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Bounds on the fundamental group of a manifold

with almost nonnegative Ricci curvature

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Several recent papers have given generalizations of the following classical theorems relating the curvature of a compact Riemannian manifold to its funda-mental group :

THEOREM 1 (Bochner). Let M be a compact Riemannian manifold. I f M has Ricci curvature nonnegative everywhere and strictly positive somewhere, then H1(M; R)=0.

THEOREM 2 (Myers). A complete Riemannian manifold with Ricci curvature bounded from below by a positive constant is compact and has finite fundamental group.

THEOREM 3 (Milnor). The fundamental group of a compact Riemannian mani-fold with nonnegative Ricci curvature has polynomial growth.
For example, Berard [2, 3] and Elworthy-Rosenberg [5] extended Bochner's theorem to the case where the Ricci curvature is arbitrarily negative, as long as the set where Ricci is negative has small enough volume ; we refer to such a set as a deep well of negative Ricci of small volume. Wu [16] generalized Myers' theorem to the case where Ricci is negative on a set of small diameter (a deep well of small diameter), and Elworthy-Rosenberg gave an extension of Myers' theorem to the case where Ricci is allowed to be a little bit negative on a set of small volume (a shallow well of small volume). Finally, Shen and Wei [13,15] generalized Milnor's theorem, allowing the lower bound on Ricci curvature to be sufficiently small in relation to the diameter and either the volume or first systole of the manifold (a shallow, wide well).
In this paper, we show that all these extensions for compact manifolds can
be essentially obtained by placing integral bounds on the negative part of the Ricci curvature (or the part of the Ricci curvature lying below a fixed constant), although in some cases there are technical differences in the hypotheses. From

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the analytic viewpoint, these L p bounds on the negative part of Ricci are a more natural definition of the notion of a deep well of small volume than the hypotheses used in [2, 5]. Specifically, we show:

THEOREM 4 (cf. Theorem 2.2). Let M be a complete Riemannian n-manifold.
If there exists w>0 satisfying

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has finite fundamental group.

§§ 1-3 state the main analytic theorems precisely. In 4, various properties
of the Gromov-Hausdorff distance and length spaces are recalled. § 5 is devoted to the observation that if a sequence of compact Riemannian manifolds with first systole bounded from below by a positive constant converges with respect to Gromov-Hausdorff distance, then the limiting metric space is a compact length space with fundamental group isomorphic to the fundamental group of almost every manifold in the sequence. This yields in § 7 a new proof of Shen and Wei's theorem using the Bishop-Gromov volume comparison theorem and its corollary, the Gromov precompactness theorem. Applying the volume esti-mate and isoperimetric inequality obtained in [17] gives extensions of Shen and Wei's theorem such as Theorem 6 in § 9. In § 10, sufficient conditions for the nonexistence of L2 harmonic sections of a Schrodinger operator -V V +V on a vector bundle over a complete Riemannian manifold are given; again, only integral bounds on the potential V are needed. Applying this to the Laplace-Beltrami operator for 1-forms gives Theorem 4. As explained in § 3, combining this theorem with Theorem 6 and [5, §~ 1-2] yields Theorem 5. Finally, the proof of Theorem 7 is given in § 11; this section is independent of the rest of the paper.
As indicated by this outline, Theorem 5 is the most involved result of the paper; its proof combines techniques from [5, 9,17]. In contrast, the proof of Theorem 7 involves only a careful use of the Bishop-Gromov comparison theorem, but imposes stronger pointwise rather than integral conditions on the well of negative curvature. We would like to thank the referee for suggesting that a result like Theorem 7 should exist.

1. Theorems on the growth of the fundamental group.

Let M be a Riemannian manifold with sectional curvature K(M), Ricci cur-vature Rc (M), diameter diam (M), and volume vol (M). Let p(x) be the lowest eigenvalue of Rc (x), let p= min xE (p(x)), and set p_(x)= min (0, p(x)) for x M. The first systole of M, sysl(M), is defined to be the infimum of the lengths of homotopically nontrivial loops in M.
The following is a theorem of Shen and Wei [13] ; a new proof is given
in §7•

THEOREM 1.1. Given n>_2 and i>0, there exists ~(n, o)>0 such that any compact Riemannian manifold M satisfying

dim M = n

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every finite cover M' of M. If r1(M) is almost solvable, then r1(M) is finite. In particular, i f r1(M) has polynomial growth, then r1(M) is finite.

Here a group is almost solvable if it contains a solvable subgroup of finite index. While the proof of the first statement is elementary, the last statement depends on Gromov's theorem that a group of polynomial growth is almost nilpotent and hence almost solvable [8].
The results of §~ 1-2 and Proposition 3.1 imply various generalizations of Myers' theorem, including the following :

COROLLARY 3.2. Given n>_3, a, a, Y, w>0, there exists o(n, a, ic), s(n, a, a, K)
>0 such that any compact manifold M satisfying

dim M = n

sys1(M) >_ 2a diam (M)

vol (M) an diam (M)n

p >_ -K2 diam (M)_2

p_ f, <_ o(n, a, ic) diam (M)-2 vol (M)

II(p-w)-II1 ~(n, a, a, K)w vol (M)

has finite fundamental group.

Here we have used Theorem 1.2 and Theorem 2.3 with q=oo. In particular, applying the Klingenberg-Cheeger estimate [10] yields a proof that a manifold with a deep, narrow well of negative Ricci curvature has finite fundamental group :

COROLLARY 3.3. Let 5L(n, K, D, V) denote the class o f Riemannian n-mani-
f olds with sectional curvature bounded below by K, diameter bounded above by D, and volume bounded below by V. Then given w>0 there exists =s(5J1, w) such that any M~ 5L with (p- w)_ f,<(51, w) has finite fundamental group.

Note that to derive the corollary we use the bound on diam (M) and !p_f1
<-_ I (p-w)-I 1•

4. Gromov-Hausdorff distance and length spaces.

We recall the definition of Gromov-Hausdorff distance [8, 9]. Let X and Y
be metric spaces and Z their disjoint union. Consider the set of all metrics o on Z such that o restricted to X and Y equals the original metrics on X and Y. The Gromov-Hausdorff distance dH(X, Y) is defined to be the infimum of

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PROPOSITION 4.3 ([9, Proposition 3.8]). The pointed Gromov-Hausdorff limit
of a convergent sequence of complete, locally compact length spaces is itself a complete, locally compact length space.

THEOREM 4.4 ([9, Theorem 1.10]). Given x and y in a locally compact, complete length space X, there exists a curve r joining x to y such that

d(x, y) = l().

DEFINITION. The curve Y is called a minimizing geodesic.

N

LEMMA 4.5. The universal cover X of a connected, locally compact, complete length space X is itself a connected, locally compact, complete length space. More-over, if the first systole of X is positive, then the natural projection X-aX is a local isometry.

The proof of Lemma 4.5 follows from the argument in [9, § 1.12]. Note
N that the length of a curve in X is simply the length of the curve projected into X.

5. Fundamental groups of Hausdorff close length spaces.

The next result is essentially a special case of a homotopy finiteness theo-
rem for convergent sequences of LGCn(p) spaces [14].

THEOREM 5.1. Let X and Y be locally compact, complete length spaces such that

sys1(X), sysl (Y) >- 2Q > 0.

If the Gromov-Hausdorff distance between X and Y is strictly less than o/9, ten 2r1(X) and 7r1(Y) are isomorphic.

PROOF. (Cf. [14, § 2].) Let d f(X, Y)=E<o /9, and let o be the distance function on XIIY described in § 4 associated to some s' > s. From now on we do not distinguish between E and s'. Pick xo~X and yo~Y be such that o(xo, yo)<E. Let rl>0 satisfy 9s+472<ax.
Given a loop r : [0, 1]-X in X based at x0, we construct a loop d based
at yo as follows. Choose 0=to<t1< <tN=1 so that l(y([t2_1, ti]))<~, 1 <_i<_N. For each i=1, ••• , N-1, there exists y1 such that o(r(t1), yi)<~. Let c : [0, 1]
- ~Y be the loop based at yo obtained by joining yti_1 to yti by a minimizing geodesic and with c(t)=y1.
First, we show that this construction associates to r a unique homotopy class of loops in Y based at yo. Let ~b be a loop in Y obtained from r using a partition to, • • • , tN and ~5' a loop obtained from r using a partition to, ..•, t'.

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PROOF OF THEOREM 1.1. Suppose that the theorem is not true. Then given
n and Q there exists a sequence of n-dimensional compact Riemannian manifolds
M1 where each manifold has diameter equal to 1, first systole bounded from
below by 2ci, the lower bound on Ricci curvature converging to zero as i-> oo, but fundamental group of larger than polynomial growth. By Theorem 7.1, there is a subsequence that converges with respect to Gromov-Hausdorff dis-tance to a compact length space X with first systole easily seen to be bounded
N from below by 2Q. Moreover, given xti E M~, by passing to a subsequence we

N N

may assume that (M1, x1)-+(X, x) in pointed Gromov-Hausdorff distance for N some x EX. By Theorem 5.1 a further subsequence contains only manifolds with fundamental group isomorphic to the fundamental group of X.
Given R>0 it follows from Proposition 7.2 that for 0<r<R,
R n li

m sup N(x~, r, R) <_ --
~-.~ r

Proposition 4.2 and Lemma 6.1 imply that the fundamental group of X has polynomial growth. This contradicts the assumption that none of the Mi has fundamental group with polynomial growth. p

8. Volume growth of geodesic balls.

The following elementary lemma replaces the Bishop-Gromov volume com-parison theorem when only L' bounds on curvature are assumed (cf. [1]) :

LEMMA 8.1. Let X be an n-dimensional complete Riemannian manifold on which there exist constants a, b, r>0 such that for any x E X and 0<s <4r,

asn <_ vol (B(s)) <_ bsn.

Then there exist constants C(n, a, b, r), k(n, a, b, r)>0 such that for any x EX and R>0,
vol (B(x, R)) <_ CekR .

PROOF. It suffices to prove the bound for R=Nr by induction on NEZ}.
The bound clearly holds for N=1.
We use a standard covering argument to obtain an upper bound on the volume of a ball with radius (N+1)r in terms of the volume of a ball with
radius Nr. Let R=Nr. Given xEX, let B(xl, r), B(xK, r) be a maximal
collection of disjoint balls contained in B(x, R). For any yEB(x, R--r) there exists xi such that d(xi, y)<2r; if not, B(y, r) would be a ball contained in B(x, R) that is disjoint from the given collection of balls. For any z E B(x, R+r), there exists yEB(x, R-r) such that d(y; z)<2r. Thus there exists xi such that d(xi, z)<4r. It follows that the balls B(xl, 4r), ••• , B(xK, 4r) cover

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9. Proof of Theorems 1.2 and 1.3.

PROOF OF THEOREM 1.3. The proof is analogous to that of Theorem 1.1.
If the theorem is not true, then given n, q> n, Q, a>0, there exists a sequence of n-dimensional compact manifolds Mi with diameter 1, volume bounded from below by an, first systole bounded from below by 2q, the Lq/2 norm of p_ converging to 0 as i-;oo, but with fundamental group of larger than polynomial growth.
By setting R=1/2 in Proposition 8.3, the assumptions of Lemma 8.1 hold with r=1/2. By Corollary 8.2 there exists a subsequence that converges in Gromov-Hausdorff distance to a compact length space X with first systole bounded from below by 2a. By Theorem 5.1 there exists a further subsequence such that the fundamental group of each manifold is isomorphic to 7r1(X). Us-ing Corollary 8.2 again, we can restrict to a subsequence, also called Mi, such

N N N

that there exist xiEMi with (Mi, xi)--~(M, x) in pointed Gromov-Hausdorff distance.

By Proposition 8.3 there exists a constant C > 0 such that for any R>0,

(9.1) lim sup N(xi, a, R) <_ CR.

As before, this implies that 2r1(X) has polynomial growth, which is a contra-diction. E

The proof of Theorem 1.2 is similar. In the argument by contradiction, the convergence of a subsequence of manifolds is obtained by picking an upper bound for diam (M) (i, e. breaking the scale invariance of Theorem 1.2) and applying Theorem 7.1. The volume growth of a large geodesic ball in the universal cover is estimated as in the proof to Theorem 1.3. Finally, for q> n

lip- it q/2 lip-ii lip- II (q-2) /q

~4/q,r2(q-2)/q diam (M)-2 vol (M)2/q

C(n, ~, diam (M))~4/q

where the last inequality follows from Bishop-Gromov. This reduces the proof of Theorem 1.2 to that of Theorem 1.3.

10. Positivity of Laplacian plus potential.

Let M be an n-dimensional complete Riemannian manifold. As in § 2, we assume that there exist constants A, B>0 such that for any f ~C(M),

(n-2)/m

(lfl2f1-2) (n) < A Q fi 2+B i f i 2.

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vol (AnB(p, D)) < ~vol (B(p, 2D))

v

fn(2KD)

.

- Therefore, if we set o=max (D, 7c/H) and

E - YnU2 f n(r*(H5, ico)ic3) f n(2 D)(f n(K(o+2D))--fn(Ko))

the corollary follows from the theorem. D

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