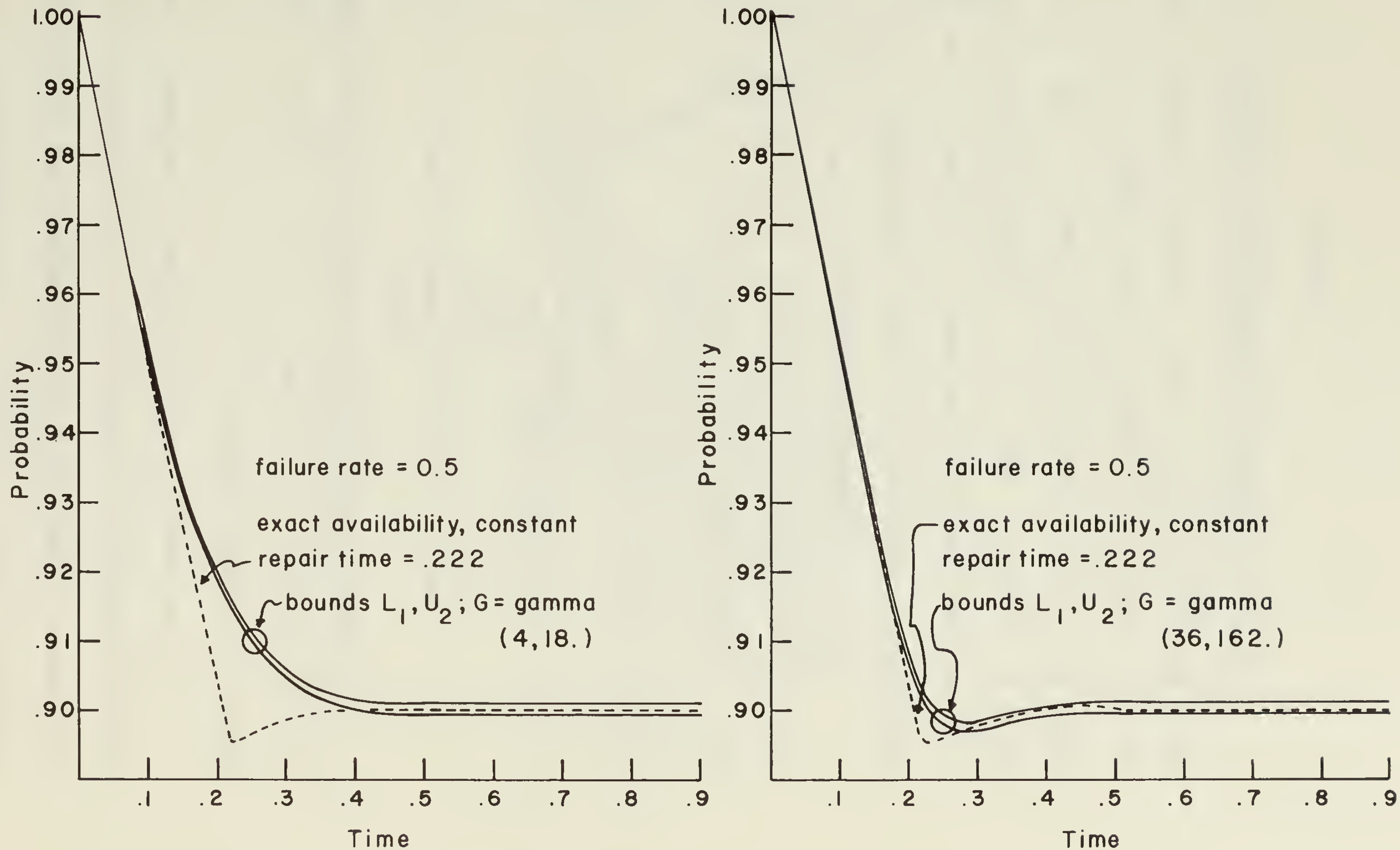
Figure 1 also shows the lower bound L_1 and the exact availability for $\alpha = 4$, $\beta = 19.6$ and $\lambda = .1$ so $\nu = 4.9$ and the steady state availability $A(\infty) = 98\%$. The accuracy of the bounds is significantly improved for these values; in fact, both L_1' and U_2 are indistinguishable from the exact availability on this graph.

When the parameters α and β tend to infinity such that $\nu = \frac{\beta}{\alpha}$ remains fixed, the gamma distribution tends to the degenerate distribution at the mean ν^{-1}. For this reason, the availability with constant repair, which is easily computed exactly, will approximate the case of gamma distributed repair when the shape parameter α is large. The question is, of course, how large must α be for the approximation to be adequate. The bounds can, within their error limits, predict how large α must be to obtain a given degree of fit by simply plotting the bounds and the exact solution for constant repair.

Figure 2 shows the bounds L_1', U_2 and the exact availability for constant repair when $\alpha = 4$, $\beta = 18.$, $\lambda = .5$ and $\alpha = 36$, $\beta = 162.$, $\lambda = .5$. The constant repair time was taken to be $\nu^{-1} = \frac{\alpha}{\beta} = .222$ so that all solutions would have the same steady state value.

FIGURE 2.

Availability for gamma repair versus constant repair.



While the degree of approximation improves as shown, it seems generally the case that α must be somewhat larger than 36 in order for the constant repair solution to approximate the gamma repair case as well as the bounds L_1' and U_1 do, particularly for times t near ν^{-1}. Computing experience not shown here verifies that the same general conclusion holds when the steady-state availability is 98% instead of 90%.

The exact solution for constant repair was found directly from the definition (not transforms) and is given here for completeness. If the repair time is denoted by ν^{-1}, and the failure rate by λ (exponential failures), then

$$A(t) = \sum_{k=0}^{n} e^{-\lambda(t-k\nu^{-1})} \frac{[\lambda(t-k\nu^{-1})]^k}{k!}$$

where $n = [t\nu]$ = greatest integer in $t\nu$.

Figure 3 shows the above solution for availability along with the bounds L_1, L_1' and U_2 for a constant repair time of .222 and a failure rate of .5.

Since repair times are frequently believed to obey a log normal distribution, some study of that situation is in order. In this case, a closed form for availability is not available. The theoretical bounds L_1 and U_2 are computable by numerical methods only and this was not undertaken. Instead, the process was simulated and the results compared to gamma repair results.

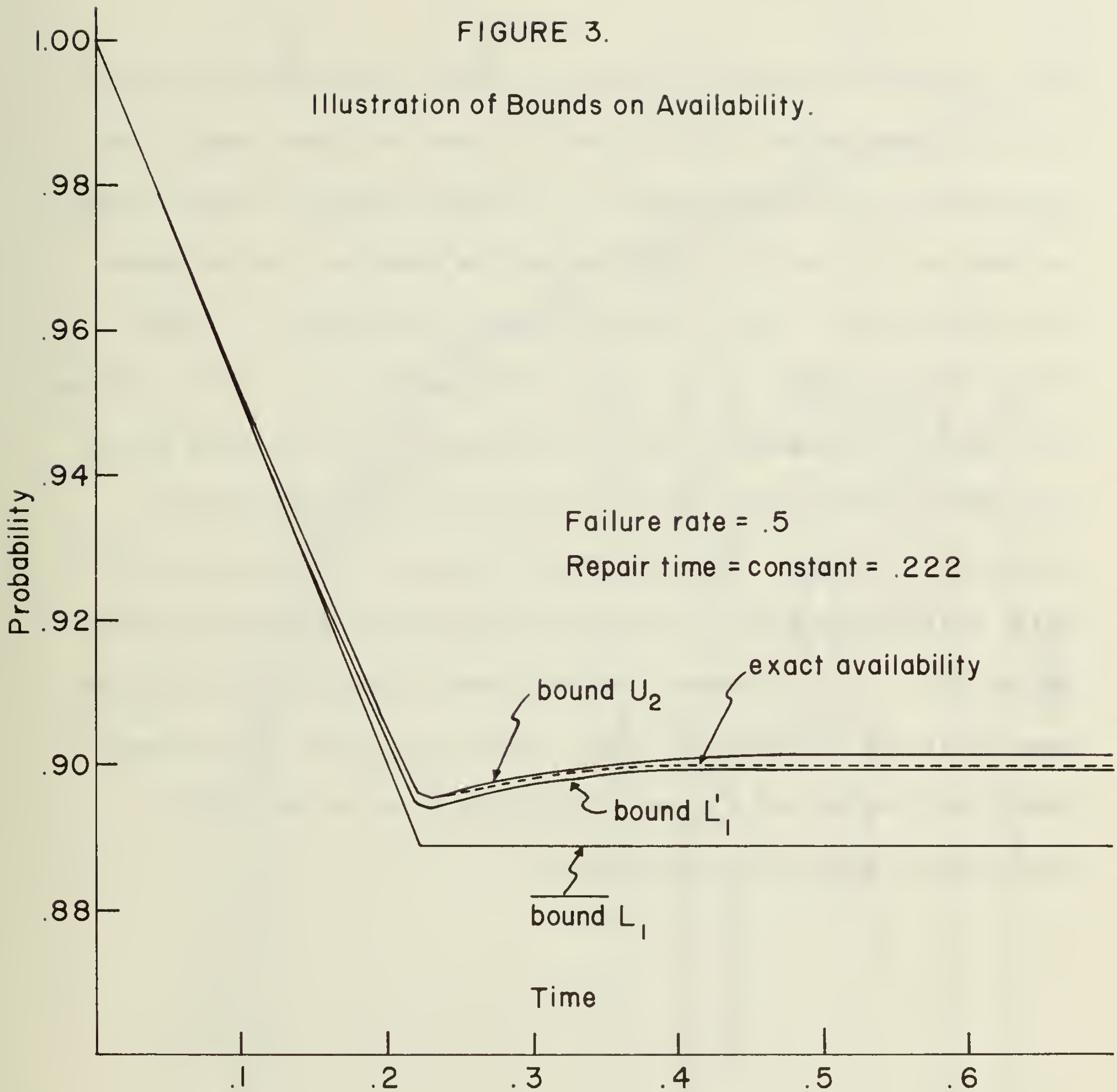


FIGURE 3.

Illustration of Bounds on Availability.

Figure 4 shows the results of 40,000 simulations using first the log normal repair distribution and then the gamma repair distribution with shape parameter 3. The other parameters were chosen so that both repair distributions had the same mean and variance; the repair rate $\nu = 4.5$ and the repair time variance = .01646. The failure rate was = .1 so that the steady state $A(\infty) = 97.83\%$. Also shown is the exact value of availability for the gamma repair distribution used. The $\pm 2\sigma$ lines refer to the gamma repair simulation averages. That is, these simulation outcomes are normally distributed with the exact availability as mean and standard deviation σ as indicated. The log normal simulation results are seen to lie in this band as well. Within the limits of stochastic modelling, the use of a gamma in place of a log normal repair distribution may well be acceptable.

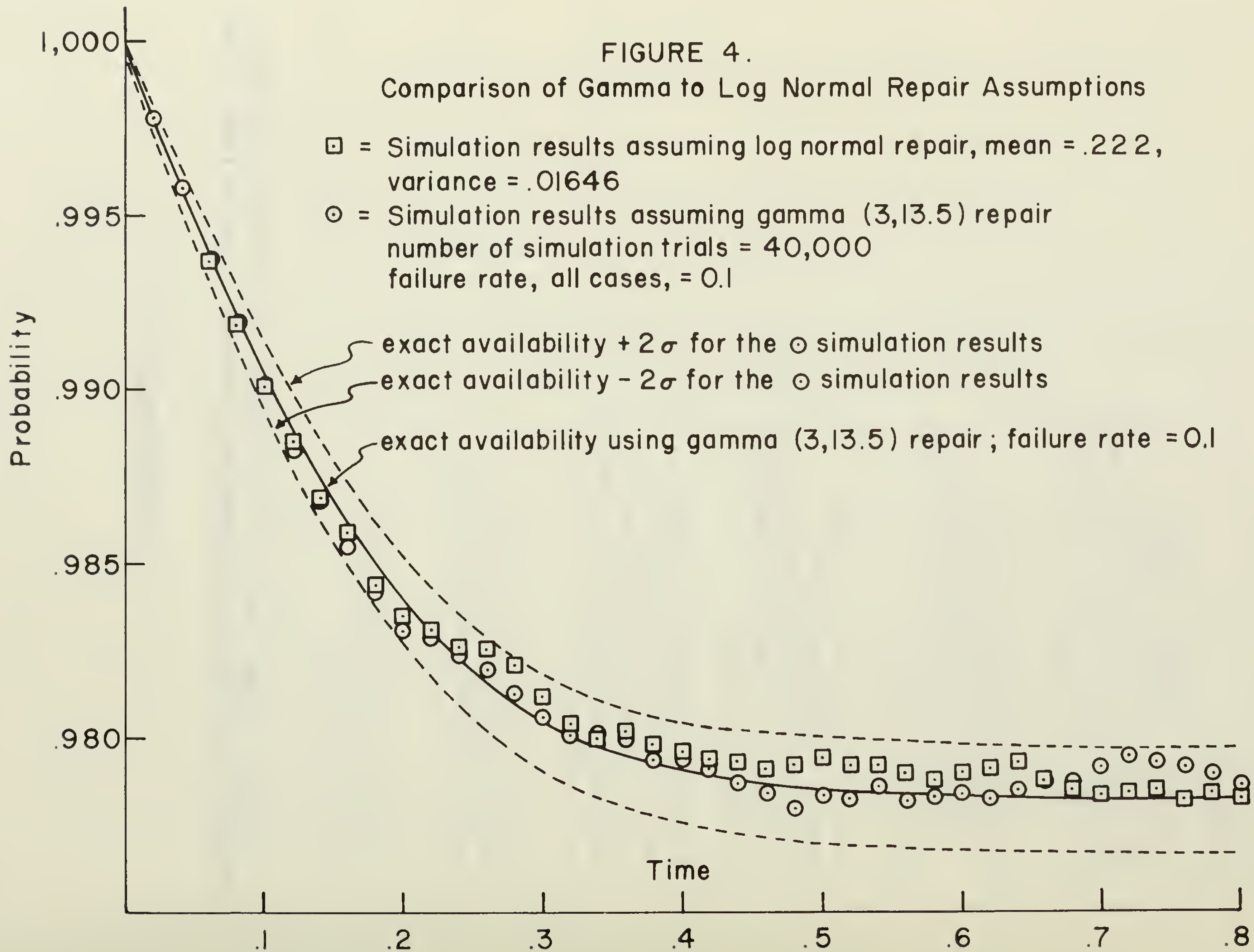


FIGURE 4.
Comparison of Gamma to Log Normal Repair Assumptions

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Dr. William Mitchell 1
Assistant Professor
California State College
Hayward, California 94542

Dr. Richard W. Butterworth 10
Department of Operations Research
and Administrative Sciences
Naval Postgraduate School
Monterey, California 93940

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13. ABSTRACT

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