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BOWEN-RUELLE MEASURES FOR CERTAIN PIECEWISE HYPERBOLIC MAPS

BY

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ABSTRACT. We consider a class of piecewise C^2 Lozi-like maps and prove the existence of invariant measures with absolutely continuous conditional measures on unstable manifolds

This work is a small step forward in the following program: Suppose a compact neighborhood is mapped into itself and the map displays some chaotic behavior. Is there a strange attractor, or more specifically, is there a Bowen-Ruelle measure? Axiom A systems aside, current techniques have hardly begun to provide answers to these questions. One of the purposes of this note is to demonstate, in a very limited way, how certain 1-dimensional results can sometimes be useful in handling dissipative systems in 2-dimension.

The 1-dimensional result we alluded to says that piecewise expanding endomorphisms of the unit interval have smooth invariant measures [**LY**]. What we prove here is an analogous statement for certain piecewise hyperbolic attractors in 2-dimension. The prime examples that motivate this study are the Lozi maps, though our analysis has little to do with the precise nature of the equations studied by Lozi or Misiurewicz [**M**]. Our main result is that these "generalized Lozi maps" have invariant measures with absolutely continuous conditional measures on unstable manifolds. As a consequence they have Bowen-Ruelle measures.

We begin by isolating several properties of the Lozi maps. They are essentially the properties upon which our proof depends. Maps satisfying these hypotheses will henceforth be called "generalized Lozi maps". (See Figure 1.)

Let $R = [0,1]\times[0,1]$ and let $f\colon R \to R$ be a continuous injective map. We assume that f or some iterate of f takes R into its interior. Let $0 < a_1 < \cdots < a_q < 1$ and let $S = \{a_1, \ldots, a_q\} \times [0,1]$. We assume that $f|(R-S)$ is a C^2 diffeomorphism onto its image with $|\mathrm{Jac}(f)| < 1$ and that both $f|(R-S)$ and $f^{-1}|f(R-S)$ have bounded second derivative. We further impose the following conditions on $f|(R-S)$ (geometric interpretations are given in parentheses).

$$\text{(H1)}\qquad \operatorname{Inf}\left\{\left(\left|\frac{\partial f_1}{\partial x}\right| - \left|\frac{\partial f_1}{\partial y}\right|\right) - \left(\left|\frac{\partial f_2}{\partial x}\right| + \left|\frac{\partial f_2}{\partial y}\right|\right)\right\} \geq 0$$

(Df preserves cones making $< 45°$ with the x-axis),

$$\text{(H2)}\qquad \operatorname{Inf}\left\{\left|\frac{\partial f_1}{\partial x}\right| - \left|\frac{\partial f_1}{\partial y}\right|\right\} = u > 1$$

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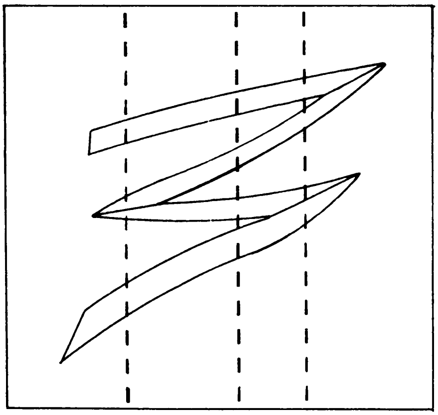

FIGURE 1. Example of a "generalized Lozi map"

(restricted to these cones the action of Df when projected onto the x-axis is uniformly expanding),

$$\text{(H3)} \qquad \operatorname{Sup}\left\{\frac{|\partial f_1/\partial y| + |\partial f_2/\partial y|}{(|\partial f_1/\partial x| - |\partial f_1/\partial y|)^2}\right\} < 1$$

(horizontal expansion dominates the action of Df on vertical vectors), and

$$\text{(H4)} \qquad \exists N \in \mathbf{Z}^+ \text{ s.t. } u^N > 2 \text{ and } f^k S \cap S = \emptyset \text{ for } 1 \leq k \leq N$$

(f expands horizontally more than it folds).

Note that (H4) is vacuous when $u > 2$. (Hence Figure 1 is legitimate.) Note also that these hypotheses are indeed satisfied by many Lozi maps. Lozi maps are usually given by

$$L(x, y) = (1 + y - a|x|, bx).$$

Changing coordinates this can be written as

$$f(x, y) = (1 + by - a|x|, x).$$

It is easy to verify that for open intervals of a and b, f takes some square $[c, c] \times [c, c]$ into itself and satisfies (H1)–(H4).[2]

DEFINITION 1. A Borel probability measure μ on R is said to have absolutely continuous conditional measures on unstable manifolds if there exist measurable partitions $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots$ of R and measurable sets $V_1 \subset V_2 \subset \cdots$ s.t.

1. $\mu V_n \uparrow 1$ as $n \to \infty$,
2. each element of $\mathcal{P}_n | V_n$ is an open subset of some unstable manifold and
3. if $\{\mu_c \colon c \in \mathcal{P}_n | V_n\}$ denotes the system of conditional measures on elements of $\mathcal{P}_n | V_n$, and m_c denotes Riemannian measure on c, then for almost every $c \in \mathcal{P}_n | V_n$, we have $\mu_c \ll m_c$.

We now state our main result.

[2] As this manuscript was being written I learned that P. Collet and Y. Levy had jointly obtained a result similar to ours for certain parameter values of the Lozi map [**CL**].

THEOREM. *If f: $R \to R$ is a generalized Lozi map, then f has an invariant Borel probability measure μ s.t.*

1. *Local unstable manifolds exist at μ-a.e. point and*
2. *μ has absolutely continuous conditional measures on unstable manifolds.*

The idea of our proof is as follows: Observe that unstable manifolds (when they exist) are piecewise smooth curves zigzagging across R (turning around at random places. Our strategy is first to construct an invariant measure μ that behaves nicely on neighborhoods of the singularity set S. This is done following a combination of the methods used by Sinai [**S**] and Lasota and Yorke [**LY**]. Since all turns are created by passing through S, we now have control over their impact as well. In particular, this allows us to construct a noninvariant measure $\tilde{\mu}$ equivalent to μ on an arbitrarily large set and having the property that its conditional measures on unstable manifolds are (obviously) absolutely continuous. This finishes the proof.

The following notations will be used: The map p: $R \to [0,1]$ denotes projection onto the first factor. Lebesgue measure on $[0,1]$ is denoted by m. For g: $[a,b] \to \mathbf{R}$, $\bigvee_a^b g$ denotes the total variation of g on $[a,b]$. If μ is a measure on R, then $f_*\mu$ is given by $f_*\mu(E) = \mu(f^{-1}E)$. If $J \subset [0,1]$ is a closed interval and α: $J \to [0,1]$ is a C^2 function with $|\alpha'| \leq 1$, then $f(\text{graph}(\alpha))$ is a union of finitely many smooth curves (H1). We denote them by $\{L_i(\alpha)\}$, dropping the α whenever no ambiguity arises. For $k > 1$, denote the smooth segments of $f^k\text{graph}(\alpha)$ by $\{L_{i_1\cdots i_k}\}$, where the indices are chosen so that $fL_{i_1\cdots i_k} = \bigcup_j L_{i_1\cdots i_k j}$. Let μ_0 be the measure on graph(α) s.t. $p_*\mu_0 =$ normalized Lebesgue measure on J. For $k = 1, 2, \dots$, define $\mu_k = (f^k)_*\mu_0$. From (H1) and (H2) we know that for each $i_1 \cdots i_k$, $p_*\mu_k | L_{i_1\cdots i_k}$ is absolutely continuous to m. We denote its density by $g_{i_1\cdots i_k}$ and the density of $p_*\mu_k$ by $\hat{g}_k$. That is, we have $\sum_{i_1\cdots i_k} g_{i_1\cdots i_k} = \hat{g}_k$. When it is convenient to consider f^N instead of f, we will write

$$f^{Nk}\text{graph}(\alpha) = \bigcup_{i_1\cdots i_k} L_{i_1\cdots i_k}^{(N)}$$

and

$$d(p_*\mu_{Nk} | L_{i_1\cdots i_k}^{(N)}) = g_{i_1\cdots i_k}^{(N)}\, dm,$$

etc.

LEMMA. *Under the hypotheses of the theorem, there exists an invariant Borel probability measure μ and a function g: $[0,1] \to [0,\infty)$ of bounded variation s.t. $d(p_*\mu) = gdm$.*

PROOF. Fix $J \subset [0,1]$ and a C^2 function α: $J \to [0,1]$ with $|\alpha'| \leq 1$. We will show that $\exists M$ s.t. $\bigvee_0^1 \hat{g}_k \leq M$ for all k. This will imply that $\bigvee_0^1 (n^{-1}\sum_{k=1}^n \hat{g}_k) \leq M$, and since $\int_0^1 (n^{-1}\sum_{k=1}^n \hat{g}_k)\, dm = 1$ the sequence $\{n^{-1}\sum_{k=1}^n \hat{g}_k\}_{n=1,2,\cdots}$ is precompact in $L^1([0,1], m)$. Choose a subsequence $\{n_i\}$ s.t. as $i \to \infty$, $(n_i)^{-1}\sum_{k=1}^{n_i} \mu_k$ converges in the weak star topology to a Borel probability measure μ and

$$\frac{1}{n_i}\sum_{k=1}^{n_i} \hat{g}_k \xrightarrow{L^1} \text{some function } g.$$

It follows immediately that μ is invariant, $d(p_*\mu) = gdm$ and that $\bigvee_0^1 g \leq M$.

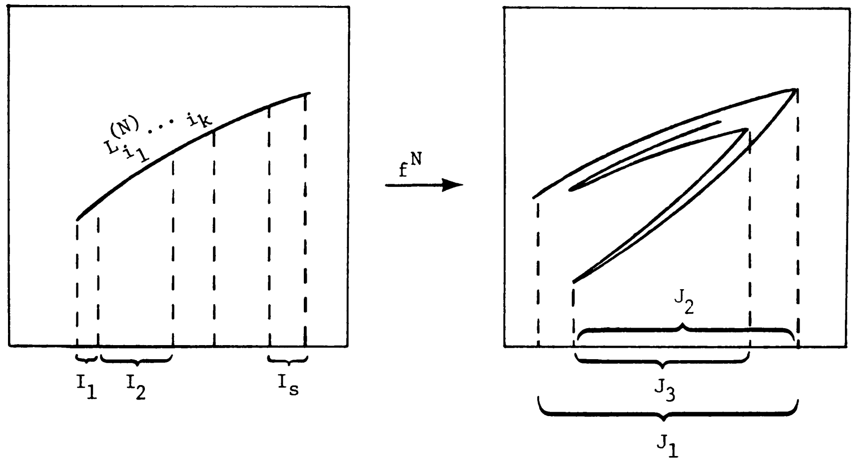

FIGURE 2

Let $N \in \mathbf{Z}^+$ satisfy (H4). It suffices to show the uniform boundedness of $\bigvee_0^1 \hat{g}_{Nk}$ for $k = 1, 2, \dots$. We will prove that $\exists A$ s.t. for every $i_1 \cdots i_k$,

$$(*) \qquad \sum_j \bigvee_0^1 g^{(N)}_{i_1\cdots i_k j} \le A \int_0^1 g^{(N)}_{i_1\cdots i_k}\, dm + \frac{2}{u^N} \bigvee_0^1 g^{(N)}_{i_1\cdots i_k}.$$

Then if $\beta_k = \sum_{i_1\cdots i_k} \bigvee_0^1 g^{(N)}_{i_1\cdots i_k}$, we would have

$$\beta_{k+1} = \sum_{i_1\cdots i_k} \left(\sum_j \bigvee_0^1 g^{(N)}_{i_1\cdots i_k j} \right) \le A + \frac{2}{u^N}\beta_k$$

so that for all k,

$$\bigvee_0^1 g_{Nk} \le \beta_k \le A \sum_{i=0}^{\infty} \left(\frac{2}{u^N}\right)^i < \infty.$$

We now fix $i_1 \cdots i_k$ and prove $(*)$. Let $pL^{(N)}_{i_1\cdots i_k} = \bigcup_{j=1}^s I_j$, where $\{f^N(x,y)\colon x \in I_j\} = L^{(N)}_{i_1\cdots i_k j}$ and $J_j = pL^{(N)}_{i_1\cdots i_k j}$. Define $\tilde{f}_j\colon I_j \to J_j$ by $\tilde{f}_j(x) = p \circ f^N(x,y)$ for $(x,y) \in L^{(N)}_{i_1\cdots i_k}$. Then each $\tilde{f}_j$ is a C^2 diffeomorphism between I_j and J_j with $|\tilde{f}_j'| \ge u^N$ ((H1), (H2)) and $|(\tilde{f}_j^{-1})''| \le Q$ for some universal Q. (This follows from (H3) by a direct computation.) Now

$$\sum_j \bigvee_0^1 g^{(N)}_{i_1\cdots i_k j} = \underbrace{\sum_j \bigvee_{J_j} \left[(g^{(N)}_{i_1\cdots i_k} \circ \tilde{f}_j^{-1}) |(\tilde{f}_j^{-1})'| \right]}_{①} + \underbrace{\sum_j \left[|\tilde{f}_j'(l_j)|^{-1} g^{(N)}_{i_1\cdots i_k}(l_j) + |\tilde{f}_j'(r_j)|^{-1} g^{(N)}_{i_1\cdots i_k}(r_j) \right]}_{②},$$

where l_j and r_j denote the left and right end points of I_j respectively. Consider one j at a time.

$$\begin{aligned}\bigvee_{J_j}(g^{(N)}_{i_1\cdots i_k}\circ\tilde f_j^{-1})|(\tilde f_j^{-1})'| &= \int_{J_j}|((g^{(N)}_{i_1\cdots i_k}\circ\tilde f_j^{-1})(\tilde f_j^{-1})')'|\,dm\\ &= \int_{J_j}|(g^{(N)}_{i_1\cdots i_k}\circ\tilde f_j^{-1})'|\,|(\tilde f_j^{-1})'|\,dm+\int_{J_j}(g^{(N)}_{i_1\cdots i_k}\circ\tilde f_j^{-1})|(\tilde f_j^{-1})''|\,dm\\ &\le \frac{1}{u^N}\bigvee_{I_j}g^{(N)}_{i_1\cdots i_k}+\frac{Q}{u^N}\int_{I_j}g^{(N)}_{i_1\cdots i_k}\,dm\end{aligned}$$

so that

$$(**)\qquad\begin{aligned}&①+|\tilde f_1'(l_1)|^{-1}g^{(N)}_{i_1\cdots i_k(l_1)}+|\tilde f_s'(r_s)|^{-1}g^{(N)}_{i_1\cdots i_k}(r_s)\\ &\le\frac{1}{u^N}\bigvee_0^1 g^{(N)}_{i_1\cdots i_k}+\frac{Q}{u^N}\int_0^1 g^{(N)}_{i_1\cdots i_k}\,dm.\end{aligned}$$

Now we claim that there is a universal $d>0$ s.t. the points $0,l_2,\dots,l_s,1$ are pairwise at least d apart. We have $|0-l_2|,|l_s-1|\ge d$ because we may assume that $f^{Nk}R\subset\operatorname{int}R$. That $|r_j-l_j|\ge$ some d for $j=2,\dots,s-1$ follows from (H4). Thus if $I=[l,r]$ is either $[0,l_1]$ or $[l_s,1]$ or $[l_j,r_j]$, $j=2,\dots,s-1$, then

$$\begin{aligned}g^{(N)}_{i_1\cdots i_k}(l)+g^{(N)}_{i_1\cdots i_k}(r)&\le 2\left(\min_I g^{(N)}_{i_1\cdots i_k}\right)+\bigvee_I g^{(N)}_{i_1\cdots i_k}\\ &\le\frac{2}{d}\int_I g^{(N)}_{i_1\cdots i_k}\,dm+\bigvee_I g^{(N)}_{i_1\cdots i_k}.\end{aligned}$$

Note that $g^{(N)}_{i_1\cdots i_k}(0)=g^{(N)}_{i_1\cdots i_k}(1)=0$, so that the terms in ② not accounted for in $(**)$

$$\le\frac{1}{u^N}\frac{2}{d}\int_0^1 g(N)_{i_1\cdots i_k}\,dm+\frac{1}{u^N}\bigvee_0^1 g^{(N)}_{i_1\cdots i_k}.$$

Thus

$$①+②\le\left(Q+\frac{2}{d}\right)\frac{1}{u^N}\int_0^1 g^{(N)}_{i_1\cdots i_k}\,dm+\frac{2}{u^N}\bigvee_0^1 g^{(N)}_{i_1\cdots i_k},$$

which completes the proof of $(*)$ and the lemma. □

Let $W^u_\delta(x)$ denote the local unstable manifold at x (assuming it exists) s.t. $pW^u_\delta(x)=[p(x)-\delta,p(x)+\delta]$. Implicit in this notation is the assertion that $W^u_\delta(x)$ contains no cusps. Let $D(S,\delta)$ denote the δ-neighborhood of S.

PROOF OF THEOREM. If μ is any invariant probability measure with $d(p_*\mu)=g\,dm$ for some bounded g, say $g\le M_0$, then for $\delta>0$,

$$\sum_{k=0}^\infty\mu(f^kD(S,\delta u^{-k}))=\sum_{k=0}^\infty\mu D(S,\delta u^{-k})\le 2\delta M_0\sum_{k=0}^\infty u^{-k}<\infty.$$

By the Borel-Cantelli lemma, μ-a.e. x is in $f^kD(S,\delta u^{-k})$ for at most finitely many k. That is, for μ-a.e. x, $\exists\delta(x)>0$ s.t. $f^{-k}x\notin D(S,\delta(x)u^{-k})$ for all $k>0$. This implies the existence of $W^u_{\delta(x)}(x)$. (For more details see [**KS**].) Let us fix one measure μ constructed as in the lemma with graph$(\alpha) = W^u_{\delta(x)}$ for some x. This guarantees that the $L_{i_1\cdots i_k}(\alpha)$ will not cross other unstable manifolds. For $\delta > 0$, let $\Lambda_\delta = \{x \in R\colon\ d(f^{-k}x, S) \geq \delta u^{-k}\ \forall k \geq 0\}$. Then each Λ_0 is closed and $\lim_{\delta\to 0} \mu\Lambda_\delta = 1$.

We now define a sequence of measurable partitions $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots$. All notations will be as in Definition 1. For $n \in \mathbf{Z}^+$, let $\{U_1, \dots, U_{2^n}\}$ be the partition of R into 2^n vertical columns of width 2^{-n}. For $x \in U_i \cap \Lambda_{2^{-n}}$, let $c(x) = W^u_{2^{-n}}(x) \cap U_i$. Let $V_n = \bigcup_{x\in\Lambda_{2^{-n}}} c(x)$ and $\mathcal{P}_n = \{c(x)\colon\ x \in V_n\} \cup \{R - V_n\}$. It suffices to show that for every $\varepsilon > 0$ and every n, there is a set $\tilde{V}_n \subset V_n$ s.t. $\mu\tilde{V}_n > \mu V_n - \varepsilon$ and condition 3 in Definition 1 is satisfied when μ is replaced by $\chi_{\tilde{V}_n}\mu$. In fact, for given ε and n, we will construct a noninvariant measure $\tilde{\mu}$ and a set $\tilde{V}$ with $\mu\tilde{V} > 1 - \varepsilon$ s.t. $\tilde{\mu} \ll \mu$, is equivalent to μ on $\tilde{V}$, and satisfies condition 3 with respect to $\mathcal{P}_{n'}$ for some $n' \geq n$. It is straightforward to verify that this implies the desired result.

Now let $n \in \mathbf{Z}^+$ and $\varepsilon > 0$ be given. Let n' be a large number to be determined later and let $\tilde{U}_1, \dots, \tilde{U}_{2^{n'}}$ be pairwise disjoint vertical columns of width $2^{-n'}$. We define a sequence of measures $\{\tilde{\mu}_k\}_{k=1,2,\cdots}$ as follows: Recall that in the definition of μ_k, μ_k is carried by $f^k(\text{graph}(\alpha))$, where $f^k(\text{graph}(\alpha))$ is a finite union of smooth curve segments. Let $\tilde{\mu}_k$ be μ_k annihilated on those parts of its support that only partially cross some $\tilde{U}_i$ (See Figure 3.) $\tilde{\mu}_k(R)$ is probably < 1.

We claim that given $\delta > 0$, $\exists n'$ such that for sufficiently large k, $\tilde{\mu}_k(R) > 1 - \delta$. Recall that $\exists M_0$ s.t. $\hat{g}_k \leq M_0\ \forall k$. If $x \in (\operatorname{supp}\mu_k - \operatorname{supp}\tilde{\mu}_k)$, then either x lies in one of the two end pieces of $f^k(\text{graph}(\alpha))$ that only partially crosses some $\tilde{U}_i$, or the horizontal distance between x and a cusp in $f^k(\text{graph}(\alpha))$ is $< 2^{-n'}$, which says that $d(f^{-i}x, S) \leq 2^{-n'}u^{-i}$ for some $1 \leq i \leq k$. Thus

$$\begin{aligned}
1 - \tilde{\mu}_k(R) &= \mu_k(\operatorname{supp}\mu_k - \operatorname{supp}\tilde{\mu}_k) \\
&= \sum_{i=1}^{k} \mu_k\{x\colon\ d(f^{-i}x, S) \leq 2^{-n'}u^{-i}\} + \mu_k\{2 \text{ end pieces}\} \\
&= \sum_{i=1}^{k} \mu_i\{x\colon\ d(x, S) \leq 2^{-n'}u^{-i}\} + \mu_k\{2 \text{ end pieces}\} \\
&\leq 2M_0 q 2^{-n'} \sum_{i=1}^{k} u^{-i} + \mu_k\{2 \text{ end pieces}\}.
\end{aligned}$$

The second term $\to 0$ as $k \to \infty$. The first term becomes arbitrarily small as $n' \uparrow \infty$. Recall also that $(n_i)^{-1}\sum_{k=1}^{n_i} \mu_k \to \mu$. Choose a subsequence $\{n_i'\}$ of $\{n_i\}$ s.t. $(n_i')^{-1}\sum_{k=1}^{n_i'} \tilde{\mu}_k \to \tilde{\mu}$ for some $\tilde{\mu}$. It is easy to verify that $\tilde{\mu}E \leq \mu E$ for every Borel set E and hence we have $\tilde{\mu} \ll \mu$ with $0 \leq d\tilde{\mu}/d\mu \leq 1$. But since $\tilde{\mu}(R)$ can be made arbitrarily near 1, we can choose $\tilde{\mu}$ s.t. $\tilde{\mu}$ is equivalent to μ except on a set of μ-measure $< \varepsilon$.

It remains to show that if $\tilde{\mu}_T$ is the transverse measure on $\mathcal{P}_{n'}$ induced by $\tilde{\mu}$, then for $\tilde{\mu}_T$-a.e. $c \in \mathcal{P}_{n'}|V_{n'}$, $\tilde{\mu}_c \ll m_c$. Recall that $g_{i_1\cdots i_k} =$ density of $p_*(\mu_k|L_{i_1\cdots i_k})$. Let $\tilde{g}_{i_1\cdots i_k} =$ density of $p_*(\tilde{\mu}_k|L_{i_1\cdots i_k})$. We claim that $\exists B > 0$ s.t. for any $1 \leq i \leq 2^{n'}$

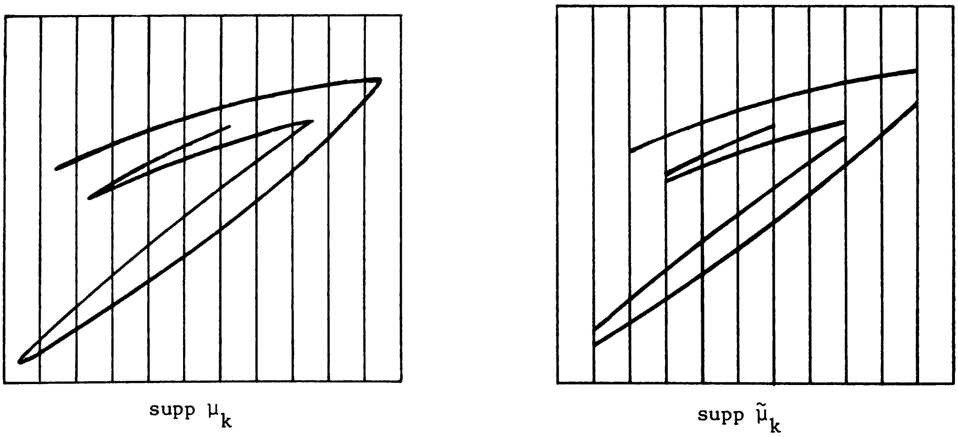

FIGURE 3

and any $i_1 \cdots i_k$, we have either

$$\tilde{g}_{i_1 \cdots i_k} \equiv 0 \quad \text{on } p\tilde{U}_i$$

or

$$\frac{\tilde{g}_{i_1 \cdots i_k}(x)}{\tilde{g}_{i_1 \cdots i_k}(y)} \leq B \quad \forall x, y \in p\tilde{U}_i.$$

This would prove that for almost every $c \in \mathcal{P}_{n'}|V_{n'}$, $d\tilde{\mu}_c = g_c dm_c$ where $\forall x, y \in c$, $g_c(x)/g_c(y) \leq B'$ for some B'. To prove the claim fix $1 \leq i \leq 2^{n'}$ and $i_1 \cdots i_k$. If $L_{i_1 \cdots i_k}$ does not cross the full width of $\tilde{U}_i$, then $g_{i_1 \cdots i_k}|p\tilde{U}_i \equiv 0$. Otherwise there are intervals $E_0, E_1, \ldots, E_k \subset [0,1]$, $E_k = p\tilde{U}_i$, and C^2 diffeomorphisms h_j: $E_{j-1} \to E_j$ s.t. $\tilde{g}_{i_1 \cdots i_k}|p\tilde{U}_i = |((h_k \circ \cdots \circ h_1)^{-1})'|$ (see lemma). While these h_j's depend strictly on i and $i_1 \cdots i_k$, they satisfy $|h_j'| \geq u > 1$ and $|h_j''| \leq Q$, where u and Q are universal constants.

The reader can easily verify that

$$\begin{aligned} &|\log|((h_k \circ \cdots \circ h_1)^{-1})'(x)| - \log|((h_k \cdots h_1)^{-1})'(y)|| \\ &\quad \leq \text{ some constant depending only on } u,\ Q \text{ and } n'. \end{aligned}$$

This completes the proof. $\square$

We have proved our result for a specific class of maps of the square into itself. It is obvious that our hypotheses are more stringent than necessary and that it is easy to make slight generalizations. We do not attempt to do that here, because we do not know what a natural general statement ought to be.

We mention a few corollaries. Since the proofs are standard, we will provide only references. The standing hypothesis for the rest of this article is that f is a generalized Lozi map.

COROLLARY 1. *Let μ be constructed as in our lemma. Then there are measurable sets $E_1, E_2, \cdots \subset R$ s.t. $f^{-1}E_i = E_i$, $\mu E_i > 0$ $\forall i$, $\mu(\bigcup E_i) = 1$ and for each i, $f|E_i$: $(E_i, \mu|E_i) \to (E_i, \mu|E_i)$ is ergodic.*

PROOF. The proof follows that in [**P**]. It uses the absolute continuity of stable manifolds [**KS**] and the fact that on most $c \in \mathcal{P}_n|V_n$, μ_c is equivalent to m_c. $\square$

DEFINITION 2. A Borel probability measure μ on R is called a *Bowen-Ruelle measure* [**B**] if there is a set $U \subset R$ of positive Lebesgue measure s.t. for every continuous function φ: $\mathbf{R} \to \mathbf{R}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi f^i x \to \int \varphi \, d\mu$$

for Lebesgue-a.e. $x \in U$.

COROLLARY 2. *Let μ be constructed as in our lemma. Then corresponding to each E_i in Corollary 1, $\mu|E_i$ normalized is a Bowen-Ruelle measure.*

PROOF. This follows from [**KS**]. □

Unlike the case of Axiom A attractors, assuming only (H1)–(H4) there can easily be more than one Bowen-Ruelle measure.

COROLLARY 3. *Let f: $R \to R$ be a generalized Lozi map. Then f has an invariant Borel probability measure μ s.t.*

$$h_\mu(f) = \int \lambda_1(x) \, d\mu(x),$$

where $\lambda_1(x)$ is the positive μ-exponent of f at x.

PROOF. See [**LS**]. □

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