

BOUNDS ON THE SPECTRAL RADIUS OF A  
HADAMARD PRODUCT OF NONNEGATIVE OR  
POSITIVE SEMIDEFINITE MATRICES\*

ROGER A. HORN<sup>†</sup> AND FUZHEN ZHANG<sup>‡</sup>

**Abstract.** X. Zhan has conjectured that the spectral radius of the Hadamard product of two square nonnegative matrices is not greater than the spectral radius of their ordinary product. We prove Zhan's conjecture, and a related inequality for positive semidefinite matrices, using standard facts about principal submatrices, Kronecker products, and the spectral radius.

**Key words.** Hadamard product, Nonnegative matrix, Positive semidefinite matrix, Positive definite matrix, Spectral radius, Kronecker product, Matrix inequality.

**AMS subject classifications.** 15A45, 15A48, 15A69.

**1. Introduction.** Many functionals in matrix analysis are submultiplicative with respect to ordinary matrix multiplication, but the spectral radius is not. However, for nonnegative or positive semidefinite matrices  $A, B \in M_n$ , the spectral radius is submultiplicative with respect to the Hadamard (entry-wise) product:

$$\rho(A \circ B) \leq \rho(A)\rho(B) \tag{1.1}$$

(see Theorem 5.3.4 and Observation 5.7.4 in [4]).

A different upper bound for the Hadamard product of nonnegative matrices,

$$\rho(A \circ B) \leq \rho(AB), \tag{1.2}$$

was conjectured recently by X. Zhan [5] and proved by K.M.R. Audenaert [2].

The bound (1.2) is not correct for positive semidefinite matrices. For example, the matrices

$$A = \begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2i \\ 2i & 2 \end{bmatrix} \tag{1.3}$$

---

\*Received by the editors November 4, 2009. Accepted for publication February 8, 2010. Handling Editor: Bit-Shun Tam.

<sup>†</sup>Mathematics Department, University of Utah, Salt Lake City, Utah 84112, USA (rhorn@math.utah.edu).

<sup>‡</sup>Division of Math, Science and Technology, Nova Southeastern University, Fort Lauderdale, Florida 33314, USA (zhang@nova.edu).

are positive definite, and  $\rho(A \circ B) = 2(2 + \sqrt{2}) = 2\rho(AB)$ . However, if  $A$  is positive semidefinite and  $B$  is positive definite, there is a bound of the form

$$\rho(A \circ B) \leq \beta\rho(AB), \quad \beta = \rho(B^{-1} \circ B). \quad (1.4)$$

For the matrices (1.3),  $\beta = 5$ .

Our matrices are all complex or real, and our notation and terminology are as in [3]. For a given positive integer  $N$ , a nonempty subset  $\alpha$  of  $\{1, \dots, N\}$ , and any  $A \in M_N$ ,  $A[\alpha] \in M_{|\alpha|}$  is the principal submatrix of  $A$  whose rows and columns are indexed by  $\alpha$ . When  $N = n^2$ , the *Hadamard index set* is  $\eta = \{1, n+1, 2n+2, \dots, n^2\}$ . For real matrices  $A = [a_{ij}], B = [b_{ij}] \in M_n$ ,  $A \geq B$  means that  $a_{ij} \geq b_{ij}$  for all  $i, j = 1, \dots, n$ ;  $A > 0$  means that every entry of  $A$  is real and positive. We say that  $A \in M_n$  is nonnegative if  $A$  is real and  $A \geq 0$ . For complex or real Hermitian matrices  $A, B \in M_n$ ,  $A \succeq B$  means that  $A - B$  is positive semidefinite;  $A \succ B$  means that  $A - B$  is positive definite.

The purpose of this paper is to provide parallel approaches to the bounds (1.2) and (1.4). In both cases, we invoke the following basic fact:

**PROPOSITION 1.1.** *The Hadamard product is a principal submatrix of the Kronecker product: if  $A, B \in M_n$ , then  $A \circ B = (A \otimes B)[\eta]$ .*

*Proof.* See [4, Lemma 5.1.1].  $\square$

**2. Nonnegative matrices.** Our approach to the bound (1.2) for nonnegative matrices relies on the following lemma:

**LEMMA 2.1.** *Let  $A, B \in M_N$  be nonnegative and let  $\alpha \subset \{1, \dots, N\}$  be given and nonempty.*

- (1) *If  $A \geq B$ , then  $\rho(A) \geq \rho(B)$ .*
- (2)  *$\rho(A[\alpha]) \leq \rho(A)$ .*
- (3)  *$A[\alpha]B[\alpha] \leq (AB)[\alpha]$ .*
- (4)  *$\rho(A[\alpha]B[\alpha]) \leq \rho((AB)[\alpha]) \leq \rho(AB)$ .*

*Proof.* For the first two assertions, see Corollaries 8.1.19 and 8.1.20 of [3]. One verifies the third assertion with a computation; the fourth assertion follows from the first three.  $\square$

For any nonnegative  $A, B \in M_n$ , use Proposition 1.1 and Lemma 2.1(3) to com-

pute

$$\begin{aligned} (A \circ B)^2 &= (A \circ B)(A \circ B) = (A \circ B)(B \circ A) \\ &= (A \otimes B)[\eta](B \otimes A)[\eta] \leq ((A \otimes B)(B \otimes A))[\eta] \\ &= (AB \otimes BA)[\eta] = AB \circ BA. \end{aligned}$$

Now invoke assertions (1) and (2) of Lemma 2.1 to obtain

$$\begin{aligned} \rho^2(A \circ B) &= \rho((A \circ B)^2) \leq \rho(AB \circ BA) = \rho((AB \otimes BA)[\eta]) \\ &\leq \rho(AB \otimes BA) = \rho(AB)\rho(BA) = \rho^2(AB). \end{aligned} \quad (2.1)$$

We have proved a little more than (1.2):

**THEOREM 2.2.** *Let  $A, B \in M_n$ . Suppose that  $A \geq 0$  and  $B \geq 0$ . Then*

$$\rho(A \circ B) \leq \rho^{1/2}(AB \circ BA) \leq \rho(AB). \quad (2.2)$$

The following lemma permits us to say something about when the inequality  $\rho(A \circ B) \leq \rho(AB)$  must be strict; for a different proof (via  $M$ -matrices), see [1, Theorem 1.7.4].

**LEMMA 2.3.** *Let  $A = [a_{ij}] \in M_n$  with  $n \geq 2$ , and suppose that  $A$  is nonnegative and irreducible. Let  $\alpha \subsetneq \{1, \dots, n\}$  be nonempty. Then  $\rho(A[\alpha]) < \rho(A)$ .*

*Proof.* The assertion follows immediately from [3, Theorem 8.4.5].  $\square$

The preceding result ensures that the second inequality in (2.1) (and hence also in (2.2)) is strict if  $AB \otimes BA$  is irreducible, in particular, if  $AB \otimes BA > 0$ .

**COROLLARY 2.4.** *Let  $A, B \in M_n$ . Suppose that  $A > 0$  and  $B > 0$ . Then*

$$\rho(A \circ B) < \rho(AB).$$

Audenaert proved (1.2) by establishing a different intermediate inequality:

$$\rho(A \circ B) \leq \rho^{1/2}((A \circ A)(B \circ B)) \leq \rho(AB). \quad (2.3)$$

The right-hand inequality in (2.3) can be verified with the tools used in our proof of Theorem 2.2:

$$\begin{aligned} (A \circ A)(B \circ B) &= (A \otimes A)[\eta](B \otimes B)[\eta] \leq ((A \otimes A)(B \otimes B))[\eta] \\ &= (AB \otimes AB)[\eta], \end{aligned}$$

and hence,

$$\rho((A \circ A)(B \circ B)) \leq \rho((AB \otimes AB)[\eta]) \leq \rho(AB \otimes AB) = \rho^2(AB).$$

The left-hand inequality in (2.3) appears to be deeper; there does not seem to be an entry-wise inequality involved. Audenaert's proof involves the characterization  $\rho(A) = \lim_{m \rightarrow \infty} (\text{tr } A^m)^{1/m}$  (for  $A > 0$ ) and an ingenious use of the Cauchy-Schwarz inequality.

**3. Positive semidefinite matrices.** Our approach to the bound (1.4) for positive semidefinite matrices relies on the following lemma:

LEMMA 3.1. *Let  $A, B, C \in M_N$  be positive semidefinite and let  $\alpha \subset \{1, \dots, N\}$  be given and nonempty.*

- (1) *If  $A \succeq B$ , then  $\rho(A) \geq \rho(B)$ .*
- (2)  *$\rho(A[\alpha]) \leq \rho(A)$ .*
- (3) *If  $B \succ 0$ , then  $\rho(A[\alpha]B[\alpha]^{-1}) \leq \rho(AB^{-1})$ .*
- (4) *If  $A \succeq B$ , then  $\rho(BC) \leq \rho(AC)$ .*

*Proof.* The first assertion is a special case of the monotonicity theorem [3, Corollary 4.3.3]: If the eigenvalues of  $A$  and  $B$ , respectively, are arranged in nondecreasing order, then each eigenvalue of  $A$  is not less than the corresponding eigenvalue of  $B$ . The second assertion is a special case of eigenvalue interlacing for Hermitian matrices [3, Theorem 4.3.15]. The third assertion follows from the Raleigh-Ritz Theorem [3, Theorem 4.2.2] and the fact that for any  $E, F \in M_n$ ,  $EF$  has the same eigenvalues as  $FE$  [3, Theorem 1.3.20]:

$$\begin{aligned} \rho(AB^{-1}) &= \rho(AB^{-1/2}B^{-1/2}) = \rho(B^{-1/2}AB^{-1/2}) \\ &= \max_{x \neq 0} \frac{x^* B^{-1/2} A B^{-1/2} x}{x^* x} = \max_{y \neq 0} \frac{y^* A y}{(B^{1/2} y)^* (B^{1/2} y)} \\ &= \max_{y \neq 0} \frac{y^* A y}{y^* B y} \geq \max_{\substack{y = [y_i] \neq 0 \\ y_i = 0 \text{ if } i \notin \alpha}} \frac{y^* A y}{y^* B y} = \max_{\substack{z \neq 0 \\ z \in \mathbb{C}^{|\alpha|}}} \frac{z^* A[\alpha] z}{z^* B[\alpha] z} \\ &= \max_{z \neq 0} \frac{z^* A[\alpha] z}{(B[\alpha]^{1/2} z)^* (B[\alpha]^{1/2} z)} = \max_{w \neq 0} \frac{w^* (B[\alpha]^{-1/2} A[\alpha] B[\alpha]^{-1/2}) w}{w^* w} \\ &= \rho(B[\alpha]^{-1/2} A[\alpha] B[\alpha]^{-1/2}) = \rho(A[\alpha] B[\alpha]^{-1}). \end{aligned}$$

The final assertion follows from the first assertion and the implication  $A \succeq B \Rightarrow SAS^* \succeq SBS^*$  for any  $S \in M_n$  [3, Observation 7.7.2]:

$$\rho(BC) = \rho(BC^{1/2}C^{1/2}) = \rho(C^{1/2}BC^{1/2}) \leq \rho(C^{1/2}AC^{1/2}) = \rho(AC). \quad \square$$

We can now prove (1.4).

**THEOREM 3.2.** *Let  $A, B \in M_n$ . Suppose that  $A \succeq 0$  and  $B \succ 0$ . Then*

$$\rho(A \circ B) \leq \beta \rho(AB), \quad \beta = \rho(B^{-1} \circ B) \geq 1. \quad (3.1)$$

*Proof.* First observe that  $B^{-1} \circ B$  is nonsingular [3, Theorem 7.5.3], and since  $B^{-1} \circ B \succeq I$  [3, Theorem 7.7.9(c)], it follows that  $\beta \geq 1$ . Moreover,  $\beta I \succeq B^{-1} \circ B$ , so  $\beta(B^{-1} \circ B)^{-1} \succeq I$  [3, Corollary 7.7.4]. Successively invoking assertions (4) and (3) of Lemma 3.1, we compute

$$\begin{aligned} \rho((A \circ B)I) &\leq \beta \rho((A \circ B)(B^{-1} \circ B)^{-1}) = \beta \rho((A \otimes B)[\eta]((B^{-1} \otimes B)[\eta])^{-1}) \\ &\leq \beta \rho((A \otimes B)(B^{-1} \otimes B)^{-1}) = \beta \rho((A \otimes B)(B \otimes B^{-1})) \\ &= \beta \rho(AB \otimes I) = \beta \rho(AB). \quad \square \end{aligned}$$

The example

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \rho(A \circ B) = 2, \quad \rho(AB) = 0$$

shows that if both  $A$  and  $B$  are positive semidefinite and singular, there need not be any inequality of the form  $\rho(A \circ B) \leq \beta \rho(AB)$ ,  $\beta > 0$ .

The example  $A = B = I$  and  $\beta = 1$  shows that equality is possible in (3.1).

#### REFERENCES

- [1] R.B. Bapat and T.E.S. Raghavan. *Nonnegative Matrices and Applications*. Encyclopedia of Mathematics and its Applications, vol. 64, Cambridge University Press, Cambridge, UK, 1997.
- [2] K.M.R. Audenaert. Spectral radius of Hadamard product versus conventional product for non-negative matrices. *Linear Algebra Appl.*, 432(1):366–368, 2010.
- [3] R. Horn and C. Johnson. *Matrix Analysis*. Cambridge University Press, New York, 1985.
- [4] R. Horn and C. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, New York, 1991.
- [5] X. Zhan. Unsolved matrix problems. Presented at Advanced Workshop on Trends and Developments in Linear Algebra, ICTP, Trieste, Italy, July 6, 2009.