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BRAESS' PARADOX IN A TWO-TERMINAL TRANSPORTATION NETWORK

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Abstract The sensitivity of a OD travel cost to changes in a flow through a new branch in the user optimizing assignment problem for a single commodity two-terminal transportation network is studied. The assignment problem and its dual are formulated as convex programming problems. Using these formulations, an intuitive characterization is derived for such paradoxical phenomenon that the creation of a new branch has the effect of increasing the OD travel cost.

1. Introduction

There are the following two broad principles, which are first enunciated by Wardrop [8], for determining the distribution of traffic in a transportation network. The first principle, which we call system optimization, is

(i) "*the total travel cost is a minimum.*"

And the second principle, which we call user optimization, is

(ii) "*the travel cost on all OD paths joining the origin and the destination actually used are equal, and less that those which would be experienced by a single user on any unused OD path.*"

The second principle is usually used to solve the traffic assignment problem.

It is well known that the user optimized flow does not always minimize the total travel cost. One of the most remarkable examples of this fact is the one presented by Braess in 1968 [1] and picked up by Murchland in 1970 [5]: Increasing the network capacity by inserting a directed branch between two initial OD paths has the effect of increasing the OD travel cost for every user on the network, which is called "Braess' paradox".

Previous studies on "Braess' paradox" (Murchland [5], Fisk [2], Stewart

[7], Frank [3]) have been concerned with the specific network presented by Braess, where the user optimization network assignment problem was formulated as the network equilibrium conditions. But as is pointed out by Murchland, "Braess' paradox" is not paradoxical in the sense that the problem can be formulated as a minimization problem whose objective function is different from the total travel cost. In this paper, we formulate the problem and its dual of a single commodity two-terminal network as convex programming problems. Using these formulations, the dual form of "Braess' paradox" is derived. And when a travel cost per unit flow associated to each branch is linear, an intuitive characterization is obtained by comparing the potential distribution of the given network and that of the linear resister network whose topological structure is the same as that of the given network and each of whose branches has resistance equal to the increase in travel cost through it resulting from a unit increase of flow in it. Another example of "Braess' paradox" is presented, where the creation of the additional branch strictly decreases the total travel cost, which is not the case in Braess' example.

2. Notations and Formulations of Flow Problems

We consider a transportation network G composed of the set of nodes $N=\{v_a|a=1,\dots,n\}$ and the set of directed branches $E=\{b_k|k=1,\dots,m\}$. The incidence relation is represented by

$$d_{ak} = \begin{cases} 1 & \text{if branch } b_k \text{ starts from node } v_a, \\ -1 & \text{if branch } b_k \text{ ends at node } v_a, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by ξ_k (≥ 0) the flow through branch b_k, and by η_k the tension across b_k. Each of the branches of G is endowed with the branch characteristic represented as

$$\eta_k = \phi_k(\xi_k) \quad \text{for } b_k \ (k=1,\dots,m),$$

where ϕ_k is a continuous function of ξ_k from $[0,\infty)$ to real numbers R, and is differentiable and monotone increasing (Fig. 1):

$$\frac{d\phi_k}{d\xi_k} > 0 \quad \text{for } \xi_k > 0. \tag{2.1}$$

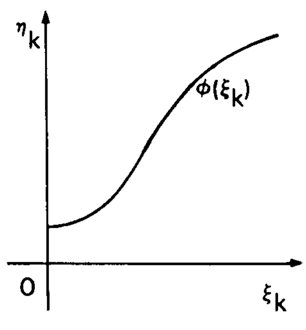

Fig. 1. Branch characteristic

$\phi_k(\xi_k)$ can be interpreted as a travel cost per unit flow through branch b_k.

The flows in G satisfy the continuity condition

$$\sum_{k=1}^{m} d_{ak}\xi_k = 0 \quad \text{for } a=1,\dots,n,$$

and the tensions in G satisfy the continuity condition

$$\sum_{a=1}^{n} d_{ak}\zeta_a = \eta_k \quad \text{for } k=1,\dots,m,$$

where ζ_a is a potential at node v_a.

In the following, as is illustrated in Fig. 2, we will consider flow problems in a two-terminal network with entrance node v_1 and exit node v_n. Let b_1 be an extra branch connecting the exit node directly to the entrance node.

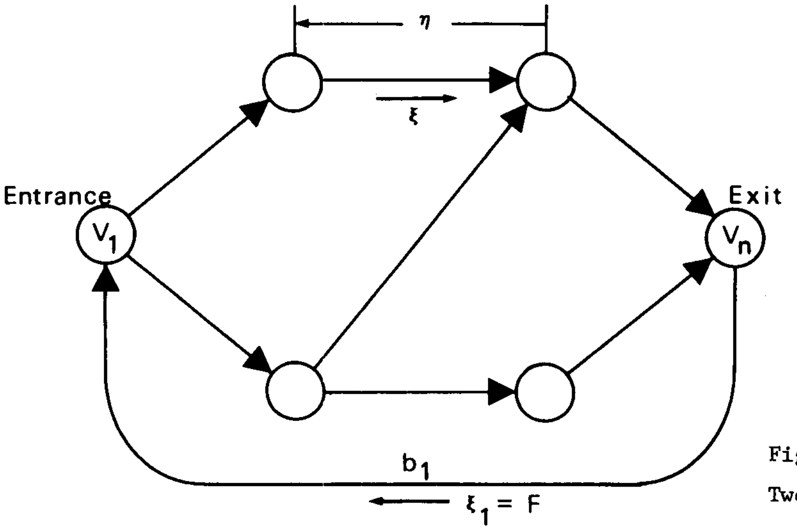

Fig. 2.
Two-terminal network

In system optimization, the problem is to determine flows ξ_k's so as to minimize the total travel cost in G which we call the system optimized cost:

[PS] Minimize

$$\sum_{k=2}^{m} \xi_k\phi_k(\xi_k)$$

subject to the constraints:

$$\sum_{k=1}^{m} d_{ak}\xi_k = 0 \quad \text{for } a=1,\dots,n,$$

$$\xi_k \geq 0 \quad \text{for } k=2,\dots,m,$$

$$\xi_1 = F,$$

where F is a given input-output flow rate and is a positive number. We call this problem a system optimization problem. We assume that PS has a feasible solution.

In user optimization, as shown in Zangwill [9], the problem can be also formulated as a minimization problem of minimizing the sum of the integral of the travel cost per unit flow of each branch of G which we call the user optimized cost. The problem is to determine flows ξ_k's so as to

[P] minimize

$$\sum_{k=2}^{m} \int_0^{\xi_k} \phi_k(\xi_k)d\xi_k$$

subject to the constraints:

$$\sum_{k=1}^{m} d_{ak}\xi_k = 0 \qquad \text{for } a=1,\dots,n,$$

$$\xi_k \ge 0 \qquad \text{for } k=2,\dots,m,$$

$$\xi_1 = F\ .$$

We call this problem a user optimization problem. We denote by $\bar{\lambda}$ the travel cost per unit flow of a solution of P along any OD path which is a connected sequence of branches joining v_1 and v_n.

The problem dual to P is the problem to determine η_k's and ζ_a's so as to

[P*] minimize

$$\eta_1 F + \sum_{k=2}^{m} \int_{-\infty}^{\eta_k} \psi_k(\eta_k)d\eta_k$$

subject to the constraints:

$$\sum_{a=1}^{n} d_{ak}\zeta_a = \eta_k \qquad \text{for } k=1,\dots,m.$$

ψ_k is the inverse of ϕ_k defined as

$$\psi_k(\eta_k) = \begin{cases} \phi_k^{-1}(\eta_k) & \text{for } \eta_k \ge \phi_k(0), \\ 0 & \text{for } \eta_k < \phi_k(0). \end{cases}$$

From (2.1), ψ_k is a continuous function from $(-\infty, \infty)$ to R and satisfies the relation

$$\text{(2.2)} \qquad \frac{d\psi_k}{d\eta_k} > 0 \qquad \text{for } \eta_k > \phi_k(0).$$

It is easily verified from (2.1) and (2.2) that the objective functions of P and P^* are convex functions of ξ and η, respectively. Therefore, from the duality theorem (Theorems 15.1, 15.2, 15.3 and 15.4 in [4]), for a solution $\bar{\xi}$ of P, there exists a solution $\bar{\eta}$ and $\bar{\zeta}$ of P* which satisfies

[W] $\bar{\eta}_k = \phi_k(\bar{\xi}_k)$ if $\bar{\xi}_k > 0$ and

$\bar{\eta}_k \leq \phi_k(0)$ if $\bar{\xi}_k = 0$ for $k=2,\ldots,m$.

It follows from the relation W that $-\eta_1$ is equal to the OD travel cost $\bar{\lambda}$ so that the relation W is equivalent to Wardrop's principle of user optimization.

Let b_{m+1} be a branch added to network G and $\tilde{G}$ be the augmented network. The user optimization flow problem $\tilde{P}$ and its dual $\tilde{P}$* of network $\tilde{G}$ can be formulated by simply changing m by m+1 in P and P*, respectively. Let $\tilde{\eta}$ and $\tilde{\zeta}$ be a solution of $\tilde{P}$* and $\tilde{\lambda}=-\tilde{\eta}_1$. Since the OD travel cost is the negative value of the tension across branch b_1, "Braess' paradox" occurs if

(2.3) $\tilde{\lambda} > \bar{\lambda}$.

We will derive the dual form of (2.3). In problem $\tilde{P}$*, we put $\eta_1=-\lambda$ and denote by $\Psi(\lambda)$ the optimal value of the objective function for a given value λ. Since $\Psi(\lambda)$ is a convex function of λ, it follows that (2.3) holds if and only if the following relation holds (see Fig. 3):

(2.4) $\frac{d\Psi}{d\lambda}(\bar{\lambda}) < 0$.

The problem dual to this problem is to determine flows ξ_k's so as to

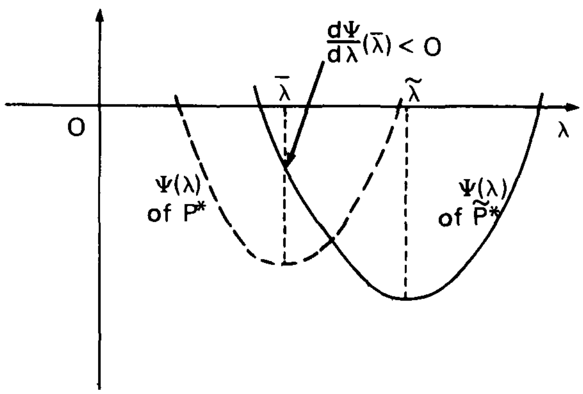

Fig. 3. Illustration of (2.4)

[$\hat{P}$] minimize

(2.5) $$-\lambda(\xi_1-F) + \sum_{k=2}^{m+1} \int_0^{\xi_k} \phi_k(\xi_k)d\xi_k$$

subject to the constraints:

$$\sum_{k=1}^{m+1} d_{ak}\xi_k = 0 \qquad \text{for } a=1,\ldots,n,$$

$$\xi_k \geq 0 \qquad \text{for } k=2,\ldots,m+1.$$

We denote by $\Phi(\lambda)$ the optimal value of (2.5) and by $\hat{F}$ the optimal flow in b_1 when $\lambda=\bar{\lambda}$.

Since the relation

$$\Phi(\lambda) + \Psi(\lambda) = 0$$

holds from the duality theorem, and the differentiation of $\Phi(\lambda)$ with respect to λ yields

$$\frac{d\Phi}{d\lambda}(\bar{\lambda}) = F - \hat{F},$$

it follows that the relation (2.3) is equivalent to the relation

$$(2.6) \qquad F - \hat{F} > 0,$$

which is the dual form of (2.3).

3. An Example of Braess' Paradox

Let us consider the flow problems of the original network of Fig. 4. The travel cost per unit flow through a branch is written beside each branch, and an input-output flow rate is 5. Because of the symmetry, the solution of the system optimization problem is a flow of 2.5 units on each of the four branches and the system optimized cost is 87.5. The solution of the user optimization problem is the same as that of the system optimization problem and

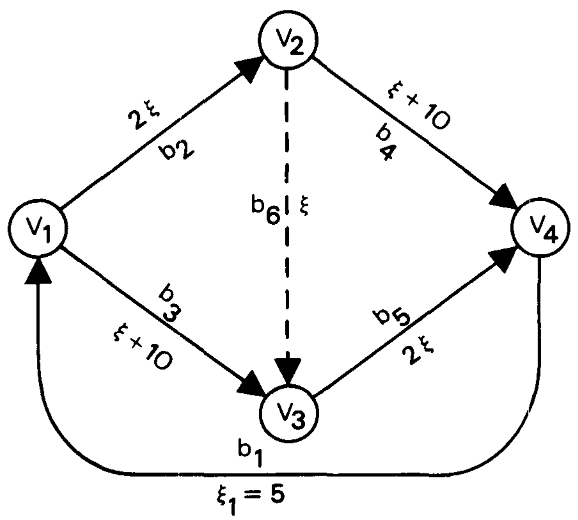

Fig. 4. Two-terminal network for an example of "Braess' paradox". The original network G is composed of branches $b_1,\ldots,b_5$, and the augmented network is $G\cup\{b_6\}$.

the user optimized cost is 68.75, and the OD travel cost is 17.5 per unit flow.

Suppose branch b_6 is inserted between v_2 and v_3 of the original network of Fig. 4. The flows of the solution of the system optimization problem of the augmented network and each OD travel cost along possible three OD paths are shown in Fig. 5. The system optimized cost in this case is 85. Therefore the addition of Branch b_6 decreases the total travel cost so that it increases the network capacity.

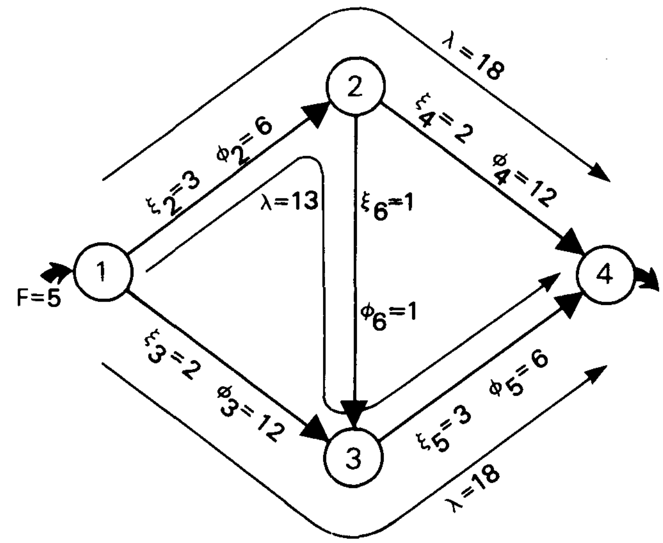

Fig. 5. Flows of the solution of the system optimization problem of the augmented network of Fig. 4.

The flows of the solution of the user optimization problem of the augmented network are shown in Fig. 6. The user optimized cost is 57.5 and the OD travel cost $\tilde{\lambda}$ is 19 per unit flow. "Braess' paradox" occurs in this example, since the addition of branch b_6 increases the OD travel cost.

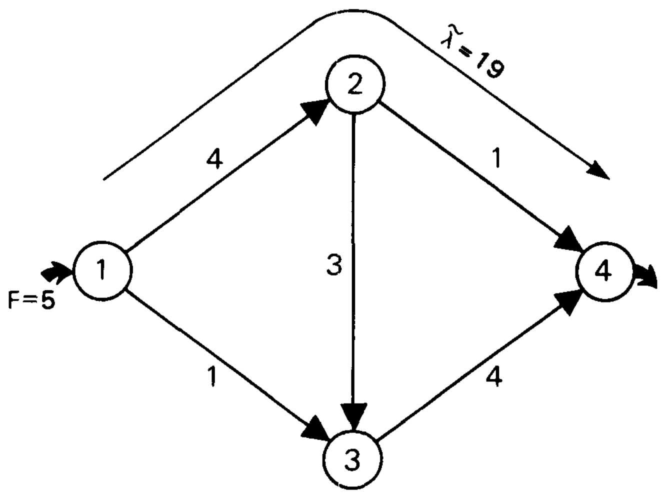

Fig. 6. Flows of the solution of the user optimization problem of the augmented network of Fig. 4.

4. A Characterization of Braess' Paradox

Let b_{m+1} be a branch added to network G defined in section 2 and $\tilde{G}$ be an augmented network. From the duality theorem, flows, tensions and potentials satisfying the following relations are solutions of P and P^*, and vice versa.

[C]

$$(4.1)\qquad \begin{aligned} &\eta_k = \phi_k(\xi_k) &&\text{if}\quad \xi_k > 0 \quad\text{and} \\ &\eta_k \le \phi_k(0) &&\text{if}\quad \xi_k = 0 \quad\text{for}\quad k=2,\dots,m, \end{aligned}$$

$$(4.2)\qquad \sum_{k=1}^{m} d_{ak}\xi_k = 0 \quad\text{for } a=1,\dots,n,$$

$$(4.3)\qquad \xi_1 = F,$$

$$(4.4)\qquad \xi_k \ge 0 \quad\text{for } k=2,\dots,m,$$

$$(4.5)\qquad \sum_{a=1}^{n} d_{ak}\zeta_a = \eta_k \quad\text{for } k=1,\dots,m.$$

Solutions $(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta})$ of $\tilde{P}$ and $\tilde{P}^*$ also satisfy the relation $\tilde{C}$ which is obtained by changing m by $m+1$ in C. In the following, we may set $\zeta_n = 0$ without loss of generality, because the corresponding flow continuity equation at exit node v_n is redundant and can be dropped in C and $\tilde{C}$. "Braess' paradox" occurs if

$$(4.6)\qquad \bar{\eta}_1 > \tilde{\eta}_1.$$

It is easily understood that when the additional branch b_{m+1} is not used, a solution of P (or P^*) is also a solution of $\tilde{P}$ (or $\tilde{P}^*$) so that $\bar{\eta}_1=\tilde{\eta}_1$. Therefore, it is necessary to (4.6) occur that ϕ_{m+1} satisfy the relation

$$(4.7)\qquad \sum_{a=1}^{n} d_{a\,m+1}\bar{\zeta}_a > \phi_{m+1}(0).$$

We now derive such characterization of "Braess' paradox" that the addition of a new branch in the opposite direction to the potential configuration of the linear resister network G_L whose topological structure is the same as that of the original network G, and whose branches have resistances $\phi'_k(\bar{\xi}_k) = d\phi_k(\bar{\xi}_k)/d\xi_k$.

We consider the case when the flow in branch b_{m+1} is sufficiently small:

$$(4.8)\qquad \xi_{k+1} = \varepsilon \ll 1.$$

From the nonamplification theorem (Theorem 16.3 in [4]), we have

(4.9) $\quad |\tilde{\xi}_k - \bar{\xi}_k| \leq \varepsilon \quad$ for $k=1,\ldots,m+1$.

Subtracting the continuity equations (4.2) from the corresponding equations in $\tilde{C}$ and using (4.3) we have

(4.10) $\quad D\Delta\xi + d_{m+1}\varepsilon = 0,$

and subtracting (4.5) from the corresponding equations in $\tilde{C}$ gives

(4.11) $\quad \Delta\eta = D^t\Delta\zeta,$

where $\Delta\xi$ and $\Delta\eta$ are (m-1)-vectors, d_k and $\Delta\zeta$ are (n-1)-vectors and D is an $(n-1)\times(m-1)$-matrix defined as

$$\begin{aligned}
\Delta\xi &= (\tilde{\xi}_2-\bar{\xi}_2, \tilde{\xi}_3-\bar{\xi}_3, \ldots, \tilde{\xi}_m-\bar{\xi}_m)^t,\\
\Delta\eta &= (\tilde{\eta}_2-\bar{\eta}_2, \tilde{\eta}_3-\bar{\eta}_3, \ldots, \tilde{\eta}_m-\bar{\eta}_m)^t,\\
\Delta\zeta &= (\tilde{\zeta}_1-\bar{\zeta}_1, \tilde{\zeta}_2-\bar{\zeta}_2, \ldots, \tilde{\zeta}_{n-1}-\bar{\zeta}_{n-1})^t,\\
d_k &= (d_{1k}, d_{2k}, \ldots, d_{n-1\ k})^t \quad \text{for } k=1,\ldots,m+1,\\
D &= (d_2, d_3, \ldots, d_m).
\end{aligned}$$

It is easily verified that matrix D defined above is of rank $n-1$, if graph G is connected.

Now, we assume that

(4.12) *the flows* $\bar{\xi}_k$*'s of* $\bar{P}$ *and* $\tilde{\xi}_k$*'s of* $\tilde{P}$ *are all positive.*

Under this assumption, (4.1) and the corresponding equations in $\tilde{C}$ hold in equality. Therefore, using (4.9), we have

(4.13) $\quad \tilde{\eta}_k-\bar{\eta}_k = \phi_k'(\bar{\xi}_k)(\tilde{\xi}_k-\bar{\xi}_k) + o(\varepsilon) \quad$ for $k=2,\ldots,m$.

Neglecting $o(\varepsilon)$ term in (4.13) and from (4.11), we have

(4.14) $\quad \Delta\xi = AD^t\Delta\zeta,$

where A is an $(m-1)\times(m-1)$ diagonal matrix whose (k-1)st diagonal element is $1/\phi_k'(\bar{\xi}_k)$. It follows from (2.1) that A is a positive definite matrix. Substituting (4.14) into (4.10) yields

$$DAD^t\Delta\zeta = -d_{m+1}\varepsilon.$$

and since DAD^t is a positive definite matrix, we have

(4.15) $\quad \Delta\zeta = -(DAD^t)^{-1}d_{m+1}\varepsilon.$

Using the relation $d_1^t\Delta\zeta = \tilde{\eta}_1-\bar{\eta}_1$, we have from (4.15) that (4.6) holds if and only if

(4.16) $\quad d_1^t(DAD^t)^{-1}d_{m+1} > 0.$

Theorem 1. Under the assumptions (4.7), (4.8) and (4.12), "Braess' paradox" occurs if and only if (4.16) holds.

Condition (4.16) has the following intuitive meaning. Consider the relation L of flows, tensions and potentials of the linear resister network G_L.

[L]

$$\eta_k = \phi_k'(\bar{\xi}_k)\xi_k \qquad \text{for } k=2,\ldots,m,$$

$$\sum_{k=1}^{m} d_{ak}\xi_k = 0 \qquad \text{for } a=1,\ldots,n,$$

$$\xi_1 = 1,$$

$$\sum_{a=1}^{n} d_{ak}\zeta_a = \eta_k \qquad \text{for } k=1,\ldots,m.$$

It is easily shown that the potential at the node which should be positively incident to b_{m+1} minus the potential at the node negatively incident to b_{m+1} is equal to $-d_{m+1}^t(DAD^t)^{-1}d_1$. Therefore, (4.16) means that the addition of a new branch opposite to the direction in which a current will flow in the linear resister network G_L causes "Braess' paradox".

The assumption (4.12) was essentially used to obtain Theorem 1, which can not be guaranteed before a user optimized flow configuration of $\tilde{G}$ is obtained. When (4.12) is weakened to

(4.17) *the flows* $\bar{\xi}_k$*'s of* P *are all positive,*

we obtain the following theorem:

Theorem 2. Under the assumptions (4.7), (4.8) and (4.17), and if we assume

(4.18) $d_1(DAD^t)^{-1}D \leq 0,$

(4.16) holds if "Braess' paradox" occurs.

Proof: Since the flow-tension relations in $\tilde{C}$ (corresponding to (4.1)) hold in inequality, we have in place of (4.14),

(4.19) $\Delta\xi \geq AD^t\Delta\zeta.$

Multiplying (4.19) by $d_1^t(DAD^t)^{-1}D$ yields

(4.20) $d_1^t(DAD^t)^{-1}D\Delta\xi \leq d_1^t(DAD^t)^{-1}DAD^t\Delta\zeta = d_1^t\Delta\zeta.$

Substituting (4.10) into the left-hand side of (4.20), we have

$$-d_1^t(DAD^t)^{-1}d_{m+1}\varepsilon \leq d_1^t\Delta\zeta = \tilde{\eta}_1 - \bar{\eta}_1,$$

and

$$\text{if} \quad \tilde{\eta}_1 - \bar{\eta}_1 < 0 \quad \text{then} \quad d_1^t(DAD^t)^{-1}d_{m+1} > 0.$$

It is easily understood that Theorems 1 and 2 remain valid assuming the branch characteristic functions ϕ_k's being linear in ξ_k for k=2,...,m, instead of the assumption (4.8).

But (4.16) is not a necessary nor sufficient condition for "Braess' paradox" to occur even if (4.7) and (4.12) hold, when the flow in b_{m+1} is not small and ϕ_k's are not linear. The networks of Figs. 7 and 8 are examples which show this situation. The topological structure of the original network and the augmented network are the same as those of Fig. 4, and branches b_2 and b_5, b_3 and b_4 have the same branch characteristic functions which are illustrated beside the branches, respectively, in Figs. 7 and 8. The solution of the user optimization problem of the original network is the same as that of Fig. 4 for each of the networks of Figs. 7 and 8.

Both (4.7) and (4.16) are satisfied in the network of Fig. 7, but the solution of the problem of the augmented network is flows of 3 units on b_2 and b_5, flows of 2 units on b_3 and b_4, a flow of 1 unit on b_6 and the OD travel cost is 17 so that "Braess' paradox" does not occur. On the other hand, (4.7) is satisfied but (4.16) is not in the network of Fig. 8, but the solution of the problem of the augmented network is the same as which is illustrated in Fig. 6 and "Braess' paradox" occurs.

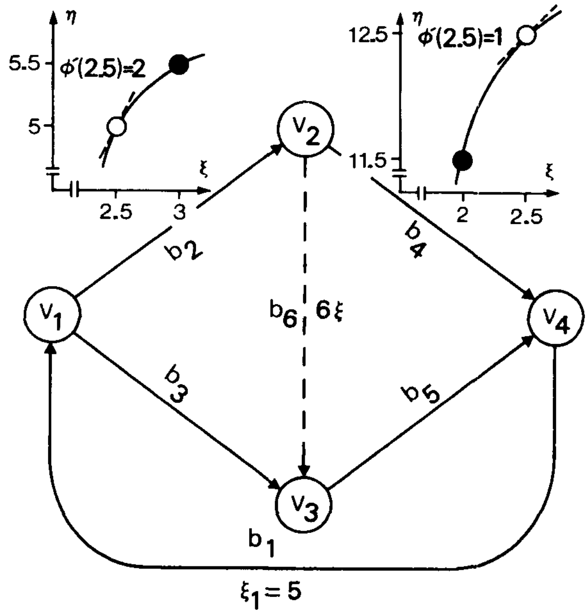

Fig. 7. Two-terminal network, where (4.7) and (4.16) are satisfied but "Braess' paradox" does not occur. ○ denotes the solution of the original network and ● denotes the solution of the augmented network.

We conclude from these examples as follows. When a branch characteristic function ϕ_k is nonlinear, there exist infinitely many curves satisfying (2.1) and connecting $(\bar{\xi}_k, \bar{\eta}_k)$ of the original network and $(\tilde{\xi}_k, \tilde{\eta}_k)$ of the augmented network (see Fig. 9). Therefore, there is no test generally efficient to forecast the occurence of "Braess' paradox", which uses only the knowledge of branch characteristics at the flow and tension configuration of the original network.

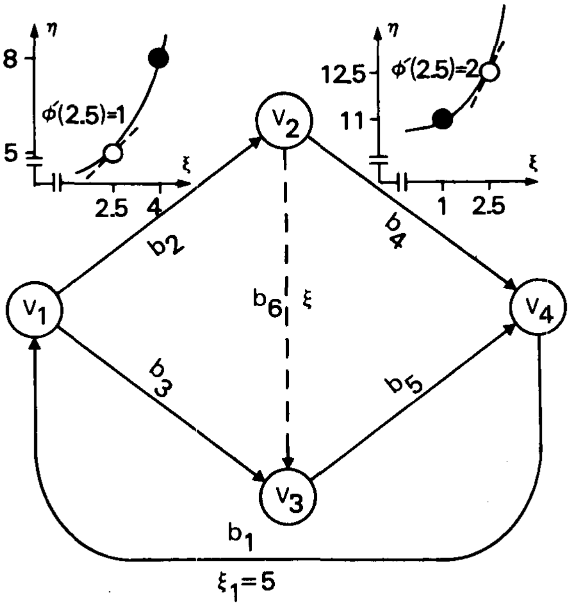

Fig. 8. Two-terminal network, where (4.7) is satisfied but (4.16) is not, but "Braess' paradox" occurs. ○ denotes the solution of the original network and ● denotes the solution of the augmented network.

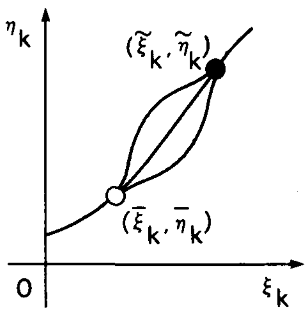

Fig. 9. Branch characteristics through $(\bar{\xi}_k, \bar{\eta}_k)$ and $(\tilde{\xi}_k, \tilde{\eta}_k)$.

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