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BOUNDS ON THE RATE OF CONVERGENCE OF MOMENTS IN THE CENTRAL LIMIT THEOREM

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We derive two-sided bounds on the rate of convergence of moments in the central limit theorem. A variety of norming constants is considered, and it is shown that a very delicate alteration to these constants can have a significant effect on the rate. Indeed, the influence of norming constants on rates of convergence of moments is of a more subtle nature than in the central limit theorem itself. We present several examples to illustrate some extreme cases.

1. Introduction and summary. The moments of a random variable are one of its most descriptive and accessible characteristics. For example, the classical form of the central limit theorem holds only for distributions with finite second moment, and the rate of convergence in this limit law is commonly described by conditions on the distribution's higher order moments. Therefore it is not surprising that one of the earliest generalisations of the Lindeberg-Feller theorem was Bernstein's (1939) discovery of necessary and sufficient conditions for the convergence of moments. Moment convergence in the central limit theorem for more general stochastic processes has also been studied; see for example Lifshits (1975), Hall (1978a) and de Acosta and Giné (1979). Brown (1969, 1970) provided an alternative proof of Bernstein's theorem. However, the *rate* of convergence of moments has received relatively little attention, in comparsion to the wealth of literature which exists on rates in the central limit theorem itself. Von Bahr (1965) obtained upper bounds for the rate of convergence of moments, and derived Edgeworth-Cramér expansions. Some more recent work of Bhattacharya and Rao (1976), in particular their Theorem 18.1, page 181, may be employed to generalise portions of von Bahr's work to the case of vector valued variables and to improve on some of his upper bounds in the scalar case. Further sharpenings have been given by Hall (1978b). See also Sweeting (1980). In view of these results it is of interest to know the full extent to which upper bounds can be tightened. The most precise information in this problem can be obtained for the case of independent and identically distributed variables, and it should consist of lower bounds on the rate of convergence. This is the context of the present paper. We shall derive upper and lower bounds which for many distributions are of the same order of magnitude.

Some results of this nature have been obtained by Osipov (1968, 1971), Rozovskii (1978a, 1978b) and Hall (1980, 1981) for the central limit theorem itself. However, the case of convergence of moments contains some very interesting surprises. For example, an extremely subtle change of norming constants can significantly improve the rate of convergence of moments. It is well known that the rate of convergence in the central limit theorem may be expedited by norming with the truncated mean and mean square rather than the true mean and mean square. The truncation may be performed at any constant multiple of $n^{1/2}$, and the value taken for the constant does not significantly affect the rate. However, in the case of convergence of moments an alteration to the constant *can* change the rate. The rate obtained with one multiple of $n^{1/2}$ can be asymptotically negligible in comparison with the rate for another multiple. Behaviour of this subtlety is perhaps best described by examples, and we shall consider the case of distributions with regularly varying tails in detail.

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Our results are presented together in Section 2. We examine moments up to the fourth in greatest depth, since this case is the simplest to present and discuss. In Theorem 1 we present upper and lower bounds of similar orders of magnitude, and in Theorem 2 we sharpen these in the special case of a distribution with regularly varying tails. Our upper and lower bounds may be used to derive characterisations of the rate of convergence; results of this type are given in Corollaries 1 and 2. Rates of convergence for different norming constants are presented in subsequent theorems. Finally, we state a result on Edgeworth-Cramér expansions for higher-order moments, which generalises Theorem 1. All our proofs are deferred until Section 3.

2. Rates of Convergence. Let $X, X_1, X_2, \cdots$ be independent and identically distributed random variables with zero mean and unit variance, and set $S_n = \sum_1^n X_i$. Let Z denote a standard normal variable, and define

$$\delta_p(n) = \begin{cases} n^{-1}E\{X^4 I(|X| \le n^{1/2})\} + E\{X^2 I(|X| > n^{1/2})\} & \text{if } 0 < p < 2 \\ n^{-1}E\{X^4 I(|X| \le n^{1/2})\} + n^{1-p/2}E\{|X|^p I(|X| > n^{1/2})\} & \text{if } 2 < p < 4. \end{cases}$$

(Here $I(E)$ denotes the indicator function of an event E.) The central limit theorem states that $S_n/n^{1/2}$ converges in distribution to Z, and if $E|X|^p < \infty$ for some $p > 0$ then $E|S_n/n^{1/2}|^p \to E|Z|^p$. We shall prove that if $0 < p < 4$ then the rate of convergence in this limit law is that of the quantity $\delta_p(n)$ to zero, up to terms of order $n^{-\min\{1,(p+1)/2\}}$. In other words, the quantities $E|S_n/n^{1/2}|^p - E|Z|^p$ and $\delta_p(n)$ have the same order of magnitude, except for terms of order $n^{-\min\{1,(p+1)/2\}}$. (If $n^{-\min\{1,(p+1)/2\}}$ is of a larger order of magnitude than $\delta_p(n)$, then of course this result is not so informative.)

THEOREM 1. *Suppose* $0 < p < 4, p \ne 2$ *and* $E|X|^p < \infty$. *Then*

(2.1) $$\limsup_{n\to\infty} |E|S_n/n^{1/2}|^p - E|Z|^p|/\{\delta_p(n) + n^{-\min\{1,(p+1)/2\}}\} < \infty$$

and

(2.2) $$\liminf_{n\to\infty}(|E|S_n/n^{1/2}|^p - E|Z|^p| + n^{-\min\{1,(p+1)/2\}})/\delta_p(n) > 0.$$

Important to the proof of this result is a lemma which generalises portions of Esséen's (1945) Lemma 3 and von Bahr's (1965) Lemma 2. We present it here because of its possible independent interest.

LEMMA 1. *Let* ϕ *be the characteristic function of a variable with finite variance. Then for any* $\delta > 0$, $\sup_x |\int_x^{x+\delta} \phi^n(t)\,dt| = O(n^{-1/2})$ *as* $n \to \infty$.

If ϕ satisfies Cramér's continuity condition, i.e.

(C) $$\limsup_{|t|\to\infty} |\phi(t)| < 1,$$

then $\sup_{x\ge\varepsilon} |\int_x^{x+\varepsilon} \phi^n(t)\,dt| = O(n^{-r})$ for each ε and $r > 0$. In this case if X has characteristic function ϕ, Theorem 1 may be slightly sharpened by replacing the term $\min\{1, (p+1)/2\}$ by 1 in (2.1) and (2.2). Similarly, the condition (2.3) below may be dropped, and some conditions in Theorems 3-5 and Corollary 2 may be relaxed.

The results of Theorem 1 may be refined if we are prepared to assume that the underlying distribution has regularly varying tails.

THEOREM 2. *Suppose the function* $R(x) = P(|X| > x)$ *is regularly varying at infinity with exponent* $-\alpha$, *where* $2 \le \alpha < 4$, *and that* $0 < p \le \alpha$ *with* $p \ne 2$. (*We continue to assume that* $E|X|^p < \infty$.) *Set*

$$K_p = 2|\sin(\tfrac{1}{2}p\pi)|\Gamma(p+1)/\pi.$$

(i) *Assume that $2 < \alpha < 4$ and $0 < p < \alpha$, and if $p < 1$, that*

$$x^{p+3}P(|X| > x) \to \infty \quad \text{as} \quad x \to \infty. \tag{2.3}$$

Then

$$|E|S_n/n^{1/2}|^p - E|Z|^p| \sim nP(|X| > n^{1/2})\alpha K_p$$
$$\times \int_0^\infty x^{-\alpha-1}dx \int_0^\infty t^{-(p+1)}\left\{\cos tx - 1 + \left(\frac{1}{2}\right)(tx)^2\right\}e^{-t^2/2}\,dt$$

as $n \to \infty$.

(ii) *If $0 < p < \alpha = 2$ then*

$$|E|S_n/n^{1/2}|^p - E|Z|^p| \sim E\{X^2I(|X| > n^{1/2})\}(K_p/2)\int_0^\infty t^{1-p}e^{-t^2/2}\,dt$$

as $n \to \infty$.

(iii) *If $2 < p = \alpha < 4$ then*

$$|E|S_n/n^{1/2}|^p - E|Z|^p| \sim n^{1-p/2}E\{|X|^pI(|X| > n^{1/2})\}K_p\int_0^\infty t^{-(p+1)}\left(\cos t - 1 + \frac{1}{2}t^2\right)dt$$

as $n \to \infty$.

In each case the right hand side can be shown to be asymptotically equivalent to a constant multiple of $\delta_p(n)$. Note that in (ii) and (iii) the right hand side is asymptotically much greater than $nP(|X| > n^{1/2})$ (see Theorem 1, page 281 of Feller (1971)). Therefore the rates of convergence derived in (ii) and (iii) are in a sense slower than in (i).

An advantage of expressing rates of convergence in the form of Theorem 1 is that they may be used to derive characterisations of rates of convergence. For example, the following two corollaries are easily proved.

COROLLARY 1. *Suppose $2 < p < 4$, $0 \le \beta < 1$ and $E|X|^p < \infty$. If $\beta < p/2 - 1$ then*

$$\Sigma n^{-1+\beta}|E|S_n/n^{1/2}|^p - E|Z|^p| < \infty. \tag{2.4}$$

If $\beta = p/2 - 1$ then (2.4) *holds if and only if*

$$E\{|X|^p(1 + |\log|X||)\} < \infty,$$

while if $p/2 - 1 < \beta < 1$, (2.4) *is equivalent to*

$$E(|X|^{2+2\beta}) < \infty.$$

COROLLARY 2. *Suppose $0 < p < 2$ and $0 < \beta < \min\{1, (p+1)/2\}$. Then the following two conditions are equivalent*:

$$\Sigma n^{-1+\beta}|E|S_n/n^{1/2}|^p - E|Z|^p| < \infty; \tag{2.5}$$

$$E(|X|^{2+2\beta}) < \infty.$$

If $0 < \beta < ½$, they are also equivalent to

$$\Sigma n^{-1+\beta}\sup_x|P(S_n/n^{1/2} \le x) - \Phi(x)| < \infty. \tag{2.6}$$

In the case $\beta = 0$, (2.5) *and* (2.6) *are each equivalent to*

$$E\{X^2(1 + |\log|X||)\} < \infty.$$

(In proving the last portion of this corollary, note the results of Heyde (1967).)

It is well known (see for example Friedman, Katz and Koopmans (1966), Egorov (1973), Heyde (1973) and Hall (1980)) that the rate of convergence in the central limit theorem may be improved by considering $(S_n - d_n)/c_n$ instead of $S_n/n^{1/2}$, where for some $\lambda > 0$,

$$c_n = [nE\{X^2 I(|X| \le \lambda n^{1/2})\}]^{1/2} \quad \text{and} \quad d_n = nE\{XI(|X| \le \lambda n^{1/2})\}.$$

(Actually only the case $\lambda = 1$ has been widely considered, but a general λ may be treated using almost identical arguments, yielding the upper and lower bounds of Hall (1980).) It is of interest to examine the effect that the same norming constants have on the rate of convergence of moments. Let us define

$$\delta_p^*(n) = n^{-1}E\{X^4 I(|X| \le n^{1/2})\} + n^{1-p/2}E\{|X|^p I(|X| > n^{1/2})\}$$

for $0 < p < 4, p \ne 2$.

THEOREM 3. *If $0 < p < 4$ and $p \ne 2$ then for any $\lambda > 0$,*

(2.7) $$\limsup_{n\to\infty} |E|(S_n - d_n)/c_n|^p - E|Z|^p|/\{\delta_p^*(n) + n^{-\min\{1,(p+1)/2\}}\} < \infty,$$

and if $2 < p < 4$,

(2.8) $$\liminf_{n\to\infty}\{|E|(S_n - d_n)/c_n|^p - E|Z|^p| + n^{-1}\}/\delta_p^*(n) > 0.$$

On comparing the results of Theorems 1 and 3 it is clear that if $2 < p < 4$, no significant improvement to the rate of convergence can be obtained by using the more complicated norming constants c_n and d_n. However if $0 < p < 2$, an improvement can be achieved in some circumstances. For example, if $R(x) \equiv P(|X| > x)$ is regularly varying at infinity with exponent -2 then $\delta_p^*(n)/\delta_p(n) \to 0$ for $0 < p < 2$, and the upper bound on the rate of convergence provided by Theorem 3 is asymptotically negligible in comparison with the lower bound given by Theorem 1. In fact, the rate of convergence can sometimes be improved still further by a judicious choice of the constant λ. This is borne out by our next theorem, which also shows that the result (2.8) cannot be extended to the case $0 < p < 2$, even if the term n^{-1} is replaced by $n^{-\min\{1,(p+1)/2\}}$.

THEOREM 4. *Suppose the function $R(x) \equiv P(|X| > x)$ is regularly varying at infinity with exponent $-\alpha$, where $2 \le \alpha < 4$, and that $0 < p < 2$. If $p < 1$, assume in addition that $x^{p+3}P(|X| > x) \to \infty$ as $x \to \infty$. Then*

$$|E|(S_n - d_n)/c_n|^p - E|Z|^p| = o\{\delta_p^*(n)\}$$

as $n \to \infty$ if and only if λ is the (unique) positive solution of the equation

$$\int_0^\lambda x^{-\alpha-1}dx \int_0^\infty t^{-(p+1)}\left\{\cos tx - 1 + \frac{1}{2}(tx)^2\right\}e^{-t^2/2}\,dt$$
$$= \int_\lambda^\infty x^{-\alpha-1}dx \int_0^\infty t^{-(p+1)}(1 - \cos tx)e^{-t^2/2}\,dt.$$

For all other positive values of λ,

$$|E|(S_n - d_n)/c_n|^p - E|Z|^p| \sim C\delta_p^*(n)$$

as $n \to \infty$, where C is a positive constant depending on α, λ and p.

It is worth recording the following consequence of Theorem 3: whenever $0 < p < 2$, $E(X) = 0$ and $E(X^2) = 1$, we have

$$\sum n^{-1}|E|(S_n - d_n)/c_n|^p - E|Z|^p| < \infty.$$

This result should be compared with a rate of convergence in the central limit theorem obtained by Friedman, Katz and Koopmans (1966); see also Egorov (1973) and Heyde (1973).

Our final result is a generalisation of Theorem 1 to higher order moments. This necessarily involves Edgeworth-Cramér expansions, and should be compared with the upper bounds derived by von Bahr (1965). For the sake of convenience and brevity we shall adopt von Bahr's notation. We assume that $E(X^{2k}) < \infty$ for a positive integer k, and define

$$\delta_{pk}(n) = \begin{cases} n^{-k}E\{X^{2k+2}I(|X| \le n^{1/2})\} + n^{1-k}E\{X^{2k}I(|X| > n^{1/2})\} & \text{if} \quad 0 < p < 2k; \\ n^{-k}E\{X^{2k+2}I(|X| \le n^{1/2})\} + n^{1-p/2}E\{|X|^p I(|X| > n^{1/2})\} & \text{if} \quad 2k < p < 2k+2. \end{cases}$$

(The case $p = 2k$, or indeed p equal to any even integer, is rather trivial.)

THEOREM 5. *Suppose $E(X^{2k}) + E|X|^p < \infty$, where $0 < p < 2k + 2$ and p is not an even integer. Then*

$$\limsup_{n\to\infty} \left| E|S_n/n^{1/2}|^p - E|Z|^p - \sum_{j=1}^{k-1} n^{-j} \int_{-\infty}^{\infty} |x|^p \, dP_{2j}(-\Phi)(x) \right| / \{\delta_{pk}(n) + n^{-\min\{k,(p+1)/2\}}\} < \infty \tag{2.9}$$

and

$$\liminf_{n\to\infty} \left\{ \left| E|S_n/n^{1/2}|^p - E|Z|^p - \sum_{j=1}^{k-1} n^{-j} \int_{-\infty}^{\infty} |x|^p \, dP_{2j}(-\Phi)(x) \right| + n^{-\min\{k,(p+1)/2\}} \right\} / \delta_{pk}(n) > 0. \tag{2.10}$$

Generalisations of the other results may also be derived. Theorem 5 effectively states that the quantities $\delta_{pk}(n)$ and

$$\left| E|S_n/n^{1/2}|^p - E|Z|^p - \sum_{j=1}^{k-1} n^{-j} \int_{-\infty}^{\infty} |x|^p \, dP_{2j}(-\Phi)(x) \right|$$

are of the same order of magnitude up to terms of order $n^{-\min\{k,(p+1)/2\}}$.

3. Proofs. The symbol C, with or without subscripts, denotes a positive generic constant, while $o(1)$ stands for a function of n which does not depend on t and which converges to zero as $n \to \infty$.

PROOF OF LEMMA 1. We may assume that ϕ is real and nonnegative (otherwise replace ϕ^n by $|\phi^2|^{n/2}$) and prove that for *some* $\delta > 0$,

$$\sup_x \int_x^{x+\delta} \phi^n(t) \, dt = O(n^{-1/2}).$$

The result for a general δ follows by adding.

Let Y denote the random variable in question. By rescaling we may suppose that $E(Y^2) = 1$. For any two real numbers t and t_0 we may write $\phi(t) = \phi(t_0) + (t - t_0)\phi'(t_0) + \frac{1}{2}(t - t_0)^2\phi''(s)$, where s lies between t and t_0. If $\alpha > 1$ is chosen so large that $E\{Y^2 I(|Y| > \alpha)\} \le \frac{1}{6}$ then

$$1 + \phi''(s) = \int_{-\infty}^{\infty} x^2(1 - \cos sx) dP(Y \le x)$$

$$\le \alpha^2 \int_{-\infty}^{\infty} (1 - \cos sx) dP(Y \le x) + 2 \int_{|x|>\alpha} x^2 dP(Y \le x) \le \alpha^2\{1 - \phi(s)\} + \frac{1}{3},$$

and so

(3.1) $$\phi(t) \le \phi(t_0) + (t - t_0)\phi'(t_0) + (t - t_0)^2[\alpha^2\{1 - \phi(s)\}/2 - \tfrac{1}{3}].$$

Let $I = I(x, \delta)$ denote the interval $[x, x + \delta]$, and let t_0 be a point in I at which ϕ attains its maximum on I. If $\phi'(t_0) \neq 0$ then t_0 must equal one of the endpoints of I. It is readily seen that if $t_0 = x$ then $\phi'(t_0) \le 0$, while if $t_0 = x + \delta$, $\phi'(t_0) \ge 0$. Therefore $(t - t_0)\phi'(t_0) \le 0$ whenever $t \in I$, and by (3.1),

(3.2) $$\phi(t) \le \phi(t_0) + (t - t_0)^2[\alpha^2\{1 - \phi(s)\}/2 - \tfrac{1}{3}].$$

Since ϕ is uniformly continuous on $(-\infty, \infty)$, there exists $\delta > 0$ such that $|\phi(u) - \phi(v)| \le 1/(6\alpha^2)$ whenever $|u - v| \le \delta$. Define I with this choice of δ. Then $|s - t_0| \le \delta$ whenever $t \in I$, and so by (3.2),

(3.3) $$\phi(t) \le 1 + (t - t_0)^2[\alpha^2\{1 - \phi(t_0)\}/2 - \tfrac{1}{4}].$$

If $\phi(t_0) \le 1 - 1/(6\alpha^2)$ then $\int_x^{x+\delta} \phi^n(t)\, dt \le \delta(1 - 1/(6\alpha^2)^n$, while if $\phi(t_0) > 1 - 1/(6\alpha^2)$ it follows from (3.3) that $\phi(t) < 1 - (t - t_0)^2/6$ whenever $t \in I$, whence

$$\int_x^{x+\delta} \phi^n(t)\, dt \le \int_{-\infty}^{\infty} \exp\{-n(t - t_0)^2/6\}\, dt = (6\pi/n)^{1/2}.$$

Lemma 1 follows from these two estimates.

LEMMA 2. *Let X be a random variable with finite variance. Then*

(3.4) $$[E\{|X|^3 I(|X| \le x)\}]^2 = O(1) + o[E\{X^4 I(|X| \le x)\}]$$

as $x \to \infty$, and for any $k \ge 1$,

(3.5) $$E\{|X|^{2k+1} I(|X| \le x)\} = O(1) + o[E\{|X|^{2k+2} I(|X| \le x)\}].$$

PROOF. The result (3.4) is trivial if $E(X^4) < \infty$, and so we may assume that $E(X^4) = \infty$. Since $E(X^2) < \infty$, there exists a symmetric function A with $A(x) \ge 1$ and $A(x) \uparrow \infty$ as $x \to \infty$, such that $E\{X^2 A^2(X)\} < \infty$. For any $y \in (0, x)$ we have

$$\begin{aligned} E\{|X|^3 I(|X| \le x)\} &\le y^3 + E\{|X|^3 I(y < |X| \le x)\} \\ &\le y^3 + E[\{|X| A(X)\}\{X^2 I(|X| \le x)\}]/A(y) \\ &\le y^3 + [E\{X^2 A^2(X)\} E\{X^4 I(|X| \le x)\}]^{1/2}/A(y), \end{aligned}$$

so that

$$\limsup_{x\to\infty}[E\{|X|^3 I(|X| \le x)\}]^2/E\{X^4 I(|X| \le x)\} \le E\{X^2 A^2(X)\}/A^2(y).$$

The result (3.4) follows on letting $y \to \infty$, and (3.5) is proved similarly.

PROOF OF THEOREMS 1 AND 5. Let $k_p = \pi/2\Gamma(p + 1)\,|\sin(\tfrac{1}{2}p\pi)|$, where $p > 0$ is not an even integer. The function $f_p(t) = \cos t - \sum_0^{[p/2]} (-1)^j t^{2j}/(2j)!$, where $[p/2]$ denotes the integer part of $p/2$, does not change sign on $(0, \infty)$. It is readily shown that

(3.6) $$\left| \int_0^\infty t^{-(p+1)} f_p(tx) dt \right| = |x|^p \left| \int_0^\infty t^{-(p+1)} f_p(t)\, dt \right| = k_p |x|^p.$$

Define the polynomials P_j by the formal expansion

$$\exp(\textstyle\sum_3^{2k} \kappa_j u^j v^{j-2}/j!) = \sum_0^\infty v^j P_j(u),$$

where κ_j is the jth cumulant of X. Let $P_j(-\Phi)(x)$ denote the function whose Fourier-Stieltjes transform equals $P_j(it)e^{-t^2/2}$. Set $H_n(x) = P(S_n \le n^{1/2}x) - \sum_{j=0}^{2k-2} n^{-j/2} P_j(-\Phi)(x)$.

With ϕ denoting the characteristic function of X, the Fourier-Stieltjes transform of H_n may be written as

$$h_n(t) = \phi^n(t/n^{1/2}) - \textstyle\sum_{j=0}^{2k-2} n^{-j/2}P_j(it)e^{-t^2/2}$$
$$= \phi^n(t/n^{1/2}) - \exp\{n \textstyle\sum_2^{2k} \kappa_j(it/n^{1/2})^j/j!\} + o(|t|^{2k}) = o(|t|^{2k})$$

as $t \to 0$. Consequently $\int_{-\infty}^{\infty} x^j dH_n(x) = 0$ for $0 \le j \le 2k$, and so in view of (3.6),

$$k_p \left| \int_{-\infty}^{\infty} |x|^p dH_n(x) \right| = \left| \int_0^{\infty} t^{-(p+1)}dt \int_0^{\infty} f_p(tx)dH_n(x) \right| = \left| \int_0^{\infty} t^{-(p+1)}\mathcal{R}\ell h_n(t)\, dt \right|.$$

But for any $\varepsilon > 0$,

$$\left| \int_{\varepsilon n^{1/2}}^{\infty} t^{-(p+1)}\mathcal{R}\ell\phi^n(t/n^{1/2})dt \right| \le n^{-p/2} \int_{\varepsilon}^{\infty} t^{-(p+1)} |\phi^n(t)|\, dt$$
$$\le n^{-p/2} \textstyle\sum_{j=1}^{\infty} (j\varepsilon)^{-(p+1)} \displaystyle\int_{j\varepsilon}^{(j+1)\varepsilon} |\phi^n(t)|\, dt = O(n^{-(p+1)/2}),$$

using Lemma 1. Consequently

$$k_p \left| \int_{-\infty}^{\infty} |x|^p dH_n(x) \right| = \left| \int_0^{\varepsilon n^{1/2}} t^{-(p+1)}\mathcal{R}\ell h_n(t)\, dt \right| + O(n^{-(p+1)/2}). \tag{3.7}$$

Next observe that

$$\log \phi(t) = \phi(t) - 1 + \textstyle\sum_{j=2}^{k}(-1)^{j+1}\{\phi(t) - 1\}^j/j + O(|t|^{2(k+1)});$$
$$\{\phi(t) - 1\}^r = \{\textstyle\sum_{j=2}^{2k} \mu_j(it)^j/j!\}^r + O(|t|^{2(r-1)+2k});$$

and

$$\textstyle\sum_{r=1}^{k} (-1)^{r+1}\{\sum_{j=2}^{2k} \mu_j(it)^j/j!\}^r/r = \sum_{j=2}^{2k} \kappa_j(it)^j/j! + \kappa'_{2k+1}(it)^{2k+1}/(2k+1)! + O(|t|^{2k+2})$$

as $t \to 0$, where $\mu_j = E(X^j)$ and κ'_{2k+1} is defined to equal the coefficient of $(it)^{2k+1}/(2k+1)!$ on the left hand side of the last expression. Therefore with $\chi(t) = \phi(t) - 1 - \sum_{j=2}^{2k} \mu_j(it)^j/j!$, we have

$$\log\phi(t) = \chi(t) + \textstyle\sum_{j=2}^{2k} \kappa_j(it)^j/j! + \kappa'_{2k+1}(it)^{2k+1}/(2k+1)! + O(|t|^{2k+2})$$

as $t \to 0$. Setting $a_n(t) = \exp\{-n \sum_2^{2k} \kappa_j(it/n^{1/2})^j/j!\}$ we have

$$a_n(t)\phi^n(t/n^{1/2}) = \exp[n\{\chi(t/n^{1/2}) + \kappa'_{2k+1}(it/n^{1/2})^{2k+1}/(2k+1)!\} + r_{n1}(t)]$$
$$= 1 + n\{\chi(t/n^{1/2}) + \kappa'_{2k+1}(it/n^{1/2})^{2k+1}/(2k+1)!\}$$
$$+ r_{n2}(t)\,|n\{\chi(t/n^{1/2}) + \kappa'_{2k+1}(it/n^{1/2})^{2k+1}/(2k+1)!\}|^2 + r_{n3}(t),$$

where $|r_{n1}(t)| \le Cn|t/n^{1/2}|^{2k+2}$,

$$|r_{n2}(t)| \le r_{n4}(t) = C\exp\{n|\chi(t/n^{1/2}) + \kappa'_{2k+1}(it/n^{1/2})^{2k+1}/(2k+1)!| + |r_{n1}(t)|\}$$

and $|r_{n3}(t)| \le |r_{n1}(t)|r_{n4}(t)$ for $|t| \le \varepsilon n^{1/2}$. If ε is chosen sufficiently small then $|r_{n1}(t)| \le Cn^{-k}|t|^{2k+2}$ and $r_{n4}(t) \le Ce^{t^2/10}$ for $|t| \le \varepsilon n^{1/2}$. (Note that $|\chi(t) + \kappa'_{2k+1}(it)^{2k+1}/(2k+1)!| = o(|t|^{2k})$ as $t \to 0$.) Furthermore, if $k \ge 2$ then

$$r_{n5}(t) \equiv n^2|\chi(t/n^{1/2}) + \kappa'_{2k+1}(it/n^{1/2})^{2k+1}/(2k+1)!|^2$$
$$\le C_1n^2([E\{|tX/n^{1/2}|^{2k+2}I(|X| \le n^{1/2}) + |tX/n^{1/2}|^{2k}I(|X| > n^{1/2})\}]^2$$
$$+ [E\{|tX/n^{1/2}|^{2k+1}I(|X| \le n^{1/2}) + |tX/n^{1/2}|^{2k-1}I(|X| > n^{1/2})\}]^2$$
$$+ |t/n^{1/2}|^{4k+2})$$

$$\leq C_2(t^{4k-2} + t^{4k+4})[n^{-k+1/2}E\{|X|^{2k+1}I(|X| \leq n^{1/2})\}$$
$$+ n^{1-k}E\{X^{2k}I(|X| > n^{1/2})\}]^2.$$

Since $E\{X^{2k}I(|X| > n^{1/2})\} \leq n^{k-p/2}E\{|X|^pI(|X| > n^{1/2})\}$ for $2k < p < 2k + 2$, we may deduce from Lemma 2 that $r_{n5}(t) \leq C(t^{4k-2} + t^{4k+4})[o(1)\delta_{pk}(n) + n^{-k}]$. In the case $k = 1$ a similar estimate yields an upper bound of

$$C(t^4 + t^8)[o(1)\delta_{p1}(n) + n^{-1}],$$

and combining the estimate above we deduce that

$$a_n(t)\phi^n(t/n^{1/2}) = 1 + n\{\chi(t/n^{1/2}) + \kappa'_{2k+1}(it/n^{1/2})^{2k+1}/(2k + 1)!\} + r_{n6}(t)$$

where $|r_{n6}(t)| \leq C(t^{2k+2} + t^{4k+4})[o(1)\delta_{pk}(n) + n^{-k}]e^{t^2/10}$ if $|t| \leq \varepsilon n^{1/2}$ and ε is sufficiently small.

Next observe that

$$\{a_n(t)\}^{-1} = e^{-t^2/2}\{1 + \textstyle\sum_1^{2k-1} P_j(it)/n^{j/2} + r_{n7}(t)\}$$

where $|r_{n7}(t)| \leq Cn^{-k}t^{2k+2}e^{t^2/10}$ if $|t| \leq \varepsilon n^{1/2}$ and ε is sufficiently small. The polynomial P_{2k-1} is odd, and so

$$\mathscr{R}\ell[\phi^n(t/n^{1/2}) - e^{-t^2/2}\{1 + \textstyle\sum_1^{2k-2} P_j(it)/n^{j/2}\}] = \mathscr{R}\ell[n\{a_n(t)\}^{-1}\chi(t/n^{1/2})] + r_{n8}(t) \tag{3.8}$$

where

$$|r_{n8}(t)| = |\mathscr{R}\ell[\{a_n(t)\}^{-1}\{n\kappa'_{2k+1}(it/n^{1/2})^{2k+1}/(2k + 1)! + r_{n6}(t)\} + r_{n7}(t)e^{-t^2/2}]| \leq Ct^{2k+2}\{o(1)\delta_{pk}(n) + n^{-k}\}e^{-t^2/5} \tag{3.9}$$

if $|t| \leq \varepsilon n^{1/2}$ and ε is sufficiently small. Now,

$$|\{a_n(t)\}^{-1} - e^{-t^2/2}| \leq e^{-t^2/2}\{n \textstyle\sum_3^{2k} |\kappa_j(t/n^{1/2})^j|/j!\}\exp\{n \sum_3^{2k} |\kappa_j(t/n^{1/2})^j|/j!\}$$
$$\leq Cn^{-1/2}|t|^3e^{-2t^2/5}$$

for $|t| \leq \varepsilon n^{1/2}$. Moreover,

$$|\chi(t/n^{1/2})| \leq E\{(|tX/n^{1/2}|^{2k+2} + |tX/n^{1/2}|^{2k+1})I(|X| \leq n^{1/2})\} + E\{(|tX/n^{1/2}|^{2k}$$
$$+ |tX/n^{1/2}|^{2k-1})I(|X| > n^{1/2})\}$$
$$\leq C(|t|^{2k-1} + t^{2k+2})[n^{-(2k+1)/2}E\{|X|^{2k+1}I(|X| \leq n^{1/2})\}$$
$$+ n^{-k}E\{X^{2k}I(|X| > n^{1/2})\}].$$

From the last two estimates and Lemma 2 we may deduce that

$$n|\{a_n(t)\}^{-1} - e^{-t^2/2}||\chi(t/n^{1/2})| \leq Ct^{2k+2}\{o(1)\delta_{pk}(n) + n^{-k}\}e^{-t^2/5},$$

and combining this with (3.8) and (3.9) we obtain

$$\mathscr{R}\ell[\phi^n(t/n^{1/2})e^{-t^2/2}\{1 + \textstyle\sum_1^{2k-2} P_j(it)/n^{j/2}\}] = \mathscr{R}\ell n\chi(t/n^{1/2})e^{-t^2/2} + r_{n9}(t)$$

where $|r_{n9}(t)| \leq Ct^{2k+2}\{o(1)\delta_{pk}(n) + n^{-k}\}e^{-t^2/5}$ for $|t| \leq \varepsilon n^{1/2}$. Substituting into (3.7) we find that

$$k_p|E|S_n/n^{1/2}|^p - E|Z|^p - \textstyle\sum_{j=1}^{k-1} n^{-j}\displaystyle\int_{-\infty}^{\infty} |x|^p\, dP_{2j}(-\Phi)(x)| + o(1)\delta_{pk}(n)$$
$$+ O(n^{-\min\{k,p+1)/2\}}) = n\left|\int_0^{\infty} t^{-(p+1)}E\{\cos(tX/n^{1/2}) - \textstyle\sum_{j=0}^{k} (-1)^j(tX/n^{1/2})^{2j}/(2j)!\}e^{-t^2/2}\, dt\right|. \tag{3.10}$$

Since

$$(3.11)\qquad Cz^{2k}\min(1, z^2) \le (-1)^{k+1}\{\cos z - \textstyle\sum_{j=0}^{k} (-1)^j z^{2j}/(2j)!\} \le z^{2k}\min(1, z^2)$$

for real z, then

$$(3.12)\qquad 0 \le (-1)^{k+1} n \int_0^\infty t^{-(p+1)} E\{\cos(tX/n^{1/2}) - \textstyle\sum_{j=0}^{k} (-1)^j (tX/n^{1/2})^{2j}/(2j)!\} e^{-t^2/2}\, dt$$

$$(3.13)\qquad \le E\{n^{-k}X^{2k+2}\int_0^{n^{1/2}/|X|} t^{2k+1-p}e^{-t^2/2}\, dt + n^{1-k}X^{2k}\int_{n^{1/2}/|X|}^{\infty} t^{2k-p-1}e^{-t^2/2}\, dt\}.$$

Now, $\int_0^z t^{2k+1-p}e^{-t^2/2}\, dt \le C \min(1, z^{2k+2-p})$, and

$$\int_z^\infty t^{2k-p-1}e^{-t^2/2}\, dt \le \begin{cases} C\min(1, z^{-2}) & \text{if } 0 < p < 2k \\ Cz^{2k-p}\min(1, z^{p-2k-2}) & \text{if } 2k < p < 2k+2, \end{cases}$$

so that the quantity in (3.13) is dominated by

$$CE\{n^{-k}X^{2k+2}I(|X| \le n^{1/2}) + n^{1-p/2}|X|^pI(|X| > n^{1/2})\} + CR_{pk}(n)$$

where $R_{pk}(n) = n^{1-k}E\{X^{2k}I(|X| > n^{1/2})\}$ if $0 < p < 2k$; 0 if $2k < p < 2k+2$. The upper bound (2.9) follows on substituting these estimates into (3.10). To derive (2.10) observe that in view of (3.11), the term on the right in (3.12) dominates a constant multiple of

$$E\{n^{-k}X^{2k+2}I(|X| \le n^{1/2})\int_0^1 t^{2k+1-p}e^{-t^2/2}\, dt +$$
$$n^{1-k}X^{2k}I(|X| > n^{1/2})\int_{n^{1/2}/|X|}^{\infty} t^{2k-p-1}e^{-t^2/2}\, dt\}.$$

An argument similar to that above may now be used to prove (2.10).

PROOF OF THEOREM 3. Assume first that $2 < p < 4$. The upper bound (2.7) may be obtained directly from Theorem 1 by noting that

$$\begin{aligned} E|(S_n - d_n)/c_n - S_n/n^{1/2}|^p &\le C_1\{(1-\sigma_n^2)^pE|S_n/n^{1/2}|^p + |d_n/n^{1/2}|^p\} \\ &\le C_2E\{X^2I(|X| > \lambda n^{1/2})\} \\ &\le C_3[n^{-1}E\{X^4I(|X| \le n^{1/2})\} \\ &\quad + n^{1-p/2}E\{|X|^pI(|X| > n^{1/2})\}]. \end{aligned}$$

In the remainder of the proof we shall make use of the fact that for any $\lambda_1 \ge \lambda_2$,

$$\begin{aligned} &n^{-1}E\{X^4I(|X| \le \lambda_1n^{1/2})\} + n^{1-p/2}E\{|X|^pI(|X| > \lambda_1n^{1/2})\} \\ &\quad \le n^{-1}E\{X^4I(|X| \le \lambda_2n^{1/2})\} + n\lambda_1^4E\{|X/\lambda_1n^{1/2}|^pI(\lambda_2n^{1/2} < |X| \le \lambda_1n^{1/2})\} \\ &\quad + n^{1-p/2}E\{|X|^pI(|X| > \lambda_1n^{1/2})\} \\ &\quad \le C[n^{-1}E\{X^4I(|X| \le \lambda_2n^{1/2})\} + n^{1-p/2}E\{|X|^pI(|X| > \lambda_2n^{1/2})\}]. \end{aligned}$$

A similar argument may be used to obtain the same inequality for $\lambda_1 < \lambda_2$. In both cases we assume that $0 < p < 4$.

We turn now to the lower bound (2.8) for $2 < p < 4$. Write $\phi(t) = \exp[-\frac{1}{2}t^2\{1 + \gamma(t)\}]$ and let $\xi_n(t) = \frac{1}{2}t^2[1 - nc_n^{-2}\{1 + \gamma(t/c_n)\}] - itd_n/c_n$. The characteristic function of $(S_n - d_n)/c_n$ may be written as

$$(3.14)\qquad \psi_n(t) = e^{-t^2/2}\{1 + \xi_n(t) + \tfrac{1}{2}\xi_n^2(t) + \tfrac{1}{6}\xi_n^3(t) + r_{n10}(t)|\xi_n^4(t)|\}$$

where $|r_{n10}(t)| \le \exp|\xi_n(t)| \le \exp[\frac{1}{2}t^2\{|1 - nc_n^{-2}| + nc_n^{-2}|\gamma(t/c_n)|\} + |td_n/c_n|]$. Since $d_n/c_n \to 0$ then if n is sufficiently large, ε sufficiently small and $|t| \le \varepsilon n^{1/2}$, $|r_{n10}(t)| \le \exp(t^2/5 + |t|) \le C\exp(t^2/4)$. Furthermore, $-\frac{1}{2}n(t/c_n)^2\{1 + \gamma(t/c_n)\} = n\log\phi(t/c_n) = n\{\phi(t/c_n) - 1\} + r_{n11}(t)$ where $|r_{n11}(t)| \le Cn^{-1}t^4$ if $|t| \le \varepsilon n^{1/2}$. Therefore with

$$\eta_n(t) = n\{\phi(t/c_n) - 1 + \tfrac{1}{2}\sigma_n^2(t/c_n)^2\} - itd_n/c_n$$

we may deduce from (3.14) that

$$\psi_n(t) = e^{-t^2/2}\left\{1 + \eta_n(t) + \frac{\eta_n^2(t)}{2} + \frac{\eta_n^3(t)}{6}\right\} + r_{n12}(t) \tag{3.15}$$

where $|r_{n12}(t)| \le C\{|\eta_n(t)|^4 + n^{-1}(t^2 + t^{16})\}e^{-t^2/2}$ for $|t| \le \varepsilon n^{1/2}$.

In view of Lemma 1, (3.6) and (3.15),

$$\begin{aligned}\Delta_n &\equiv k_p|E|(S_n - d_n)/c_n|^p - E|Z|^p| \\ &= \left|\int_0^\infty t^{-(p+1)}\mathscr{R}\ell[\psi_n(t) + \tfrac{1}{2}nt^2\{(1-\sigma_n^2)/c_n^2 + \mu_n^2/\sigma_n^2\} - e^{-t^2/2}]\,dt\right| \\ &= \left|\int_0^{\varepsilon n^{1/2}} t^{-(p+1)}\mathscr{R}\ell[\eta_n(t) + \tfrac{1}{2}\eta_n^2(t) + \tfrac{1}{6}\eta_n^3(t)\right. \\ &\qquad \left. + \tfrac{1}{2}nt^2\{(1-\sigma_n^2)/c_n^2 + \mu_n^2/\sigma_n^2\}e^{t^2/2}]e^{-t^2/2}dt\right| + r_{n13}\end{aligned} \tag{3.16}$$

where

$$|r_{n13}| \le C\left\{\int_0^{\varepsilon n^{1/2}} t^{-(p+1)}|\eta_n(t)|^4 e^{-t^2/4}dt + n^{-1}\right\}. \tag{3.17}$$

Now,

$$\begin{aligned}&\mathscr{R}\ell\eta_n(t) + \tfrac{1}{2}nt^2\{(1-\sigma_n^2)/c_n^2 + \mu_n^2/\sigma_n^2\}e^{-t^2/2} \\ &\quad = nE[\{\cos(tX/c_n) - 1 + \tfrac{1}{2}(tX/c_n)^2\}I(|X| \le \lambda n^{1/2}) \\ &\qquad + \{\cos(tX/c_n) - 1 + \tfrac{1}{2}(tX/c_n)^2e^{t^2/2}\}I(|X| > \lambda n^{1/2})] + \tfrac{1}{2}t_n^2e^{t^2/2}n\mu_n^2/\sigma_n^2,\end{aligned}$$

and

$$\begin{aligned}\mathscr{R}\ell\eta_n^2(t) &= (nE[\{\cos(tX/c_n) - 1 + \tfrac{1}{2}(tX/c_n)^2\}I(|X| \le \lambda n^{1/2}) \\ &\quad + \{\cos(tX/c_n) - 1\}I(|X| > \lambda n^{1/2})])^2 \\ &\quad - (n[E\{\sin(tX/c_n) - (tX/c_n)\} - t\mu_n/c_n])^2 \\ &= -t^2n\mu_n^2/\sigma_n^2 + r_{n14}(t),\end{aligned}$$

where in view of Lemma 2,

$$\begin{aligned}|r_{n14}(t)| &\le C_1([nE\{|tX/c_n|^4I(|X| \le \lambda n^{1/2}) + |tX/c_n|^2I(|X| > \lambda n^{1/2})\}]^2 \\ &\quad + [nE\{|tX/c_n|^3I(|X| \le \lambda n^{1/2}) + |tX/c_n|^2I(|X| > \lambda n^{1/2})\}]^2 \\ &\quad + |t\mu_n/c_n|n^2E\{|tX/c_n|^3I(|X| \le \lambda n^{1/2}) + |tX/c_n|^{\min(p,3)}I(|X| > \lambda n^{1/2})\}) \\ &\le C_2(|t|^{p+1} + t^4 + t^8) \\ &\quad \cdot[o(1)\delta_p^*(n) + n^{-1/2}E\{X^2I(|X| > \lambda n^{1/2})\}E\{|X|^3I(|X| < \lambda n^{1/2})\} + n^{-1}].\end{aligned}$$

But by Lemma 2, since $E\{X^2I(|X| > \lambda n^{1/2})\} \le C\delta_p^*(n)$,

$$\begin{aligned}[E\{X^2I(|X| > \lambda n^{1/2})\}n^{-1/2}E\{|X|^3I(|X| \le \lambda n^{1/2})\}]^2 &= o(1)\delta_p^*(n)[n^{-1} + n^{-1}E\{X^4I(|X| \\ &\quad \le \lambda n^{1/2})\}] = o[\{n^{-1} + \delta_p^*(n)\}^2].\end{aligned}$$

Therefore

$$|r_{n14}(t)| \le C(|t|^{p+1} + t^4 + t^8)\{o(1)\delta_p^*(n) + n^{-1}\}.$$

A similar argument leads to the estimate

$$|\mathcal{R}\ell\eta_n^3(t)| \le |\mathcal{R}\ell\eta_n(t)|^3 + 3|\mathcal{R}\ell\eta_n(t)||\mathcal{I}m\eta_n(t)|^2 \le C(t^4 + t^{16})\{o(1)\delta_p^*(n) + n^{-1}\}.$$

Combining the results from (3.16) down we may deduce that

$$\Delta_n = n\int_0^\infty t^{-(p+1)}(E[\{\cos(tX/c_n) - 1 + \tfrac{1}{2}(tX/c_n)^2\}I(|X| \le \lambda n^{1/2})$$
$$+ \{\cos(tX/c_n) - 1 + \tfrac{1}{2}(tX/c_n)^2 e^{t^2/2}\}II|X| > \lambda n^{1/2})]$$
$$+ \tfrac{1}{2}t^2(e^{t^2/2} - 1)n\mu_n^2/\sigma_n^2)e^{-t^2/2}dt + o(1)\delta_p^*(n) + O(n^{-1}) + r_{n13}.$$

The estimate (3.17) and techniques like those used just above may be used to prove that $r_{n13} = o(1)\delta_p^*(n) + O(n^{-1})$, whence

$$\Delta_n + o(1)\delta_p^*(n) + O(n^{-1}) \ge n\int_o^\infty t^{-(p+1)}E\{\cos(tX/n^{1/2}) - 1 + \tfrac{1}{2}(tX/n^{1/2})^2\}e^{-t^2/2}\,dt.$$

The lower bound (2.8) follows from this estimate in the same way that we proved (2.10) from (3.10).

We now consider $0 < p < 2$. In this case we may use (3.6) and the estimates from (3.15) down to prove that

$$(3.18)\qquad k_p|E|(S_n - d_n)/c_n|^p - E|Z|^p| = \left|\int_0^\infty t^{-(p+1)}\{\mathcal{R}\ell\psi_n(t) - e^{-t^2/2}\}\,dt\right.$$
$$= \left|\int_0^{\epsilon n^{1/2}} t^{-(p+1)}\mathcal{R}\ell\eta_n(t)e^{-t^2/2}dt\right| + r_{n14}$$

where $|r_{n14}| \le C\{\int_0^{\epsilon n^{1/2}} t^{-(p+1)}|\eta_n(t)|^2 e^{-t^2/2}dt + n^{-\min\{1,(p+1)/2\}}\}$. Now,

$$\eta_n(t) = nE[\{\cos(tX/c_n) - 1 + \tfrac{1}{2}(tX/c_n)^2\}I(|X| \le \lambda n^{1/2})$$
$$- \{1 - \cos(tX/c_n)\}I(|X| > \lambda n^{1/2})]$$
$$+ inE[\{\sin(tX/c_n) - (tX/c_n)\}I(|X| \le \lambda n^{1/2}) + \sin(tX/c_n)I(|X| > \lambda n^{1/2})],$$

and so

$$|\eta_n(t)|^2 \le C([nE\{(|tX/c_n|^3 + |tX/c_n|^4)I(|X| \le \lambda n^{1/2})\}]^2$$
$$+ n^2E\{|tX/c_n|^2 I(|X| > \lambda n^{1/2})\}P(|X| > \lambda n^{1/2})).$$

Since $nP(|X| > \lambda n^{1/2}) \le C\delta_p^*(n)$ then $|\eta_n(t)|^2 \le (t^2 + t^8)o(1)\delta_p^*(n)$, and we may now deduce from (3.18) that

$$k_p|E|(S_n - d_n)/c_n|^p - E|Z|^p| + o(1)\delta_p^*(n) + O(n^{-\min\{1,(p+1)/2\}})$$
$$(3.19)\qquad = \left|n\int_0^\infty t^{-(p+1)}E[\{\cos(tX/c_n) - 1 + \tfrac{1}{2}(tX/c_n)^2\}I(|X| \le \lambda n^{1/2})\right.$$
$$\left. - \{1 - \cos(tX/c_n)\}I(|X| > \lambda n^{1/2})]e^{-t^2/2}dt\right|.$$

To obtain the upper bound (2.7) observe that

$$n \int_0^\infty t^{-(p+1)} E[\{\cos(tX/c_n) - 1 + \tfrac{1}{2}(tX/c_n)^2\} I(|X| \le \lambda n^{1/2})] e^{-t^2/2} dt$$
(3.20)
$$\le nE\{(X/c_n)^4 I(|X| \le \lambda n^{1/2})\} \int_0^\infty t^{3-p} e^{-t^2/2}\, dt$$

while

$$n \int_0^\infty t^{-(p+1)} E[\{1 - \cos(tX/c_n)\} I(|X| > \lambda n^{1/2})] e^{-t^2/2}\, dt$$
$$\ge n c_n^{-p} \int_0^\infty t^{-(p+1)} E\{(1 - \cos tX) I(|X| > \lambda n^{1/2})\}\, dt$$
(3.21)
$$= n c_n^{-p} E\{|X|^p I(|X| > \lambda n^{1/2})\} \int_0^\infty t^{-(p+1)} (1 - \cos t)\, dt.$$

Therefore the left sides of (3.20) and (3.21) are both dominated by $\delta_p^*(n)$, and (2.7) now follows from (3.19).

Proof of Theorem 4. Observe first that

$$I_n \equiv \int_0^\infty t^{-(p+1)} E[\{\cos(tX/c_n) - 1 + \tfrac{1}{2}(tX/c_n)^2\} I(|X| \le \lambda n^{1/2})] e^{-t^2/2} dt$$
$$= \int_0^{\lambda n^{1/2}/c_n} G(x) dP(|X| \le c_n x)$$
$$= \int_0^{\lambda n^{1/2}/c_n} g(x) P(|X| > c_n x) dx - G(\lambda n^{1/2}/c_n) P(|X| > \lambda n^{1/2}),$$

where $G(x) = \int_o^\infty t^{-(p+1)}\{\cos tx - 1 + \tfrac{1}{2}(tx)^2\} e^{-t^2/2} dt$ and $g(x) = G'(x)$. Now, $n^{1/2} \sim c_n$ as $n \to \infty$, and so in view of Theorem 2.7, page 66 of Seneta (1976),

$$\int_0^{\lambda n^{1/2}/c_n} g(x) P(|X| > c_n x) dx \sim P(|X| > n^{1/2}) \int_0^\lambda g(x) x^{-\alpha} dx.$$

Therefore

$$I_n \sim P(|X| > n^{1/2}) \int_0^\lambda g(x) x^{-\alpha} dx - G(\lambda) P(|X| > \lambda n^{1/2})$$
$$\sim \alpha P(|X| > n^{1/2}) \int_0^\lambda G(x) x^{-\alpha-1} dx.$$

A similar argument based on Seneta's Theorem 2.6 shows that

$$\int_0^\infty t^{-(p+1)} E[\{1 - \cos(tX/c_n)\} I(|X| > \lambda n^{1/2})] e^{-t^2/2} dt$$
$$\sim \alpha P(|X| > n^{1/2}) \int_\lambda^\infty x^{-\alpha-1} dx \int_0^\infty t^{-(p+1)} (1 - \cos tx) e^{-t^2/2}\, dt.$$

It is easily proved from Seneta's Theorems 2.6 and 2.7 that under the conditions of Theorem 4, $\delta_p^*(n) \sim CnP(|X| > n^{1/2})$. These estimates together with (3.19) imply that

$$k_p|E|(S_n - d_n)/c_n|^p - E|Z|^p| + o\{nP(|X| > n^{1/2})\} + O(n^{-\min\{1,(p+1)/2\}})$$

$$= \alpha nP(|X| \ge n^{1/2})\left| \int_0^\lambda x^{-\alpha-1}dx \int_0^\infty t^{-(p+1)}\{\cos tx - 1 + \tfrac{1}{2}(tx)^2\}e^{-t^2/2}\,dt - \int_\lambda^\infty x^{-\alpha-1}dx \int_0^\infty t^{-(p+1)}(1 - \cos tx)e^{-t^2/2}dt \right|,$$

from which follows Theorem 4.

PROOF OF THEOREM 2. In the notation used just above, observe that

$$J_n \equiv \int_0^\infty t^{-(p+1)}E\{\cos(tX/n^{1/2}) - 1 + \tfrac{1}{2}(tX/n^{1/2})^2\}e^{-t^2/2}dt$$

$$= \int_0^\infty G(x)dP(|X| \le n^{1/2}x) = \int_0^\infty g(x)P(|X| > n^{1/2}x)dx.$$

Now for positive x,

$$g(x) = \int_0^\infty t^{-p}(tx - \sin tx)e^{-t^2/2}dt \le x^3 \int_0^{1/x} t^{3-p}e^{-t^2/2}dt + 2x\int_{1/x}^\infty t^{1-p}e^{-t^2/2}dt$$

$$\le \begin{cases} C\{x^3\min(1, x^{p-4}) + xe^{-1/4x^2}\} & \text{if } 0 < p < 2 \\ C\{x^3\min(1, x^{p-4}) + x^{p-1}e^{-1/4x^2}\} & \text{if } 2 < p < 4. \end{cases}$$

Therefore whenever $2 \le \alpha \le 4$ and $0 < p \le \alpha$ there exists $\eta > 0$ such that

$$\int_0^\lambda x^{-\alpha-\eta}g(x)dx < \infty \quad \text{for} \quad \lambda > 0,$$

and if $2 < \alpha < 4$ and $0 < p < \alpha$ there exists $\eta > 0$ such that

$$\int_\lambda^\infty x^{-\alpha+\eta}g(x)dx < \infty \quad \text{for} \quad \lambda > 0.$$

In the latter case we may deduce from Theorems 2.6 and 2.7 of Seneta (1976) that

$$J_n \sim \alpha P(|X| > n^{1/2})\int_0^\infty G(x)x^{-\alpha-1}dx. \tag{3.22}$$

If $\alpha = 2$ and $0 < p < 2$ then

$$J_n = E\{G(X/n^{1/2})\} = E\{G(X/n^{1/2})I(|X| > n^{1/2})\} + O[n^{-2}E\{X^4I(|X| \le n^{1/2})\}];$$

$$n^{-2}E\{X^4I(|X| \le n^{1/2})\} = O\{P(|X| > n^{1/2})\} = o[n^{-1}E\{X^2I(|X| > n^{1/2})\}];$$

$$E\{G(X/n^{1/2})I(|X| > n^{1/2})\} = (1/2n)E\{X^2I(|X| > n^{1/2})\}\int_0^\infty t^{1-p}e^{-t^2/2}\,dt - \int_0^\infty t^{-(p+1)}E[\{1 - \cos(tX/n^{1/2})\}I(|X| > n^{1/2})]e^{-t^2/2}\,dt;$$

and if $0 < \varepsilon < 2 - p$,

$$\int_0^\infty t^{-(p+1)} E[\{1 - \cos(tX/n^{1/2})\} I(|X| > n^{1/2})] e^{-t^2/2}\, dt$$

$$\leq E\{|X/n^{1/2}|^{p+\varepsilon} I(|X| > n^{1/2})\} \int_0^\infty t^{\varepsilon - 1} e^{-t^2/2}\, dt = o[n^{-1} E\{X^2 I(|X| > n^{1/2})\}].$$

Therefore

$$J_n \sim (1/2n) E\{X^2 I(|X| > n^{1/2})\} \int_0^\infty t^{1-p} e^{-t^2/2}\, dt. \tag{3.23}$$

Finally, if $2 < p = \alpha < 4$ then arguing as just above we may prove that for any $\lambda > 0$,

$$J_n = E\{G(X/n^{1/2}) I(|X| > \lambda n^{1/2})\} + O\{P(|X| > n^{1/2})\}.$$

It is easily shown that $G(x) \sim |x|^p \int_0^\infty t^{-(p+1)}(\cos t - 1 + \frac{1}{2} t^2)\, dt$ as $|x| \to \infty$, and therefore

$$E\{G(x/n^{1/2}) I(|X| > \lambda n^{1/2})\}$$

$$= \{1 + \varepsilon(\lambda, n)\} E\{|X/n^{1/2}|^p I(|X| > \lambda n^{1/2})\} \int_0^\infty t^{-(p+1)} \left(\cos t - 1 + \frac{1}{2} t^2\right) dt$$

where $\varepsilon(\lambda, n)$ may be made arbitrarily small uniformly in n by choosing λ sufficiently large. The function $E\{|X|^p I(|X| > x)\}$ is slowly varying at infinity and so

$$E\{|X/n^{1/2}|^p I(|X| > \lambda n^{1/2})\} \sim E\{|X/n^{1/2}|^p I(|X| > n^{1/2})\} \gg P(|X| > n^{1/2}),$$

whence

$$J_n \sim n^{-p/2} E\{|X|^p I(|X| > n^{1/2})\} \int_0^\infty t^{-(p+1)} \left(\cos t - 1 + \frac{1}{2} t^2\right) dt. \tag{3.24}$$

Theorem 2 follows on combining (3.10) in the case $k = 1$ with (3.22) through (3.24).

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