Bounds on Information Propagation in Disordered Quantum Spin Chains

Christian K. Burrell and Tobias J. Osborne
Department of Mathematics, Royal Holloway University of London, Egham, Surrey, TW20 0EX, UK

We investigate the propagation of information through the disordered XY model. We find, with a probability that increases with the size of the system, that all correlations, both classical and quantum, are suppressed outside of an effective lightcone whose radius grows at most polylogarithmically with $|t|$.

PACS numbers: 75.10.Pq, 72.15.Rn, 03.67.-a, 75.40.Mg

How fast can information propagate through a locally interacting system? For classical systems an essentially universal answer to this question is that the velocity of information propagation is bounded (often only approximately) by an effective *speed of light*. It is a more subtle issue to formulate equivalent velocity bounds for quantum systems because they can encode *quantum information* in the form of *qubits* and therefore might be able to exploit quantum interference to propagate information faster. However, for quantum spin networks this is not the case: the *Lieb-Robinson bound* limits the velocity at which correlations can propagate [1].

The Lieb-Robinson bound implies that there is an effective light cone for two-point dynamical correlations, i.e., apart from an exponentially suppressed tail, two-point correlations propagate no faster than a speed of light. Simplified and alternative proofs of the Lieb-Robinson bound have been subsequently discovered [2–5]. More recently, it has been realised that the Lieb-Robinson bound is strong enough to bound not only the propagation of two-point correlations but of any local encoding of information: regardless of the encoding no information (either quantum or classical) can propagate faster than the speed of light for the system [6].

There are many consequences of the Lieb-Robinson bound. Apart from the aforementioned bounds on the velocity of information propagation, it has been realised that the Lieb-Robinson bound can be used to provide a method to efficiently simulate the properties of low-dimensional spin networks [7–11]. Additionally, using the Lieb-Robinson bound, dynamical entropy area laws for quantum spin systems can be obtained [6, 12].

While the Lieb-Robinson bound is extremely general — it relies only on the ultraviolet cutoff imposed by lattice structure — it is, as a consequence, relatively weak. Thus, it is extremely desirable to develop stronger bounds constraining the propagation of quantum information through systems where more is known about the structure of the interactions. One setting where one would expect stronger bounds to be available is that of a quantum spin system with disordered interactions. Such systems have attracted a large amount of interest as they can exhibit the striking phenomenon of *Anderson localisation* [13], which means that information is essentially frozen: a quantum particle placed anywhere within a localised system diffuses only slightly, even for extremely large times. Thus, exploiting the parallels between bounds on information propagation and Lieb-Robinson bounds, we are motivated to conjecture that interacting spin systems with disordered interactions satisfy stronger bounds on correlation propagation (see Fig. 1). More specifically, we conjecture that for quantum spin networks with disordered interactions all correlations, both quantum and classical, are suppressed outside of a light cone whose radius grows at most *polylogarithmically* in time. (Contrast this with the light cone supplied by the Lieb-Robinson bound: it has a radius which grows linearly with time.)

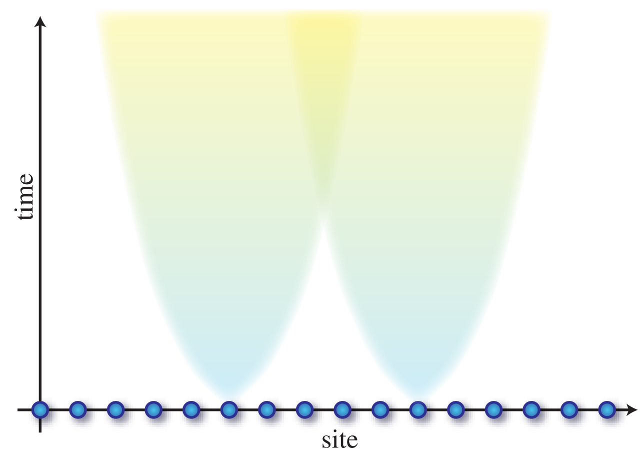

FIG. 1. Schematic illustration of the conjectured *logarithmic light cone* for disordered systems: as time progresses information is exponentially attenuated outside of a light cone whose radius grows at most logarithmically with time.

In this Letter we study a setting where the dynamics of a class of disordered interacting spin systems can be shown to satisfy our polylogarithmic light-cone conjecture; we study the XY spin chain with disordered interactions in a disordered magnetic field and show that information, and hence correlations, are attenuated outside of a light cone whose radius grows polylogarithmically with time. The main result of this Letter, a *polylogarithmic light cone* for the disordered XY model, can by summarised with the following bound on the dynamic two-point correlation functions:

$$\left\|\left[A_j, e^{itH}B_ke^{-itH}\right]\right\| \leq c_\zeta n^2|t|e^{-\gamma_\zeta|j-k|^\zeta}, \quad (1)$$

which holds for any $0 < \zeta < 1$ with n larger than a constant depending on ζ and γ_ζ, with probability $p \geq 1 - e^{-n^\zeta}$, where A_j and B_k are local operators acting nontrivially only on spins j and k respectively, and γ_ζ and c_ζ are constants proportional to ζ and the second moment of the disorder distribution. We apply our new bound to study the structure of the propagator for large times and the scaling of the entropy of a block of spins in the evolving system. As a consequence, we prove the entropy saturation numerically observed by De Chiara *et. al.* [24]. Our results also constitute a proof of a conjecture raised in [14]: namely, if two parties, Alice and Bob, have access to a bounded region at either end of the chain, respectively, then it is impossible for Alice to send any information to Bob, regardless of how Alice encodes the information in the single- and higher-excitation sectors.

We consider a one-dimensional chain of n spin-$1/2$ particles with XY-model type interactions between nearest-neighbouring spins in an additional transverse field (e.g. a magnetic B-field). We allow the coefficients of the couplings and the transverse field strength to vary from site to site within the spin chain. Thus, we study the evolution of the chain under the Hamiltonian

$$H = \sum_{j=1}^{n-1} \mu_j \left(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y \right) + \sum_{j=1}^{n} \nu_j \sigma_j^Z, \qquad (2)$$

where μ_j and ν_j are drawn from probability distributions $\mathbb{P}_\mu$ and $\mathbb{P}_\nu$ respectively, and where σ_j^α ($\alpha \in \{X, Y, Z\}$) is a Pauli spin operator acting on the spin at site j. Typically, $\mu_j = -J$ for all j, however this is not necessary and we deal with the more general case here.

We solve this system using the *Jordan-Wigner transform* (for an introduction to the Jordan-Wigner transform see [15]) which, when combined with some exact results from the theory of localisation, allows us to bound the dynamics of our spin chain. We note that it is not immediate that the dynamics of the disordered XY model satisfy a logarithmic light cone: while the XY model is equivalent to a free fermion model which is the original Anderson model, and is localised, the Jordan-Wigner transform is a *nonlocal* operation and might confound the localisation occurring in the free fermion system.

Let's start by applying the Jordan-Wigner transform, which maps a system of interacting qubits into a system of free fermions. The Jordan-Wigner transform defines annihilation operators $a_j = (\sigma_1^Z \cdots \sigma_{j-1}^Z)\sigma_j$ (where $\sigma_j = |0\rangle \langle 1|$ acts on site j) and the corresponding creation operators $a_j^\dagger$, which satisfy the canonical fermionic anticommutation relations. Using this we are able to rewrite the system Hamiltonian as $H = \sum_{j,k=1}^n M_{jk} a_j^\dagger a_k$, where the tridiagonal matrix M is defined via $M_{j,k} = 2\mu_k \delta_{j,k+1} + 2\mu_j \delta_{j,k-1} - 2\nu_j \delta_{j,k}$.

It is now possible (following the method described in [15]) to diagonalise H. After doing so we find the dynamics for the annihilation operators in the Heisenberg picture, with $a_j(t) = e^{itH} a_j e^{-itH}$:

$$a_j(t) = \sum_{k=1}^{n} v_{jk}(t) a_k, \qquad (3)$$

where $v_{jk}(t) = \left(e^{-iMt}\right)_{j,k}$. We now concentrate on bounding the $v_{jk}(t)$, which in turn bounds the dynamics of the system.

The quantity v_{jk} has been well studied in the physical and mathematical literature. At the level of physical rigour it is typically argued that v_{jk} decays, with probability increasing with n, exponentially with separation. That is, $|v_{jk}| \leq ce^{-\gamma|j-k|}$, where c and v are constants depending only on γ, a constant proportional to the (assumed finite) *second moment* of the probability distribution $\mathbb{P}_\nu$ (which we assume is i.i.d.). The mathematical literature hasn't yet achieved results as good as this (although this situation is recently changing, see [16] for recent progress). Instead, the best currently available result is obtained via *bootstrap multiscale analysis* (see, e.g., [17]), and reads

$$|v_{jk}| \leq c_\zeta e^{-\gamma_\zeta |j-k|^\zeta}, \qquad (4)$$

which holds for any $0 < \zeta < 1$ when n is greater than a constant depending only on γ_ζ, with probability $p \geq 1 - e^{-n^\zeta}$, where c_ζ and γ_ζ are constants depending only on ζ and the second moment of the disorder distribution. These results are typically obtained for infinite lattices, however the proof technique may be adapted to show the result in the finite-size case that concerns us here [18].

The inequality Eq. (4) is a quantitative statement of the result that the modulus of the diagonal matrix elements of e^{-iMt} are large, while the modulus of the off diagonal matrix elements decay with distance from the diagonal. This means that $a_j(t)$ is effectively a linear combination of only a small number of a_k operators — namely those for which $|j - k|$ is small. It is this fact which inhibits the spread of operators on the chain, giving rise to the light cone we derive below.

We now turn to the proof of the improved Lieb-Robinson bound for our system. We begin by bipartitioning the spin chain into two sections, A and B, where we assume the boundary between partitions is between spins m and $m+1$. We then attempt to write e^{itH} as a product of e^{itH_A} and e^{itH_B}. Clearly this won't be exact and so we introduce an operator $V(t)$ which bridges the boundary between A and B, and which is designed to compensate for any errors introduced:

$$e^{itH} = e^{it(H_A + H_B)} V(t). \qquad (5)$$

The operator $V(t)$ acts nontrivially on all spins in the chain, however, we now show that $V(t)$ can be well approximated by another operator, which we call $V^\Omega(t)$, which acts only on a small number $|\Omega|$ of spins. The reason we can do this is that $V(t)$ acts strongly on spins which are close to the boundary and progressively weaker on spins as we move away from the boundary. To prove this approximation is valid, we use the following differential equation for $V(t)$:

$$\frac{dV}{dt} = iV(t)h_m(t), \tag{6}$$

where $h_m(t) = e^{-itH}h_m e^{itH}$ and h_m is the interaction term in the Hamiltonian which bridges the boundary. We let Ω denote a set of $|\Omega|$ spins centred on the boundary between the partitions A and B. We also define $h_m^{\Omega}(t) = e^{-itH_\Omega}h_m e^{itH_\Omega}$ where H_Ω contains only those interactions in H which act on sites in Ω. We then define $V^{\Omega}(t)$ via

$$\frac{d}{dt}V^{\Omega}(t) = iV^{\Omega}(t)h_m^{\Omega}(t). \tag{7}$$

Clearly the operator $V^{\Omega}(t)$ acts nontrivially only on Ω.

The error between $V(t)$ and $V^{\Omega}(t)$ is bounded by

$$\left\|V(t) - V^{\Omega}(t)\right\| \leq \int_0^{|t|} \left\|h_m(s) - h_m^{\Omega}(s)\right\| ds. \tag{8}$$

Calculating $\|h_m(t) - h_m^{\Omega}(t)\|$ is a lengthy but straightforward task, and we begin by using the Jordan Wigner transform to write this quantity in terms of the a_j operators:

$$h_m = 2\mu_m \left(a_m^\dagger a_{m+1} - a_m a_{m+1}^\dagger\right) + \frac{\nu_m}{2}\left(a_m a_m^\dagger - a_m^\dagger a_m\right) + \frac{\nu_{m+1}}{2}\left(a_{m+1}a_{m+1}^\dagger - a_{m+1}^\dagger a_{m+1}\right). \tag{9}$$

When we calculate $\|h_m(t) - h_m^{\Omega}(t)\|$ we'll have to deal with terms such as $\|a_m^\dagger(t)a_{m+1}(t) - a_m^{\Omega\dagger}(t)a_{m+1}^{\Omega}(t)\|$, which can be bounded as follows

$$\begin{aligned}\left\|a_m^\dagger(t)a_{m+1}(t) - a_m^{\Omega\dagger}(t)a_{m+1}^{\Omega}(t)\right\| \leq \\ \left\|a_m^\dagger(t)\right\| \left\|a_{m+1}(t) - a_{m+1}^{\Omega}(t)\right\| \\ + \left\|a_m^\dagger(t) - a_m^{\Omega\dagger}(t)\right\| \left\|a_{m+1}^{\Omega}(t)\right\|.\end{aligned} \tag{10}$$

Now $\|a_m(t)\| = \|a_m^{\Omega}(t)\| = 1$, so we've reduced the problem of bounding $\|h_m(t) - h_m^{\Omega}(t)\|$ to bounding $\|a_m(t) - a_m^{\Omega}(t)\|$. The operator $a_m^{\Omega}(t)$ is given by

$$a_m^{\Omega}(t) = \sum_{k\in\Omega} v_{mk}(t)a_k. \tag{11}$$

Hence we arrive at

$$\left\|a_m(t) - a_m^{\Omega}(t)\right\| = \left\|\sum_{k\notin\Omega} v_{mk}a_k\right\| \leq \sum_{k\notin\Omega} nc_\zeta e^{-\gamma_\zeta|m-k|^\zeta}, \tag{12}$$

where we've used our bound Eq. (4) on $|v_{jk}|$ and the fact that $\|a_k\| = 1$.

Since Ω is a set centred on the boundary between partitions A and B of the chain, we have that $|m-k| \geq |\Omega|/2$ for all $k \notin \Omega$. Hence

$$\left\|a_m(t) - a_m^{\Omega}(t)\right\| \leq c_\zeta n\left(n - |\Omega|\right)e^{-\gamma_\zeta|\Omega|^\zeta/2} \tag{13}$$

and so we are finally able to conclude that $\|h_m(t) - h_m^{\Omega}(t)\| \leq cn^2 e^{-\gamma_\zeta|\Omega|^\zeta/2}$ and that

$$\left\|V(t) - V^{\Omega}(t)\right\| \leq c_\zeta|t|n^2 e^{-\gamma_\zeta|\Omega|^\zeta/2}, \tag{14}$$

(here we have redefined the constant γ_ζ). In particular, given $\epsilon \geq 0$, choosing $|\Omega|^\zeta \geq 2\log\left(c_\zeta|t|n^2/\epsilon\right)/\gamma_\zeta$ ensures that $\|V(t) - V^{\Omega}(t)\| \leq \epsilon$. Even $\zeta = 1/2$ gives a polylogarithmic light cone whose width grows as the square of a logarithm of $|t|$. This may be improved arbitrarily by choosing larger ζ at the expense of worse constants. That is, given any $\epsilon \geq 0$ we can choose Ω to be a large enough set such that $V^{\Omega}(t)$ approximates $V(t)$ to within ϵ. This enables us to write

$$e^{itH} = e^{it(H_A+H_B)}V^{\Omega}(t) + \mathcal{O}(\epsilon) \tag{15}$$

Following [7] we recursively apply the above partitioning procedure to find $e^{itH} = Q(t) + \mathcal{O}(\epsilon)$, where

$$Q(t) \equiv \left(\bigotimes_{j=1}^{n/|\Omega|} e^{itH_{\Omega_j}}\right)\left(\bigotimes_{k=0}^{n/|\Omega|} V^{\Omega_k'}(t)\right), \tag{16}$$

and where $\mathcal{P}_1 = \{\Omega_j\}$ is a partition of the chain into $\frac{n}{|\Omega|}$ blocks, each containing $|\Omega|$ spins and where $\mathcal{P}_2 = \{\Omega_k'\}$ is a partition of the chain obtained by shifting $\mathcal{P}_1$ along by $\frac{|\Omega|}{2}$ sites (note that Ω_0' and $\Omega_{n/|\Omega|}'$ are half-size blocks of $\frac{|\Omega|}{2}$ sites each). This is our fundamental structure result for the dynamics of the disordered XY spin chain.

A Lieb-Robinson bound is an upper bound on quantities such as $\|[A, B(t)]\|$. We now show how the above structure result implies a version of the Lieb-Robinson bound which is substantially stronger than the original. Define $\widetilde{B}(t)$ to be the operator which arises when we evolve B according to the approximation $Q(t)$ of e^{itH}, namely, $\widetilde{B}(t) = Q(t)BQ^\dagger(t)$. This enables us to write $B(t) = \widetilde{B}(t) + \mathcal{O}(\epsilon)$. Note that $\widetilde{B}(t)$ acts trivially on all sites which are a distance greater than $3|\Omega|/2$ away from those sites on which B acts. Thus, if $d(A,B) \geq 3|\Omega|/2$, where $d(A,B)$ is the distance between A and B, then $[A, \widetilde{B}(t)] = 0$, and so for a given $|\Omega|$:

$$\begin{aligned}\|[A, B(t)]\| &= \left\|\left[A, \widetilde{B}(t)\right] + [A, \mathcal{O}(\epsilon)]\right\| \\ &\leq 2\|A\|\|\mathcal{O}(\epsilon)\| \\ &\leq c_\zeta n^2|t|e^{-\gamma_\zeta|\Omega|^\zeta/2} \qquad (17) \\ &\leq c_\zeta n^2|t|e^{-\gamma_\zeta d(A,B)^\zeta}. \qquad (18)\end{aligned}$$

where we've redefined our constants. This is the polylogarithmic light cone for the two-point dynamical correlation functions. Compare this to the original Lieb-Robinson bound, which reads

$$\|[A, B(t)]\| \leq ce^{k_1|t|}e^{-k_2 d(A,B)}. \tag{19}$$

To conclude we'd like to mention two consequences of our light cone for the disordered XY model. The first is a proof of the conjecture that two parties, Alice and Bob, with access to only bounded regions A and B at either end of the chain, respectively, cannot use the dynamics of the disordered model to send information from Alice to Bob. We follow the argument of [6], appropriately modified to take account of our stronger bound.

Let $C = L \setminus (A \cup B)$, where L is the chain, be the region that Alice and Bob cannot access. The most general way Alice can encode her message is via a set of unitary operators $\{U_A^k \,|\, k = 1, 2, \ldots, m\}$ on her system, where k is varied according to the message she wants to send. After a time t has elapsed the system has evolved from an initial state ρ_0 to $\rho(t) = e^{-iHt}\rho_0 e^{iHt}$. We interpret this as a quantum channel with input $\rho_{ABC}^k = U_A^k \rho_0 {U_A^k}^\dagger$ and output $\rho_B^k(t) = \mathrm{tr}_{AC}(U_A^k(t)\rho_0 {U_A^k}^\dagger(t))$. As argued in [6], the output states are all very close together, as measured in trace norm:

$$\|\rho_B^k(t) - \rho_B(t)\|_1 \leq c_\varsigma n^2 |t| e^{-\gamma_\varsigma d(A,B)^\varsigma},$$

where $\rho_B(t) = \mathrm{tr}_{AC}(e^{-iHt}\rho_0 e^{iHt})$.

If Alice applies the unitaries $\{U_A^k\}$ according to the probability distribution $\{p_k\}$, the amount of information that is sent through the channel is given by the Holevo capacity:

$$\chi(t) = S\left(\sum_{k=1}^m p_k \rho_B^k(t)\right) - \sum_{k=1}^m p_k S(\rho_B^k(t)),$$

where $S(\cdot)$ is the von Neumann entropy. Applying Fannes inequality [19] we find that

$$\chi(t) \leq 2\epsilon(|B| - \log_2(\epsilon)),$$

where $\epsilon = c_\varsigma n^2 |t| e^{-\gamma_\varsigma d(A,B)^\varsigma}$. That is, Bob has to wait a subexponentially long time (in $d(A, B)$) before a nontrivial amount of information can arrive. The optimal encoding for Alice to adopt was investigated in [20] and [21] and completely solved in the single-use case in [22].

The second consequence of the polylogarithmic light cone bound is that the entropy of any contiguous block B of spins in a dynamically evolving product state $|\psi(t)\rangle = e^{itH}|00\cdots 0\rangle$ is bounded. Indeed, applying the argument of [12, 23], we find that $S(\rho_B(t)) \leq c_1 + c_2 \log_2^{\frac{1}{\varsigma}}(n|t|)$ as $|B| \to \infty$, where c_1 and c_2 are constants. This provides a theoretical explanation for the phenomenon numerically observed by De Chiara *et. al.* [24]. It seems nontrivial to adapt the argument of [6] to prove the same result because their proof can't be simply modified to make use of the presence of disorder.

Acknowledgments—Helpful conversations with Jens Eisert are gratefully acknowledged. This work was supported by the EPSRC and the Nuffield Foundation.

[1] E. H. Lieb and D. W. Robinson, Commun. math. Phys. **28**, 251 (1972).
[2] B. Nachtergaele and R. Sims, Commun. Math. Phys. **265**, 119 (2006), math-ph/0506030.
[3] M. B. Hastings, Phys. Rev. B **69**, 104431 (2004), cond-mat/0305505.
[4] M. B. Hastings and T. Koma, Commun. Math. Phys. **265**, 781 (2006), math-ph/0507008.
[5] T. J. Osborne (2006), `www.lri.fr/qip06/slides/osborne.pdf`.
[6] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006), quant-ph/0603121.
[7] T. J. Osborne, Phys. Rev. Lett. **97**, 157202 (2006), quant-ph/0508031.
[8] T. J. Osborne (2006), cond-mat/0605194.
[9] T. J. Osborne, Phys. Rev. A **75**, 032321 (2007), quant-ph/0601019.
[10] T. J. Osborne, Phys. Rev. A **75**, 042306 (2007), quant-ph/0603137.
[11] M. B. Hastings, Phys. Rev. B **73**, 085115 (2005), cond-mat/0508554.
[12] J. Eisert and T. J. Osborne, Phys. Rev. Lett. **97**, 150404 (2006), quant-ph/0603114.
[13] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[14] J. P. Keating, N. Linden, J. C. F. Matthews, and A. Winter (2006), quant-ph/0606205.
[15] M. A. Nielsen (2005), `http://www.qinfo.org/people/nielsen/blog/archive/notes/fermions_and_jordan_wigner.pdf`.
[16] V. Chulaevsky (2011), arXiv:1102.5059
[17] F. Germinet and A. Klein, Commun. Math. Phys. **222**, 415 (2001).
[18] A. Klein, private communication.
[19] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, 2000).
[20] T. J. Osborne and N. Linden, Phys. Rev. A **69**, 052315 (2003), quant-ph/0312141.
[21] D. Burgarth and S. Bose, New J. Phys. **7**, 135 (2005), quant-ph/0502186.
[22] H. L. Haselgrove, Phys. Rev. A **72**, 062326 (2005), quant-ph/0404152.
[23] G. R. Grimmett, T. J. Osborne, and P. F. Scudo (2007), arXiv:0704.2981.
[24] G. De Chiara, S. Montangero, P. Calabrese, and R. Fazio, J. Stat. Mech. **06**, P03001 (2006), cond-mat/0512586.