Bounds on the labelling numbers of chordal graphs

Peter Che Bor Lam[1], Guohua Gu[2], Wensong Lin[2] and Ping-Tsai Chung[3]

1. Department of mathematics, Hong Kong Baptist University, Hong Kong, P. R. China.
2. Department of Mathematics, Southeast University, Nanjing, P. R. China
3.Department of Computer Science, Long Island University, Brookville, New York, USA.

Abstract

Motivated by the conjecture on the $L(2,1)$-labelling number $\lambda(G)$ of a graph G by Griggs and Yeh [2] and the question: "Is the upper bound $(\Delta+3)^2/4$ for $\lambda(G)$ for chordal graphs with maximum degree Δ is sharp?", posed by Sakai [3], we study the bounds for $\lambda(G)$ for chordal graphs in this paper. Let G be a chordal graph on n vertices with maximum degree Δ and maximum clique number ω. We improve the upper bound $(\Delta+3)^2/4$ on $\lambda(G)$ and the upper bound $(\Delta+2d-1)^2/4$ on $\lambda_d(G)$ with $d \geq 2$, answering question of Sakai and improving results of Chang et al. Finally, we study the labelling numbers of r-power paths P_n^r on n vertices. We obtain $\lambda_d(P_n^r)$ for small integers $d \geq 2$ and $r \geq 2$, and give a better bound of $\lambda_d(P_n^r)$ for large integers d and r.

Key words. channel assignment problem, distance two labelling, chordal graph

1. Introduction

The problem of labelling vertices of a graph with a condition at distance two is a variation of the channel assignment problem introduced by Hale and Roberts, respectively in [6] and [7], where "close" transmitters receive different channels and "very close" transmitters must receive channels at least two apart. This problem was first introduced and formulated as a graph labelling problem by Griggs and Yeh [2].

Given a graph G with vertex set V and edge set E, for any $u, v \in V$, let $d_G(u,v)$ denote the distance (the length of a shortest path) between u and v in G. An $L(2,1)$ - *labelling* f is an integer assignment $f: V \to \{0,1,2,\dots\}$ such that if $uv \in E$, then $|f(u)-f(v)| \geq 2$; and if $d_G(u,v)=2$, then $|f(u)-f(v)| \geq 1$. The number assigned to each vertex under f will be called f-labels, or simply *labels*. The *span* of an $L(2,1)$-labelling f, denoted span(f), is the absolute difference between the maximum and minimum labels. Clearly, we may assume the smallest and the largest label of each labelling f of G is 0 and span(f) respectively. The $L(2,1)$-labelling number, $\lambda(G)$, is the minimum of span(f) over all $L(2,1)$-labellings of G. In [5], the authors considered a generalization of $L(2,1)$-labelling, namely, $L(d,1)$-labelling of graphs. For a positive integer d, an $L(d,1)$-labelling of G is an integer assignment $f: V \to \{0,1,2,\dots\}$ such that if $uv \in E$, then $|f(u)-f(v)| \geq d$; and if $d_G(u,v)=2$, then $|f(u)-f(v)| \geq 1$. The $L(d,1)$-labelling number of G, $\lambda_d(G)$, is the minimum of span(f) over all $L(d,1)$-labellings of G. Clearly, any complete graph of n vertices has the $L(d,1)$-labelling number $(n-1)d$.

The $L(2,1)$-labelling and the L(d,1)-labelling of graphs have been extensively studied in the past decade ([2]). For any graph G, we shall denote by $\Delta(G)$, or simply by Δ, its maximum degree. Griggs and Yeh [2] proved that $\lambda(G) \leq \Delta^2+2\Delta$ and conjectured that $\lambda(G) \leq \Delta^2$; Chang and Kuo [4] improved this bound to $\Delta^2+\Delta$. Chang et al [5] proved that $\lambda_d(G) \leq \Delta^2+(d-1)\Delta$ for any integer $d \geq 2$. In [3], Sakai investigated the $L(2,1)$-labelling of chordal graphs and unit interval graphs and proved that $\lambda(G) \leq (\Delta(G)+3)^2/4$ for any chordal graph G and asked whether this upper bound is sharp.

In Section 2, we improved the above upper bound. By similar arguments, we also improve the upper bound $(2d+\Delta-1)^2/4$ for $\lambda_d(G)$ of any chordal graph G and any integer $d \geq 2$, presented by Chang et al [5]. In Section 3, we determine $\lambda(G)$ and give an upper bound of $\lambda_d(G)$, where G are r-power paths for any positive integers $d, r \geq 2$. Also, we correct a result on power paths by Chang et al [5].

2. General Chordal Graphs

In this paper, all graphs are finite, simple and undirected. For undefined terms and concepts in graph theory, we refer to [1]. Let $G=(V,E)$ be a graph with vertex set V and edge set E. The number of vertices, denoted by $n=|V|$, is called the *order* of G. Two vertices $u, v \in V$ are said to be *adjacent* if edge $uv \in E$. A vertex u is a *neighbor* of vertex v if u is adjacent to v. For any set $S \subseteq V$, $|S|$ denotes the number of vertices of S and $G[S]$ denote the subgraph of G induced by S. Also, $G-S$ and $G-v$ denote the induced subgraph $G[V \setminus S]$ and $G[V \setminus \{v\}]$ respectively. A *clique* of order k , denoted by K_k, is a complete subgraph induced by k vertices. We use $P_k=[v_1,\dots,v_k]$ to denote a path of length $k-1$ from v_1 to v_k; and $C_k=[v_1,\dots,v_k,v_1]$ a cycle of length k in G. A *chord* of cycle C_k is an edge joining two non-consecutive vertices on the cycle C_k. A subset $S \subset V$ is a *separator* or *vertex cutset* of G if there are two vertices x and y in the same component of G such that they are in two distinct components of $G-S$. We say S separates G and S is an *xy-separator*. The set S is a *minimal separator* of G if S is a separator and no proper subset of S separates G. Likewise, S is a *minimal xy-separator* if S is an xy-separator and no proper subset of S separates x and y into distinct components.

A graph is chordal if every cycle of length greater than three has a chord. Clearly, any induced subgraph of a chordal graph is also chordal. A *simplicial vertex* of a graph G is a vertex such that its neighbors induce a clique in G. A *perfect elimination ordering* of G on n vertices is a vertex ordering $v_1, v_2, \dots, v_n$ of G such that for $1 \leq i \leq n$, the vertex v_i is a simplicial vertex of the subgraph induced by $\{v_i, v_{i+1}, \dots, v_n\}$ in G. Dirac proved that every chordal graph has a simplicial vertex and characterized chordal graphs by minimal vertex separators, as shown below in Lemma 2.1 and Theorem 2.2.

Lemma 2.1 [8,9] Every chordal graph G has a simplicial vertex. If G is not complete, then it has two nonadjacent simplicial vertices.

Theorem 2.2 [8] A graph G is chordal if and if every minimal separator of G induces a complete subgraph in G.

The following lemma is an useful fact that for any connected chordal graph G, the induced subgraph of G obtained by removing some simplicial vertices from the graph remains connected and chordal.

Lemma 2.3 Let G be a connected chordal graph and let S be a set of simplicial vertices of G. Then $G-S$ is also a connected chordal graph.

Proof. Clearly, $G-S$ is chordal. If $G-S$ is not connected, then S contains a minimal xy-separator S' for some vertices x and y. By Theorem 2.2, $G[S']$ is complete in G. Let G_x and G_y be the connected of $G-S'$ containing x and y, respectively. Since S' is minimal, each vertex $v \in S'$ is adjacent to some vertex in G_x and some vertex in G_y; otherwise $S' \setminus \{v\}$

would be an xy-separator, contrary to the minimality of S'. Then each vertex in S' can not be simplicial, a contradiction. Hence $G - S$ is connected. $\square$

We now consider the labellings of chordal graphs. Sakai obtained [3] an upper bound of $\lambda(G) \le (\Delta+3)^2/4$ for any chordal graph G and asked whether this upper bound is sharp. Theorem 2.5 below answers her question in the negative. We shall need the following proposition by Griggs and Yeh [2].

Proposition 2.4 Suppose P_n denotes a path on n vertices. Then $\lambda(P_2) = 2$, $\lambda(P_3) = \lambda(P_4) = 3$ and $\lambda(P_n) = 4$ for all $n \ge 5$.

Theorem 2.5 If G is a chordal graph, then $\lambda(G) \le \lfloor (\Delta+3)^2/4 \rfloor - 1$.

Proof. We may assume without loss of generality that G is connected and chordal. The theorem clear holds if G is complete or if $\Delta \le 2$, so we may also assume that G is not complete and $\Delta \ge 3$. We proceed by induction on the number of vertices n of G.

The case where $n = 1$ is trivial. Suppose that the theorem holds for all chordal graphs with fewer than n vertices, where $n > 1$. By Lemma 2.1, G must have two nonadjacent simplicial vertices. Moreover, for any simplicial vertex v of G, $G_v) = G \setminus \{v\}$ is a chordal graph with fewer vertices than G; hence by induction $\lambda(G_v) \le \lfloor (\Delta+3)^2/4 \rfloor - 1$. Suppose that $\lambda(G) \ge \lfloor (\Delta+3)^2/4 \rfloor$. With a series of claims about simplicial vertices of G, we shall reach a contradiction.

Let $d(v) = k$ and $v_1, v_2, \ldots, v_k$ its neighbors in G. Then, v, together with vertices v_i's, form a maximal clique $K_{k+1}(v)$. For $i = 1, 2, \ldots, k$, we let $M(v_i)$ be the set of vertices adjacent to v_i but not in $K_{k+1}(v)$. Let f be a labelling of G_v with $\mathrm{span}(f) = \lfloor (\Delta+3)^2/4 \rfloor - 1$.

Claim 1 $k = \lfloor (\Delta+3)/2 \rfloor$ or $k = \lfloor (\Delta+3)/2 \rfloor + 1$.

Proof. Note that v is distance one away from k vertices v_i, $i = 1, 2, \ldots, k$ and distance two away from neighbors of each vertex in $M(v_i)$, $i = 1, 2, \ldots, k$. Therefore, when we try to label v with the numbers in $\{0, 1, \ldots, \lfloor (\Delta+3)^2/4 \rfloor - 1\}$, there are at most $3k + k(\Delta - k)$ numbers used by f to be avoided. However, the quadratic function $3k + k(\Delta - k)$ attends its maximum $(\Delta+3)^2/4$ when $k = (\Delta+3)/2$. Therefore, if $k \ne \lfloor (\Delta+3)/2 \rfloor$ and $k \ne \lfloor (\Delta+3)/2 \rfloor + 1$, then $3k + k(\Delta - k) \le \lfloor (\Delta+3)^2/4 \rfloor - 1$, and there is at least one number in $\{0, 1, \ldots, \lfloor (\Delta+3)^2/4 \rfloor - 1\}$ to be assigned to v, contrary to the assumption on G. This completes the proof of Claim 1. $\square$

Claim 2 For $1 \le i \le k$, $d(v_i) = \Delta$, and for $i \ne j$, $M(v_i) \cap M(v_j) = \emptyset$. Also $|M(v_i)| = \Delta - k$ for $i = 1, 2, \ldots, k$.

Proof. If any of Claim 2 does not hold, then the simplicial vertex v is distance two away from at most $k(\Delta - k) - 1$ vertices. By the similar arguments in Claim 1, we can label vertex v with a number in $\{0, 1, \ldots, \lfloor (\Delta+3)^2/4 \rfloor - 1\}$, contradicting the assumption on G. So Claim 2 holds. $\square$

Claim 3 Each v_i, $i = 1, 2, \ldots, k$ is a minimal separator of G.

Proof. Without loss of generality, suppose that G_{v_1} is connected. By Claim 1, $k \ge 3$. Since G is not complete, so by Claim 2, $M(v_i)$ is non-empty for $i = 1, 2, \ldots, k$. Take a vertex x in $M(v_1)$, and then consider some shortest path $P_{r+1} = [x = x_0, x_1, \ldots, x_r = v_2]$ from x to v_2 in G_{v_1}. By Claim 2, $r \ge 2$. Adding two edges v_1v_2 and v_1x to P_{r+1}, we get a cycle $C_{r+2} = [v_1, x_0, x_1, \ldots, x_r, v_1]$ in G. Since G is chordal and C_{r+2} is a cycle of length greater than three, C_{r+2} must have a chord. But the edge v_1v_{r-1} is only one possible chord in C_{r+2}, which is contrary to Claim 2. Thus Claim 3 follows. $\square$

Claim 4 If u and v are two distinct simplicial vertices of G, then $d(u, v) \geq 3$.

Proof. Let v_1, v_2 be two distinct simplicial vertices of G with $d(v_1) = l_1$ and $d(v_2) = l_2$. Also let $K(v_1)$ and $K(v_2)$ be two maximal cliques containing v_1 and v_2, respectively. By Claim 3, each neighbor of a simplicial vertex of G is not simplicial, so v_1 and v_2 are not adjacent. If v_1 and v_2 are at distance two, then there is a vertex u adjacent to each of them. By Claim 1, $l_i > \Delta/2$, $i = 1, 2$. By Claim 3, u is a minimal separator and so $K(v_1) \cap K(v_2) = \{u\}$. Thus $d(u) \geq l_1 + l_2 > \Delta$, a contradiction. □

Let S be the set of all simplicial vertices of G. By Claim 4, the induced subgraph $G - S$ is nonempty. By Lemma 2.3, $G - S$ is a connected and chordal graph. Let x be a simplicial vertex of $G - S$. If x is not adjacent to any vertex in S, then x is simplicial in G and we have a contradiction. So by Claim 4 again, there is one and only one simplicial vertex v in S such that v is adjacent to x in G. Suppose $d(v) = k$. Then $x = v_i$ for some i, $1 \leq i \leq k$. Clearly, $\{v_1, v_2, \ldots, v_k\} \cup (\cup_{i=1}^{k} M(v_i)) \subseteq V(G - S)$. Using Claims 1, 2 and 3, we can show that x cannot be a simplicial vertex of $G - S$, a contradiction. This completes the proof of Theorem 2. □

Based on the same arguments in the proof of Theorem 2.5 above, we have the following theorem about the upper bound of the $L(d, 1)$-labelling number of chordal graphs, improving the results by Chang et al [5].

Theorem 2.6 Suppose $d \geq 2$ is a positive integer and G is a chordal graph. Then $\lambda_d(G) \leq \lfloor (\Delta + 2d - 1)^2/4 \rfloor - 1$.

We are unable to find any example in which the upper bound in Theorem 2.5 is attained. By Proposition 2.4, $\lambda(P_n) = 4 = \lfloor (\Delta + 3)^2/4 \rfloor - 2$ for paths on $n \geq 5$ vertices. So we think that the upper bound of $(\Delta + 3)^2/4$ might be decreased only to $\lfloor (\Delta + 3)^2/4 \rfloor - 2$ for $\lambda(G)$ for a general chordal graph G.

3. Power Paths

In this section, we consider a kind of special chordal graphs. Let $r \geq 1$ be an integer. The *r-power path* on n vertices, denoted by P_n^r, is the graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$ and the edge set $\{v_i v_j \colon 1 \leq |i - j| \leq r\}$. A 1-path on n vertices is an ordinary path denoted as P_n in the first paragraph of Section 2.

In [5], the authors studied the $L(d, 1)$-labelling of the r-power paths and claimed that for two integers $r \geq 2$ and $d \geq 2$,

$$\lambda_d(P_n^r) = \begin{cases} (n-1)d, & n \leq r+1, \\ rd+1, & r+2 \leq n \leq 2r+2, \\ rd+2, & n \geq 2r+3. \end{cases}$$

There is a mistake in the proof of the above claim when $n \geq 2r + 3$. It is straight-forward to check that $\lambda_3(P_{10}^2)$ is 9 not 8, and so $\lambda_3(P_n^2)$ is at least 9 for $n \geq 10$. Thus the 2-power path P_n^2 for $n \geq 10$ is a counterexample to their claim for the case $n \geq 2r + 3$.

In this section, we also study the $L(d, 1)$-labelling of the r-power paths. We try to determine all $L(d, 1)$-labelling numbers of P_n^r, but we find that it seems difficult to compute the exact value of $\lambda_d(P_n^r)$ for large integers r and d. First, we have the following theorem which gives a upper bound for $\lambda_d(P_n^r)$ for all integers $r \geq 2$, $d \geq 2$ and $n \geq 1$.

Theorem 3.1 Let $r \geq 2$ and $d \geq 2$ be two integers. If $l = \min\{d-1, r\}$, then

$$\lambda_d(P_n^r) \leq \begin{cases} rd + i, & i(r+1) < n \leq (i+1)(r+1),\ 0 \leq i \leq l, \\ rd + l + 1, & n > (l+1)(r+1). \end{cases}$$

Proof. Suppose $i(r+1) < n \leq (i+1)(r+1),\ 0 \leq i \leq l$. We have the following $L(d,1)$-labelling for $P^r_{(i+1)(r+1)}$:

$$i, d+i, 2d+i, \ldots, rd+i, i-1, d+(i-1), \ldots, rd+(i-1), \ldots, 0, d, 2d, \ldots, rd.$$

Therefore $\lambda_d(P_n^r) \leq \lambda_d(P^r_{(i+1)(r+1)}) \leq rd+i$.

Suppose $n > (l+1)(r+1)$. If we label the vertices of P_n^r one by one with elments from the cyclic sequence:

$$0, d, 2d, \ldots, rd, (r+1)d, 1, d+1, 2d+1, \ldots, rd+1, 0, d, 2d, \ldots, rd, (r+1)d, 1, \cdots,$$

until all vertices of P_n^r have been labelled, we get an $L(d,1)$-labelling of P_n^r with span $rd+d$.

If instead we use the cyclic sequence:

$$0, d+1, 2d+2, \ldots, rd+r, 1, d+2, 2d+3, \ldots, rd+r+1, 0, d+1, 2d+2, \cdots$$

we get an $L(d,1)$-labelling of P_n^r with span $rd+r+1$.

Therefore $\lambda_d(P_n^r) \leq \min\{rd+d, rd+r+1\} = rd+l+1$. $\square$

The following simple lemma is useful to compute the $L(d,1)$-labelling numbers of r-power paths.

Lemma 3.2 Let d, r, s be three integers with $d, r \geq 2$ and $0 \leq s \leq d$. Let K_{r+1} be a clique of size $r+1$. If f is an $L(d,1)$-labelling of clique K_{r+1} with $\text{span}(f) = rd+s$, then there are $r+1$ integers $\varepsilon_i, i = 0, 1, 2, \ldots, r$, $0 \leq \varepsilon_0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_r \leq s$, such that $f(K_{r+1})$, the label set of K_{r+1}, is $\{id + \varepsilon_i \mid i = 0, 1, 2, \ldots, r\}$.

Proof. Suppose f is an $L(d,1)$-labelling of K_{r+1} with $\text{span}(f) = rd+s$. First, sort $r+1$ labels of $f(K_{r+1})$ in an ascending order as $0 \leq f_0 < f_1 < \ldots < f_r \leq rd+s$. Then order $r+1$ vertices of K_{r+1} as $v_0, v_1, v_2, \ldots, v_r$, such that $f(v_i) = f_i$, $i = 0, 1, 2, \ldots, r$.

We now can see that $id \leq f_i \leq id+d$, $i = 0, 1, 2, \ldots, r$. To the contrary, suppose that there is some i, $f_i < id$ or $f_i > id+d$. If $f_i < id$, then the clique of $i+1$ vertices induced by $\{v_0, v_1, \ldots, v_i\}$ would have the $L(d,1)$-labelling number less than id(since $f_i - f_0 < id$), a contradiction. Similarly, if $f_i > id+d$, then the clique of $r-i+1$ vertices induced by $\{v_i, v_{i+1}, \ldots, v_r\}$ would have the $L(d,1)$-labelling number less than $(r-i)d$(since $f_r - f_i < (r-i)d$), which is the same contradiction.

Thus for each i, $i = 0, 1, 2, \ldots, r$, there exists an integer ε_i, $0 \leq \varepsilon_i \leq s$, such that $f_i = id + \varepsilon_i$. If there is some index i such that $\varepsilon_i > \varepsilon_{i+1}$, then $f_{i+1} - f_i = (i+1)d + \varepsilon_{i+1} - (id + \varepsilon_i) = d + (\varepsilon_{i+1} - \varepsilon_i) < d$, contradicting that f is an $L(d,1)$-labelling of K_{r+1}. Therefore, $0 \leq \varepsilon_0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_r \leq s$, and then

$$f(K_{r+1}) = \{f_i \mid i = 0, 1, 2, \ldots, r\} = \{id + \varepsilon_i \mid i = 0, 1, 2, \ldots, r\}. \quad \square$$

Although we can not determine completely the values of $\lambda_d(P_n^r)$ for all $d, r \geq 2$ and n, with Lemma 3.2 , we prove the following weaker result.

Theorem 3.3 For integers $r \geq 2$ and $d \geq 2$, let $l = \min\{d-1, r\}$, then

$$\lambda_d(P_n^r) = \begin{cases} (n-1)d, & n \leq r+1, \\ rd+1, & r+1 < n \leq 2r+2, \\ rd+2, & 2r+2 < n \leq 3r+3, \end{cases}$$

and for $n > 3r+3$, $rd+2 \leq \lambda_d(P_n^r) \leq rd+l+1$.

Proof. If $n \leq r+1$, then P_n^r is a complete graph of n vertices. Thus $\lambda_d(P_n^r) = (n-1)d$.

If $r+1 < n \leq 2r+2$, then by Theorem 3.1, $\lambda_d(P_n^r) \leq rd+1$. Notice that every $r+1$ consecutive vertices induce a clique and any $k(r+1 \leq k \leq 2r+1)$ consecutive vertices induce an r-power subpath P_k^r with diameter two in P_n^r. Thus, if $\lambda_d(P_n^r) < rd+1$, then $\lambda_d(P_n^r) = rd$. Let $f$be an $L(d,1)$-labelling of P_n^r with span$(f) = rd$. Then by Lemma 3.2, every clique of size $r+1$ has an unique label set $\{id | i = 0, 1, 2, \ldots, r\}$. This implies that the label of the first vertex of P_n^r is the same as that of $(r+2)$th vertex of P_n^r, contradicting that f is an $L(d,1)$-labelling, since these two vertices are distance two away from each other. Therefore, $\lambda_d(P_n^r) = rd+1$.

If $2r+2 < n \leq 3r+3$, then by Theorem 3.1, $\lambda_d(P_n^r) \leq rd+2$. Let $V(P_n^r) = \{v_1, v_2, \ldots, v_n\}$. Since the r-power path P_{2r+2}^r induced by $\{v_1, v_2, \ldots, v_{2r+2}\}$ is a subgraph of P_n^r, then $\lambda_d(P_n^r) \geq \lambda_d(P_n 2r+2^r) = rd+1$. If $\lambda_d(P_n^r) < rd+2$, then $\lambda_d(P_n^r) = rd+1$. Let f be an $L(d,1)$-labelling of P_n^r with span $rd+1$. Then f is also an $L(d,1)$-labelling of P_{2r+2}^r. Let $V_1 = \{v_1, v_2, \ldots, v_{r+1}\}$ and $V_2 = \{v_{r+2}, v_{r+3}, \ldots, v_{2r+2}\}$. By Lemma 3.2,

$$f(V_1) = \{id + \varepsilon_i^1 \mid 0 \leq i \leq r, 0 \leq \varepsilon_0^1 \leq \varepsilon_1^1 \leq \ldots \leq \varepsilon_r^1 \leq 1\}$$

$$f(V_2) = \{id + \varepsilon_i^2 \mid 0 \leq i \leq r, 0 \leq \varepsilon_0^2 \leq \varepsilon_1^2 \leq \ldots \leq \varepsilon_r^2 \leq 1\}$$

If $\varepsilon_r^1 = \varepsilon_r^2$, then $\varepsilon_r^1 = \varepsilon_r^2 = 1$, otherwise $f(V_1) = f(V_2) = \{id \mid i = 0, 1, 2, \ldots, r\}$, a contradiction. Moreover, $f(v_1) = f(v_{2r+2}) = rd+1$. It follows that there is no label of the form $rd + \varepsilon_r$ in the label sets of any cliques formed by $r+1$ consecutive vertices except vertices v_1 and v_{2r+2} in P_{2r+2}^r, which contradicts Lemma 3.2. Hence $\varepsilon_r^1 \neq \varepsilon_r^2$. Similarly, $\varepsilon_0^1 \neq \varepsilon_0^2$. Then either $\varepsilon_0^1 = \varepsilon_r^1 = 0$ and $\varepsilon_0^2 = \varepsilon_r^2 = 1$, or $\varepsilon_0^1 = \varepsilon_r^1 = 1$ and $\varepsilon_0^2 = \varepsilon_r^2 = 0$. For the former case, $f(V_1) = \{id \mid i = 0, 1, 2, \ldots, r\}$ and $f(V_2) = \{id+1 \mid i = 0, 1, 2, \ldots, r\}$, and then by Lemma 3.2, it is no difficult to see that

$$f(v_i) = \begin{cases} (r+1-i)d, & i = 1, 2, \ldots, r+1, \\ (2r+2-i)d+1, & i = r+2, r+3, \ldots, 2r+2. \end{cases}$$

For the later case, $f(V_1) = \{id+1 \mid i = 0, 1, 2, \ldots, r\}$ and $f(V_2) = \{id \mid i = 0, 1, 2, \ldots, r\}$, and then by Lemma 3.2,

$$f(v_i) = \begin{cases} (i-1)d+1, & i = 1, 2, \ldots, r+1, \\ (r+2-i)d, & i = r+2, r+3, \ldots, 2r+2. \end{cases}$$

Therefore, for each case, there is no label in $\{0, 1, 2, \ldots, rd+1\}$, which can be used to label the vertex v_{2r+3}, and then we have reached a contradiction. Hence $\lambda_d(P_n^r) = rd+2$.

By Theorem 3.1 and the result in the previous paragraph, the last part of the theorem follows. □

Corollary 3.4 Let $r \geq 2$ be an integer, then

$$\lambda(P_n^r) = \begin{cases} 2(n-1), & n \leq r+1, \\ 2r+1, & r+1 < n \leq 2r+2, \\ 2r+2, & n > 2r+2. \end{cases}$$

Proof. In Theorem 3.3, we let $d = 2$. Then $l = 1$. For the case $n > 3r + 3$, $2r + 2 \leq \lambda(P_n^r) \leq 2r + l + 1 = 2r + 2$, thus $\lambda(P_n^r) = 2r + 2$. Hence the corollary holds. □

Corollary 3.5 Let $d \geq 3$ be an integer, then

$$\lambda_d(P_n^2) = \begin{cases} (n-1)d, & n \leq 3, \\ 2d+1, & 3 < n \leq 6, \\ 2d+2, & 6 < n \leq 9, \\ 2d+3, & n > 9. \end{cases}$$

Proof. In Theorem 3.3, let $r = 2$. Since $d \geq 3$, then $l = r = 2$. Thus it suffices to show that the fourth case of the corollary is true.

By Theorem 3.3, for $n > 9$, $2d+2 \leq \lambda_d(P_n^2) \leq 2d+l+1 = 2d+3$. Suppose $\lambda_d(P_n^2) < 2d+3$, then $\lambda_d(P_n^2) = 2d + 2$. Let f be an $L(d,1)$-labelling of P_n^2 with span $2d + 2$. Similar to the arguments in the proof of Theorem 3.3, we consider the subgraph P_9^2 induced by the first 9 vertices of P_n^2. Let $V(P_n^2) = \{v_1, v_2, \dots, v_n\}$, then $V(P_9^2) = \{v_1, v_2, \dots, v_9\}$. By Lemma 3.2, the label set of any clique formed by 3 consecutive vertices must be one of the sets $\{id + \varepsilon_i \mid i = 0, 1, 2, 0 \leq \varepsilon_0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 2\}$. A tedious analysis by cases shows that only eight $L(d,1)$-labellings of P_9^2 are possible.

$\{2d, d, 0, 2d+1, d+1, 1, 2d+2, d+2, 0\}$ $\{2d, d, 0, 2d+1, d+1, 1, 2d+2, d+2, 2\}$
$\{2d+2, d, 0, 2d+1, d+1, 1, 2d+2, d+2, 0\}$ $\{2d+2, d, 0, 2d+1, d+1, 1, 2d+2, d+2, 2\}$
$\{0, d+2, 2d+2, 1, d+1, 2d+1, 0, d, 2d\}$ $\{2, d+2, 2d+2, 1, d+1, 2d+1, 0, d, 2d\}$
$\{0, d+2, 2d+2, 1, d+1, 2d+1, 0, d, 2d+2\}$ $\}2, d+2, 2d+2, 1, d+1, 2d+1, 0, d, 2d+2\}$

But whenever one of the $L(d,1)$-labellings above occurs, we can not find a number in $\{0, 1, 2, \dots, 2d+2\}$ to label the vertex v_{10}, contradicting that f is an $L(d,1)$-labelling of P_n^2. Therefore, $\lambda_d(P_n^2) = 2d + 3$ for $n > 9$. □

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