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Abstract

Gupta and Kumar established that the per node throughput of ad hoc networks with multi-pair unicast traffic scales with an increasing number of nodes n as $\lambda(n) = \Theta(1/\sqrt{n \log n})$, thus indicating that network performance does not scale well. However, Gupta and Kumar did not consider the possibility of network coding and broadcasting in their model, and recent work has suggested that such techniques have the potential to greatly improve network throughput.

Here, for multiple unicast flows in a random topology under the protocol communication model of Gupta and Kumar [1], we show that for arbitrary network coding and broadcasting in a $2D$ random topology that the throughput scales as $\lambda(n) = \Theta(1/nr(n))$ where n is the total number of nodes and $r(n)$ is the transmission radius. When $r(n)$ is set to ensure connectivity, $\lambda(n) = \Theta(1/\sqrt{n \log n})$, which is of the same order as the lower bound for the throughput without network coding and broadcasting; in other words, network coding and broadcasting at most provides a constant factor improvement in the throughput. This result is also extended to other dimensional random deployment topologies, where it is shown that $\lambda(n) = \Theta(1/n)$ for the $1D$ topology, $\lambda(n) = \Theta(\frac{1}{\sqrt[3]{n \log^2 n}})$ for $3D$ networks and $\lambda_C(n) \leq \Theta(\frac{W}{\sqrt[k]{n \log^{k-1} n}})$ in kD, combined with the lower bounds of non-coding throughput, we show coding&broadcasting provide no order difference improvement.

Of course, in practice the constant factor of improvement is important; thus, to more precisely characterize the benefit of network coding for multi-pair communication in wireless networks, we derive tight bounds for the throughput benefit ratio - the ratio of optimal network coding scheme throughput to the optimal non-coding flow scheme throughput. We show that the improvement factor is $\frac{1+\Delta}{1+\Delta/2}$ for $1D$ random networks, where $\Delta > 0$ is a parameter of the wireless medium that characterizes the intensity of the interference. We obtain this by giving tight bounds (both upper and lower) on the throughput of the coding and flow schemes. For $2D$ networks, we obtain an upper bound for the throughput benefit ratio as $\alpha(n) \leq 2c_{\Delta 4}\sqrt{\pi}\frac{1+\Delta}{\Delta}$ for large n, where $c_{\Delta 4} = \max\{2, \sqrt{\Delta^2 + 2\Delta}\}$. This is obtained by finding a tighter upper bound for the coding scheme throughput and a tighter lower bound for the flow scheme throughput.

We then consider the more general physical communication model as in Gupta and Kumar [1]. We show that the coding scheme throughput under the physical model is upper bounded by $\Theta(\frac{1}{n})$ for the $1D$ random network and by $\Theta(\frac{1}{\sqrt{n}})$ for the $2D$ case. We also show the flow scheme throughput for the $1D$ case can achieve the same order throughput as the coding scheme. Combined with previous work on a $2D$ lower bound [2], we conclude that the throughput benefit ratio under the physical model is also bounded by a constant. Thus we have shown for both the protocol and physical model, the coding benefit on throughput is a constant factor.

Finally, we evaluate the potential coding gain from another important perspective - total energy efficiency - and show that the factor by which the total energy is decreased is upper bounded by 3.

I. INTRODUCTION

Multi-hop wireless networks have been intensively studied in recent years for both commercial and government applications. Such networks, static or mobile, have the potential to serve as either a self-contained network that provides communication without the presence of an established infrastructure, or as a ubiquitous bridge between end users and the high speed wired infrastructure. Two representative applications are wireless sensor networks and wireless mesh networks. Multi-hop wireless sensor networks can be deployed randomly in geographic regions to collect large volume of environment data and provide distributed query services. Wireless mesh networks can be potentially deployed in the streets of big cities, campuses, conference centers, combat fields, etc. Hence, issues of the connectivity and capacity of such networks are of interest.

Even though these self-contained wireless mesh networks alone may not be enough to sustain all of the communication among users, they probably will be mandatory for supporting the last several hops to the end users, and thus serve as a glue between the end users at any corner and the wired infrastructure. For either case, one major concern with such wireless networks is scalability. Under a traditional communication model without network coding, Gupta&Kumar [1] shows that the per node throughput of such random networks scales as $\lambda(n) = \Theta(\frac{1}{\sqrt{n \log n}})$ where n is the total number of nodes in the network and each node needs to send to a randomly chosen destination with a rate of $\lambda(n)$. This result shows that as the total number of nodes increases, the many to many throughput decreases polynomially. However, recent work by Ahlswede, Cai, Li and Yeoung [3] introduces the concept of network coding (NC), and there has been tremendous interest in applying network coding in both wired [4] and wireless networks [5] [6] [7]. For the wired case, the benefit of network coding in terms of throughput and capacity is often limited. Specifically, for networks with bidirectional links that can be modelled as an arbitrary undirected graph, [4] shows that the throughput improvement is upper bounded by a factor of 2 for the single multicast case, and upper bounded by one (no benefit) for the single unicast or broadcast case, and it is conjectured that there is no throughput benefit for the multi-pair unicast case; this is called the Li&Li conjecture, which is still open with no counter-examples found yet.

For wireless networks, network coding, combined with wireless broadcasting, can potentially improve the performance on throughput [6] [7], energy efficiency and congestion control [5] [6]. In addition, recent work by Katti et al. [8] demonstrates the potential throughput benefit of applying network coding to wireless networks through constructive examples and experiments. Since network coding was not taken into consideration in Gupta&Kumar's original work [1] and the related works that followed, an interesting question raised after [8] is how much throughput benefit can it provide to such networks. Answering this question will help us to better understand not only the benefit and limitations of network coding on the capacity of wireless networks and networks in general, but also the degree of scalability of random wireless networks, thus providing design guidelines for the coverage ratio between wireless mesh net and the infrastructure wired net in hybrid networks.

The idea of [8] is to broadcast combined information (coded) of intersecting flows, and then each flow's next hop relaying node is able to decode its relaying flow's traffic based on all of the broadcasts that it has received as well as on local information (source data generated locally). In this way a node can potentially deliver to multiple neighboring nodes multiple data flows with a single broadcast transmission. An example of this is shown in Fig. 1. Without network coding and broadcasting, four transmissions are required; however, when the opportunistic algorithm of [8] is applied, with the middle node broadcasting the XOR of a and b, only three transmissions are required to move packets a and b forward two hops.

Thus, intuitively it appears that there could be a throughput benefit ratio proportional to the expected number of neighbors, $\Theta(\log n)$ in the case of uniformly random deployed networks. However, here we demonstrate that such a large improvement is not possible; in fact, only a constant improvement in throughput can be achieved. Since bandwidth is a scarce resource in wireless networks, and also because we may not use the ad-hoc mesh network alone to support the end to end traffic, the significance of the constant factor matters. So we further provide bounds on the scalar throughput benefit ratio and extend the results to more realistic communication models.

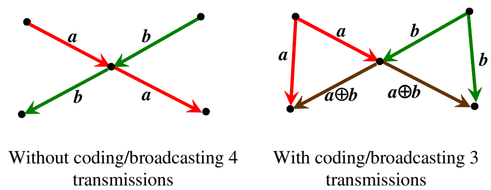

Fig. 1. An example demonstrating the benefit of Katti etc. [8]'s opportunistic coding scheme

More formally, we first derive a tighter upper bound on the throughput of schemes aided by network coding and broadcasting. We obtain this by analyzing the information rate across a sparsity cut of the network. By exploiting the geometric constraints on the receivers' locations, we derive a fairly tight upper bound on the maximum number of simultaneous transmissions across any sparsity cut. This, combined with an information compression coding rate constraint, yields the throughput upper bound for the coding scheme. Next, we derive a tighter lower bound for the non-coding scheme throughput in a semi-constructive way by showing that there exist flow schemes within a constructive framework that can achieve the lower bound throughput rate. With these two bounds together, we show that in the $2D$ case, the constant factor improvement that network coding can provide for multi-pair concurrent unicast throughput is bounded as $\alpha(n) \leq 2c_{\Delta 4}\sqrt{\pi}\frac{1+\Delta}{\Delta}$ for large n, where $\Delta > 0$ is a parameter of the wireless medium that characterizes the intensity of the interference and $c_{\Delta 4} = \max\{2, \sqrt{\Delta^2 + 2\Delta}\}$. For the $1D$ case, we obtain a tight benefit ratio as $\frac{1+\Delta}{1+\Delta/2}$, i.e., we show that $\frac{1+\Delta}{1+\Delta/2}$ throughput improvement is the maximum achievable. The reason that we can get this tighter result for the $1D$ case is because of the lower geometric complexity in $1D$ than in the higher dimensional cases.

The protocol communication model of [1] is a nice abstraction of the wireless communication channel. A more general and realistic refined model is the physical communication model [1]. To make our results more persuasive, we also analyze the throughput benefit of network coding under the physical model. We show that the coding scheme throughput is upper bounded by $\Theta(\frac{1}{n})$ for the $1D$ case and by $\Theta(\frac{1}{\sqrt{n}})$ for the $2D$ case. We also show a same order lower bound for the $1D$ flow scheme throughput. Combining this and [2]'s previous result on the $2D$ throughput lower bound, we demonstrate that the throughput benefit ratio of coding schemes is also at most a constant under the physical model.

At first, our results might seem counter-intuitive, since, with network coding and broadcasting, each node can send information to all neighbors with one transmission as demonstrated in [8]. However, essentially each node still needs to receive information one transmission at a time. In other words, whereas there is simultaneous transmission, there is no simultaneous receptions. Since the information flow rate across any node also needs to be conserved, the incoming information rate will be a bottleneck for the throughput improvement.

Another concern with multi-hop wireless networks is energy consumption. For either static or mobile nodes in the network, the power is mostly supported by a self-carried battery which has limited energy. This is especially true for low cost cheap sensors. [9] studies the problem of minimizing the total energy cost of collecting correlated data at a single sink and proves that a special joint coding/routing scheme can achieve the minimum cost. In this work, we evaluate the energy efficiency for multiple source-sink pair unicast traffic. We assume the data (messages) from different sources are independent of each other. Without network coding, the minimum cost scheme is to straightforwardly send each source to its sink independently along a shortest path (in terms of energy cost) route. We focus on the further coding gain based on the minimum cost flow scheme. For both the coding scheme and flow scheme, we ignore the delay and throughput aspects and merely ask the question of how much energy could be further saved by exploiting network coding and broadcasting. [10] gives coding gain bounds for the limited case where overhearing is not used for coding, we conisder the more general case where coding is based on a node's current possessed info. including all the overhearings and show that this energy benefit is at most a factor of 3.

Main Contributions Our main contributions are summarized as follows:

- We show that, contrary to the common intuition, for multi-pair unicast traffic in wireless multi-hop networks, the benefit of network coding and broadcasting on the concurrent throughput rate is upper bounded by a constant factor for both the protocol model and the physical model. This is true for random deployed networks in $1D$, $2D$, $3D$ and in general kD Euclidean space.
- We further bound the constant factor as $\alpha(n) \leq 2c_{\Delta 4}\sqrt{\pi}\frac{1+\Delta}{\Delta}$ for $2D$ networks and $\frac{1+\Delta}{1+\Delta/2}$ for the $1D$ case.
- We first characterize the throughput order of the coding scheme on such networks and also derive tight bounds for the constants. We also develop tighter lower bounds for the non-coding scheme on such networks.
- We show that the coding gain for energy efficiency is bounded by a constant factor of 3 for the COPE [10] type of point to point codings.

II. MODEL FORMULATION

For both static and mobile ad-hoc networks, we consider the network model of Gupta&Kumar [1], where n nodes are randomly located, i.e., independently and uniformly distributed, in a region of fixed area. The reason that this model is often justified is two-fold: first, in reality, deploying a large volume of static nodes is costly, and random deployment is the most economic option; second, even if the nodes are mobile, the mobility pattern is often either dynamically random or not available for use, or, even if the knowledge is present, the cost to use the mobility information may not be worth its benefit(e.g. imagine mobile users in the streets of NYC); in addition, often the speed of the node movement is much far below the wireless signal transmitting speed, and thus the mobile case can be treated as a static random network for the lifetime of a packet.

Gupta&Kumar [1] studies two types of regions: a disk and the surface of a sphere, both with an area of one meter squared. More generally, we do not limit the shape of the region or its dimension. However, for simplicity of presentation, we derive our results based on a unit square in two dimensions ($2D$), a unit line segment in one dimension ($1D$), and a unit cube in three dimensions ($3D$).

There are n source-destination pairs in the network. Each node i in the network is a data source that needs to route its data through multi-hop wireless communications to a destination node that is independently and uniformly randomly chosen. The same protocol and physical communication models as in Gupta&Kumar [1] are employed.

For the protocol model, as shown in Fig. 2, a transmission from node i to j is successful iff the distance between them satisfies $|X_i - X_j| \leq r(n)$ and any other simultaneously transmitting node k satisfies $|X_k - X_j| \geq (1+\Delta)r(n)$. Here, X_i is node i's location, $r(n)$ is the transmission radius and $\Delta > 0$ ensures a safety zone that limits the interference; in particular, Δ is a constant that depends on the properties of the wireless medium. In addition, there is a finite bandwidth limit of W bits/sec for each transmission. In order to ensure connectivity, the fixed transmission radius for the protocol model needs to be at least $r(n) = \Theta(\frac{\sqrt{\log n}}{\sqrt{n}})$ [1].

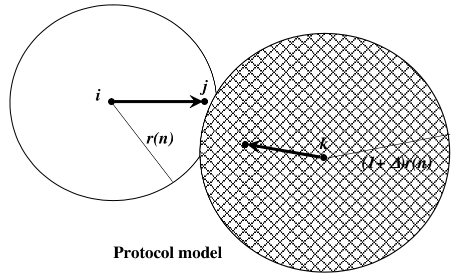

Fig. 2. The protocol communication model

The physical communication model [1] differs from the protocol model in that there is no fixed transmission radius. Each transmission has a fixed power P, a transmission from node i to j is successful if the signal to noise ratio is above a threshold:

$$\frac{\frac{P}{r_{i,j}^{\gamma}}}{N + \Sigma_{k \in K} \frac{P}{r_{k,j}^{\gamma}}} \geq \beta \tag{1}$$

where K is the node set of all other nodes that are simultaneously transmitting, N is the ambient noise level and $\beta > 0$ is the threshold.

As in Gupta&Kumar [1], attention here is focused on the many to many throughput of the network, i.e. the data rate at which each node can send to its destination node. A throughput $\lambda(n)$ (bits/sec) is *feasible* if there exists a scheme that achieves $\lambda(n)$ on average. The *throughput capacity* of such a random network is defined as the maximum throughput that is feasible with high probability.

Here, transmission schemes correspond to the same type of "spatial and temporal scheduling schemes that operate the network in a multi-hop fashion and buffers at intermediate nodes when awaiting transmissions" as in [1]. Two types of schemes are considered: a *flow scheme* and a *coding scheme*. A flow scheme is a non-coding scheme where data are routed as commodity flows (duplication, forwarding, but no coding) and thus the broadcast nature of the wireless medium is not helpful for the flow scheme for the unicast task. Gupta&Kumar[1] focus on the throughput of flow schemes. A coding scheme is one that allows all of the operations in a flow scheme, along with allowing messages received at each node to be decoded/recoded; in other words, intermediate nodes can send the results obtained from applying arbitrary functions to all previously received bits and its own source data as long as each destination node is able to decode the data intended for it from all of its received bits and local data. Thus all possible benefits of combining network coding and wireless broadcasting as demonstrated in Katti etc. [8] are incorporated in the considered coding schemes. The throughput capacity is denoted as $\lambda_F(n)$ for flow schemes and $\lambda_C(n)$ for coding schemes. The *throughput benefit ratio* of the coding scheme is denoted as $\alpha(n) = \frac{\lambda_F(n)}{\lambda_C(n)}$. As in Gupta&Kumar [1], all packets are independent of each other whether they are from different sources or the same source; in other words, there are no spatial or temporal correlations among the source data.

III. Throughput Benefit of Coding Schemes under the Protocol Model

In this section, we show that under the protocol model, coding schemes provide at most a constant factor improvement in throughput over flow schemes. In other words, there exists some constant c (i.e. not dependent on n) such that $\alpha(n) \leq c$. After this, we derive tight bounds for the constant factor.

A. Sparsity Cut for a Random Network

Before we present the throughput improvement results, we redefine *sparsity cut* for a random network.

In general, a cut Γ is defined as a partition of the nodes in a graph, the cut capacity is the sum of the links' bandwidths crossing the cut, and the sparsity cut is a cut where the cut capacity divided by the traffic demand is the minimum over all cuts. Since the network studied here is a random network embedded in an Euclidean space and transmissions are between neighboring nodes, attention can be focused on a narrow class of cuts that are induced by a line segment (or a plane in the $3D$ case) that cuts the region into two regions. The *cut length* l_Γ is defined as the length of the cut line segment. The cut lines that we consider have zero width measuring such that no nodes lie on it. Denote the two subregions divided by the cut as Γ_1 and Γ_2. A *sparsity cut* for a random network is defined as a cut induced by the line segment with the minimum length that separates the region into two equal area subregions. For the square deployment region illustrated in Fig. 3, the line segment AB induces a sparsity cut Γ_{AB}. Since nodes are uniformly randomly deployed in a random network, such a sparsity cut captures the traffic bottleneck of these random networks on average. The *cut capacity* is defined as $(\Lambda_{\Gamma_{1,2}}, \Lambda_{\Gamma_{2,1}})$ where $\Lambda_{\Gamma_{1,2}}$ equals the transmission bandwidth W times the maximum possible number of simultaneous transmissions (broadcast or non-broadcast) across the cut from Γ_1 to Γ_2 [1]; and $\Lambda_{\Gamma_{2,1}}$ equals the same quantity from Γ_2 to Γ_1. This cut capacity constrains the information rate that the nodes from one side of the cut as a whole can deliver to the nodes at the other side as a whole. The number of sources in Γ_1 whose destinations are in Γ_2 is denoted as $n_{\Gamma_{1,2}}$.

B. Throughput Order of Coding Scheme

The cut capacity is bounded by deriving an upper bound on the maximum number of simultaneous transmissions across the cut. It is easy to see that all of the direct receivers of transmissions across a cut Γ in one direction lie in the shaded rectangle region with area $l_\Gamma \times r(n)$ as shown in Fig. 3. In [1], disks of radius $\frac{\Delta r(n)}{2}$ centered at each receiver are disjoint[2]. However, [1] does not exploit broadcast transmissions while a coding scheme does. As shown in Fig. 4, with the consideration of broadcast and network coding, observe that such disks centered at receivers of the same sender (broadcast transmission) could overlap, disks centered at receivers of different senders are still disjoint. In other words, we have the following Observation:

Observation 1: The union of disks (with radius $\frac{r(n)}{2}$) centered at the receivers of one transmission should be disjoint from the union of disks centered at the receivers of another transmission.

1) $2D$ case::

Lemma 1: The capacity of a cut Γ for a $2D$ region has an upper bound of $\frac{c_\Delta l_\Gamma W}{r(n)}$ where $c_\Delta = \max\{\frac{16}{\pi\Delta^2}, \frac{\sqrt{3}}{\Delta}\}$

Proof: The cut capacity is bounded by deriving an upper bound on the maximum number of simultaneous transmissions across the cut. It is easy to see that all of the direct receivers of all transmissions across a cut Γ in one direction lie in the shaded rectangle region with area $l_\Gamma \times r(n)$ as shown in Fig. 3. When $\Delta < 2$, Observation 1 means each transmission across the cut consumes at least an area of $\frac{1}{4}\pi(\frac{\Delta r(n)}{2})^2$ of the shaded region in Fig.3, with the minimum achieved when receiver lies in the corner of the shaded region. Thus, the maximum number of simultaneous transmissions across the cut is upper bounded by the area of the shaded region divided by $\frac{1}{4}\pi(\frac{\Delta r(n)}{2})^2$, which is $\frac{16 l_\Gamma}{\pi\Delta^2 r(n)}$.

[1]The maximum number of bits of information per second that can be transmitted across the cut from one side (Γ_1) to the other (Γ_2)

[2]Otherwise some sender is within $(1+\Delta)r(n)$ of some other sender's receiver.

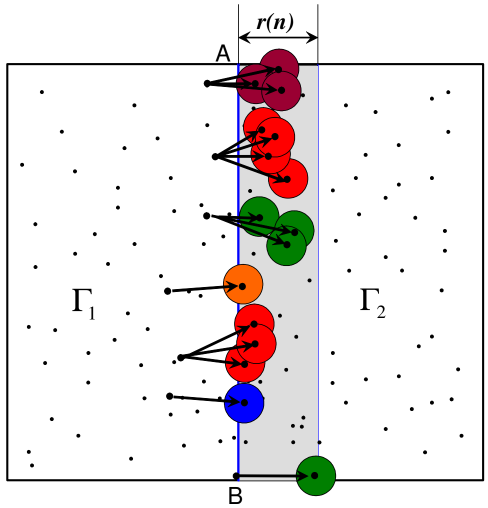

Fig. 3. Cut Capacity in 2D

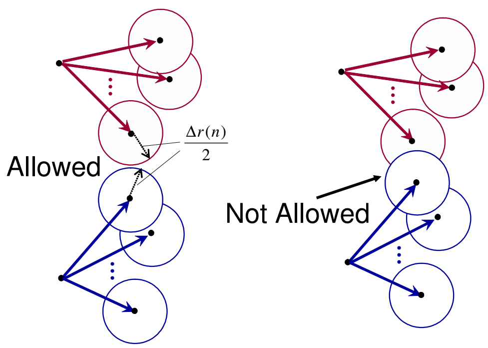

Fig. 4. Interference of coding schemes in 2D

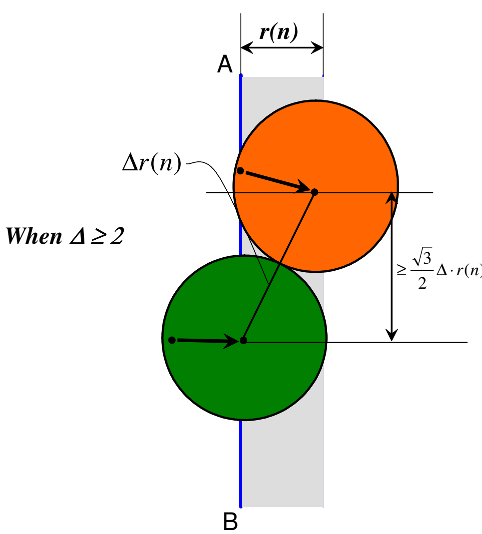

Fig. 5. $2D$ Cut capacity: $\Delta \geq 2$ case

When $\Delta \geq 2$, as shown in Fig. 5, any two receivers of two different transmissions require a $\frac{\sqrt{3}}{2}\Delta r(n)$ difference in their coordinates along the cut line. Thus, there can be at most $\frac{l_\Gamma}{\sqrt{3}\Delta r(n)/2}+1 \leq \frac{\sqrt{3}l_\Gamma}{\Delta r(n)}$ simultaneous transmissions across the cut.

Since each transmission is able to send W bits/sec, combining the two cases above, the cut capacity is upper bounded by $\Lambda_{\Gamma_{1,2}} \leq \frac{c_\Delta l_\Gamma W}{r(n)}$ and $\Lambda_{\Gamma_{2,1}} \leq \frac{c_\Delta l_\Gamma W}{r(n)}$, where $c_\Delta = \max\{\frac{16}{\pi\Delta^2}, \frac{\sqrt{3}}{\Delta}\}$. ■

Corollary 1: The sparsity cut capacity of a $2D$ *random network has an upper bound of* $\frac{c_\Delta W}{r(n)}$.

Proof: Regardless of the shape of the unit area region, there exists a sparsity cut for each orientation of the cut line. If we rotate the cut line, there has to be at least one sparsity cut with cut length $l_\Gamma \leq 1$. Hence, from Lemma 1, we derive the corollary. ■

Next an upper bound for the throughput of coding schemes in a $2D$ random network is derived.

Theorem 1: The throughput of coding schemes in a $2D$ *random network is upper bounded by* $\Theta(\frac{W}{nr(n)}) = \Theta(\frac{W}{\sqrt{n \log n}})$

Proof: Assume the coding throughput of the n node random network is $\lambda_C(n)$. Then, by its definition, with high probability (w.h.p.) there exists some coding scheme that, for some $T < \infty$, during each time interval $[(i-1)T, iT]$ (in seconds), every node can send $T\lambda_C(n)$ bits of information to its corresponding destination node. For a sparsity cut Γ_{AB} in the middle, by a Chernoff bound [11] argument we have that w.h.p. there are $\Theta(n)$ pairs of source-destination nodes that need to cross Γ_{AB} in one direction, i.e., $n_{\Gamma_{1,2}} = n_{\Gamma_{2,1}} = \Theta(n)$ w.h.p.. Now we view all of the nodes lying on the right side of AB as a super node, and treat all of the distinct messages it receives from the left side of AB within the time interval $[(i-1)T, iT]$ as a single 'meta' message M. We denote the number of bits of M as B_M. According to the definition of our coding scheme, this meta message M can be arbitrarily coded but with only one coding constraint: by Shannon's data compression theorem [12], in order for the right side destination nodes to decode the original data from the left side sources which are independent of each other, M has to satisfy $B_M \geq Tn_{\Gamma_{1,2}}\lambda_C(n)$ or $B_M \geq T\Theta(n)\lambda_C(n)$ w.h.p..

At the same time, our work above has provided a capacity constraint that upper bounds B_M. A broadcast transmission across the cut to multiple receivers delivers identical information to the receivers; however, by the definition of M, the identical messages will only be counted once in M for one of the receivers. Also by the definition of cut capacity and Corollary 1, we get $B_M \leq \frac{c_\Delta W}{r(n)}T$. Combined with the coding constraint above, we derive $\lambda_C(n) \leq \frac{c_\Delta W}{\Theta(n)r(n)}$ w.h.p.. Since $r(n)$ is at least $\Theta(\frac{\sqrt{\log n}}{\sqrt{n}})$ to ensure connectivity [1], and throughput is defined as a high probability quantity, we have $\lambda_C(n) \leq \frac{c_\Delta W}{\Theta(\sqrt{n \log n})}$. ■

Theorem 2: The $2D$ *throughput benefit ratio is upper bounded by a constant:*

$$\alpha(n) = \Theta(1)$$

Proof: Gupta&Kumar [1] already establishes a lower bound for the throughput of flow schemes, $\lambda_F(n) \geq \Theta(\frac{c_1 W}{(1+\Delta)^2\sqrt{n \log n}})$ where $c_1 > 0$ is a constant. Combined with Theorem 1, we get $\alpha(n) = \frac{\lambda_C(n)}{\lambda_F(n)} \leq \Theta(1)$. Meanwhile, since coding scheme is a superset of flow scheme, $\alpha(n) \geq 1$, thus $\alpha(n) = \Theta(1)$. ■

This constant throughput benefit ratio is true for a random network deployed in any arbitrarily shaped region. First, the upper bound for the throughput of the coding scheme still holds. Second, the constructive lower bound of Gupta&Kumar [1] can in fact be extended to arbitrarily shaped regions, even though the asymmetry may cause the constructive scheme in [1] to have a skewed load distribution for some cut. Since the region is of fixed area, the throughput loss due to the asymmetric shape will be a fixed constant factor as n increases. Thus, we have shown that network coding combined with wireless broadcast provides no order-different improvement on the throughput of a random network deployed in any arbitrarily shaped $2D$ region.

2) 1D case: The $1D$ case is easier to deal with than the $2D$ case. Traffic either goes left or right along one line. However, $1D$ differs from $2D$ in that the transmission radius does not affect the order of throughput. Thus, we are able to give a constructive lower bound for the $1D$ throughput that does not have a $\log n$ factor as in the $2D$ and $3D$ cases.

First we make the following straightforward observation.

Observation 2: Under the protocol, for any cut in the $1D$ line, there can be at most one transmission across the cut (including both directions) at any given point in time.

Lemma 2: The throughput of the coding scheme on a $1D$ random network is upper bounded by

$$\lambda_C(n) \leq \frac{2W}{n}$$

Proof: We prove this by showing that for any given constant $\epsilon > 0$ that is arbitrarily small, $\lambda_C(n) \leq \frac{2W}{(1-\epsilon)n}$ for large n.

Consider the sparsity cut Γ_m that cuts the line segment in the middle. Using a Chernoff bound [11], it is easy to show that the number of sources that need to send data across Γ_m from left to right is larger than $(1-\epsilon)\frac{n}{4}$ w.h.p., and that the same is true for the number of sources crossing the cut from right to left.

Applying the same technique as in Theorem 1 and by Observation 2, we have $\lambda_C(n)2(1-\epsilon)\frac{n}{4} \leq W$, which yields the desired result. ■

To our knowledge, there is no constructive lower bound for the throughput of a $1D$ random network yet. In this paper, we first construct a lower bound for a $1D$ random network and show that its throughput capacity is on the order of $\Theta(\frac{W}{n})$.

Lemma 3: The throughput of flow schemes on a $1D$ random network is lower bounded by

$$\lambda_F(n) \geq \frac{c_{\Delta_2} W}{n}$$

where $c_{\Delta_2} = \min\{\frac{1}{\lceil 2\Delta \rceil + 2.75}, \frac{1}{4}\}$.

Proof: We prove this by showing that for any given constant $\epsilon > 0$ that is arbitrarily small, $\lambda_F(n) \geq \frac{c_{\Delta_2} W}{(1+\epsilon)n}$ for large n.

We choose a transmission radius $r(n) = \frac{40 \log n}{n}$, divide the line deployed region into bins each of length $\frac{r(n)}{2} = \frac{20 \log n}{n}$ and all together $\frac{n}{20 \log n}$ bins. Then by the same union bound argument as in [13], w.h.p. every bin contains at least one node. Furthermore, a node in one bin can reach any node in an adjacent bin directly.

Routing consists of hopping from one bin to the next bin in direction of the destination unless the destination is in the same bin as the source. Any node in the next bin can be a relay but for the last bin the algorithm will choose the destination node itself.

Next we show that there exists a spatial&temporal scheduling scheme that on average allows each bin a chance to transmit W bits to each of its two neighboring bins every $4/c_{\Delta_2}$ seconds. This is done by mapping the bin-hopping transmissions to a new graph. For each bin we construct two virtual vertices in a new graph, one for transmissions from this bin going to the right, one for transmissions going left. We connect any two vertices in the new graph with an edge if the transmissions that they represent could potentially interfere with each other when occurring simultaneously. When $\Delta < 1$, each vertex in the new graph will have a degree of at most 15. By the graph coloring theorem [14], $15 + 1$ colors are enough to color the vertices s.t. no two interfering vertices (transmissions) have the same color. This gives a schedule of length 16 where each bin gets a chance to transmit to both directions. When $\Delta \geq 1$, we count the potential interfering bin transmissions and it is not hard to see that the vertex degree is at most $4\lceil 2\Delta \rceil + 10$, so there is a schedule of length $4\lceil 2\Delta \rceil + 11$. Combining these two cases, we have on average that every $4/c_{\Delta_2}$ seconds each bin will get a chance to transmit one second (W bits) for both directions.

There are altogether $K = \frac{n}{20 \log n}$ bins. Denote the sum of the number of source and destination nodes in each bin as $b_1, b_2, \ldots, b_K$. Then, using the Chernoff bound and the union bound of probability, we have $b_i < \frac{50 \log n}{n}$ for all $i = 1, \ldots, K$ simultaneously w.h.p.. Also, using the Chernoff bound and union bound, for any cut, the number of sources that need to send traffic across the cut is upper bounded by $(1+\epsilon)\frac{n}{4}$ for each of the two directions, where $\epsilon > 0$ is an arbitrarily small constant. Thus a throughput of $\lambda(n) = \frac{W}{4/c_{\Delta_2}((1+\epsilon)\frac{n}{4} + \frac{50 \log n}{n})}$ is achievable w.h.p., because there exists a schedule that can deliver $\lambda(n)(1+\epsilon)\frac{n}{4}$ bits/sec across any cut w.h.p.. This throughput is just $\lambda(n) = \frac{c_{\Delta_2} W}{((1+\epsilon)n + \frac{200 \log n}{n})} = \frac{c_{\Delta_2} W}{(1+\epsilon+\frac{200 \log n}{n^2})n}$. Since $\frac{200 \log n}{n^2} \to 0$ as $n \to \infty$, it can be absorbed into the ϵ term, yielding the lemma. ■

Theorem 3: The $1D$ throughput improvement of the coding scheme over the flow scheme is at most a constant factor; more specifically:

$$\alpha(n) \leq \frac{2}{c_{\Delta_2}}$$

where $c_{\Delta_2} = \min\{\frac{1}{\lceil 2\Delta \rceil + 2.75}, \frac{1}{4}\}$.

Proof: This follows directly from Lemmas 2 and 3. ■

From Theorem 3, we see that, as in $2D$, coding schemes provide no order different throughput improvement as in $1D$. One thing to notice is that in $1D$ there is no $\log n$ factor in the throughput. The reason that such can be achieved is because in $1D$ the transmission radius has no order difference effect on the throughput. Intuitively, in $2D$, as we increase the transmission radius $r(n)$, the number of hops and the relaying traffic for each node drops linearly, while the spatial multiplexing drops quadratically, and thus the joint effect on throughput will be decreasing linearly on $r(n)$. However, for $1D$, as we increase $r(n)$, the spatial multiplexing also drops linearly; thus, there is no order difference effect on throughput no matter what $r(n)$ we choose, and the $\log n$ factor can be eliminated.

In [2], percolation theory is employed to construct a lower bound of the throughput in $2D$ that removes the $\log n$ factor. Essentially, this is achieved by using smaller $r(n)$ for the majority of the transmissions. For $1D$, the percolation result does not hold, but our result shows that the percolation technique is not required to remove the $\log n$ factor.

3) 3D case: The $3D$ case is similar to the $2D$ situation. Applying the same technique as in $2D$, similar results follows as below.

Theorem 4: The throughput of the coding scheme on a $3D$ random network is upper bounded by $\lambda_C(n) \leq \Theta(\frac{W}{\sqrt[3]{n \log^2 n}})$. More specifically, $\lambda_C(n) \leq \frac{c_{\Delta_3} W}{n r(n)^2}$ where $c_{\Delta_3} = \min\{\frac{192}{\pi \Delta^3}, \frac{256}{\sqrt{3}\pi \Delta^2}\}$.

Proof: The proof uses the same technique as in the $2D$ and $1D$ cases. One difference is now that, when $\Delta < 2$, each transmission occupies at least a volume of $\frac{4}{3}\pi(\frac{\Delta r(n)}{2})^3 \frac{1}{8}$, i.e. one eighth of a sphere of radius $\frac{\Delta r(n)}{2}$; when $\Delta \geq 2$, each transmission will occupy at least one fourth of a circle with radius $\frac{\sqrt{3}\Delta r(n)}{4}$ on the cut plane. Another difference is that ensuring connectivity in $3D$ requires that $r(n) = \Theta(\sqrt[3]{\frac{\log n}{n}})$. The rest of the argument is the same. ■

Theorem 5: The $3D$ throughput benefit ratio is upper bounded by a constant:

$$\alpha(n) = \Theta(1)$$

Proof: Gupta&Kumar have already shown a constructive lower bound for the throughput of flow schemes in [15], $\lambda_F(n) \geq \Theta(\frac{W}{(n\log^2 n)^{\frac{1}{3}}})$. Combined with Theorem 4, we get $\alpha(n) = \Theta(1)$. ■

In general, we can show that for a random network in an abstract kD ($k > 1$) Euclidean space, the coding scheme provides no order different benefit on throughput. More specifically, the coding throughput is upper bounded by $\lambda_C(n) \leq \Theta(\frac{W}{\sqrt[k]{n\log^{k-1} n}})$, and there exists a flow scheme achieving the same order throughput.

C. Bounds on the throughput benefit ratio α

Now we bound the constant as tight as we can. The technique we used is similar to those we used in Subsection III-B, but now we want to derive tighter bounds for the coding scheme and the flow scheme. We do this for the $1D$ and $2D$ case.

1) 1D throughput improvement: The $1D$ situation is different from $2D$ or $3D$. Traffic either goes left or right along one line. Also in $1D$ the transmission radius does not affect order of the throughput. For $1D$, we derive a tight bound, which is an upper bound as well as a lower bound, for the coding scheme throughput and a tight bound for the flow scheme throughput, thus a tight bound for the throughput benefit ratio $\alpha(n)$. The reason that we are able to do this is because the simpler geometry of $1D$ node deployment and we can choose larger transmission radius in $1D$ without sacrificing throughput.

From Theorem 3, we already know an upper bound of at least 8 for α. This is actually a quite loose bound. To get a tighter bound, we first present two tighter upper bounds for the flow scheme and coding scheme, then show that there exist corresponding schemes achieve these two bounds asymptotically w.h.p., thus make them also lower bounds.

Theorem 6: The throughput of the flow scheme on a $1D$ random network is upper bounded by

$$\lambda_F(n) \leq \frac{2W}{(1+\Delta)n}.$$

Proof: We prove this by showing that for any given constant $\epsilon > 0$ that is arbitrarily small, $\lambda_F(n) \leq \frac{2W}{(1-\epsilon)n}\frac{1}{1+\Delta}$ for large n.

We use the same method as in Lemma 2. Additionally now we show for large n, the usage rate of capacity of a sparsity cut is at most $\frac{1}{1+\Delta}$ for almost all the traffic crossing the cut in any flow scheme.

As argued in Lemma 2, w.h.p. there are at least $(1-\epsilon)\frac{n}{4}$ sources need to cross the sparsity cut from left to right, and the same for the other way around. Of all these sources, w.h.p. $(1-\epsilon)\frac{n}{4} - \Theta(\log n)$ of them are from sources that are more than $50r(n)$ away from the cut. We evaluate the cut capacity usage for traffic from these sources.

We first look at the case when $\Delta < 1$. As shown in Fig. 6, the top transmission $S_1 \to R_1$ is a transmission across the sparsity cut and the traffic is originally from sources $50r(n)$ away from the cut. Since the sender S_1 is a relaying node, there has to be some supporting transmission $S_2 \to S_1$ to satisfy the flow conservation constraint. Due to the wireless interference, the spatial gap between $S_2 \to S_1$ and another transmission to its right is at least $\Delta r(n)$. Since there are at most $\Theta(\log n)$ sources within $50r(n)$ from the cut w.h.p., the transmission to pair up with

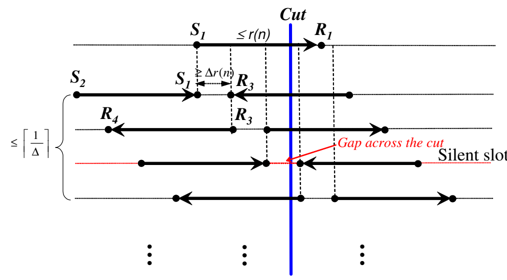

Fig. 6. Tighter Bound of $1D$ flow scheme

$S_2 \to S_1$ will be a transmission of traffic from sources $50r(n)$ away from the cut as well w.h.p.. More specifically, for $\lambda_F(n)(1-\epsilon)\frac{n}{2} - \lambda_F(n)\Theta(\log n)$ of bits per second cross the sparsity cut, the pair up transmission sent to R_3 also need to be supported by other transmission (to be forwarded in this case). And this pattern continues, we will see that there is a drifting spatial gap of size at least $\Delta r(n)$ between transmissions(see Fig. 6). In the end, this gap will hit the cut and there will be a zero bits crossing period (silent period) for the cut, and the length of the silent period equals the time slot length for $S_1 \to R_1$.

From Fig. 6, we can easily see that there will be one silent slot for the cut at most every $\lceil \frac{1}{\Delta} \rceil + 1$ slots. When $\frac{1}{\Delta}$ is not an integer, we can actually approach its value arbitrarily close using arbitrarily large periodic patterns. So we have shown that averagely for $\lambda_F(n)(1-\epsilon)\frac{n}{2} - \lambda_F(n)\Theta(\log n)$ bits traffic that cross the sparsity cut every second, the time needed is at least $t_1 = \frac{\lambda_F(n)(1-\epsilon)\frac{n}{2} - \lambda_F(n)\Theta(\log n)}{W\frac{1/\Delta}{1+1/\Delta}}$ seconds. The rest of the traffic that needs to cross the cut is the $\lambda_F(n)\Theta(\log n)$ part, and it needs at least $\lambda_F(n)\Theta(\log n)/W$ seconds. On average, the two parts traffic has to cross the cut in one second, so $t_1 + t_2 \le 1$, then we get $\lambda_F(n) \le \frac{2W}{(1-\epsilon)n - \frac{2\Delta}{1+\Delta}\Theta(\log n)}\frac{1}{1+\Delta}$ w.h.p.. The $\Theta(\log n)$ component will get absorbed in ϵn, thus we have $\lambda_F(n) \le \frac{2W}{(1-\epsilon)n}\frac{1}{1+\Delta}$ for arbitrarily small $\epsilon > 0$, thus $\lambda_F \le \frac{2W}{(1+\Delta)n}$.

For the $\Delta \ge 1$ case, the proof is almost the same. The only difference is now the gap is bigger and we are drifting the transmissions to fill the gap. The cut usage will be bounded as one transmission across it at least every $\frac{\Delta}{1} + 1$ slots, or equivalently at most one transmission can across it every $\frac{\Delta}{1} + 1$ slots. Again, this means a usable bandwidth of $\frac{W}{1+\Delta}$ across the sparsity cut. ■

Theorem 7: The throughput of the coding scheme on a $1D$ *random network is upper bounded by*

$$\lambda_C(n) \le \frac{2W}{(1+\frac{\Delta}{2})n}$$

where $\epsilon > 0$ *is an arbitrary small constant.*

Proof: We prove this by showing that for any given constant $\epsilon > 0$ that is arbitrarily small, $\lambda_C(n) \le \frac{2W}{(1-\epsilon)n}\frac{1}{1+\frac{\Delta}{2}}$ for large n.

The proof differs from Theorem 6 in only one regard. For the coding scheme, one transmission could deliver information to both directions. Thus as we can see in Fig. 7, the gap drifting pace is decreased, and so is the frequency of the silent slot. Now we have one gap cross the cut at most every $1 + 2\frac{1}{\Delta}$ when $\Delta < 1$. Arguing in the same way as in Theorem 6 we derive $\lambda_C(n) \le \frac{2W}{(1-\epsilon)n}\frac{1}{1+\frac{\Delta}{2}}$. Similarly we can derive the same bound for the case of $\Delta \ge 1$. ■

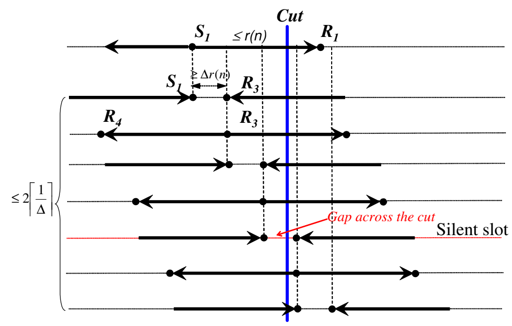

Fig. 7. Tighter Bound of $1D$ coding scheme

Another subtle issue for the coding scheme is the flow conservation argument. Since the source data are independent and by the throughput of the coding scheme we mean the throughput of the optimal coding scheme, there should be no redundancy in messages of the coding scheme. At the same time, any source information S_1 delivers to R_1 that is not originally from S_1 has to be received by S_1 beforehand, and since information are all maximally compressed, the receiving will consume an equal amount of time slots as the transmission of $S_1 \to R_1$. The difference from the flow scheme is that now the pairing is rather between the time fraction of receiving and sending of a node than between transmissions. The actual receiving of the information forwarded could be more than one transmissions from different senders, but the time fraction is still the same so the argument still holds.

Next we will show w.h.p. there exist schemes that can achieve a throughput arbitrarily close to the above two bounds and thus the ratio of the above two bounds is a tight bound for the throughput benefit ratio.

Theorem 8: The throughput of the flow scheme on a $1D$ *random network is lower bounded by*

$$\lambda_F(n) \geq \frac{2W}{(1+\Delta)n}.$$

Proof: We prove this by showing that for any given constant $\epsilon > 0$ that is arbitrarily small, $\lambda_F(n) \geq \frac{2W}{(1+\epsilon)n}\frac{1}{1+\Delta}$ for large n. We use the same binning technique as in Lemma 3 with the same bin size of $\frac{20 \log n}{n}$ but a larger $r(n) = 80\frac{\log^2 n}{n}$.

Since w.h.p. there are at least one node in each bin, we randomly choose a node in each bin as a representative relaying node, which we refer as *bin relaying node*.

The flow scheme that we construct has two phases, the scheduling phase and the routing phase. Every second will be split for the two phases. The first phase's goal (scheduling) is to deliver the traffic generated by all the nodes within this second to their corresponding bin relaying nodes using one hop transmissions, we denote the time assigned to this phase as t_1. The second phase (routing) is to relay traffic mostly among bin relaying nodes, denote the time assigned as t_2. The only exception in the second phase is when the destination is in the same bin as the scheduled bin relaying node, we will route to the destination node directly in this case.

We design the routing phase to pack the transmissions in a pattern as shown in Fig. 6 but a bin approximating version. What we want ideally is each transmission travels a distance of $r(n)$ and there is a gap of exactly $\Delta r(n)$ between two adjacent transmissions. Here we tolerate a small disturbing of twice the bin size, $40\frac{\log n}{n}$. Thus in our routing each transmission travels a distance in the interval of $(r(n) - 40\frac{\log n}{n}, r(n))$, each gap is in the interval of $(\Delta r(n), \Delta r(n) + 40\frac{\log n}{n})$.

By Lemma 3, there are at most $\Theta(\log n)$ source nodes in each bin w.h.p.. Also for the scheduling phase, each transmission has a degree of confliction with other bins as $\Theta(\log n)$ due to the increasing of $r(n)$. Thus by the vertex coloring argument as in Lemma 3, we know there is a scheduling scheme that can achieve the scheduling phase's goal in time $t_1 = \frac{\lambda(n)\Theta(\log^2 n)}{W}$.

For the routing phase, also by Lemma 3, we know w.h.p. there are at most $\lambda(n)(1+\epsilon)\frac{n}{2}$ bits/second of data that needs to cross any cut. By the bin approximation of the scheduling as in Fig. 6, the cut capacity usage is at least $\frac{u}{1+u} = \frac{1-\frac{2}{\log n}}{1+\Delta}$ where $u = \frac{r(n)-\frac{r(n)}{2\log n}}{\Delta r(n)+\frac{r(n)}{2\log n}}$. Thus w.h.p. this scheme can support any throughput that satisfies $t_2 W\frac{1-\frac{2}{\log n}}{1+\Delta} \geq \lambda(n)(1+\epsilon)n/2$. Applying $t_2 = 1-t_1$ yields $\lambda(n) \leq \frac{2W(1-\frac{2}{\log n})}{n(1+\epsilon)(1+\Delta)+2\Theta(\log^2 n)(1-\frac{2}{\log n})}$.

Again, the $\Theta(\log^2 n)$ component gets absorbed in the ϵn term. Thus we get for large n any throughput of $\lambda(n) = \frac{2W}{(1+\epsilon)n}\frac{1}{1+\Delta}$ is achievable w.h.p. where $\epsilon > 0$ is an arbitrary small constant. ■

Theorem 9: The throughput of the flow scheme on a $1D$ random network is

$$\lambda_F(n) = \frac{2W}{(1+\Delta)n}.$$

Proof: This follows directly from Theorem 6 and 8. ■

Theorem 10: The throughput of the coding scheme on a $1D$ random network is lower bounded by

$$\lambda_C(n) \geq \frac{2W}{(1+\frac{\Delta}{2})n}.$$

Proof: We prove this by showing that for any given constant $\epsilon > 0$ that is arbitrarily small, $\lambda_C(n) \geq \frac{2W}{(1+\epsilon)n}\frac{1}{1+\frac{\Delta}{2}}$ for large n.

Use the same technique as in Theorem 8. Schedule the flows as Fig. 7. Simply choose 'XOR' as the coding operation. Each transmission will then broadcast to two receivers on both sides a 'XOR' result of the two flows relayed at the sender going to opposite directions. ■

Theorem 11: The throughput of the coding scheme on a $1D$ random network is

$$\lambda_C(n) = \frac{2W}{(1+\frac{\Delta}{2})n}.$$

Proof: This follows directly from Theorem 7 and 10. ■

Theorem 12: The throughput benefit ratio on a $1D$ random network is $\alpha(n) = \frac{1+\Delta}{1+\frac{\Delta}{2}}$.

Proof: This is a straightforward conclusion from Theorem 9 and 11. ■

2) 2D throughput improvement: The $2D$ case is harder to derive a tight bound for α. The reason is two fold: we do not have the $1D$ flexibility of choosing arbitrary large transmission radius without sacrificing throughput; packing non-conflict transmissions is more complex in the $2D$ case than $1D$. We will first give a tighter upper bound for the capacity of the sparsity cut and thus the coding scheme throughput $\lambda_C(n)$, then we construct a tighter lower bound for the flow scheme throughput $\lambda_F(n)$.

We first define a *two way cut capacity* which differs from the previous cut capacity in that it evaluates the maximum possible bits that can cross the cut concurrently, regardless of in which of the two directions. Note that an upper bound for the two way cut capacity is automatically an upper bound for the one way cut capacity. In fact, being able to bound the maximum packing of two-way instead of one-way transmissions across the cut makes all the upper bounds we derived for the throughput a factor of 2 tighter. Before we describe the upper bound, we derive some basic properties for a transmission across a cut.

Lemma 4: For any two transmissions across a cut, $S_1 \rightarrow R_1$ *and* $S_2 \rightarrow R_2$ [3]*, the line segments* S_1R_1 *and* S_2R_2 *have no intersection point and* R_1R_2 *can not be vertical to the cut line.*

Proof: The first property is actually universally true for any two transmissions, not necessarily only for transmissions across a cut.

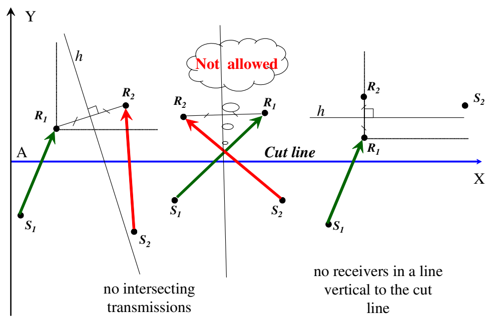

Fig. 8. Geometric properties of transmissions across a cut

As shown in Fig. 8, connect R_1R_2, draw the perpendicular bisector h of R_1R_2. By the protocol communication model, we know $|S_1R_1| \leq |S_1R_2|$ and $|S_2R_2| \leq |S_2R_1|$. Thus S_1 and R_1 lie on one side of h and S_2 and R_2 lie on the opposite side. So there can never be an intersecting point between S_1R_1 and S_2R_2. The middle case in Fig. 8 can not occur.

Also if R_1R_2 is vertical to the cut line, as we see in Fig. 8, h is parallel to the cut line. Then one of the transmissions could never cross the cut, contradicted with the assumption. Thus this is impossible for any two transmissions across a cut. ■

Next, we construct a coordinate system for a cut AB. Let A be the origin, the line of AB be the X axis, and a line vertical to AB be the Y axis (see Fig. 9). We denote a node R's X coordinates by $R(x)$. Order all of the simultaneous transmissions across the cut by their intersecting points with the X axis (the cut line), from small X coordinates to big ones. Label the sender-receiver pair of the ordered transmissions as $S_1 \rightarrow R_1$, $S_2 \rightarrow R_2$, $\ldots$, $S_m \rightarrow R_m$, where m is the total number of scheduled simultaneous transmissions across the cut. Denote $X_k = \max\{S_k(x), R_k(x)\}$ for $1 \leq k \leq m$ and $X_0 = 0$.

Lemma 5: $X_k - X_{k-1} \geq \Delta r(n)$ *for all* $2 \leq k \leq m$*.*

Proof: W.l.o.g., consider the first two consecutive transmissions in the ordered list $S_1 \rightarrow R_1$ and $S_2 \rightarrow R_2$. By Lemma 4, we know $R_2(x) > R_1(x)$; otherwise the ordering should be switched.

Senario 1: We first consider scenario 1 as shown in Fig. 9 that the senders are on one side of the cut and receivers on the other side. Within this, Fig. 9 shows the case for $S_1(x) \leq R_1(x)$. There are two possibilities for $R_2(y)$: either $R_2(y) \leq R_1(y)$ or $R_2(y) > R_1(y)$.

When $R_2(y) \leq R_1(y)$, from the facts that $|S_1R_2| \geq (1+\Delta)r(n)$, $|S_1R_1| \leq r(n)$, $S_1(x) \leq R_1(x)$ and $S_1(y) < R_1(y)$ we can easily get $R_2(x) - R_1(x) > \Delta r(n)$ thus $X_2 - X_1 > \Delta r(n)$.

When $R_2(y) > R_1(y)$ as shown in Fig. 9, we look at the triangle $R_1R_2S_2$. By the protocol model, $|R_1S_2| \geq (1+\Delta)r(n)$, $|R_2S_2| \leq r(n)$. From R_2 draw $R_2F \perp R_1S_2$ and intersect R_1S_2 at F, then $|R_1F| \geq |R_1S_2| - |R_2S_2| \geq \Delta r(n)$. Note that F has to be between R_1 and S_2 because $\phi < 90^o$ (because $|R_1S_2| > |R_2S_2|$). Then from R_1

[3] Note that each transmission could have multiple receivers, we just pick any one of them.

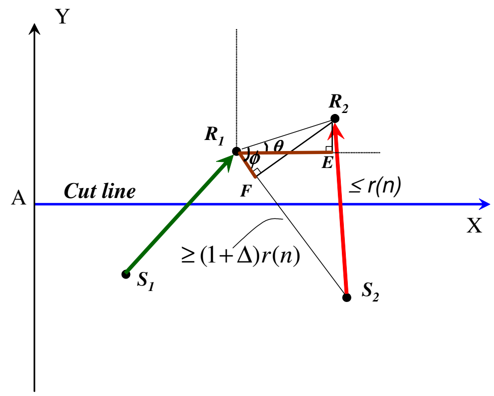

Fig. 9. Tighter bound for packing transmissions across a cut: Scenario 1

draw a line R_1E parallel to the X axis, from R_2 draw R_2E vertical to R_1E and intersect with it at E. Easy to see $|R_1E| = R_2(x) - R_1(x)$. Since $S_2(y) < R_1(y)$, $\theta < \phi$. Thus we get $|R_1E| > |R_1F|$, then $R_2(x) - R_1(x) > \Delta r(n)$ and $X_2 - X_1 > \Delta r(n)$.

Combining these two cases, we have shown $X_2 - X_1 > \Delta r(n)$ for the situation when $S_1(x) \leq R_1(x)$. Then, apply two transformations, both separately and simultaneously: mirror mapping the two transmissions by the cut line; reverse the sender receiver role of the two transmissions. We automatically show $X_2 - X_1 > \Delta r(n)$ for all situations of two senders at one side and two receivers at the other side.

Senario 2: The other scenario is when senders are on different sides. There are two cases for this scenario as well.

Case a): When we have $S_2(y) > R_1(y)$ as shown in Fig. 10, using the same triangle technique we can get $|R_1E| > |R_1F| > \Delta r(n)$ thus $X_2 - X_1 > \Delta r(n)$.

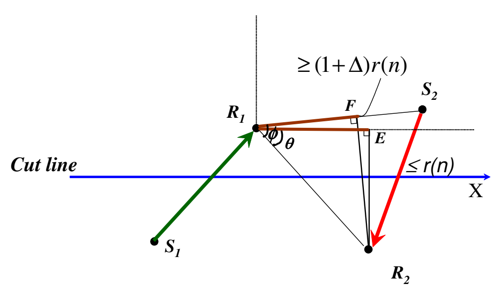

Fig. 10. Tighter bound for packing transmissions across a cut: Scenario 2 case a

Case b): Fig. 11 shows the other case when $S_2(y) \leq R_1(y)$. As it shows that now we have $S_2(x) - R_1(x) \geq \sqrt{\Delta^2 + 2\Delta} r(n) > \Delta r(n)$.

Also by the axis symmetry of the cut line and sender-receiver symmetry we show $X_2 - X_1 > \Delta r(n)$ for this scenario. Combining these two scenarios, and noting that S_1R_1 and S_2R_2 can be an adjacent pair of transmissions anywhere in the ordered list S_kR_k, $1 \leq k \leq m$, the Lemma is proved. ■

Lemma 6: The two way capacity of a cut Γ *for a* $2D$ *region has an upper bound of* $W(\frac{l_\Gamma}{\Delta r(n)} + 1)$

Proof: Applying Lemma 5 to the fact that $\sum_{1 \leq k \leq m}(X_k - X_{k-1}) \leq l_\Gamma$ yields this. ■

Theorem 13: The throughput of the coding scheme on a $2D$ *square random network is upper bounded by*

$$\lambda_C(n) \leq \frac{2W}{n}(\frac{1}{\Delta r(n)} + 1).$$

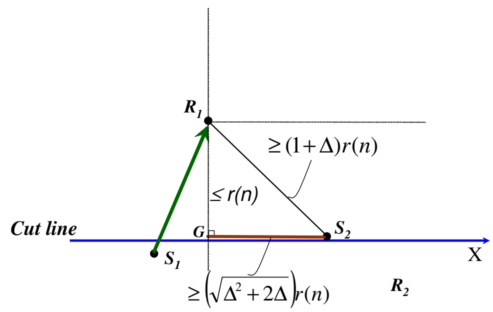

Fig. 11. Tighter bound for packing transmissions across a cut: Scenario 2 case b)

Proof: The proof is almost the same as Theorem 1, except that now we use the tighter bound for cut capacity in Lemma 6, and We prove this by showing that for any given constant $\epsilon > 0$ that is arbitrarily small, $\lambda_C(n) \leq \frac{2W}{n(1-\epsilon)}(\frac{1}{\Delta r(n)} + 1)$ for large n. ■

Theorem 14: The throughput of the flow scheme on a $2D$ square random network is lower bounded by

$$\lambda_F(n) \geq \frac{W}{c_{\Delta 4}\sqrt{\pi}(1+\Delta)nr(n)}$$

where $c_{\Delta 4} = \max\{2, \sqrt{\Delta^2 + 2\Delta}\}$

Proof: We prove this by showing that for any given constant $\epsilon > 0$ that is arbitrarily small, $\lambda_F(n) \geq \frac{W}{c_{\Delta 4}\sqrt{\pi}(1+\Delta)(1+\epsilon)nr(n)}$ for large n.

For the case of $\Delta < 1$, we choose a larger $r'(n)$ for the actual transmission radius according to the given $r(n)$ which already ensures connectivity as $r'(n) = \frac{\sqrt{\pi}}{\sqrt{\Delta^2+2\Delta}} r(n)$. Then divide the square region into grid cells with each cell as a square of size $\sqrt{\pi}r(n) \times \sqrt{\pi}r(n)$. Then $r(n)$ is a valid transmission radius, by the same union bound argument as in [13], w.h.p. each square cell contains at least one node.

Define a X axis as one direction of the square edge, Y as the other dimension. We construct a scheme that is a natural extension of the $1D$ scheme in Theorem 8. Set the transmission radius to $r'(n)$, route along cells that lies in a line parallel to either the X axis or Y axis. The scheduling and routing within one line is the same as the $1D$ case. To deal with the interference between lines, assign half of the time routing along lines in X, half along Y; also schedule the transmissions along the X lines or Y lines as every other line works simultaneously while the other half of lines keep silent. Each source routes its data along X axis first until it reaches a cell that is in the same Y line as the destination cell, then it starts to route along the Y line.

By Chernoff bound and union bound, the load among lines is asymptotically balanced(equally loaded), more specifically, w.h.p. the ratios of the load between lines is arbitrarily close to 1 for large n. Thus based on Theorem 8 we can evaluate the cut capacity usage level and derive the a lower bound as $\lambda_F(n) \geq \frac{W}{2\sqrt{\pi}(1+\Delta)(1+\epsilon)nr(n)}$ for large n.

For the case of $\Delta > 1$, we choose $r'(n) = \sqrt{\pi}r(n)$ and now schedule one line to transmit every $\sqrt{\Delta^2 + 2\Delta}$ lines instead of every two lines. Thus for this case a throughput of $\frac{W}{\sqrt{\Delta^2+2\Delta}\sqrt{\pi}(1+\Delta)(1+\epsilon)nr(n)}$ is achievable. Combining these two cases we prove the theorem. ■

Theorem 15: The throughput improvement of the flow scheme on a $2D$ square random network is upper bounded *by*

$$\alpha(n) \le 2c_{\Delta 4}\sqrt{\pi}\frac{(1+\Delta)}{\Delta}$$

for large n, where $c_{\Delta 4} = \max\{2, \sqrt{\Delta^2 + 2\Delta}\}$

Proof: From Theorem 13 and Theorem 14 we get this result. ■

IV. Physical Model

In this section we extend our constant benefit ratio results for the protocol model to the physical model. Denote the coding scheme throughput and flow scheme throughput under the physical model as $\lambda_C^p(n)$ and $\lambda_F^p(n)$ correspondingly.

A. $1D$ case:

Observation 3: Under the physical model, for any cut in the $1D$ line, there can be at most one transmission across the cut (including both directions) at any given point in time.

Theorem 16: The coding scheme throughput of the $1D$ random network under the physical model is upper bounded as:

$$\lambda_C^p(n) \le \Theta\left(\frac{W}{n}\right)$$

Proof: As shown in [1], (1) implies that

$$r_{k,j} \ge (1+\Delta)r_{i,j} \tag{2}$$

for all $k \in K$, $\Delta = \beta^{\frac{1}{\gamma}} - 1$. Since $\beta > 1$, we always have $\Delta > 0$. Also Observation 3 still holds, thus, our previous result for the $1D$ throughput upper bound holds. ■

Theorem 17: For any wireless medium with $\gamma > 1$, the flow scheme throughput of the $1D$ random network under the physical model is lower bounded as:

$$\lambda_F^p(n) \ge \Theta\left(\frac{W}{n}\right)$$

Proof: Normally wireless mediums satisfy $\gamma > 1$, then $\Sigma_{i=1}^{\infty}\frac{1}{i^{\gamma}}$ converges to some constant c. The binning technique that we used for the protocol model also works for the physical model, and we just need to schedule the bin transmissions so as to guarantee the signal to noise ratio. This can be done by the following: first use the previous protocol model schedule for a mapped protocol model from the physical model with $\Delta = \beta^{\frac{1}{\gamma}} - 1$, and then make this schedule a constant factor -$c^{\frac{1}{\gamma}}$- sparser. This guarantees the SNR with a constant factor lower throughput but achieves the same order throughput as the upper bound. ■

Theorem 18: The throughput benefit ratio of coding schemes for the $1D$ random network under the physical model is a constant factor.

$$\alpha(n) = \Theta(1) \text{ and } \lambda_F^p(n) = \lambda_C^p(n) = \Theta(W/n)$$

Proof: This follows from Theorems 16 and 17.[4] ■

[4]Implicitly, we also use the fact that $\lambda_C^p(n) \ge \lambda_F^p(n)$ because any flow scheme is also a coding scheme in the trivial sense.

B. 2D case:

Before we derive the $2D$ throughput bounds under the physical model, we show some geometric properties that the transmissions across a cut under the physical model need to satisfy.

Under the physical model, Lemma 4 is only partially true: receivers could lie in a line vertical to the cut line. However, the no crossing property for any two transmissions is still valid, and two senders of transmissions across the cut in one direction cannot be in a line vertical to the cut line for the physical model. We have the following lemma, which is actually also true under the protocol model.

Lemma 7: Under both the physical and the protocol model, for any two transmissions across a cut, $S_1 \to R_1$ and $S_2 \to R_2$ [5], the line segments S_1R_1 and S_2R_2 have no intersection point and S_1S_2 can not be vertical to the cut line.

Proof: The proof is similar to the proof of Lemma 4. Now we connect S_1S_2, draw the perpendicular bisector h of S_1S_2. For any communication model, protocol or physical, $|S_1R_1| < |S_2R_1|, |S_2R_2| < |S_1R_2|$ is always true. Thus S_1 and R_1 lie on one side of h and S_2 and R_2 lie on the opposite side. So there can never be an intersecting point between S_1R_1 and S_2R_2[6]. Also if S_1S_2 is vertical to the cut line, h is parallel to the cut line. Then one of the transmissions could never cross the cut, because if both cross the cut line, one of them has to cross h as well which is impossible from above. Thus this is impossible for any two transmissions across a cut. ■

Theorem 19: The coding throughput of a $2D$ random network under the physical model is upper bounded by

$$\lambda_C^p(n) \leq \Theta\left(\frac{W}{\sqrt{n}}\right)$$

for large n.

Proof: We show that the maximum number of simultaneous transmissions across a sparsity cut in one direction under the physical model is upper bounded by $\Theta(\sqrt{n})$, and then use the same argument as in Theorem 13 to derive the upper bound.

Under the physical model, nodes could transmit with any hop distance r so long as the signal to noise ratio is satisfied.

We order all the transmissions across the cut in one direction in the same way as in Lemma 5. Now, senders are on one side since we focus on the one way capacity (crossing the cut in one direction). By Lemma 7, we can argue in a similar way as Lemma 6 and apply (2), we have $S_j(x) - S_{j-1}(x) \geq \Delta \min\{r_j, r_{j-1}\}$ for $j = 2, \ldots, m$ where m is the total number of transmissions across the cut in that direction and $r_j = |S_jR_j|$. The sparsity cut line has unit length; thus, we have

$$\Sigma_{j=2}^{m} \min\{r_j, r_{j-1}\} \leq \frac{1}{\Delta}. \tag{3}$$

Consider a band region of size $\frac{2}{\sqrt{n}} \times |AB|$ with the cut line in the center; by the Chernoff bound, we know that w.h.p. there are less than or equal to $3\sqrt{n}$ nodes in this band region; thus, there are at most $3\sqrt{n}$ transmissions with a radius less than $\frac{1}{\sqrt{n}}$ across the cut. Then there are at least $m - 9\sqrt{n}$ transmissions crossing the cut such that any one of them has $S_{j'}R_{j'}$ satisfying $\min\{r_{j'}, r_{j'-1}, r_{j'+1}\} \geq \frac{1}{\sqrt{n}}$. Then, by (3), we have

$$(m - 9\sqrt{n})\frac{1}{\sqrt{n}} \leq \frac{1}{\Delta} \Rightarrow m \leq \Theta(\sqrt{n}).$$

[5]Note that each transmission could have multiple receivers, we just pick any one of them.

[6]Thus this is in general true for any two transmissions, not necessarily two across a cut.

Thus we obtain an order upper bound for the sparsity cut capacity, and then derive an upper bound for the coding throughput in the same way as we did for the protocol model in Theorem 13. ■

Theorem 20: Franceschetti et al. [2] The flow throughput of a $2D$ *physical random network is lower bounded by*

$$\lambda_F^p(n) \geq \Theta\left(\frac{W}{\sqrt{n}}\right)$$

for large n*.*

Theorem 21: The throughput benefit ratio of a $2D$ *physical random network is a constant, and* $\lambda_F^p(n) = \lambda_C^p(n) = \Theta(W/\sqrt{n})$.

Proof: This follows from Theorem 19 and Theorem 20. ■

In summary, we have extended the constant throughput improvement of coding schemes over flow scheme to physical communication models.

V. Bounds on gains in energy consumption

In this section, we evaluate the coding benefit from a different perspective, and ask the question of how much the coding gain is in terms of the total energy consumption. We assume the COPE [10] type of coding is applied. More precisely, in this section we consider the type of coding where the coded bits that each node sends out can and will be decoded before recoding at the next hop, i.e., a node will not send out some information from other nodes that it can not decode, refer to this type of coding as *point to point coding*. We assume packets are decoded at the receiver side based on all its possessed information which includes all the overhearing information.

We define the coding gain as the ratio of minimum total energy cost of the flow scheme to that of the coding scheme. Since nodes are uniformly randomly distributed, the expected per hop distance is uniformly the same across the deployed region. We assume transmissions use the same power then the total energy cost is the total number of transmissions multiplied by a constant depending on the transmission power but is the same for both coding and flow scheme, so the coding gain equals the ratio of the total number of transmissions of the flow scheme to the coding scheme[7]. In [10], the coding gain is bounded for the case of no opportunistic listening is allowed, we now derive bounds for the general case when opportunistic listening is allowed, which is the major benefit that the opportunistic coding relies on.

Before presenting the result, we denote the average one hop transmission distance as $r'(n)$, which is less than the overhearing range $r(n)$. Thus, $r'(n)$ is the employed one hop range while $r(n)$ characterizes the real capable range of the signal transmitting power. In other words, even though nodes may choose a smaller distance for one hop, they could potentially use a stronger transmission power, i.e. larger $r(n)$ than $r'(n)$, to increase opportunities for overhearing. We define the *degree* of a coded packet as the number of original source packets it combines (coded into one packet)[8], this coded packet aids the decoding of the original packets at their corresponding next hop receivers.[9]

Lemma 8: The coding gain is lower bounded by 2.

Proof: A coding gain of 2 is achievable even without overhearing. [10] shows that when traffic is going in opposite directions along a path, the number of transmissions can be cut in half. Here we argue that w.h.p. there

[7]This is the definition of coding gain in [10].

[8]This is defined in the context of point to point coding, the non-coding packet has a degree of 1.

[9]Receivers may not decode only based on this coded packet but can rely on all history data.

exists a scheme that reduces the number of transmissions by half. First, the optimal flow scheme uses shortest path routing (in terms of energy cost); second, because the nodes are uniformly random distributed, by the Chernoff bound the traffic load ratio will be asymptotically one between the two opposite directions of any path. Thus we can reduce the total number of transmissions of the whole network by one-half. ■

For upper bounds, we first analyze the case when $r(n) = r'(n)$.

Lemma 9: The coding gain is upper bounded by 2 when $r(n) = r'(n)$.

Proof: For any transmission of the coding scheme from S to R of degree $d > 2$, by the definition of point to point coding, the information sources combined in this packet all come from points that lie on a circle of radius $r'(n)$ centered at S. Since $r(n) = r'(n)$, the original flow is uniformly distributed, and packets are still following the shortest path route of the flow scheme; on average, R can overhear at most $\frac{d}{3}$ packets of information of the sources coded in $S \to R$. Since the unicast sources are mutually independent, in order to decode this degree d packet at R, it has to receive at least $\frac{2d}{3}$ packets sent from S. Thus, to deliver the original d packets outgoing from S, it has to send at least $\frac{2d}{3}$ packets, leading to a coding gain of at most 1.5 for $d > 2$. Thus, the $d = 2$ coding gain of 2 is the upper bound. ■

Next, we study the case when the overhearing range is larger than the actual one hop distance. We show that the overhearing gain can not compensate much of the loss due to higher transmission power. We also assume a node can decode everything it overhears, since we expect in practice that if a node can decode in this random environment, it must be that the distributed random coding is designed to guarantee that the expected number of overheard bits exceeds or equals the expected entropy of overheard sources.

Lemma 10: The coding gain is upper bounded by 3 for $r(n) > r'(n)$, and 2 when $r(n) \gg r'(n)$.

Proof: Let the total number of sources that is overheard (in any coded format) by a node B be x. We draw a circle of radius $r(n)$ centered at B. Denote the average intersection length between the circle and the path for any of the x sources as $\bar{l}$. Then by point to point coding, on average, each of the x sources is involved in $\bar{l}/r'(n)$ transmissions overheard by B. Then the total number of original flow transmissions that are overheard by B is $x\bar{l}/r'(n)$. Let the average degree of the coding scheme's transmissions be $\bar{d}$, since one of the x sources is from B itself, in order for B to decode all of the x sources, we need

$$\frac{x\bar{l}/r'(n)}{\bar{d}} \geq x - 1.$$

Let $\bar{l} = qr(n)$. This means $\bar{d} \leq q\frac{xr(n)}{(x-1)r'(n)}$. Since $\bar{d}$ is also the expected ratio of the number of original flow transmissions to the number of coding transmissions, with consideration of the energy efficiency loss because of using small $r'(n)$ to go $r(n)/r'(n)$ multiple hops when the nodes can actually make one hop with the same power, the overall coding gain for all the transmissions overheard by B is upper bounded by $\frac{r'(n)}{r(n)}\bar{d} \leq q\frac{x}{x-1} \leq 2\frac{x}{x-1}$. [10] Thus, when $x \geq 3$, the coding gain is upper bounded by 3; since we also have $\bar{d} \leq x + 1$ at the same time, so when $x \leq 2$, the coding gain is upper bounded by $\frac{r'(n)}{r(n)}\bar{d} \leq x + 1 \leq 3$ as well. When $r(n) \gg r'(n)$, $x \gg 1$, then the coding gain is bounded by $q < 2$. Since this is true for any node picked in the network and this is a uniform sampling for the flow transmissions, we know on average the coding gain for the whole network is upper bounded by 3. ■

[10] The latter inequality is because $q < 2$ since each source traverses the network along a shortest path.

Theorem 22: The coding gain in terms of total energy cost in a random network in kD Euclidean space ($k = 1, 2, 3, \dots$) is upper bounded by 3.

Proof: This follows from Lemma 10 and 22. ■

In fact, If the $2D$ deployed region is a sphere surface, then $q = 4/\pi$, the coding gain then has an tighter upper bound as $4/\pi + 1$. If we do not assume the sphere surface or other torus type surface, then we have to deal with the boundary effect. Even in this case, q can be tighter bounded by $\sqrt{3}$, thus a tighter bound for coding gain as $\sqrt{3} + 1$. Certainly, for any situation, including other dimensional spaces, $q < 2$ is always right, and thus 3 is a general bound of coding gain for any dimension and any type of regions.

VI. Discussions

Our work, combined with the previous work of [4] on the throughput benefit of network coding for wired undirected networks, seems to be not positive about network coding's utility for improving network capacity, while we believe this does not damage network coding's overall benefit to improve the performance of information networks in general. In the future, to improve the performance of any information networks, we still should check if network coding will help. From previous successful/unsuccessful examples of network coding and our work, we believe that what we learned is not that network coding is not helpful but rather better guidance on where network coding's true advantage lies. In fact, we think the key benefit of network coding lies in its ability to blur the information's identities. So where there is a need for information blurring, network coding can have much utility. For example, communication in lossy, unstable, dynamic environments (e.g. Delay/Disruption Tolerant Networks (DTN)), distributed storage/recoveries in disaster, fault-tolerant situations (e.g. growth codes [16]) etc. For these scenarios, network coding's ability to blur the information identities can help to balance out the risks and redistribute them uniformly across all packets, which could benefit the delay, reliability and robustness of the system. Also, the benefit factor is often in the order of $\Theta(\log n)$ because of the coupon collector effect. If we think about the capacity benefit of network coding on directed graphs, the benefit also essentially comes from its ability to blur the information identity. For our case of bidirectional wireless networks modelled as using lossless communication channels without a random unreliable factor, even though network coding is able to blur the information identities, the overall information content cannot be compressed with network coding and the capacity gain for this case is thus limited.

The techniques that we used to bound the throughput benefit ratio, is to match the order of the upper bound of coding throughput with a lower bound of the flow scheme throughput. We derive the upper bound for coding throughput by analyzing the information flow across the sparsity cut of the network, bounding the maximum number of transmissions from the receiving constraint and bounding the throughput by the coding constraint.

The well known Li&Li conjecture [4] on the benefit of network coding for undirected pint to point link networks (no broadcasting) is related to our problem, it conjectures that network coding provides no throughput benefit for such networks. There has been no counter examples found yet, people tend to believe it is true and there has been some approach trying to prove it [17] but until today we only know it can be reduced to a long-standing open question in Input/Output (I/O) complexity [18]. Also inspired by the coding scheme proposed in Katti etc.'s paper [8], we conjecture that the only possible network coding scheme that could improve the throughput for wireless networks is to do *point-by-point coding*, packets are decoded at the receiver side based on all the overhearing. Proving this may involved solving the hardcore of Li&Li's conjecture, but assuming this is true, it is possible that we can get tighter constant upper bound for the throughput benefit ratio based on point-by-point coding.

VII. Conclusion and future work

We study the benefit of network coding & broadcasting on the many to many throughput of wireless networks under the framework proposed by Gupta and Kumar [1]. We show that the benefit is upper bounded by a constant both for the protocol model and the physical model. Further, we develop tight bounds for these constants. We also show that the energy saving factor in such unicast random networks is upper bounded by 2.

The tighter constant bound that we find for the throughput benefit ratio $\alpha(n)$ of the $2D$ random network is still loose, and we suspect that the constant factor is even smaller; specifically, we conjecture it is 2, and proving it even partially may involve solving the well known Li&Li conjecture [4], which is still open. Thus, part of the future work is to develop tighter constant bounds for the coding throughput and flow throughput. The future work also includes studying possibilities of improving the throughput with other forms of coordination among the wireless nodes and studying the impact of network coding on delay and buffer size.

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