TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 6, pp. 1575-1587, December 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

BOUNDS ON THE NUMBER OF VERTEX INDEPENDENT SETS IN A GRAPH

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Abstract. We consider the number of vertex independent sets $i(G)$. In general, the problem of determining the value of $i(G)$ is NP-complete. We present several upper and lower bounds for $i(G)$ in terms of order, size or independence number. We obtain improved bounds for $i(G)$ on restricted graph classes such as the bipartite graphs, unicyclic graphs, regular graphs and claw-free graphs.

1. NOTATION

We denote by G a graph of *order* $n = |V(G)|$ and *size* $m = |E(G)|$. For a vertex x in $V(G)$ let $\deg_G(x)$ denote its *degree*. A *leaf* is a vertex of degree one and a *stem* is a vertex adjacent to a leaf. P_n denotes a *path* on n vertices and C_n a *cycle* on n vertices. The *diameter* of a graph G is the maximum distance between two vertices in G. The *complement* of G is denoted by $\overline{G}$. The *complete* graph on n vertices is denoted by K_n, while $\overline{K_n}$ denotes the graph consisting of n isolated vertices. By $K_{1,n-1}$ we denote the *star* consisting of one center vertex adjacent to $n-1$ leaves. A *corona graph* G is a graph in which each vertex is a leaf or is a stem adjacent to exactly one leaf. If H is a graph, then $H \circ K_1$ denotes the corona graph constructed from H by attaching precisely one leaf at each vertex of H. A graph is called *unicyclic* if it is connected and contains exactly one cycle. The *Fibonacci numbers*, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34,... are defined recursively by $F(0) = 0, F(1) = 1$, and for $n \geq 2$, $F(n) = F(n-2) + F(n-1)$. The *Lucas numbers* are $L(n) = F(n-1) + F(n+1)$ for $n \geq 1$. Given a graph G, a subset $S \subseteq V(G)$ is called independent if no two vertices of S are adjacent in G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of a largest independent set S in G. The set of independent sets in G is denoted by $I(G)$.

Received April 20, 2004, revised May 18, 2005.
Communicated by Xuding Zhu.
2000 *Mathematics Subject Classification*: 05C69, 05C05.
Key words and phrases: Independent sets; Merrifield-Simmons; Hoyosa; Fibonacci number; Claw-free graphs; Unicyclic graphs; Bipartite graphs.

The empty set is independent. The set of independent sets in G which contains the vertex x is denoted by $I_x(G)$, while $I_{-x}(G)$ denotes the set of independents sets which do not contain x. The number of independent sets in G is denoted by $i(G)$. The number of edge independent sets in G is denoted by $i'(G)$. In the chemical literature the graph parameter $i(G)$ is referred to as the *Merrifield-Simmons index* [14] while $i'(G)$ is referred to as the *Hoyosa index* [6].

2. Introduction

After the first papers by Miller and Muller [15] and Moon and Moser [16] about maximal independent sets, Prodinger and Tichy [17] gave impetus to the study of the number $i(G)$ of independent sets in a graph. A survey by Chou and Chang [2] and several other references are listed at the end. The problem of counting the number of independent sets in a graph is NP-complete (see for instance Roth [18]). However, for certain types of graphs the problem of determining i is polynomial. For instance, Prodinger and Tichy [17] proved, by induction, that $i(P_n)$ and $i(C_n)$, respectively, is the sequence of Fibonacci and Lucas numbers.

Theorem 2.1. ([17])

$$\begin{aligned} \forall n \in \mathbb{N}: \quad & i(P_n) &=& \; F(n+2). \\ \forall n \in \mathbb{N}_{\geq 3}: \quad & i(C_n) &=& \; L(n) = F(n-1) + F(n+1). \end{aligned}$$

We list some useful facts.

Fact (i) For a spanning proper subgraph H of G we have $i(G) < i(H)$.

Fact (ii) Let G have components $G_1, G_2, \ldots, G_k$. Then $i(G) = \prod_{i=1}^{k} i(G_i)$. That implies that if $i(G)$ is a prime number then G is connected.

Fact (iii) Let G be any graph of order n. Then $1 + n \leq i(G) \leq 2^n$. The lower bound is attained precisely for $G \simeq K_n$ and the upper bound precisely for $G \simeq \overline{K_n}$ ([17]).

Fact (iv) Let G be any connected graph of order n. Then

$$1 + n \leq i(G) \leq 2^{n-1} + 1.$$

The lower bound is attained precisely for complete graphs K_n and the upper bound precisely for stars $K_{1,n-1}$.

Fact (v) Let T be any tree of order n. Then $F(n+2) \leq i(T) \leq 1 + 2^{n-1}$ ([17]). By induction it can be proven that the lower bound is attained precisely when T is a path P_n and the upper bound precisely when T is a star $K_{1,n-1}$ ([10]).

Fact (vi) Let T be any tree of order $n \geq 7$ distinct from P_n. Then

$$i(T) \geq 2F(n) + 3F(n-3).$$

Equality holds if and only if T is the tree $T = Y_n$ obtained by identifying one endvertex of a P_{n-4} with the center-vertex of a P_5 ([10]). Lin and Lin [10] also characterized all trees T of order $n \geq 8$ satisfying $2^{n-2}+7 \leq i(T) \leq 2^{n-1}+1$.

Fact (vii) If T is a tree of order n and diameter at least k, then $i(T) \leq F(k)+2^{n-k}F(k+1)$. Equality occurs if and only if $T \simeq B_{n,k,1}$, where $B_{n,k,1}$ denotes the graph constructed from the k-path by attaching $n-k$ leaves to one endvertex of the k-path ([20,021]).

Let $H_{n,k}$ denote the graph constructed from the k-cycle by attaching $n-k$ leaves to one vertex of the k-cycle.

Fact (viii) If G is a connected graph which contains at least one cycle, then $i(G) \leq 3 \cdot 2^{n-3} + 1$. Equality holds if and only if G is a 4-cycle or $G \simeq H_{n,3}$ ([21]).

Fact (ix) If G is a unicyclic graph not isomorphic to $H_{n,3}$, then $i(G) \leq 5 \cdot 2^{n-4} + 2$. Equality occurs if and only if $G \simeq H_{n,4}$ ([21]).

3. Lower Bounds for $i(G)$

Our first result gives a lower bound of $i(G)$ in terms of the order of G and the independence number of G.

Theorem 3.1. *Let G be any graph of order n with independence number $\alpha = \alpha(G)$. Then*

$$i(G) \geq 2^{\alpha} + n - \alpha,$$

and equality occurs if and only if G can be constructed by joining each vertex in a $K_{n-\alpha}$ to each vertex in a $\overline{K_{\alpha}}$.

Proof. Let S denote a maximum independent set in G. Then every subset of S is an independent set in G and every singleton of $V(G-S)$ is an independent set. Consequently, $i(G) \geq 2^{|S|} + |V(G) - S| = 2^{\alpha} + n - \alpha$. Assume equality occurs, then any independent set of G is either a subset of S or a singleton in $V(G) - S$. It follows that every two vertices of $V(G) - S$ are adjacent and every vertex of $V(G) - S$ is adjacent to every vertex of S. Thus G can be constructed as claimed. The converse is just as obvious. ∎

For t-regular graphs we have $\alpha(G) \geq \frac{n}{t+1}$ and $n - \alpha \geq t$, since a vertex in an independent set $S, |S| = \alpha(G)$, has all its t neighbours in $G - S$. Thus we obtain

Corollary 3.2. *Let G be a t-regular graph of order n. Then $i(G) \geq 2^{\frac{n}{t+1}} + t$, and equality occurs if and only if $G \simeq K_n$.*

The following lower bound is an improvement of Fact (iv).

Theorem 3.3. *Let G denote a graph with n vertices and let $m(\overline{G})$ denote the number of edges in the complement $\overline{G}$. Then $i(G) \geq 1 + n + m(\overline{G})$, and equality occurs if and only if $\alpha(G) \leq 2$, that is, $\overline{G}$ is triangle-free.*

Proof. The empty set and the singletons of G are in $I(G)$. Every edge in $\overline{G}$ corresponds to an independent set in G and so $I(G)$ contains precisely $m(\overline{G})$ sets each with two elements. Thus, $i(G) \geq 1 + n + m(\overline{G})$. If $\alpha(G) > 2$, then by definition $i(G) > 1 + n + m(\overline{G})$. Hence $i(G) = 1 + n + m(\overline{G})$ implies $\alpha(G) \leq 2$. The converse is just as obvious. ∎

Corollary 3.4. *Let G denote a graph of order n and let t denote the number of components in $\overline{G}$. Then $i(G) \geq 2n + 1 - t$, and equality occurs if and only if $\overline{G}$ is a forest.*

Proof. We simply observe that $m(\overline{G}) \geq n - t$ and so Theorem 3.3 implies $i(G) \geq 2n + 1 - t$. If $i(G) = 2n + 1 - t$, then we must have $m(\overline{G}) = n - t$, that is, $\overline{G}$ is a forest and consequently also triangle-free. ∎

4. Upper Bounds for $i(G)$

Theorem 4.1. *Let G be a graph without isolated vertices and let S be an independent set in G. Then*

$$i(G) \leq 2^{\alpha}(i(G - S) - 1) + 1,$$

and equality occurs precisely if $G \simeq K_{1,n-1}$ and S is the set of its leaves.

Proof. An independent set A in G can be written as the union of two sets $B = A \cap S$ and $C = A \cap V(G - S)$. Any $B \neq \varnothing$ has a vertex with a neighbour in $G - S$, so at least one independent set in $G - S$ cannot be used as C together with B. This gives at most $(2^{|S|} - 1)(i(G - S) - 1)$ independent sets while $B = \varnothing$ can be combined with $i(G - S)$ sets, that adds up to $i(G) \leq 2^{|S|}(i(G - S) - 1) + 1 \leq 2^{\alpha}(i(G - S) - 1) + 1$ as claimed. Clearly $K_{1,n-1}$ with S taken to be its leaves gives equality. Let conversely S be an independent set in G such that

$i(G) = 2^\alpha(i(G-S)-1)+1$. Then $|S| = \alpha(G)$, because otherwise $|S| < \alpha(G)$ would imply $i(G) \leq 2^{|S|}(i(G-S)-1)+1 < 2^\alpha(i(G-S)-1)+1$. Also $G-S = \{x\}$ because assume otherwise $\{x,y\} \subseteq G-S$, then both x and y would have a neighbour in S, so a set B in S containing these two neighbours could only be combined with at most $i(G-S)-2$ sets C in $G-S$ and we would get strict inequality for $i(G)$. Therefore $G-S = \{x\}$ and all vertices in S are joined precisely to x giving $G = K_{1,n-1}$. ■

Theorem 4.2. *In the graph G, let x be a vertex of degree $t \geq 1$. Then $i(G) \leq 2i(G-x)-t$, and equality holds precisely if each neighbour of x is adjacent to every other vertex of G.*

Proof. Let $y_1, y_2, \ldots, y_t$ denote the neighbours of x. We may write $I(G) = I_x(G) \cup I_{-x}(G)$. Observe that $|I_{-x}(G)| = i(G-x)$. Every set $S-\{x\}$ with $S \in I_x(G)$ is also a member of $I_{-x}(G)$, and so $|I_x(G)| \leq |I_{-x}(G)|$. But the t singletons $\{y_i\}, 1 \leq i \leq t$, are in $I_{-x}(G)$ and correspond to no set $S-\{x\}$ with $S \in I_x(G)$. Thus, $|I_{-x}(G)| \leq |I_x(G)|-t$ which implies $i(G) = |I_x(G)|+|I_{-x}(G)| \leq 2|I_{-x}(G)|-t$ and the desired inequality is established. If $i(G) = 2i(G-x)-t$ we have $y_iz \in E(G)$ for each $z \in V(G) \setminus \{x, y_i\}$ for every $i \in \{1,\ldots,t\}$, because if $zy_i \notin E(G)$ then $\{z, y_i\} \in I_{-x}(G)$ and $\{z, y_i, x\} \notin I_x(G)$ which would imply $|I_x(G)| < |I_{-x}(G)|-t$ and hence $i(G) < 2i(G-x)-t$, a contradiction. The converse is easily seen and Theorem 4.2 is proven. ■

Theorem 4.3. *Let G be a graph on $n \geq 1$ vertices without isolated vertices. Then $i(G) \leq 2^{n-1}+1$ and $i(G) = 2^{n-1}+1$ precisely if $G = K_{1,n-1}$ or $G = 2K_2$.*

Proof. For any edge e in G we have $i(G) < i(G-e)$. We may thus assume that removal of any edge results in a graph with at least one isolated vertex, consequently each component is a non-trivial star $K_{1,r}$ with $i(K_{1,r}) = 2^r+1$ and Lemma 4.4 below gives the result. ■

Lemma 4.4. *Let $s \geq 2$ denote an integer and $r_1, \ldots, r_s$ denote positive integers. Then*

$$(2^{r_1}+1)(2^{r_2}+1)\ldots(2^{r_s}+1) \leq 2^{r_1+r_2+\cdots+r_s+s-1}+1,$$

and equality occurs if and only if $s = 2$ and $r_1 = 1 = r_2$.

Proof.

(i) Suppose $s = 2$. If $r_1 = 1 = r_2$, then equality occurs. Suppose that at least one of r_1 and r_2 is greater than one, say $r_1 > 1$. Then

$$3 \leq (2^{r_1}-1)(2^{r_2}-1) = 2^{r_1+r_2}-2^{r_1}-2^{r_2}+1,$$

which implies $2^{r_1} + 2^{r_2} < 2^{r_1+r_2}$. Applying this result, we obtain

$$(2^{r_1} + 1)(2^{r_2} + 1) = 2^{r_1+r_2} + 2^{r_1} + 2^{r_2} + 1 < 2 \cdot 2^{r_1+r_2} + 1,$$

and so we have strict inequality.

(ii) Suppose $s \geq 3$. Then, by induction on

$$(2^{r_1} + 1) \cdots (2^{r_{s-1}} + 1) \leq 2^{r_1+\cdots+r_{s-1}+(s-1)-1} + 1,$$

it follows that

$$(2^{r_1} + 1) \cdots (2^{r_{s-1}} + 1)(2^{r_s} + 1) \leq (2^{r_1+\cdots+r_{s-1}+(s-1)-1} + 1)(2^{r_s} + 1). \tag{1}$$

Now we apply the result of Case (i) with $r_1' := r_1 + \cdots + r_{s-1} + (s-1) - 1 \geq 2s - 3 > 1$, $r_2' := r_s$ and $s' := 2$, and we obtain

$$\begin{aligned} (2^{r_1+\cdots+r_{s-1}+(s-1)-1} + 1)(2^{r_s}+1) &< 2^{r_1+\cdots+r_{s-1}+(s-1)-1+r_s+2-1}+1 \\ &= 2^{r_1+\cdots+r_{s-1}+r_s+s-1} + 1. \end{aligned} \tag{2}$$

Thus, the desired inequality follows from (1) and (2), and the lemma follows by induction. ■

One might be inclined to expect that more edges would imply fewer independent sets. However, Corollary 4.5 below shows that this is not true in general. For instance, the graph rK_2 ($r \geq 3$), which has order $2r$ and size r, has fewer independent sets than the graph $K_{1,2r-1}$, which has order $2r$ and size $2r - 1$.

From Theorem 4.3 one easily obtains the following result.

Corollary 4.5. *If G is a graph on $n \geq 1$ vertices and $m \geq 1$ edges, then $i(G) \leq 2^{n-1} + 2^t$, where t denotes the number of isolated vertices in G. Equality occurs precisely when $G \simeq K_{1,n-t-1} \cup \overline{K_t}$ or $t = n - 4$ and $G \simeq 2K_2 \cup \overline{K_t}$.*

5. Independent Sets in Forests

Lin and Lin [10] proved that for any forest F on n vertices, $F(n+2) \leq i(F) \leq 2^n$; and moreover that $i(F) = F(n + 2)$ if and only if $F \simeq P_n$, and $i(F) = 2^n$ if and only if $F \simeq \overline{K_n}$.

By adding the condition that the forest F contains no isolated vertex, we obtained the following bounds.

Theorem 5.1. *Let F be a forest on n vertices, none of which are isolated. Then $F(n + 2) \leq i(F) \leq 1 + 2^{n-1}$ and moreover, $i(F) = F(n + 2)$ if and only if $F \simeq P_n$, and $i(F) = 1 + 2^{n-1}$ if and only if $F \simeq K_{1,n-1}$ or $F \simeq 2K_2$.*

Proof. If nessecary, add edges to F to obtain a tree T. Then the left inequality follows from $i(F) \geq i(T) \geq F(n+2)$. Assume $i(F) = F(n+2)$, then $i(T) = F(n+2)$ and, by Fact (v), $T \simeq P_n$. If we added edges to F in order to obtain T, then $i(F) > i(T) = F(n+2)$, a contradiction. Hence $F \simeq T \simeq P_n$. The upper bound and the remainder of the statement follows from Theorem 4.3. ■

6. Independent Sets in Bipartite Graphs

Theorem 6.1. *Let G be a bipartite graph of order n with no isolated vertex. Then $2^{\frac{n}{2}+1} - 1 \leq i(G) \leq 1 + 2^{n-1}$. The lower bound is attained precisely for $G \simeq K_{n/2,n/2}$, and the upper bound precisely for $G \simeq K_{1,n-1}$ or $G \simeq 2K_2$.*

Proof. Let V_1, V_2 be bipartition classes for G, $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \varnothing$. Let $n_i = |V_i|$ for $i = 1, 2$. We note that $0 \leq (2^{n_1/2} - 2^{n_2/2})^2 = 2^{n_1} + 2^{n_2} - 2^{(n_1+n_2)/2+1}$ implies $2^{n/2+1} \leq 2^{n_1} + 2^{n_2}$. All subsets of V_i, $i = 1, 2$, are independent, so counting the empty set only once we have $i(G) \geq 2^{n_1} + 2^{n_2} - 1 \geq 2^{n/2+1} - 1$. For equality to occur we must have $n_1 = n_2 = n/2$ and $G \simeq K_{n/2,n/2}$. The upper bound and its extremal graphs follow from Theorem 4.3. ■

7. Unicyclic Graphs

We have in [21] proven that among all unicyclic graphs of order n the smallest number of independent subsets is obtained for two graphs, namely for C_n and for the graph constructed by placing $n-4$ subdivision vertices on an edge pendent to a 3-circuit, the largest number of independent subsets is obtained for $H_{n,k}$, and when $n = 4$ also for C_4 (Fact (viii)). Specifying order and circuit length we obtain

Theorem 7.1. ([21]) *If G is a unicyclic graph of order n and circuit length k, then $i(G) \leq 2^{n-k}F(k+1) + F(k-1)$. Equality occurs if and only if $G \simeq H_{n,k}$.*

We shall here sharpen this bound by not only prescribing order n and circuit length k but also demanding that at least one vertex is at distance h from the circuit.

Theorem 7.3. *Let $n \geq k \geq 3$, $h \geq 0$ be integers and let G be a unicyclic graph of order n with circuit length k and maximum distance h from a vertex in G to the circuit. Then*

$$i(G) \leq F(k-1)F(h+1) + 2^{n-k-h}F(k+1)F(h+2).$$

Equality holds if and only if either (i) $h = 0$, i.e. G is a circuit, or (ii) $h \geq 1$ and G can be obtained from a circuit of length k by attaching to the same vertex $n-k-h$ pendent edges and one more pendent edge having $h-1$ subdivision vertices.

Proof. If (i) occurs, i.e. if $h = 0$, we have $k = n$ and $G \simeq C_n$. From Theorem 2.1 we have that $i(C_n) = F(n-1) + F(n+1)$ and that is the statement of Theorem 7.2. If (ii) occurs, let x be the unique stem on C. Then $G - N[x]$ consists of two disjoint paths P_{k-3} and P_{h-1}, so $|I_x(G)| = F(k-1)F(h+1)$ while $G - x$ consists of $n - h - k$ isolated vertices and the disjoint paths P_{k-1} and P_h, so $|I_{-x}(G)| = 2^{n-k-h}F(k+1)F(h+2)$ yielding $i(G) = |I_x(G)| + |I_{-x}(G)| = F(k-1)F(h+1) + 2^{n-k-h}F(k+1)F(h+2)$ as stated in Theorem 7.2. We shall now prove the main statement. For $h = 0$ we have just seen that $G \simeq C_n$ and Theorem 7.2 holds. For $h = 1$ the inequality is $i(G) \leq F(k-1) + 2^{n-k}F(k+1)$. That and the characterization of the extremal graph is proven in [21]. We shall proceed to prove the theorem by induction on n. The theorem is true for small values of n, since it holds for $h = 0, 1$ and all values of k and n. So, let a unicyclic graph G of order n, with a circuit C of length k and with $h \geq 2$ be given. Assume the theorem holds for all unicyclic graphs having order smaller than n, we shall then prove it for G. We shall consider three cases depending on the maximum distance in $G - N[x]$ from a vertex to C, where x is a vertex at maximum distance from C in G.

Let $x_0 \in V(C)$ and let $x_0x_1 \dots x_h$ be a longest path in $G - E(C)$. Let $\deg_G(x_{h-1}) = 2 + t, t \geq 0$, i.e. x_{h-1} is a stem with $t + 1$ leaves. We shall write $x = x_h$ below.

Case 1, $G - N[x]$ has a vertex at distance h from C. By the induction hypothesis we have

$$|I_{-x}(G)| \leq F(k-1)F(h+1) + 2^{n-k-h-1}F(k+1)F(h+2),$$

$$|I_x(G)| \leq 2^t \left(F(k-1)F(h+1) + 2^{n-k-h-t-2}F(k+1)F(h+2)\right).$$

As $G - N[x]$ has order $n-2$ and contains t isolated vertices, the circuit C of length k and at least a path with h further vertices we have $t \leq n - k - h - 2$ and hence

$$2^tF(k-1)F(h+1) < 2^{n-k-h-2}F(k+1)F(h+2).$$

That implies

$$|I_x(G)| < 2^{n-k-h-1}F(k+1)F(h+2)$$

and thus

$$i(G) = |I_{-x}(G)| + |I_x(G)| < F(k-1)F(h+1) + 2^{n-k-h}F(k+1)F(h+2),$$

giving strict inequality in the theorem.

Case 2, in $G - N[x]$ the maximum distance from a vertex to C is $h - 1$.

(i) Assume $t \geq 1$. By the induction hypothesis we have as in Case 1

$$|I_{-x}(G)| \leq F(k-1)F(h+1) + 2^{n-k-h-1}F(k+1)F(h+2)$$

$$|I_x(G)| \leq 2^t \Big(F(k-1)F(h) + 2^{n-k-h-t-2}F(k+1)F(h+2)\Big)$$

There are in $N[x]$ two vertices and $G - N[x]$ contains a component with at least $k+h-1$ vertices, namely C together with a path of length $h-1$. That implies $1 \leq t \leq n-k-h-1$. Using $2^t \leq 2^{n-k-h-1}$, $F(k-1) < F(k+1)$ and $F(h) + F(h+1) = F(h+2)$ we obtain

$$|I_x(G)| < 2^{n-k-h-1}F(h+1)F(h+2)$$

giving the desired strict inequality

$$i(G) = |I_{-x}(G)| + |I_x(G < F(k-1)F(h+1) + 2^{n-k-h}F(k+1)F(h+2).$$

(ii) Assume $t = 0$.

From the induction hypothesis we now have

$$|I_x(G)| \leq F(k-1)F(h) + 2^{n-k-h}F(k+1)F(h+1)$$
$$|I_{-x}(G)| \leq F(k-1)F(h+1) + 2^{n-k-h-1}F(k+1)F(h+1)$$
$$\leq F(k-1)F(h+1) + 2^{n-k-h}F(k+1)F(h+2)$$
$$-F(k-1)F(h-1) - 2^{n-k-h}F(k+1)F(h).$$

Therefore

$$i(G) = |I_{-x}(G)| + |I_x(G)| \leq F(k-1)F(h+1) + 2^{n-h-k}F(k+1)F(h+2)$$
$$+F(k-1)F(h-2) - 2^{n-h-k-1}F(k+1)F(h-2).$$

As $n-h-k-1 \geq 0$ we have for $h > 2$ that $F(h-2) > 0$, giving strict inequality in the theorem, while $h = 2$ gives equality.

In case 2 an extremal graph can occur only if $h = 2$ and $t = 0$.

If we for $h = 2$ have equality, i.e. if $i(G) = 2F(k-1) + 2^{n-k-2}F(k+1) \cdot 3$ then we must have equality throughout so that in particular

$$|I_{-x}(G)| = F(k-1) + 2^{n-1-k}F(k+1).$$

In [21] it is proven that this implies that $G - x$ consists of a circuit of length k with $n-1-k$ leaves attached to one of its vertices. That in turn implies that G is as decribed in the theorem.

Case 3, in $G - N[x]$ the maximum distance from a vertex to C is $h-2$.

(i) Assume $t \geq 1$.

By induction

$$\begin{aligned}
|I_{-x}(G)| &\leq F(k-1)F(h+1) + 2^{n-k-h-1}F(k+1)F(h+2) \\
&= F(k-1)F(h+1) + 2^{n-h-k}F(k+1)F(h+2) \\
&\quad -2^{n-h-k-1}F(k+1)F(h+2) \\
|I_x(G)| &\leq 2^t\left(F(k-1)F(h-1) + 2^{n-k-h-t}F(k+1)F(h)\right). \\
&= 72^tF(k-1)F(h-1) + 2^{n-k-h}F(k+1)F(h). \\
i(G) &= |I_{-x}(G)| + |I_x(G)| \\
&\leq F(k-1)F(h+1) + 2^{n-h-k}F(k+1)F(h+2) \\
&\quad -2^{n-h-k-1}F(k+1)(2F(h) + F(h-1)) \\
&\quad +2^tF(k-1)F(h-1) + 2^{n-h-k}F(k+1)F(h) \\
&= F(k-1)F(h+1) + 2^{n-h-k}F(k+1)F(h+2) \\
&\quad +(2^tF(k-1) - 2^{n-h-k-1}F(k+1))F(h-1)
\end{aligned}$$

and as $n-k-h \geq t \geq 1$, $h \geq 2$, $F(h-1) > 0$ we have proven strict inequality for the theorem. By induction

(ii) Assume $t = 0$.

$$\begin{aligned}
i(G) = |I_{-x}(G)| + |I_x(G)| &\leq F(k-1)F(h) + 2^{n-h-k}F(k+1)F(h+1) \\
&\quad +F(k-1)F(h-1) + 2^{n-h-k}F(k+1)F(h) \\
&= F(k-1)F(h) + 2^{n-h-k}F(k+1)F(h+2).
\end{aligned}$$

Thus the inequality is proven.

If equality holds, then $i(G-x) = |I_{-x}(G)| = F(k-1)F(h) + 2^{n-h-k}F(k+1)F(h+1)$ and $G-x$ has order $n-1$, circuit length k, maximum distance $h-1$ from a vertex to the circuit, so by induction hypothesis $G-x$ consists of C with a vertex x_0 having $(n-1)-k-(h-1)$ pendent edges and one further pendent edge x_0x_{h-1} having $h-2$ subdivision vertices $x_1, x_2, \ldots, x_{h-2}$. To obtain G from $G-x$ we must join $x = x_h$ to x_{h-1} to obtain distance h from x to C. ∎

8. Independent Sets in Claw-free Graphs

In this section we determine the bounds for $i(G)$ on the class of claw-free graphs. A graph is said to be claw-free if it does not contain the star $K_{1,3}$ as an induced subgraph.

Lemma 8.1. *Every endvertex x of a longest induced path in a non-complete connected claw-free graph G has the property that both $G - x$ and $G - N[x]$ are connected.*

Proof. Let $P = x_0x_1 \dots x_k$ denote a longest induced path in G, then $k \geq 2$ as G is not complete.

If x_k were a cut-vertex of G, one component G_1 of $G - x_k$ contains x_0 and another component G_2 contains a vertex w adjacent to x_k. But then $x_0x_1 \dots x_kw$ would be a longer induced path in G, a contradiction.

$G - N[x_k]$ is connected because assume otherwise G_1 is a component of $G - N[x_k]$ containing x_0 and G_2 is another component of $G - N[x_k]$. G_2 has a vertex w adjacent to a vertex $z \in N(x_k)$. If w were adjacent to x_{k-1} then G would contain the claw $\{x_{k-2}, x_{k-1}, x_k, w\}$ with center x_{k-1}, a contradiction.

Thus w is adjacent to $z \in N(x_k) \setminus \{x_{k-1}\}$. Note that $x_{k-2}z \notin E(G)$ as otherwise $\{x_{k-2}, x_k, w, z\}$ would be a claw with center z. But then either $x_0 \dots x_{k-1}x_kzw$ (if $zx_{k-1} \notin E(G)$) or $x_0 \dots x_{k-1}zw$ (if $zx_{k-1} \in E(G)$) is a longer induced path in G. This contradiction proves Lemma 8.1. ∎

Theorem 8.2. *If G is a connected claw-free graph of order n, then $1 + n \leq i(G) \leq F(n + 2)$. The lower bound is attained if and only if $G \simeq K_n$, while the upper bound is attained if and only if $G \simeq P_n$.*

Proof. By Fact (iv) the lower bound holds for any connected graph and is attained precisely for K_n which is claw-free. We prove the upper bound by induction on n. If $n \leq 3$, then G is a path and so the desired statement follows from Theorem 2.1. Suppose $n \geq 4$. If G is a complete graph, then $i(G) = n + 1 < F(n + 2)$. Hence we may assume that G is not complete. Now, by Lemma 8.1, there exists a vertex $x \in V(G)$ such that both $G - x$ and $G - N[x]$ are connected. Since any induced subgraph of a claw-free graph is itself claw-free, the induction hypothesis applies to both $G - x$ and $G - N[x]$. Since $n(G - x) = n - 1$ and $n(G - N[x]) \leq n - 2$, we obtain

$$i(G) = i(G - x) + i(G - N[x]) \leq F(n + 1) + F(n) = F(n + 2).$$

If $i(G) = F(n + 2)$, then we must have $i(G - x) = F(n + 1)$ and $i(G - N[x]) = F(n)$. Therefore, by induction, $G - x$ and $G - N[x]$ must be paths of order $n - 1$ and $n - 2$, respectively. This is only possible if G itself is a path. The converse, that is, $i(P_n) = F(n + 2)$ is stated in Theorem 2.1. ∎

It is remarkable that the Fibonacci numbers again occur as a bound for the graph parameter i, that is, if G is a connected claw-free graph on n vertices, then $i(G) \leq F(n + 2)$, while if G is a tree on n vertices, then $i(G) \geq F(n + 2)$.

Since every line graph is a claw-free graph, Theorem 8.2 also gives a bound for the number of independent sets in a line graph. Moreover, the bound is optimal since the extremal graphs i.e. the paths are line graphs.

Acknowledgment

We thank M. Kouider and B. Randerath for helpful suggestions and an anonymous referee for substantial shortening of theorems.

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