BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

BOUNDS ON THE EIGENVALUES OF THE LAPLACE AND SCHROEDINGER OPERATORS

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Communicated by I. M. Singer, April 19, 1976

If Ω is an open set in $\mathbf{R}^n$, and if $\widetilde{N}(\Omega, \lambda)$ is the number of eigenvalues of $-\Delta$ (with Dirichlet boundary conditions on $\partial\Omega$) which are $\leqslant \lambda$ ($\lambda \geqslant 0$), one has the *asymptotic* formula of Weyl [1], [2]: $\lim_{\lambda\to\infty} \lambda^{-n/2}\widetilde{N}(\Omega, \lambda) = C_n|\Omega|$. Here $|\Omega|$ is the volume of Ω and $C_n = (4\pi)^{-n/2}\Gamma(1 + n/2)^{-1}$. The same holds [3] if $\mathbf{R}^n$ is replaced by a Riemannian manifold, M, with $|\Omega|$ being the Riemannian volume and Δ being the Laplace-Beltrami operator. One purpose of this note is to state that there often exist bounds of the form

$$\widetilde{N}(\Omega, \lambda) \leqslant D_n\lambda^{n/2}|\Omega|, \forall\lambda \geqslant 0, \forall\Omega \subset M, \tag{1a}$$

$$\widetilde{N}(\Omega, \lambda) \leqslant (D_n\lambda^{n/2} + E_n)|\Omega|, \quad \forall\lambda \geqslant 0, \forall\Omega \subset M, \tag{1b}$$

with D_n, E_n *independent of* λ *and* Ω and depending only on M. (1a) holds for noncompact M if condition (8), below, holds. In particular, (1a) holds for $\mathbf{R}^n$ and for homogeneous spaces with curvature $\leqslant 0$. (1b) always holds for compact M, and it also holds for noncompact M if condition (9) holds.

REMARK. There is an asymptotic formula [4], [5]: $\widetilde{N}(\Omega, \lambda) = C_n\lambda^{n/2}|\Omega| + O(\lambda^{(n-1)/2})$. While this has the correct limiting constant, C_n, the remainder, $O(\cdot)$, can get very large if Ω is very irregular. The remainder is not bounded by a universal constant times $|\Omega|\lambda^{(n-1)/2}$ or even $|\Omega|\lambda^{n/2}$. Our emphasis is different. By introducing $D_n \geqslant C_n$ we have a bound which is universal in the sense that it depends on M but *not* on $\Omega \subset M$; in particular, (1) is applicable to unbounded Ω.

A second, closely related problem is to estimate $N_\alpha(V)$ = number of nonpositive eigenvalues of the Schroedinger operator $-\Delta + V(x)$ on $L^2(M)$ which are $\leqslant \alpha \leqslant 0$. There exists an asymptotic formula [6], [7], [8] for suitably regular V:

$$\lim_{\gamma\to\infty} \gamma^{-n/2}N_{\gamma\alpha}(\gamma V) = C_n \int_M [V(x) - \alpha]_-^{n/2}\, dx \tag{2}$$

AMS (MOS) subject classifications (1970). Primary 58G99, 35J05, 35J10, 35P15, 35P20; Secondary 47F05, 81A09, 81A45.

[1] Work supported by U. S. National Science Foundation grant MCS 75-21684.

where $V_- = \frac{1}{2}(|V| - V)$, and dx is the Riemannian volume element. Our new, *nonasymptotic result* is

$$N_\alpha(V) \leqslant L_n \int_M [V(x) - \alpha]_-^{n/2}\, dx, \quad \forall \alpha, V \tag{3}$$

when M satisfies (8) and $\dim(M) \geqslant 3$. [(3) was obtained simultaneously and independently by M. Cwikel [9]; his estimate for L_n is not as sharp as ours, however. When $n = 3$, our $L_3 = .116$ and it is known that $L_3 \geqslant .078$.]

The connection between $\widetilde{N}(\Omega, \lambda)$ and $N_\alpha(V)$ is the following elementary consequence of the min-max principle:

$$\widetilde{N}(\Omega, \lambda) \leqslant N_\alpha((\alpha - \lambda)\chi_\Omega), \quad \forall \alpha \leqslant 0 \tag{4}$$

where χ_Ω is the characteristic function of Ω. Thus (3) implies $\widetilde{N}(\Omega, \lambda) \leqslant L_n \lambda^{n/2} |\Omega|$ for $\dim(M) \geqslant 3$. Another important consequence of the min-max principle is

$$N_\alpha(V) \leqslant N_{\alpha+\beta}(-(V + \beta)_-), \qquad 0 \leqslant \beta \leqslant -\alpha. \tag{5}$$

Consequently, one need consider only the case $V = -V_-$ in (3).

The asymptotic formula (2) has been extended to $V_- \in L^{n/2+\epsilon} \cap L^{n/2-\epsilon}$ by Simon **[10]**. Using his methods and (3), one easily extends (2) to all $V_- \in L^{n/2}$.

Results (1) and (3) are corollaries of the following

THEOREM. *Let f: $[0, \infty) \longrightarrow [0, \infty)$ be convex and polynomially bounded at infinity and satisfy*

$$\int_0^\infty t^{-1} e^{-t} f(t)\, dt = 1. \tag{6}$$

For $t > 0$, let $G(x, y; t)$ be the kernel of $e^{t\Delta}$, i.e. the fundamental solution of the heat equation on the Riemannian manifold M. Then, for $\alpha \leqslant 0$,

$$N_\alpha(V) \leqslant \int_M dx \int_0^\infty t^{-1} e^{\alpha t} G(x, x; t) f(tV_-(x))\, dt. \tag{7}$$

Our proof of this theorem uses the Wiener integral in an essential way and will be published elsewhere.

To apply (7) we choose $f(t) = 0$, $t \leqslant a$, $f(t) = b(t - a)$, $t \geqslant a$, for some $a, b > 0$ such that (6) holds. To prove (3), we assume

$$G(x, x; t) \leqslant At^{-n/2}, \quad \forall x \in M, \forall t > 0. \tag{8}$$

This holds for $\mathbf{R}^n$ ($A = (4\pi)^{-n/2}$) and for homogeneous spaces with curvature $\leqslant 0$. Next we use (5) with $\beta = -\alpha$ and then (7) with $\alpha = 0$.

To prove (1a) we assume (8). For (1b) we require a bound of the form

$$G(x, x; t) \leqslant At^{-n/2} + B, \quad \forall x \in M, \forall t > 0, \tag{9}$$

which always holds for compact M, for example. In either case, using (4) and (7) with $\alpha = -\lambda$,

$$\widetilde{N}(\Omega, \lambda) \leqslant \int_\Omega dx \int_0^\infty t^{-1} e^{-t/2} G(x, x; (2\lambda)^{-1} t) f(t)\, dt.$$

The author is most grateful to B. Simon for many helpful suggestions during the course of this work. He is also indebted to I. M. Singer for remarks on Riemannian manifolds.

ADDED IN PROOF. I have recently become aware of the paper of G. V. Rozenbljum, Dokl. Akad. Sci. SSSR 202 no. 5 (1972), 1012–1015 (MR 45 #4216) in which a proof of (3) is announced. Rozenbljum's method is completely different, however, and his estimate for L_n does not appear to be as sharp.

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