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BOWMAN-BRADLEY TYPE THEOREM FOR FINITE MULTIPLE ZETA VALUES

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Abstract. The multiple zeta values are multivariate generalizations of the values of the Riemann zeta function at positive integers. The Bowman-Bradley theorem asserts that the multiple zeta values at the sequences obtained by inserting a fixed number of twos between $3, 1, \dots, 3, 1$ add up to a rational multiple of a power of π. We show that an analogous theorem holds in a very strong sense for finite multiple zeta values, which have been investigated by Hoffman and Zhao among others and recently recast by Zagier.

1. Introduction.

1.1. Finite multiple zeta values. The *multiple zeta values* and *multiple zeta-star values* are real numbers defined by

$$\zeta(k_1, \dots, k_n) = \sum_{m_1 > \dots > m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

$$\zeta^\star(k_1, \dots, k_n) = \sum_{m_1 \geq \dots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$$

for positive integers $k_1, \dots, k_n$ with $k_1 \geq 2$. They are generalizations of the values of the Riemann zeta function at positive integers and are known to be related to number theory, algebraic geometry, combinatorics, knot theory, and quantum field theory among others. Research on these numbers has mainly been focused on their numerous linear and algebraic relations; see for example [3, 12] and the references therein for an introduction.

Hoffman [4] and Zhao [11] among others developed a theory of modulo p values, for primes p, of the finite truncations of the above-mentioned infinite sums, where the indices of summation are all restricted to be less than p. Following an idea of Zagier [7], we look at the truncations in the ring $\mathcal{A} = (\prod_p \mathbb{Z}/p\mathbb{Z})/(\bigoplus_p \mathbb{Z}/p\mathbb{Z})$, where p runs over all primes; the elements of $\mathcal{A}$ are of the form $(a_p)_p$, where $a_p \in \mathbb{Z}/p\mathbb{Z}$, and two elements (a_p) and (b_p) are identified if and only if $a_p = b_p$ for all but finitely many primes p. Note that $\mathcal{A}$ is a $\mathbb{Q}$-algebra. We shall simply write a_p for (a_p) since no confusion is likely.

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DEFINITION 1.1. For positive integers $k_1, \dots, k_n$, we define

$$\zeta_{\mathcal{A}}(k_1, \dots, k_n) = \sum_{p > m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} \in \mathcal{A},$$

$$\zeta^{\star}_{\mathcal{A}}(k_1, \dots, k_n) = \sum_{p > m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} \in \mathcal{A},$$

and call them *finite multiple zeta(-star) values* in this paper.

The finite multiple zeta(-star) values are similar to multiple zeta(-star) values in many respects as we shall see in this paper. They do, however, have some differences, of which one of the most striking is the following:

PROPOSITION 1.2 ([4, Theorem 4.3]). *We have $\zeta_{\mathcal{A}}(k) = \zeta^{\star}_{\mathcal{A}}(k) = 0$ for all positive integers k.*

PROOF. Let p be an arbitrary prime larger than $k+1$. Taking a primitive root a modulo p, we have

$$\sum_{m=1}^{p-1} \frac{1}{m^k} \equiv \sum_{i=0}^{p-2} \frac{1}{a^{ik}} \equiv \frac{1 - a^{-k(p-1)}}{1 - a^{-k}} \equiv 0 \pmod{p}.$$

Since we have proved that $\sum_{m=1}^{p-1} m^{-k} \equiv 0 \pmod{p}$ for all but finitely many primes p, it follows that $\zeta_{\mathcal{A}}(k) = \zeta^{\star}_{\mathcal{A}}(k) = 0$ in $\mathcal{A}$. □

1.2. Bowman-Bradley theorem. Bowman and Bradley [1] proved that the multiple zeta values at the sequences obtained by inserting a fixed number of twos between 3,1,...,3,1 add up to a rational multiple of a power of π; Kondo, Tanaka, and the first author [9] obtained the same result for multiple zeta-star values. Let $\{a_1, \dots, a_l\}^m$ denote the m times repetition of the sequence $a_1, \dots, a_l$:

$$\{a_1, \dots, a_l\}^m = \underbrace{a_1, \dots, a_l, \dots, a_1, \dots, a_l}_{lm}.$$

For the empty sequence $\emptyset$, we conventionally set $\zeta(\emptyset) = \zeta^{\star}(\emptyset) = 1$.

THEOREM 1.3 ([1, 9]). *For all nonnegative integers m and n, we have*

$$\sum_{\substack{\sum_{i=0}^{2m} n_i = n \\ n_0, \dots, n_{2m} \geq 0}} \zeta(\{2\}^{n_0}, 3, \{2\}^{n_1}, 1, \{2\}^{n_2}, \dots, 3, \{2\}^{n_{2m-1}}, 1, \{2\}^{n_{2m}}) \in \mathbb{Q}\pi^{4m+2n},$$

$$\sum_{\substack{\sum_{i=0}^{2m} n_i = n \\ n_0, \dots, n_{2m} \geq 0}} \zeta^{\star}(\{2\}^{n_0}, 3, \{2\}^{n_1}, 1, \{2\}^{n_2}, \dots, 3, \{2\}^{n_{2m-1}}, 1, \{2\}^{n_{2m}}) \in \mathbb{Q}\pi^{4m+2n}.$$

The theorem is a common generalization of the previously known results that

$$\zeta(\{3,1\}^m), \zeta^{\star}(\{3,1\}^m) \in \mathbb{Q}\pi^{4m}, \qquad \zeta(\{2\}^n), \zeta^{\star}(\{2\}^n) \in \mathbb{Q}\pi^{2n}$$

for all nonnegative integers m and n.

For finite multiple zeta(-star) values, Hoffman [4, Equation (15)] proved that

$$\zeta_{\mathcal{A}}(\{c\}^n) = \zeta^{\star}_{\mathcal{A}}(\{c\}^n) = 0$$

for all positive integers c and n, and Zhao [11, Theorem 3.18] proved that

$$\zeta_{\mathcal{A}}(\{a,b\}^m) = \zeta^{\star}_{\mathcal{A}}(\{a,b\}^m) = 0$$

for all odd positive integers a and b and for all positive integers m, of which the special case $a = 3$ and $b = 1$ was conjectured by Kaneko [6]. Our aim in this paper is to generalise Zhao's result by giving the following Bowman-Bradley type theorem, which is a corollary of our main theorem:

THEOREM 1.4. *If a and b are odd positive integers and c is an even positive integer, then for all nonnegative integers m and n with $(m,n) \neq (0,0)$, we have*

$$\begin{aligned}
&\sum_{\substack{\sum_{i=0}^{2m} n_i = n \\ n_0,\dots,n_{2m}\geq 0}} \zeta_{\mathcal{A}}(\{c\}^{n_0}, a, \{c\}^{n_1}, b, \{c\}^{n_2}, \dots, a, \{c\}^{n_{2m-1}}, b, \{c\}^{n_{2m}}) \\
&= \sum_{\substack{\sum_{i=0}^{2m} n_i = n \\ n_0,\dots,n_{2m}\geq 0}} \zeta^{\star}_{\mathcal{A}}(\{c\}^{n_0}, a, \{c\}^{n_1}, b, \{c\}^{n_2}, \dots, a, \{c\}^{n_{2m-1}}, b, \{c\}^{n_{2m}}) \\
&= 0\,.
\end{aligned}$$

Setting $n = 0$ in Theorem 1.4 gives Zhao's result.

1.3. Statement of the main theorem. To state our main theorem, we find it convenient to use an algebraic setup, due to Hoffman [2] in the case of ζ and $\zeta^{\star}$. Let $\mathfrak{H}^1 = \mathbb{Q}\langle z_1, z_2, \dots\rangle$ denote the noncommutative polynomial algebra in countably many variables. The product $\widetilde{ш}$ on $\mathfrak{H}^1$, due to Muneta [10], is the $\mathbb{Q}$-bilinear map $\widetilde{ш}\colon \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ defined inductively by

$$1 \mathbin{\widetilde{ш}} w = w \mathbin{\widetilde{ш}} 1 = w\,, \quad z_k w \mathbin{\widetilde{ш}} z_{k'} w' = z_k(w \mathbin{\widetilde{ш}} z_{k'} w') + z_{k'}(z_k w \mathbin{\widetilde{ш}} w')$$

for $w, w' \in \mathfrak{H}^1$ and $k, k' \in \mathbb{Z}_{\geq 1}$.

EXAMPLE 1.5. We have

$$z_k \mathbin{\widetilde{ш}} z_l = z_k z_l + z_l z_k\,, \qquad z_k \mathbin{\widetilde{ш}} z_l z_{l'} = z_k z_l z_{l'} + z_l z_k z_{l'} + z_l z_{l'} z_k$$

for $k, l, l' \in \mathbb{Z}_{\geq 1}$.

Define $\mathbb{Q}$-linear maps $Z_{\mathcal{A}}, \bar{Z}_{\mathcal{A}}\colon \mathfrak{H}^1 \to \mathcal{A}$ by setting

$$Z_{\mathcal{A}}(1) = \bar{Z}_{\mathcal{A}}(1) = 1\,,$$

$$Z_{\mathcal{A}}(z_{k_1}\cdots z_{k_l}) = \zeta_{\mathcal{A}}(k_1,\dots,k_l)\,, \qquad \bar{Z}_{\mathcal{A}}(z_{k_1}\cdots z_{k_l}) = \zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_l)\,.$$

For $(m,n) \in \mathbb{Z}^2_{\geq 0} \setminus \{(0,0)\}$, let $I_{m,n}$ denote the set of all sequences

$$\boldsymbol{a} = (a_1,\dots,a_m; b_1,\dots,b_m; c_1,\dots,c_n)$$

where $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ are odd positive integers and $c_1, \ldots, c_n$ are even positive integers. For $\boldsymbol{a} = (a_1, \ldots, a_m; b_1, \ldots, b_m; c_1, \ldots, c_n) \in I_{m,n}$, set

$$\begin{aligned} z_{\boldsymbol{a}} &= \sum_{\substack{\sigma,\tau\in\mathfrak{S}_m \\ \rho\in\mathfrak{S}_n}} z_{a_{\sigma(1)}} z_{b_{\tau(1)}} \cdots z_{a_{\sigma(m)}} z_{b_{\tau(m)}} \,\widetilde{\text{Ш}}\, z_{c_{\rho(1)}} \cdots z_{c_{\rho(n)}} \\ &= \sum_{\sigma,\tau\in\mathfrak{S}_m} z_{a_{\sigma(1)}} z_{b_{\tau(1)}} \cdots z_{a_{\sigma(m)}} z_{b_{\tau(m)}} \,\widetilde{\text{Ш}}\, z_{c_1} \,\widetilde{\text{Ш}} \ldots \widetilde{\text{Ш}}\, z_{c_n} \in \mathfrak{H}^1 , \end{aligned}$$

where $\mathfrak{S}_l$ is the symmetric group of degree l.

THEOREM 1.6 (Main theorem). *For all $(m, n) \in \mathbb{Z}^2_{\geq 0} \setminus \{(0, 0)\}$ and $\boldsymbol{a} \in I_{m,n}$, we have $Z_{\mathcal{A}}(z_{\boldsymbol{a}}) = \bar{Z}_{\mathcal{A}}(z_{\boldsymbol{a}}) = 0$.*

PROOF THAT THEOREM 1.6 IMPLIES THEOREM 1.4. Put $\boldsymbol{a} = (a, \ldots, a; b, \ldots, b; c, \ldots, c) \in I_{m,n}$. Then since

$$z_{\boldsymbol{a}} = \sum_{\substack{\sigma,\tau\in\mathfrak{S}_m \\ \rho\in\mathfrak{S}_n}} z_a z_b \cdots z_a z_b \,\widetilde{\text{Ш}}\, z_c \cdots z_c = m!^2 n! (z_a z_b)^m \,\widetilde{\text{Ш}}\, z_c^n ,$$

Theorem 1.6 shows that $Z_{\mathcal{A}}\big((z_a z_b)^m \,\widetilde{\text{Ш}}\, z_c^n\big) = \bar{Z}_{\mathcal{A}}\big((z_a z_b)^m \,\widetilde{\text{Ш}}\, z_c^n\big) = 0$, which is equivalent to Theorem 1.4. □

2. Proof of the main theorem.

2.1. Outline of the proof. For $(m, n) \in \mathbb{Z}^2_{\geq 0} \setminus \{(0, 0)\}$, write $P_{m,n}$ for the statement that $Z_{\mathcal{A}}(z_{\boldsymbol{a}}) = \bar{Z}_{\mathcal{A}}(z_{\boldsymbol{a}}) = 0$ for all $\boldsymbol{a} \in I_{m,n}$. Then the main theorem says that $P_{m,n}$ is true for all $(m, n) \in \mathbb{Z}^2_{\geq 0} \setminus \{(0, 0)\}$. Our proof consists of the following four lemmas:

LEMMA 2.1. *The statement $P_{0,n}$ is true for all positive integers n.*

LEMMA 2.2. *Suppose that m is a positive integer such that $P_{m,0}$ is true. Then $P_{m,n}$ is true for all nonnegative integers n.*

LEMMA 2.3. *Suppose that m is a positive integer such that $P_{m',n}$ is true whenever m' is a positive integer less than m and n is a nonnegative integer. Then $Z_{\mathcal{A}}(z_{\boldsymbol{a}}) + \bar{Z}_{\mathcal{A}}(z_{\boldsymbol{a}}) = 0$ for all $\boldsymbol{a} \in I_{m,0}$.*

LEMMA 2.4. *Suppose that m is a positive integer such that $P_{m',n}$ is true whenever m' is a positive integer less than m and n is a nonnegative integer. Then $Z_{\mathcal{A}}(z_{\boldsymbol{a}}) = \bar{Z}_{\mathcal{A}}(z_{\boldsymbol{a}})$ for all $\boldsymbol{a} \in I_{m,0}$.*

It is easy to see that the lemmas imply the main theorem. Indeed, $P_{1,0}$ follows from Lemmas 2.3 and 2.4 because $m = 1$ vacuously satisfies the assumption; Lemma 2.2 then shows that $P_{1,n}$ is true for all nonnegative integers n; it follows that $m = 2$ satisfies the assumption of Lemmas 2.3 and 2.4, and so $P_{2,0}$ is true; induction proceeds in this manner.

2.2. Proof of Lemma 2.1. Although Lemma 2.1 is a direct consequence of [4, Theorem 4.4], we give a rather detailed proof of the lemma for the convenience of the reader, partly because our notation differs from that of [4] and partly because some of the concepts introduced will also be necessary afterwards.

DEFINITION 2.5. The *harmonic products* $*$ and $\bar{*}$ on $\mathfrak{H}^1$ are the $\mathbb{Q}$-bilinear maps $*, \bar{*}\colon \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ defined inductively by

$$1 * w = w * 1 = w\,, \qquad z_k w * z_{k'} w' = z_k(w * z_{k'} w') + z_{k'}(z_k w * w') + z_{k+k'}(w * w')\,,$$
$$1 \mathbin{\bar{*}} w = w \mathbin{\bar{*}} 1 = w\,, \qquad z_k w \mathbin{\bar{*}} z_{k'} w' = z_k(w \mathbin{\bar{*}} z_{k'} w') + z_{k'}(z_k w \mathbin{\bar{*}} w') - z_{k+k'}(w \mathbin{\bar{*}} w')$$

for $w, w' \in \mathfrak{H}^1$ and $k, k' \in \mathbb{Z}_{\geq 1}$.

EXAMPLE 2.6. We have

$$z_k * z_l = z_k z_l + z_l z_k + z_{k+l}\,, \qquad z_k \mathbin{\bar{*}} z_l = z_k z_l + z_l z_k - z_{k+l}$$

for $k, l \in \mathbb{Z}_{\geq 1}$.

We remark that $\mathfrak{H}^1$ is a commutative $\mathbb{Q}$-algebra with either $*$ or $\bar{*}$ as its product.

As illustrated by

$$\begin{aligned} Z_{\mathcal{A}}(z_k)Z_{\mathcal{A}}(z_l) = \zeta_{\mathcal{A}}(k)\zeta_{\mathcal{A}}(l) &= \Bigg(\sum_{p>m\geq 1}\frac{1}{m^k}\Bigg)\Bigg(\sum_{p>n\geq 1}\frac{1}{n^l}\Bigg) \\ &= \Bigg(\sum_{p>m>n\geq 1} + \sum_{p>n>m\geq 1} + \sum_{p>m=n\geq 1}\Bigg)\frac{1}{m^k n^l} \\ &= \zeta_{\mathcal{A}}(k,l) + \zeta_{\mathcal{A}}(l,k) + \zeta_{\mathcal{A}}(k+l) = Z_{\mathcal{A}}(z_k z_l + z_l z_k + z_{k+l}) \\ &= Z_{\mathcal{A}}(z_k * z_l)\,, \end{aligned}$$

the harmonic products have been defined so that $Z_{\mathcal{A}}$ and $\bar{Z}_{\mathcal{A}}$ are respectively a $*$- and $\bar{*}$-homomorphism:

PROPOSITION 2.7. *The maps* $Z_{\mathcal{A}}, \bar{Z}_{\mathcal{A}}\colon \mathfrak{H}^1 \to \mathcal{A}$ *are respectively a* $*$- *and* $\bar{*}$-*homomorphism, i.e.* $Z_{\mathcal{A}}(w * w') = Z_{\mathcal{A}}(w)Z_{\mathcal{A}}(w')$ *and* $\bar{Z}_{\mathcal{A}}(w \mathbin{\bar{*}} w') = \bar{Z}_{\mathcal{A}}(w)\bar{Z}_{\mathcal{A}}(w')$ *for all* $w, w' \in \mathfrak{H}^1$.

Recall that a *partition* of a set X is a family of pairwise disjoint nonempty subsets of X with union X.

PROPOSITION 2.8 ([4, Theorem 4.4]). *Let* $k_1, \dots, k_n$ *be positive integers. Then*

$$Z_{\mathcal{A}}(z_{k_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{k_n}) = \bar{Z}_{\mathcal{A}}(z_{k_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{k_n}) = 0\,.$$

PROOF. Observe that

$$z_{k_1} * \cdots * z_{k_n} = \sum_{\Pi \text{ is a partition of } \{1,\dots,n\}} \widetilde{\coprod_{A\in\Pi}} z_{\sum_{i\in A} k_i}\,;$$

apply $Z_{\mathcal{A}}$ and use Propositions 1.2 and 2.7 to obtain

$$\sum_{\Pi \text{ is a partition of } \{1,\dots,n\}} Z_{\mathcal{A}}\Bigg(\widetilde{\coprod_{A\in\Pi}} z_{\sum_{i\in A} k_i}\Bigg) = 0\,.$$

This shows by induction on n that $Z_{\mathcal{A}}(z_{k_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{k_n}) = 0$ whenever $k_1, \ldots, k_n$ are positive integers. The other equation $\bar{Z}_{\mathcal{A}}(z_{k_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{k_n}) = 0$ can be proved in a similar fashion by using $\bar{*}$ instead of $*$. □

PROOF OF LEMMA 2.1. Immediate from Proposition 2.8. □

2.3. Proof of Lemma 2.2. Before presenting a proof for general m, we look at the simple case of $m = 1$. We prove by induction on n that $Z_{\mathcal{A}}(z_a z_b \mathbin{\widetilde{ш}} z_{c_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{c_n}) = 0$ for all $(a; b; c_1, \ldots, c_n) \in I_{1,n}$, assuming the base case $n = 0$. Let $n \geq 1$ and suppose that the claim is true if n is smaller. Let $(a; b; c_1, \ldots, c_n) \in I_{1,n}$. Apply $Z_{\mathcal{A}}$ to the identity

$$\begin{aligned} z_a z_b * (z_{c_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{c_n}) &= z_a z_b \mathbin{\widetilde{ш}} z_{c_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{c_n} \\ &\quad + \sum_{j=1}^{n} \Bigl(z_{a+c_j} z_b \mathbin{\widetilde{ш}} \mathop{\widetilde{Ш}}_{k \neq j} z_{c_k} \Bigr) + \sum_{j=1}^{n} \Bigl(z_a z_{b+c_j} \mathbin{\widetilde{ш}} \mathop{\widetilde{Ш}}_{k \neq j} z_{c_k} \Bigr) \\ &\quad + \sum_{i \neq j} \Bigl(z_{a+c_i} z_{b+c_j} \mathbin{\widetilde{ш}} \mathop{\widetilde{Ш}}_{k \neq i,j} z_{c_k} \Bigr) \end{aligned}$$

and use the inductive hypothesis to obtain

$$0 = Z_{\mathcal{A}}(z_a z_b) Z_{\mathcal{A}}(z_{c_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{c_n}) = Z_{\mathcal{A}}(z_a z_b \mathbin{\widetilde{ш}} z_{c_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{c_n});$$

here the inductive hypothesis applies because adding an even integer does not change parity. The key to the proof for general m given below is to find a generalization of the above identity for $m \geq 2$.

PROOF OF LEMMA 2.2. We prove $P_{m,n}$ by induction on n, assuming the base case $n = 0$. Let $n \geq 1$ and assume $P_{m,n'}$ for all integers n' with $0 \leq n' < n$. We only prove that $Z_{\mathcal{A}}(z_{\boldsymbol{a}}) = 0$ for all $\boldsymbol{a} = (a_1, \ldots, a_m; b_1, \ldots, b_m; c_1, \ldots, c_n) \in I_{m,n}$, because $\bar{Z}_{\mathcal{A}}(z_{\boldsymbol{a}}) = 0$ can be proved in a similar fashion.

Let G be a spanning subgraph, with all degrees at most 1, of the complete bipartite graph on the vertex set $\{a_1, b_1, \ldots, a_m, b_m\} \cup \{c_1, \ldots, c_n\}$; the $2m + n$ vertices are regarded as distinct even if some of them are equal as integers. Define $a'_i = a_i$ if the vertex a_i is isolated; $a'_i = a_i + c_k$ if the vertices a_i and c_k are adjacent. Define b'_j in a similar manner. Write $c'_1, \ldots, c'_l$ for the isolated vertices among $c_1, \ldots, c_n$. Then we have

$$z_{a_1} z_{b_1} \cdots z_{a_m} z_{b_m} * (z_{c_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{c_n}) = \sum_{G} (z_{a'_1} z_{b'_1} \cdots z_{a'_m} z_{b'_m} \mathbin{\widetilde{ш}} z_{c'_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{c'_l}),$$

where G runs over all such subgraphs.

Replacing a_i with $a_{\sigma(i)}$ and b_j with $b_{\tau(j)}$, and summing over all $\sigma, \tau \in \mathfrak{S}_m$, we obtain

$$z_{(a_1, \ldots, a_m; b_1, \ldots, b_m; \emptyset)} * (z_{c_1} \mathbin{\widetilde{ш}} \cdots \mathbin{\widetilde{ш}} z_{c_n}) = \sum_{G} z_{(a'_1, \ldots, a'_m; b'_1, \ldots, b'_m; c'_1, \ldots, c'_l)} .$$

Let us see what happens when we apply $Z_{\mathcal{A}}$ to this equation. The left-hand side is obviously 0. In the right-hand side, the graph G with no edge yields $Z_{\mathcal{A}}(z_{\boldsymbol{a}})$ and all the other terms vanish by the inductive hypothesis because $(a'_1, \dots, a'_m; b'_1, \dots, b'_m; c'_1, \dots, c'_l) \in I_{m,l}$ with $l < n$ when G has at least one edge. Hence we conclude that $Z_{\mathcal{A}}(z_{\boldsymbol{a}}) = 0$. □

2.4. Proof of Lemma 2.3.

PROPOSITION 2.9 ([4, Theorem 4.5]). *Let $k_1, \dots, k_n$ be positive integers. Then*

$$\zeta_{\mathcal{A}}(k_n, \dots, k_1) = (-1)^{k_1+\cdots+k_n} \zeta_{\mathcal{A}}(k_1, \dots, k_n)\,,$$
$$\zeta^{\star}_{\mathcal{A}}(k_n, \dots, k_1) = (-1)^{k_1+\cdots+k_n} \zeta^{\star}_{\mathcal{A}}(k_1, \dots, k_n)\,.$$

PROOF. We have

$$\begin{aligned}
\zeta_{\mathcal{A}}(k_n, \dots, k_1) &= \sum_{p > m_n > \cdots > m_1 \geq 1} \frac{1}{m_n^{k_n} \cdots m_1^{k_1}} \\
&= \sum_{p > \tilde{m}_1 > \cdots > \tilde{m}_n \geq 1} \frac{1}{(p - \tilde{m}_n)^{k_n} \cdots (p - \tilde{m}_1)^{k_1}} \\
&= (-1)^{k_1+\cdots+k_n} \sum_{p > \tilde{m}_1 > \cdots > \tilde{m}_n \geq 1} \frac{1}{\tilde{m}_1^{k_1} \cdots \tilde{m}_n^{k_n}} \\
&= (-1)^{k_1+\cdots+k_n} \zeta_{\mathcal{A}}(k_1, \dots, k_n)\,.
\end{aligned}$$

The other equation can be proved in the same manner. □

DEFINITION 2.10. Define a $\mathbb{Q}$-linear transformation $d\colon \mathfrak{H}^1 \to \mathfrak{H}^1$ by setting $d(1) = 1$ and

$$d(z_{k_1} \cdots z_{k_n}) = \sum_{m=1}^{n} \sum_{0 = i_0 < i_1 < \cdots < i_m = n} z_{k_{i_0+1}+\cdots+k_{i_1}} \cdots z_{k_{i_{m-1}+1}+\cdots+k_{i_m}}$$

for positive integers $k_1, \dots, k_n$.

EXAMPLE 2.11. We have $d(z_k) = z_k$ and $d(z_k z_l) = z_k z_l + z_{k+l}$.

As illustrated by

$$\begin{aligned}
\bar{Z}_{\mathcal{A}}(z_k z_l) = \zeta^{\star}_{\mathcal{A}}(k, l) &= \sum_{p > m \geq n \geq 1} \frac{1}{m^k n^l} = \Bigg(\sum_{p > m > n \geq 1} + \sum_{p > m = n \geq 1} \Bigg) \frac{1}{m^k n^l} \\
&= \zeta_{\mathcal{A}}(k, l) + \zeta_{\mathcal{A}}(k + l) = Z_{\mathcal{A}}(z_k z_l + z_{k+l}) = Z_{\mathcal{A}}\big(d(z_k z_l)\big)\,,
\end{aligned}$$

the transformation d has been defined so that $\bar{Z}_{\mathcal{A}} = Z_{\mathcal{A}} \circ d$:

PROPOSITION 2.12. *We have $\bar{Z}_{\mathcal{A}} = Z_{\mathcal{A}} \circ d$, i.e. $\bar{Z}_{\mathcal{A}}(w) = Z_{\mathcal{A}}\big(d(w)\big)$ for all $w \in \mathfrak{H}^1$.*

LEMMA 2.13. *Let $k_1, \dots, k_l$ be positive integers, where $l \geq 1$. Then*

$$\sum_{j=0}^{l} (-1)^j d(z_{k_j} \cdots z_{k_1}) * z_{k_{j+1}} \cdots z_{k_l} = 0\,.$$

PROOF. The lemma is proved in [8, Proposition 7.1]; it also follows from [5, Proposition 6], where our d is denoted by S and the coefficient $(-1)^j$ is missing. □

REMARK 2.14. When $l = 0$, the left-hand side of the equation in Lemma 2.13 should naturally be interpreted as 1 rather than 0, hence the odd-looking assumption that $l \geq 1$.

For $k \in \mathbb{Z}_{\geq 0}$, we write $[k] = \{i \in \mathbb{Z} \mid 1 \leq i \leq k\}$. For sets X and Y of the same cardinality, we write $\mathrm{Bij}(X, Y)$ for the set of all bijections from X to Y.

PROOF OF LEMMA 2.3. Let $\boldsymbol{a} = (a_1, \dots, a_m; b_1, \dots, b_m; \emptyset) \in I_{m,0}$. Then for each $(\sigma, \tau) \in \mathfrak{S}_m^2$, applying Lemma 2.13 to $l = 2m$ and $(k_1, \dots, k_l) = (a_{\sigma(1)}, b_{\tau(1)}, \dots, a_{\sigma(m)}, b_{\tau(m)})$ gives

$$
\begin{aligned}
&\sum_{i=0}^{m} d(z_{b_{\tau(i)}} z_{a_{\sigma(i)}} \cdots z_{b_{\tau(1)}} z_{a_{\sigma(1)}}) * z_{a_{\sigma(i+1)}} z_{b_{\tau(i+1)}} \cdots z_{a_{\sigma(m)}} z_{b_{\tau(m)}} \\
&\quad - \sum_{i=1}^{m} d(z_{a_{\sigma(i)}} z_{b_{\tau(i-1)}} z_{a_{\sigma(i-1)}} \cdots z_{b_{\tau(1)}} z_{a_{\sigma(1)}}) * z_{b_{\tau(i)}} z_{a_{\sigma(i+1)}} z_{b_{\tau(i+1)}} \cdots z_{a_{\sigma(m)}} z_{b_{\tau(m)}} \\
&= 0\,.
\end{aligned}
$$

By summing over all $(\sigma, \tau) \in \mathfrak{S}_m^2$ and applying $Z_{\mathcal{A}}$, we obtain

$$
\begin{aligned}
&\sum_{i=0}^{m} \sum_{\sigma,\tau \in \mathfrak{S}_m} \zeta_{\mathcal{A}}^{\star}(b_{\tau(i)}, a_{\sigma(i)}, \dots, b_{\tau(1)}, a_{\sigma(1)}) \zeta_{\mathcal{A}}(a_{\sigma(i+1)}, b_{\tau(i+1)}, \dots, a_{\sigma(m)}, b_{\tau(m)}) \\
&\quad - \sum_{i=1}^{m} \sum_{\sigma,\tau \in \mathfrak{S}_m} \zeta_{\mathcal{A}}^{\star}(a_{\sigma(i)}, b_{\tau(i-1)}, a_{\sigma(i-1)}, \dots, b_{\tau(1)}, a_{\sigma(1)}) \\
&\qquad\qquad \times \zeta_{\mathcal{A}}(b_{\tau(i)}, a_{\sigma(i+1)}, b_{\tau(i+1)}, \dots, a_{\sigma(m)}, b_{\tau(m)}) \\
&= 0\,.
\end{aligned}
$$

For simplicity, we write the left-hand side as $\sum_{i=0}^{m} P_i - \sum_{i=1}^{m} Q_i$. Since $P_0 = Z_{\mathcal{A}}(z_{\boldsymbol{a}})$ and $P_m = \bar{Z}_{\mathcal{A}}(z_{\boldsymbol{a}})$ by Proposition 2.9, it suffices to show that $P_i = 0$ for $i = 1, \dots, m-1$ and $Q_i = 0$ for $i = 1, \dots, m$.

For $i = 1, \dots, m-1$, we have

$$
\begin{aligned}
P_i &= \sum_{\sigma,\tau \in \mathfrak{S}_m} \zeta_{\mathcal{A}}^{\star}(b_{\tau(i)}, a_{\sigma(i)}, \dots, b_{\tau(1)}, a_{\sigma(1)}) \zeta_{\mathcal{A}}(a_{\sigma(i+1)}, b_{\tau(i+1)}, \dots, a_{\sigma(m)}, b_{\tau(m)}) \\
&= \sum_{\substack{A,B \subset [m] \\ \#A=\#B=i}} \left(\sum_{\substack{\sigma' \in \mathrm{Bij}([i],A) \\ \tau' \in \mathrm{Bij}([i],B)}} \zeta_{\mathcal{A}}^{\star}(b_{\tau'(i)}, a_{\sigma'(i)}, \dots, b_{\tau'(1)}, a_{\sigma'(1)}) \right) \\
&\qquad \times \left(\sum_{\substack{\sigma'' \in \mathrm{Bij}([m-i],[m]\setminus A) \\ \tau'' \in \mathrm{Bij}([m-i],[m]\setminus B)}} \zeta_{\mathcal{A}}(a_{\sigma''(1)}, b_{\tau''(1)}, \dots, a_{\sigma''(m-i)}, b_{\tau''(m-i)}) \right) \\
&= 0
\end{aligned}
$$

by the hypothesis. In a similar fashion, for $i = 1, \dots, m$, we have

$$\begin{aligned}
Q_i &= \sum_{\sigma,\tau\in\mathfrak{S}_m} \zeta^\star_{\mathcal{A}}(a_{\sigma(i)}, b_{\tau(i-1)}, a_{\sigma(i-1)}, \ldots, b_{\tau(1)}, a_{\sigma(1)})\zeta_{\mathcal{A}}(b_{\tau(i)}, a_{\sigma(i+1)}, b_{\tau(i+1)}, \ldots, a_{\sigma(m)}, b_{\tau(m)}) \\
&= \sum_{\substack{A,B\subset[m]\\ \#A=i\\ \#B=i-1}} \Biggl(\sum_{\substack{\sigma'\in\mathrm{Bij}([i],A)\\ \tau'\in\mathrm{Bij}([i-1],B)}} \zeta^\star_{\mathcal{A}}(a_{\sigma'(i)}, b_{\tau'(i-1)}, a_{\sigma'(i-1)}, \ldots, b_{\tau'(1)}, a_{\sigma'(1)}) \Biggr) \\
&\qquad \times \Biggl(\sum_{\substack{\sigma''\in\mathrm{Bij}([m-i],[m]\setminus A)\\ \tau''\in\mathrm{Bij}([m-i+1],[m]\setminus B)}} \zeta_{\mathcal{A}}(b_{\tau''(1)}, a_{\sigma''(1)}, b_{\tau''(2)}, \ldots, a_{\sigma''(m-i)}, b_{\tau''(m-i+1)}) \Biggr) \\
&= 0
\end{aligned}$$

because of Proposition 2.9 and the assumption that $a_1, \ldots, a_m, b_1, \ldots, b_m$ are all odd, and the proof is complete. □

2.5. Proof of Lemma 2.4. Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ be positive integers, and write X for the multiset consisting of the $2m$ positive integers. For $P \subset X$, denote by $s(P)$ the sum of the elements of P; denote by $\mu_a(P)$ and $\mu_b(P)$ the numbers of a's and b's contained in P respectively; define $|P| = \mu_a(P)!\mu_b(P)!$.

Write $\mathcal{P}$ for the set of all partitions Π of X such that $|\mu_a(P) - \mu_b(P)| \le 1$ for every $P \in \Pi$. For $\Pi \in \mathcal{P}$, write $\Pi = \{A_1, \ldots, A_k, B_1, \ldots, B_k, C_1, \ldots, C_l\}$ where $\mu_a(A_i) - \mu_b(A_i) = 1$, $\mu_a(B_i) - \mu_b(B_i) = -1$, and $\mu_a(C_j) = \mu_b(C_j)$, and define

$$z_\Pi = \Biggl(\prod_{P\in\Pi} |P|\Biggr) \sum_{\sigma,\tau\in\mathfrak{S}_k} z_{s(A_{\sigma(1)})} z_{s(B_{\tau(1)})} \cdots z_{s(A_{\sigma(k)})} z_{s(B_{\tau(k)})} \,\widetilde{\text{ш}}\, z_{s(C_1)} \,\widetilde{\text{ш}} \cdots \widetilde{\text{ш}}\, z_{s(C_l)} \,.$$

EXAMPLE 2.15. If $m = 1$, then $\mathcal{P}$ consists of the following two elements:

- Π_1 consisting of $C_1 = \{a_1, b_1\}$, for which $z_{\Pi_1} = z_{a_1+b_1}$;
- Π_2 consisting of $A_1 = \{a_1\}$ and $B_1 = \{b_1\}$, for which $z_{\Pi_2} = z_{a_1} z_{b_1}$.

We thus have

$$\sum_{\Pi\in\mathcal{P}} z_\Pi = z_{a_1} z_{b_1} + z_{a_1+b_1} = d(z_{a_1} z_{b_1}) \,.$$

If $m = 2$, then $\mathcal{P}$ consists of the following 12 elements:

- Π_1 consisting of $C_1 = \{a_1, b_1, a_2, b_2\}$, for which $z_{\Pi_1} = 4z_{a_1+b_1+a_2+b_2}$;
- Π_2 consisting of $A_1 = \{a_1, b_1, a_2\}$ and $B_1 = \{b_2\}$, for which $z_{\Pi_2} = 2z_{a_1+b_1+a_2} z_{b_2}$;
- Π_3 consisting of $A_1 = \{a_1, b_2, a_2\}$ and $B_1 = \{b_1\}$, for which $z_{\Pi_3} = 2z_{a_1+b_2+a_2} z_{b_1}$;
- Π_4 consisting of $A_1 = \{a_1\}$ and $B_1 = \{b_1, a_2, b_2\}$, for which $z_{\Pi_4} = 2z_{a_1} z_{b_1+a_2+b_2}$;
- Π_5 consisting of $A_1 = \{a_2\}$ and $B_1 = \{b_1, a_1, b_2\}$, for which $z_{\Pi_5} = 2z_{a_2} z_{b_1+a_1+b_2}$;
- Π_6 consisting of $C_1 = \{a_1, b_1\}$ and $C_2 = \{a_2, b_2\}$, for which $z_{\Pi_6} = z_{a_1+b_1} \,\widetilde{\text{ш}}\, z_{a_2+b_2}$;
- Π_7 consisting of $C_1 = \{a_1, b_2\}$ and $C_2 = \{a_2, b_1\}$, for which $z_{\Pi_7} = z_{a_1+b_2} \,\widetilde{\text{ш}}\, z_{a_2+b_1}$;
- Π_8 consisting of $A_1 = \{a_1\}$, $B_1 = \{b_1\}$, and $C_1 = \{a_2, b_2\}$, for which $z_{\Pi_8} = z_{a_1} z_{b_1} \,\widetilde{\text{ш}}\, z_{a_2+b_2}$;
- Π_9 consisting of $A_1 = \{a_1\}$, $B_1 = \{b_2\}$, and $C_1 = \{a_2, b_1\}$, for which $z_{\Pi_9} = z_{a_1} z_{b_2} \,\widetilde{\text{ш}}\, z_{a_2+b_1}$;

- Π_{10} consisting of $A_1 = \{a_2\}$, $B_1 = \{b_1\}$, and $C_1 = \{a_1, b_2\}$, for which $z_{\Pi_{10}} = z_{a_2} z_{b_1} \,\widetilde{\text{ш}}\, z_{a_1+b_2}$;
- Π_{11} consisting of $A_1 = \{a_2\}$, $B_1 = \{b_2\}$, and $C_1 = \{a_1, b_1\}$, for which $z_{\Pi_{11}} = z_{a_2} z_{b_2} \,\widetilde{\text{ш}}\, z_{a_1+b_1}$;
- Π_{12} consisting of $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $B_1 = \{b_1\}$, and $B_2 = \{b_2\}$, for which $z_{\Pi_{12}} = z_{a_1} z_{b_1} z_{a_2} z_{b_2} + z_{a_1} z_{b_2} z_{a_2} z_{b_1} + z_{a_2} z_{b_1} z_{a_1} z_{b_2} + z_{a_2} z_{b_2} z_{a_1} z_{b_1}$.

We thus have

$$
\begin{aligned}
z_{\Pi_1} &= \sum_{\sigma,\tau \in \mathfrak{S}_2} z_{a_{\sigma(1)}+b_{\tau(1)}+a_{\sigma(2)}+b_{\tau(2)}}\,, \\
z_{\Pi_2} + z_{\Pi_3} &= \sum_{\sigma,\tau \in \mathfrak{S}_2} z_{a_{\sigma(1)}+b_{\tau(1)}+a_{\sigma(2)}} z_{b_{\tau(2)}}\,, \\
z_{\Pi_4} + z_{\Pi_5} &= \sum_{\sigma,\tau \in \mathfrak{S}_2} z_{a_{\sigma(1)}} z_{b_{\tau(1)}+a_{\sigma(2)}+b_{\tau(2)}}\,, \\
z_{\Pi_6} + z_{\Pi_7} &= \sum_{\sigma,\tau \in \mathfrak{S}_2} z_{a_{\sigma(1)}+b_{\tau(1)}} z_{a_{\sigma(2)}+b_{\tau(2)}}\,, \\
z_{\Pi_8} + \cdots + z_{\Pi_{11}} &= \sum_{\sigma,\tau \in \mathfrak{S}_2} (z_{a_{\sigma(1)}+b_{\tau(1)}} z_{a_{\sigma(2)}} z_{b_{\tau(2)}} + z_{a_{\sigma(1)}} z_{b_{\tau(1)}+a_{\sigma(2)}} z_{b_{\tau(2)}} \\
&\qquad + z_{a_{\sigma(1)}} z_{b_{\tau(1)}} z_{a_{\sigma(2)}+b_{\tau(2)}})\,, \\
z_{\Pi_{12}} &= \sum_{\sigma,\tau \in \mathfrak{S}_2} z_{a_{\sigma(1)}} z_{b_{\tau(1)}} z_{a_{\sigma(2)}} z_{b_{\tau(2)}}
\end{aligned}
$$

and so

$$
\sum_{\Pi \in \mathcal{P}} z_\Pi = \sum_{\sigma,\tau \in \mathfrak{S}_2} d(z_{a_{\sigma(1)}} z_{b_{\tau(1)}} z_{a_{\sigma(2)}} z_{b_{\tau(2)}})\,.
$$

LEMMA 2.16. *We have*

$$
\sum_{\Pi \in \mathcal{P}} z_\Pi = \sum_{\sigma,\tau \in \mathfrak{S}_m} d(z_{a_{\sigma(1)}} z_{b_{\tau(1)}} \cdots z_{a_{\sigma(m)}} z_{b_{\tau(m)}})\,.
$$

PROOF. Succinctly speaking, the left-hand side is the expansion of the right-hand side. To be more precise, for each $\Pi \in \mathcal{P}$, each monomial w that appears in the expansion of z_Π appears in the right-hand side exactly as many times as there are pairs $(\sigma, \tau) \in \mathfrak{S}_m^2$ for which $d(z_{a_{\sigma(1)}} z_{b_{\tau(1)}} \cdots z_{a_{\sigma(m)}} z_{b_{\tau(m)}})$ gives rise to the monomial w; the number of such σ is $\prod_{P \in \Pi} \mu_a(P)!$ and the number of such τ is $\prod_{P \in \Pi} \mu_b(P)!$, from which it follows that the number of such pairs (σ, τ) is

$$
\prod_{P \in \Pi} \mu_a(P)! \cdot \prod_{P \in \Pi} \mu_b(P)! = \prod_{P \in \Pi} |P|\,.
$$

This proves the lemma. □

PROOF OF LEMMA 2.4. Let $\boldsymbol{a} = (a_1, \ldots, a_m; b_1, \ldots, b_m; \emptyset) \in I_{m,0}$. Then Lemma 2.16 shows that

$$\bar{Z}_{\mathcal{A}}(z_{\boldsymbol{a}}) = Z_{\mathcal{A}}\big(d(z_{\boldsymbol{a}})\big) = \sum_{\Pi \in \mathcal{P}} Z_{\mathcal{A}}(z_{\Pi}).$$

If $\Pi = \big\{\{a_1\}, \ldots, \{a_m\}, \{b_1\}, \ldots, \{b_m\}\big\}$, then $z_{\Pi} = z_{\boldsymbol{a}}$; otherwise, z_{Π} is an integer multiple of $z_{\boldsymbol{b}}$ for some $\boldsymbol{b} \in \bigcup_{1 \le m' < m} \bigcup_{n \ge 0} I_{m',n}$, and so $Z_{\mathcal{A}}(z_{\Pi}) = 0$ by the hypothesis. It follows that $\bar{Z}_{\mathcal{A}}(z_{\boldsymbol{a}}) = Z_{\mathcal{A}}(z_{\boldsymbol{a}})$. □

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