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Bragg scattering and wave-power extraction by an array of small buoys

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Future designs of systems for power extraction from ocean waves will likely involve a periodic array of absorbing units. We report an asymptotic theory of scattering and radiation by a linear array of heaving buoys in a channel and attached to power-takeoff devices. The spacing between buoys is assumed to be comparable to the incident wavelength and sea depth but much greater than the buoy size. The effects of extraction rate on the buoy motion, transmission and reflection coefficients for a range of frequencies in and outside the band gap are studied. It is found that strong reflection for frequencies inside the band gap of Bragg resonance reduces the extraction efficiency significantly. For comparison an alternate theory for the efficiency away from the band gap is derived by using Froude-Krylov approximation. The predictions confirms and complements the asymptotic theory.

Keywords: Periodic buoy array, Multiple scattering and radiation, Bragg resonance, Wave power extraction, Homogenisation.

1. Introduction

Extensive theoretical studies have been devoted to the potential of power extraction from sea waves by an isolated unit such as a buoy, a raft or an oscillating water column (see reviews by Newman (1979); Falnes (2002); Mei *et al.* (2005); Cruz (2008); McCormick (1980)). To achieve power output comparable to a conventional power plant or a wind-turbine farm, a large array of absorbing units is necessary. The possible effects of hydrodynamic interactions among units in any geometrical deployment are therefore of design interest.

Several methods have been developed to compute the scattering and radiation by a finite number of stationary or floating bodies. In particular Falnes (1980); Falnes and Budal (1982); Falnes (1984) have examined the case of large separation where hydrodynamic interactions between bodies are weak. Kagemoto and Yue (1986) have used eigenfunction expansions and addition theorems of Bessel functions to derive a numerical method to treat scattering by a few fixed vertical cylinders of circular cross sections. Infinite and semi-infinite lines of vertical cylinders have been treated semi-numerically by Linton and Evans (1992); Linton and Mclver (1996) using multipole expansions. Numerical studies of a finite number of cylinders have been reported by Linton and Mclver (1996); Chamberlain (2007); Peter *et al.* (2006); Peter and Meylan (2007); McIver (2002); Siddorn and Eatock Taylor (2008) and Mavrakos and McIver (1997) who also considered wave power extraction

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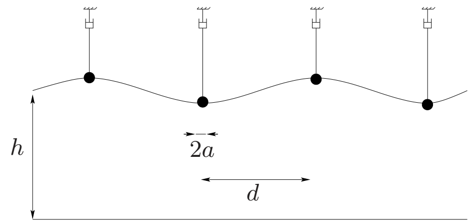


Figure 1. A sparse array of buoys attached to energy absorbers.

without giving numerical results. These methods can in principle deal with several cylinders of different sizes, but become computationally intensive for a large number of cylinders.

It is well known that, if resonated, a single buoy with one degree of freedom (e.g., heave) can absorb all the energy within the length of $1/k$ of the incoming wave front. If roll is also allowed and optimized then the length can be doubled. Thus for high efficiency the ideal spacing d between adjacent buoys is $1/k$, i.e., $kd = 1$. For environmental and navigational considerations, future wave-power farms will likely involve two-dimensional arrays with much larger spacing. Now it is known in general physics that when $kd = n\pi$ where n is an integer, waves are strongly scattered due to Bragg resonance. It is therefore of interest to predict the effect of Bragg resonance on energy absorption and buoy response.

In this article, we shall first extend the multiple-scale analysis of Li and Mei (2007*a*) for a periodic array of fixed slender piles to small movable buoys which scatter, radiate and absorb energy by its heave motion only. To simulate their potential for power absorption we assume that each buoy is attached to a linear device which converts the mechanical energy of the buoy to electricity. No phase control of the power-takeoff system is assumed. Analytical results for the scattering coefficients, energy absorption rate and buoy motion will be derived and discussed.

2. Scales and normalization

Consider a linear array of small buoys in a long channel of constant width d and mean depth h, as shown in figure 1. Simple harmonic waves arrive from one end of the channel $x \sim -\infty$. In the framework of potential theory for inviscid and incompressible fluid, the mathematical problem is equivalent to an infinitely long strip of buoys in an rectangular lattice, attacked by a plane wave with crests parallel to the edge of the strip.

The geometry has three sharply different length scales: the small radius a of the buoy, the water depth h, and the large horizontal extent of the array. These scales will be respectively referred to as the micro-scale for the near field, the meso-scale for the intermediate field, and the macro-scale for the far field. In all three fields the vertical displacements of the free surface (η) and of the buoys (ζ) are characterized by the incident wave amplitude A. Let us first introduce the following dimensionless variables, distinguished by primes, for the intermediate field,

$$\Phi = A\sqrt{gh}\Phi', \qquad p = \rho g A\, p', \qquad (\eta, \zeta) = A(\eta', \zeta'), \tag{2.1}$$

$$(x, y, z, d) = h(x', y', z', d'), \qquad t = t'\sqrt{\frac{h}{g}}. \tag{2.2}$$

We shall assume that the wavelength, mean depth and buoy separation are of the same order of magnitude, so that

$$k' = kh = O(1), \qquad d' = \frac{d}{h} = O(1), \tag{2.3}$$

are both of order unity but the buoy radius and drafts are small:

$$\frac{a}{h} \equiv \mu \ll 1, \qquad \text{and} \qquad \frac{H}{h} = O(\mu) \ll 1. \tag{2.4}$$

In the near field of a buoy the physics is controlled by the much smaller radius a, hence it is proper to employ the micro-scale coordinates, distinguished by bars and defined by

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{a}(x, y, z) = \frac{1}{\mu}(x', y', z'). \tag{2.5}$$

It is known that the scattered wave from a small cylinder is smaller than the incident wave by a factor of order $(ka)^2$, as in the case sound (see, e.g., Mei *et al.* (2005)). It will also be shown in the next section that a small heaving buoy extracts a fraction $(ka)^2$ of the incoming energy. Therefore, significant scattering and radiation effects can be expected to accumulate in an array with $O(1/\mu^2)$ buoys, or after a distance of h/μ^2. We therefore define the following macro-scale coordinates for the far-field to describe the evolution of wave envelopes:

$$(X, Y, T) = \mu^2(x', y', t'). \tag{2.6}$$

3. Away from Bragg resonance

As a reference for later comparison, we first study the performance of an array of buoys well separated from each other without Bragg resonance. In this case, interaction between buoys can be neglected as a first approximation. Consider one small buoy in a plane incident wave of frequency ω and amplitude A. The potential of the incoming wave is

$$\Phi_I = \phi_I e^{-\mathrm{i}\omega t} \quad \text{with} \qquad \phi_I = \frac{Ag}{\mathrm{i}\omega}\frac{\cosh k(z+h))}{\cosh(kh)}\mathrm{e}^{\mathrm{i}kx}, \tag{3.1}$$

where ω and k are related by the dispersion relation

$$\omega^2 = gk\tanh(kh). \tag{3.2}$$

Because of the small size of buoys, the scattered and radiated waves are negligible. Froude-Krylov approximation can be applied so that the hydrodynamic pressure on each buoy is dominated by the undisturbed incoming wave (see Newman (1979)). The vertical exciting force on the buoy is therefore

$$\mathrm{i}\rho\omega \iint_{S_B} \phi_I(0,0,0)\,\mathrm{d}S = \rho g A \pi a^2. \tag{3.3}$$

Let us assume that an energy extraction device is attached to each buoy and exerts a load force $\mathrm{i}\omega\lambda_g\zeta$ where λ_g denotes the extraction rate. Since the added buoyancy force due to heave is $-\pi a^2 \rho g \zeta$, Newton's law gives

$$-M\omega^2\zeta = \pi\rho g a^2 A + \mathrm{i}\omega\lambda_g\zeta - \pi\rho g a^2\zeta, \tag{3.4}$$

where $M = \rho\pi a^2 H$ by Archimedis principle. It follows that

$$\frac{\zeta}{A} = \frac{1}{1 - \frac{\mathrm{i}\omega\lambda_g}{\rho g\pi a^2} - \frac{\omega^2 H}{g}} = \frac{1}{1 - \frac{\mathrm{i}\omega\lambda_g}{\rho g \pi a^2}} + O(\mu). \tag{3.5}$$

Use has been made of the fact that $\omega^2 H/g \sim kH = O(\mu) \ll 1$. Thus the inertia of a small buoy is relatively unimportant. As a consequence, the draft of the buoy H is much less relevant than its lateral dimension a. The time-averaged rate of energy extraction by a single buoy is given by

$$\overline{\lambda_g \left[\frac{\partial}{\partial t}\mathrm{Re}\left(\zeta e^{-\mathrm{i}\omega t}\right)\right]^2} = \frac{1}{2}\omega^2\lambda_g|\zeta|^2. \tag{3.6}$$

The fraction of power extracted by one buoy from the incident energy flux across a channel width d is then

$$\mathfrak{E} = \frac{E}{E_{inc}} = \frac{\frac{1}{2}\omega^2\lambda_g|\zeta^2|}{\frac{1}{2}\rho g A^2 C_g d}, \tag{3.7}$$

where E_{inc} denotes the rate of energy influx over a width d and

$$C_g = \frac{\omega}{2k}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{3.8}$$

is the group velocity of the incident wave.

Let the normalised frequency and extraction rate be denoted by

$$\omega' \equiv \frac{\omega}{\sqrt{\frac{g}{h}}}, \qquad \lambda_g' \equiv \frac{\lambda_g}{\pi\rho a^2\sqrt{gh}},$$

then in dimensionless form

$$\omega'^2 = k'\tanh(k'), \qquad C_g' = \frac{\omega'}{2k'}\left(1 + \frac{2k'}{\sinh 2k'}\right),$$

and

$$\zeta' = \frac{1}{1 - \mathrm{i}\omega'\lambda_g'} + O(\mu) \tag{3.9}$$

Consider now a large number of buoys along the centreline of a channel and spaced at the same distance d. On the macro-scale of the total array, the mathematical effect of many point absorbers can be replaced by a continuous distribution.

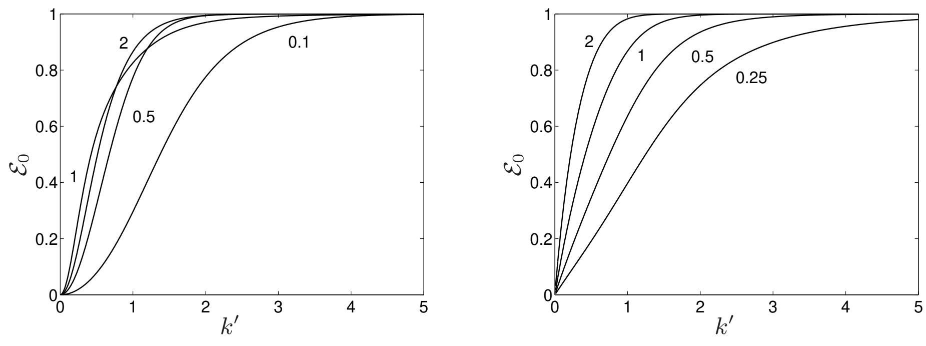


Figure 2. Energy extraction efficiency $\mathcal{E}_0$ vs. k' for a square array with $d' = 1$. (a) Effects of the extraction rates $\lambda'_g = 0.1, 0.5, 1, 2$ for $L = 1$ (b) Effects of complete optimization for every frequency using (3.13) for $L = 0.25, 0.5, 1, 2$.

The fraction of the incoming energy absorbed within a unit distance of the macro-scale coordinates must be

$$\frac{h}{d}\frac{\pi a^2}{h^2}\frac{\lambda'^2_g\omega'^2|\bar{\zeta}|^2}{d'C'_g} = \mu^2\frac{\pi\bar{a}}{d'^2C'_g}\frac{\lambda'_g\omega'^2}{1+\omega'^2\lambda'^2_g} \equiv \mu^2\mathfrak{D}. \tag{3.10}$$

In terms of the macro coordinate $X = \mu^2 x'$, the spatial rate of change of $\mathfrak{E}(X)$ is

$$\frac{\mathrm{d}\mathfrak{E}}{\mathrm{d}X} = -\mathfrak{D}\,\mathfrak{E}, \qquad \text{i.e.} \qquad \mathfrak{E}(X) = \mathrm{e}^{-\mathfrak{D}X}. \tag{3.11}$$

Thus the fraction of energy remaining at the end of the array $X = L$ (i.e. $x' = L/\mu^2$) is $\mathfrak{E}(L) = \mathrm{e}^{-\mathfrak{D}L}$ and the extraction efficiency is

$$\mathcal{E}_0 \equiv 1 - \mathfrak{E}(L) = 1 - \mathrm{e}^{-\mathfrak{D}L}. \tag{3.12}$$

Clearly the efficiency depends on the number of buoys $N = L/(\mu^2 d')$, the frequency of the incoming wave through ω' and C'_g, and the extraction rate λ'_g. For sufficiently large L, $\mathcal{E}_0$ approaches unity. The dependence of efficiency on k' and the extraction rate is plotted in figure 2(a) for a fixed $L = 1$. For a given L, the optimal extraction rate for maximum efficiency is found from

$$\frac{\mathrm{d}\mathcal{E}_0}{\mathrm{d}\lambda'_g} = 0, \qquad \text{or equivalently} \qquad \frac{\mathrm{d}\mathfrak{D}}{\mathrm{d}\lambda'_g} = 0,$$

which gives the optimal extraction rate

$$\lambda'_{g,opt} = \frac{1}{\omega'}. \tag{3.13}$$

Thus the optimum extraction rate should be higher for longer waves. Figure 2(b) shows the efficiency for a few L/d', when the extraction rate is optimized for every k'. One sees that at high frequencies, a long enough array can extract all the incoming energy, and a larger array is better for low-frequency waves.

We now turn to the physics of Bragg resonance.

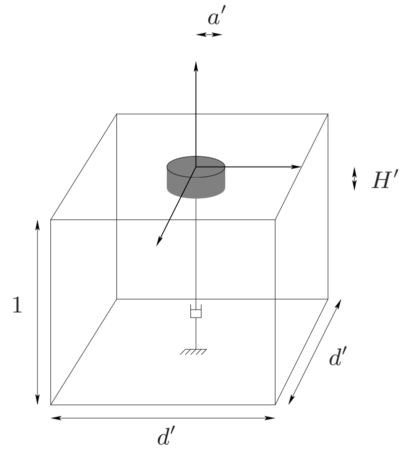


Figure 3. A unit cell surrounding a buoy.

4. Scattering by an array of fixed buoys

(a) *Linearised dimensionless equations*

As it is standard in linearised theories, the problem of wave-body interaction is equivalent to the sum of two hydrodynamic problems : scattering by stationary bodies and radiation by body motion. The two problems are coupled by the dynamics of the floating bodies. Let us first study the diffraction of a wave-train by fixed buoys. The following symbols for different parts of the physical domains are employed: Ω_F for the fluid domain, S_F for the free surface, S_W for the lateral surface of the buoys, and S_B for the bottom surface of the buoys. The seabed is the entire horizontal plane $z' = -1$. In terms of the meso-scale coordinates, the dimensionless governing equations for the scattering potential Φ' is

$$\Delta'\Phi \equiv \frac{\partial^2\Phi'}{\partial x'^2} + \frac{\partial^2\Phi'}{\partial y'^2} + \frac{\partial^2\Phi'}{\partial z'^2} = 0, \qquad \boldsymbol{x}' \in \Omega_F, \tag{4.1a}$$

$$\left(\frac{\partial}{\partial z'} + \frac{\partial^2}{\partial t'^2}\right)\Phi' = 0, \qquad \boldsymbol{x}' \in S_F, \tag{4.1b}$$

$$\frac{\partial\Phi'}{\partial z'} = 0, \qquad \boldsymbol{x}' \in S_B, \tag{4.1c}$$

$$\frac{\partial\Phi'}{\partial r'} = 0, \qquad \boldsymbol{x}' \in S_W, \tag{4.1d}$$

$$\frac{\partial\Phi'}{\partial z'} = 0, \qquad z' = -1, \tag{4.1e}$$

$$\frac{\partial\Phi'}{\partial y'} = 0, \qquad y' = \pm\frac{d'}{2}. \tag{4.1f}$$

We shall now find the law for the slow evolution of the envelope from the fast variations between buoys, by the method of multiple scales (homogenization).

(*b*) *Envelope equations by multiple-scale analysis*

Substituting into (4.1) the following expansion

$$\Phi'_S = \left(\phi_0 + \mu^2\phi_1 + \cdots\right) \mathrm{e}^{-\mathrm{i}\omega' t'}, \tag{4.2}$$

with $\phi_n,\ n = 0, 1, 2, \cdots$ being dimensionless functions of $(x', y', z'; X; T)$, we get from (4.1a) and (4.1b) the following equations

$$\left(\Delta' + 2\mu^2 \frac{\partial^2}{\partial X \partial x'} + \mu^4 \frac{\partial^2}{\partial X^2}\right)\left(\phi_0 + \mu^2\phi_1 + \cdots\right) = 0, \quad \boldsymbol{x}' \in \Omega_F, \tag{4.3a}$$

$$\left(\frac{\partial}{\partial z'} - \omega'^2 - 2\mu^2 \mathrm{i}\omega' \frac{\partial}{\partial T} - \mu^4 \frac{\partial^2}{\partial T^2}\right)\left(\phi_0 + \mu^2\phi_1 + \cdots\right) = 0, \quad \boldsymbol{x}' \in S_F. \tag{4.3b}$$

From successive orders of these and the remaining equations in (4.1) we obtain boundary-value problems for the wave-scale variations in a unit cell. Because each cell is one period in vary large array, we invoke Bloch's theorem (Ashcroft and Mermin (1976)) which states that the solution ϕ_n should be of the form $\mathrm{e}^{\pm \mathrm{i}kx'} f(\boldsymbol{x}')$ where $f(\boldsymbol{x}')$ is periodic in x' with the period d'. Since we shall focus on the state of Bragg resonance with $kd' = \pi$, Bloch's theorem implies

$$\phi_n\left(-\frac{d'}{2}, y'\right) = \phi_n\left(\frac{d'}{2}, y'\right), \quad \text{and} \quad \frac{\partial \phi_n}{\partial x'}\left(-\frac{d'}{2}, y'\right) = -\frac{\partial \phi}{\partial x'}\left(\frac{d'}{2}, y'\right). \tag{4.4}$$

(i) *Leading order*

At the leading order $O(\mu^0)$, the governing equations are homogeneous:

$$\Delta'\phi_0 = 0, \qquad \boldsymbol{x}' \in \Omega_F, \tag{4.5a}$$

$$\left(\frac{\partial}{\partial z'} - \omega'^2\right)\phi_0 = 0, \qquad \boldsymbol{x}' \in S_F, \tag{4.5b}$$

$$\frac{\partial \phi_0}{\partial z'} = 0, \qquad z' = -1, \tag{4.5c}$$

$$\frac{\partial \phi_0}{\partial y'} = 0, \qquad y' = \pm d'/2. \tag{4.5d}$$

As reasoned in Li and Mei (2007*a*,*b*), since $\mu = a/h \ll 1$, $H' \ll 1$ and $d' = O(1)$, the areas of the buoy surfaces S_B and S_W are of order $O(\mu^2)$. Hence the buoys have negligible effects on the waves at the leading order, i.e., the boundary conditions on S_B and S_W are ineffective until at higher orders. The formal solution is the sum of left- and right-going plane waves in free space

$$\phi_0 = \alpha^+(X, T)Z(z')\mathrm{e}^{\mathrm{i}k'x'} + \alpha^-(X, T)Z(z')\mathrm{e}^{-\mathrm{i}k'x'}, \tag{4.6}$$

where

$$Z(z') = \frac{1}{\mathrm{i}\omega'} \frac{\cosh(k'(z'+1))}{\cosh(k')}. \tag{4.7}$$

The no-flux boundary conditions at $y = \pm d/2$ and Bloch conditions (4.4) are trivially satisfied. The envelope functions $\alpha^\pm(X, T)$ are yet to be found.

(ii) *First order*

From (4.3) the first order $O(\mu)$ meso-scale problem in the unit cell is inhomogeneous, and governed by

$$\Delta'\phi_1 = -2\frac{\partial^2\phi_0}{\partial X\partial x'}, \qquad \boldsymbol{x}'\in\Omega_F, \tag{4.8a}$$

$$\left(\frac{\partial}{\partial z'} - \omega'^2\right)\phi_1 = 2\mathrm{i}\omega'\frac{\partial\phi_0}{\partial T}, \qquad \boldsymbol{x}'\in S_F, \tag{4.8b}$$

$$\frac{\partial\phi_1}{\partial z'} = -\frac{1}{\mu^2}\frac{\partial\phi_0}{\partial z'}, \qquad \boldsymbol{x}'\in S_B, \tag{4.8c}$$

$$\frac{\partial\phi_1}{\partial r'} = -\frac{1}{\mu^2}\frac{\partial\phi_0}{\partial r'}, \qquad \boldsymbol{x}'\in S_W, \tag{4.8d}$$

$$\frac{\partial\phi_1}{\partial z'} = 0, \qquad z' = -1, \tag{4.8e}$$

$$\frac{\partial\phi_1}{\partial y'} = 0, \qquad y' = \pm\frac{d'}{2}, \tag{4.8f}$$

together with the Bloch conditions (4.4). Despite the large factor μ^{-2}, the integrated effects of the boundary value in (4.8c) and (4.8d) are of order 1 since the area of the buoy surface is of order μ^2. An important estimate of ϕ_1 near and on the buoy may be deduced. In terms of the meso-scale coordinates, we have, in the neighborhood of the buoy,

$$\frac{\partial\phi_1}{\partial n'} \sim \frac{\phi_1}{\mu} \gg \phi_1.$$

Now (4.8c) and (4.8d) imply that

$$\frac{\partial\phi_1}{\partial n'} \sim \frac{1}{\mu^2}, \tag{4.9}$$

hence

$$\phi_1 \sim \frac{1}{\mu} \tag{4.10}$$

in the neighbourhood of the buoy.

We next apply Green's formula to ϕ_1 and the homogeneous solutions

$$\psi^{\mp} \equiv Z(z')\exp(\pm\mathrm{i}k'x') \tag{4.11}$$

over the volume of cell Ω_F:

$$\iiint_{\Omega_F}\left[\phi_1\Delta'\psi^{\mp} - \psi^{\mp}\Delta'\phi_1\right]\mathrm{d}V' = \iint_{\partial\Omega_F}\left[\phi_1\frac{\partial\psi^{\mp}}{\partial n'} - \psi^{\mp}\frac{\partial\phi_1}{\partial n'}\right]\mathrm{d}S', \tag{4.12}$$

which can be rewritten after using the boundary conditions in (4.8) as

$$2\iiint_{\Omega_F} Z(z')\mathrm{e}^{\mp\mathrm{i}k'x'}\frac{\partial^2\phi_0}{\partial X\partial x'}\mathrm{d}V' = -2\mathrm{i}\omega'\iint_{S_F}\psi^{\mp}\frac{\partial\phi_0}{\partial T}\mathrm{d}S' + \iint_{S_B\cup S_W}\left[\frac{1}{\mu^2}\psi^{\mp}\frac{\partial\phi_0}{\partial n'} + \phi_1\frac{\partial\psi^{\mp}}{\partial n'}\right]\mathrm{d}S'. \tag{4.13}$$

This is the solvability condition for the inhomogeneous problem of ϕ_1 and gives a constraint on ϕ_0. To the leading order, the left hand side of (4.13) can be simplified to

$$2\iiint_{\Omega_F} = \pm 2\mathrm{i}k'd'^2\frac{\partial\alpha^\pm}{\partial X}\int_{-1}^{0} Z(z')^2\,\mathrm{d}z' + O(\mu^3).$$

The integral over the free surface on the right hand side gives

$$\iint_{S_F} = -\frac{2d'^2}{\mathrm{i}\omega'}\frac{\partial\alpha^\pm}{\partial T} + O(\mu^2).$$

In view of (4.10), the second term in the surface integral over the buoy can be neglected with an error of $O(\mu)$. The remaining surface integral on the buoy is

$$\begin{aligned}\iint_{S_B\cup S_W} \approx& \frac{1}{\mu^2}\iint_{S_B\cup S_W}\psi^\mp\frac{\partial\left[\alpha^+\psi^+ + \alpha^-\psi^-\right]}{\partial n'}\,\mathrm{d}S' \\ &= -\frac{1}{\mu^2}\iint_{S_W}\mathrm{i}k'\psi^\mp\left(\alpha^+\psi^+ - \alpha^-\psi^-\right)\boldsymbol{e}_r.\boldsymbol{e}_x\,\mathrm{d}S' \\ &\quad + \frac{1}{\mu^2}\iint_{S_B}\psi^\mp\frac{\mathrm{d}Z(z')}{\mathrm{d}z'}\left(\alpha^+\mathrm{e}^{\mathrm{i}k'x'} + \alpha^-\mathrm{e}^{-\mathrm{i}k'x'}\right)\mathrm{d}S'.\end{aligned}$$

Let us now use the fact that the buoy draft is small $H' \equiv H/h = O(\mu)$ so that

$$Z \approx \frac{1}{\mathrm{i}\omega'} \qquad \text{and} \qquad \frac{\mathrm{d}Z}{\mathrm{d}z'} \approx -\mathrm{i}\omega' \qquad \text{on} \qquad S_W\cup S_B \tag{4.14}$$

from (4.7). In terms of the micro-scale variables $\bar{r} = r/a$ and $\bar{z} = z/a$ so that $dS'/\mu^2 = d\bar{S}$, we have

$$\begin{aligned}\iint_{S_B\cup S_W} &\approx -(\alpha^+ - \alpha^-)\int_0^{2\pi}\int_{-\bar{H}}^{0}\mathrm{i}k'\cos(\theta)\,d\bar{z}\,\mathrm{d}\theta \\ &\quad - (\alpha^+ + \alpha^-)\int_0^{2\pi}\int_0^1 \bar{r}\,\mathrm{d}\bar{r}\,\mathrm{d}\theta \\ &\approx -\pi(\alpha^+ + \alpha^-).\end{aligned}$$

Hence (4.13) becomes

$$\frac{2d'^2}{\mathrm{i}\omega'}\frac{\partial\alpha^\pm}{\partial T} \pm 2\mathrm{i}k'd'^2\frac{\partial\alpha^\pm}{\partial X}\int_{-1}^{0} Z(z)^2\,\mathrm{d}z = -\pi\left(\alpha^+ + \alpha^-\right),$$

which can be simplified to the coupled-mode equations on the macro-scale

$$\boxed{\frac{\partial\alpha^\pm}{\partial T} \pm C_g'\frac{\partial\alpha^\pm}{\partial X} = -\mathrm{i}\Omega_0\left(\alpha^+ + \alpha^-\right)} \tag{4.15}$$

where Ω_0 is the dimensionless coupling constant

$$\Omega_0 = \frac{\pi\omega'}{2d'^2}. \tag{4.16}$$

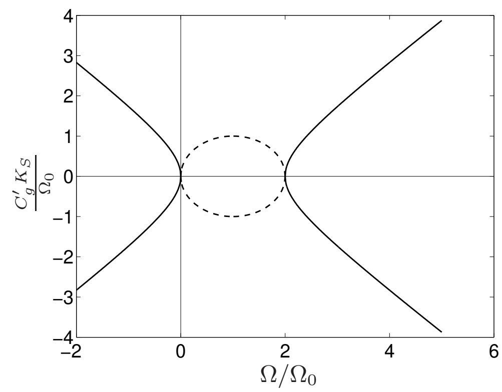


Figure 4. Dispersion relation in an infinite periodic array. Real values are plotted in solid lines, and imaginary values in dashed line.

Equation (4.15) can be rewritten in physical coordinates as

$$\frac{\partial \alpha^{\pm}}{\partial t} \pm C_g \frac{\partial \alpha^{\pm}}{\partial x} = -\mathrm{i}\frac{\omega a^2}{2d^2}\left(\alpha^+ + \alpha^-\right) \tag{4.17}$$

These equations govern the envelopes of the forward and backward waves in an array of fixed small buoys, and are similar to those for wave propagation in an array of slender piles or over a periodic seabed (see Li and Mei (2007*a*,*b*); Naciri and Mei (1988)). The difference with Li and Mei (2007*a*,*b*) is due to the fact that in the present problem the scattering effect comes from the bottom of the buoy and not from the lateral wall.

(*c*) *Macro-scale dispersion relation and band gap*

As the first application of (4.15) let us consider periodically modulated wave envelopes in an infinite array,

$$\alpha^{\pm} \propto \mathrm{e}^{\mathrm{i}(\pm K_S X - \Omega T)}, \tag{4.18}$$

where Ω corresponds to a frequency detuning of $\mu^2\Omega$. The homogeneous equations (4.15) have non-trivial solutions only if

$$C_g'^2 K_S^2 = \Omega(\Omega - 2\Omega_0), \tag{4.19}$$

which is plotted in figure 4. For real Ω three regions can be distinguished. In either $\Omega \leq 0$ or $\Omega \geq 2\Omega_0$, K_S is real, hence waves propagate. However, inside the band gap defined by $0 \leq \Omega \leq 2\Omega_0$, K_S is imaginary, hence propagation is forbidden. The wave train can only decay in distance. The spatial rate of amplitude decay is proportional to $\mathrm{Im}(K_S)$, which is maximum at $\Omega = \Omega_0$.

The presence of the band gap and its limits have been confirmed by numerical computation of the band structure but without the small-buoy assumption, using a standard numerical approach of solid-state physics (Garnaud (2009)).

(*d*) *Scattering by an array of finite width*

Let there be a finite array of fixed buoys in $0 < X < L$, where L corresponds to the physical width $(L/\mu^2)h$. An incoming wave slightly detuned from Bragg resonance

$$\phi_{in} = Z(z')\mathrm{e}^{\mathrm{i}k'x'}\mathrm{e}^{\mathrm{i}(K_0X-\Omega T)} \tag{4.20}$$

arrives from $X \sim -\infty$ with $K_0 \equiv C_g'\Omega$. Let the scattering potential be

$$\phi_0 = \left(\mathrm{e}^{\mathrm{i}k'x'}\mathrm{e}^{\mathrm{i}K_0X} + C_R\mathrm{e}^{-\mathrm{i}k'x'}\mathrm{e}^{-\mathrm{i}K_0X}\right)Z(z')\mathrm{e}^{-\mathrm{i}\Omega T}, \qquad X<0, \tag{4.21}$$

on the incidence side and

$$\phi_0 = C_T\mathrm{e}^{\mathrm{i}k'x'}\mathrm{e}^{\mathrm{i}K_0(X-L)}Z(z')\mathrm{e}^{-\mathrm{i}\Omega T}, \qquad X>L, \tag{4.22}$$

on the transmission side. C_R and C_T are respectively the complex reflection and transmission coefficients. The potential inside the array is

$$\phi_0 = Z(z')\left(\widehat{\alpha}^+(X)\mathrm{e}^{\mathrm{i}k'x'} + \widehat{\alpha}^-(X)\mathrm{e}^{-\mathrm{i}k'x'}\right)\mathrm{e}^{-\mathrm{i}\Omega T}, \tag{4.23}$$

where $\widehat{\alpha}^\pm$ are defined by

$$\alpha^\pm = \widehat{\alpha}^\pm\mathrm{e}^{-\mathrm{i}\Omega T}. \tag{4.24}$$

By differentiating (4.15), we find

$$\frac{\mathrm{d}^2\widehat{\alpha}^\pm}{\mathrm{d}x^2} + K_S^2\widehat{\alpha}^\pm = 0, \tag{4.25}$$

with

$$K_S \equiv \frac{\Omega_S}{C_g'} \qquad \text{and} \qquad \Omega_S \equiv \sqrt{\Omega(\Omega-2\Omega_0)}. \tag{4.26}$$

Requiring continuity of the leading-order pressure and horizontal velocity at the edges of the array, we must have

$$\phi_0|_{X=0^-} = \phi_0|_{X=0^+}, \qquad \phi_0|_{X=L^-} = \phi_0|_{X=L^+}, \tag{4.27a}$$

$$\left.\frac{\partial\phi_0}{\partial x'}\right|_{X=0^-} = \left.\frac{\partial\phi_0}{\partial x'}\right|_{X=0^+}, \qquad \left.\frac{\partial\phi_0}{\partial x'}\right|_{X=L^-} = \left.\frac{\partial\phi_0}{\partial x'}\right|_{X=L^+}, \tag{4.27b}$$

where L^+ means slightly greater than L and L^- slightly less that L. Using (4.21) and (4.23) in (4.27) we obtain

$$1 + C_R = \widehat{\alpha}^+(0) + \widehat{\alpha}^-(0), \qquad C_T = \widehat{\alpha}^+(L) + \widehat{\alpha}^-(L), \tag{4.28a}$$

$$1 - C_R = \widehat{\alpha}^+(0) - \widehat{\alpha}^-(0), \qquad C_T = \widehat{\alpha}^+(L) - \widehat{\alpha}^-(L). \tag{4.28b}$$

It follows that

$$\widehat{\alpha}^-(L) = 0, \qquad \widehat{\alpha}^+(0) = 1. \tag{4.29a}$$

The general solutions of (4.25) are of the form

$$\widehat{\alpha}^\pm = C_1^\pm\cos(K_SX) + C_2^\pm\sin(K_SX).$$

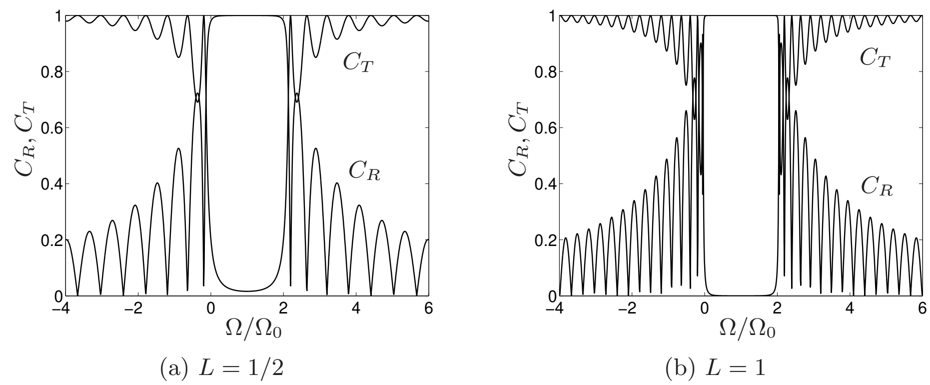


Figure 5. Coefficients of reflection and transmission for fixed buoys as functions of the detuning frequency, for two array lengths.

The six coefficients $C_R, C_T, C_1^{\pm}$ and $C_2^{\pm}$ can be found from the four conditions (4.28a), (4.28b) and the coupled equations (4.15). Finally the amplitudes inside the array are given by

$$\widehat{\alpha}^{+}(X) = \frac{\mathrm{i}(\Omega - \Omega_0)\sin(K_S(X-L))) + \Omega_S\cos(K_S(X-L))}{-\mathrm{i}(\Omega-\Omega_0)\sin(K_S L) + \Omega_S\cos(K_S L)} \tag{4.30a}$$

and

$$\widehat{\alpha}^{-}(X) = \frac{\mathrm{i}\Omega_0\sin(K_S(X-L))}{-\mathrm{i}(\Omega-\Omega_0)\sin(K_S L) + \Omega_S\cos(K_S L)} \tag{4.30b}$$

In the open sea, the transmission and reflection coefficients are

$$C_T = \widehat{\alpha}^{+}(L) = \frac{\Omega_S}{-\mathrm{i}(\Omega-\Omega_0)\sin(K_S L) + \Omega_S\cos(K_S L)}, \tag{4.31a}$$

and

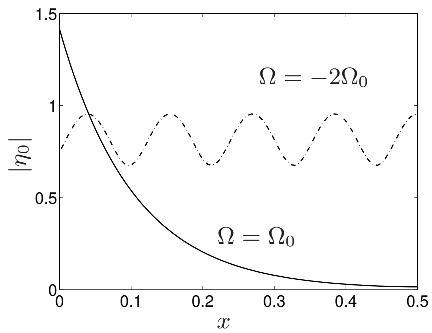
$$C_R = \widehat{\alpha}^{-}(0) = \frac{-\mathrm{i}\Omega_0\sin(K_S L)}{-\mathrm{i}(\Omega-\Omega_0)\sin(K_S L) + \Omega_S\cos(K_S L)}. \tag{4.31b}$$

The numerical results are qualitatively similar to those for vertical piles. As shown in figure 5, the scattering coefficients depend strongly on the detuning frequency Ω. Outside the band gap, $\Omega/\Omega_0 < 0$ or $\Omega/\Omega_0 > 2$, the scattering effects of the array are weak, as C_R is small and C_T is close to unity. Both are oscillatory in Ω/Ω_0. Inside the band gap, $0 \leq \Omega/\Omega_0 \leq 2$, propagation is inhibited. The reflection coefficient is close to 1 and the transmission coefficient close to 0. Inside the array the free surface profile is oscillatory in space if Ω is outside the gap and exponentially attenuating if inside, as shown in figure 6.

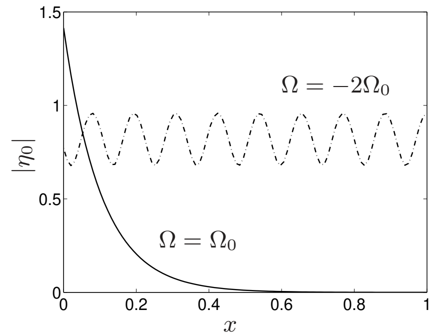
Next let us examine the effects of buoy motion on waves.

5. Envelope of radiated waves

In this section, we allow the small buoys to heave either freely or partially constrained by energy-absorbing devices. Since $k'd' = \pi$, if one buoy goes up then its immediate neighbours must go down, and vice versa. Across the wide array of many buoys, the amplitude of waves and buoy displacements will be a slowly



(a) $L = 1/2$



(b) $L = 1$

Figure 6. Wave amplitude inside the buoy array ($0 < X < L$) around Bragg resonance for $d' = 1$ and $\Omega = \Omega_0$ (inside the band gap) and $\Omega = -2\Omega_0$ (outside of the band gap).

varying function of space. The normalised heave amplitude of the m^{th} buoy can therefore be expressed as

$$\zeta'^{(m)} = (-1)^m \left(\zeta_0(X,T) + \mu^2\zeta_1(X,T) + \cdots\right) \mathrm{e}^{-\mathrm{i}\omega' t'}. \tag{5.1}$$

The buoy displacement $\zeta_0(X,T)$ is yet unknown and will be found later by additional account of the buoy dynamics. Let the centre of the m^{th} buoy be located at $x'_m = md'$. For mathematical convenience, we express the term $(-1)^m$ as

$$(-1)^m = \mathrm{e}^{\mathrm{i}k' x'_m} = \mathrm{e}^{\mathrm{i}m\pi} \tag{5.2}$$

in the kinematic boundary conditions on the buoys. Denoting the radiation potential by Φ_R and expanding

$$\Phi'_R = \left(\Phi_0 + \mu^2\Phi_1 + \cdots\right) e^{-\mathrm{i}\omega' t'}, \tag{5.3}$$

the governing equations become

$$\left(\Delta' + 2\mu^2 \frac{\partial^2}{\partial X \partial x'} + \mu^4 \Delta\right)\left(\Phi_0 + \mu^2\Phi_1 + \cdots\right) = 0, \qquad \boldsymbol{x}' \in \Omega_F, \tag{5.4a}$$

$$\left(\frac{\partial}{\partial z'} - \omega'^2 - 2\mu^2 \mathrm{i}\omega' \frac{\partial}{\partial T} - \mu^4 \frac{\partial^2}{\partial T^2}\right)\left(\Phi_0 + \mu^2\Phi_1 + \cdots\right) = 0, \qquad \boldsymbol{x}' \in S_F, \tag{5.4b}$$

$$\frac{\partial\left(\Phi_0 + \mu^2\Phi_1 + \cdots\right)}{\partial z'} + \mathrm{i}\omega' \mathrm{e}^{\mathrm{i}k' x'_m}\left(\zeta_0 + \mu^2\zeta_1 + \cdots\right) = 0, \quad \boldsymbol{x}' \in S_B^{(m)}, \tag{5.4c}$$

$$\frac{\partial\left(\Phi_0 + \mu^2\Phi_1 + \cdots\right)}{\partial z'} = 0, \qquad z' = -1, \tag{5.4d}$$

$$\left(\frac{\partial}{\partial r'} + \mu^2 \frac{\partial}{\partial R}\right)\left(\Phi_0 + \mu^2\Phi_1 + \cdots\right) = 0, \quad \boldsymbol{x}' \in S_W^{(m)}. \tag{5.4e}$$

where $S_B^{(m)}$ and $S_W^{(m)}$ are respectively the bottom and lateral boundaries of the m^{th} buoy. As shown in figure 3, we require the radiation potential to be anti-periodic in x' with the period d',

$$\Phi_i(x'+d', y', z') = -\Phi_i(x', y', z'), \quad \frac{\partial \Phi_i}{\partial x'}(x'+d', y', z') = -\frac{\partial \Phi_i}{\partial x'}(x', y', z'). \tag{5.5}$$

for $i = 1, 2, \cdots$.

From (5.4), it is evident that the zeroth order radiation potential is of the form

$$\Phi_0 = \left[\beta^+(X,T)\mathrm{e}^{\mathrm{i}k'x'} + \beta^-(X,T)\mathrm{e}^{-\mathrm{i}k'x'}\right] Z(z'), \tag{5.6}$$

where the long-scale functions $\beta^\pm$ represent the unknown amplitudes of the propagating waves. The first-order problem is governed by

$$\Delta'\Phi_1 = -2\frac{\partial^2\Phi_0}{\partial X\partial x'}, \qquad \boldsymbol{x}' \in \Omega_F, \tag{5.7a}$$

$$\frac{\partial\Phi_1}{\partial z'} - \omega'^2\Phi_1 = 2\mathrm{i}\omega'\frac{\partial\Phi_0}{\partial T}, \qquad \boldsymbol{x}' \in S_F, \tag{5.7b}$$

$$\frac{\partial\Phi_1}{\partial z'} = \frac{-\mathrm{i}\omega'\zeta_0\mathrm{e}^{\mathrm{i}k'x'_n}}{\mu^2} - \frac{1}{\mu^2}\frac{\partial\Phi_0}{\partial z'}, \qquad \boldsymbol{x}' \in S_B^{(m)}, \tag{5.7c}$$

$$\frac{\partial\Phi_1}{\partial z'} = 0, \qquad z' = -1, \tag{5.7d}$$

$$\frac{\partial\Phi_1}{\partial r'} = -\frac{1}{\mu^2}\frac{\partial\Phi_0}{\partial r'}, \qquad \boldsymbol{x}' \in S_W^{(m)}. \tag{5.7e}$$

Note that as the buoy radii and drafts are of order μ, we have $x' = x'_m + O(\mu)$ for $\boldsymbol{x}' \in S_W^{(m)} \cup S_B^{(m)}$. The problem (5.7) is similar to the scattering problem (4.8) except for the additional term proportional to the buoy displacement, which is also anti-periodic with a period d'.

As in the scattering problem, we derive the solvability condition for Φ_1 by applying Green's formula to Φ_1 and $\psi^\pm = Z(z')\exp(\pm\mathrm{i}k'x')$ over a cell Ω_F.

It is easily checked that the only change is in the surface integral over the buoys

$$\begin{aligned}
\iint_{S_B\cup S_W} &\approx \frac{1}{\mu^2}\iint_{S_B\cup S_W} \psi^\mp\frac{\partial\left[\beta^+\psi^+ + \beta^-\psi^-\right]}{\partial n'}\,\mathrm{d}S' \\
&\quad - \frac{1}{\mu^2}\iint_{S_B} \psi^\mp\mathrm{i}\omega'\zeta_0\mathrm{e}^{\mathrm{i}k'x'_m}\,\mathrm{d}S' \\
&= -\frac{1}{\mu^2}\iint_{S_W} \mathrm{i}k'\psi^\mp\left(\beta^+\psi^+ - \beta^-\psi^-\right)\boldsymbol{e}_r.\boldsymbol{e}_x\,\mathrm{d}S' \\
&\quad + \frac{1}{\mu^2}\iint_{S_B} \psi^\mp\frac{\mathrm{d}Z(z')}{\mathrm{d}z'}\left(\beta^+\mathrm{e}^{\mathrm{i}k'x'_m} + \beta^-\mathrm{e}^{-\mathrm{i}k'x'_m}\right)\mathrm{d}S' \\
&\quad - \frac{1}{\mu^2}\iint_{S_B} \psi^\mp\mathrm{i}\omega'\zeta_0\mathrm{e}^{\mathrm{i}k'x'_m}\,\mathrm{d}S'.
\end{aligned} \tag{5.8}$$

where $S_B \equiv \cup_m S_B^{(m)}$. Using (4.14), we get

$$\begin{aligned}
\iint_{S_B\cup S_W} &\approx -(\beta^+ - \beta^-)\int_0^{2\pi}\int_{-\bar{H}}^0 \mathrm{i}k'\cos(\theta)\,d\bar{z}\,\mathrm{d}\theta \\
&\quad - (\beta^+ + \beta^- - \zeta_0)\int_0^{2\pi}\int_0^1 \bar{r}\,\mathrm{d}\bar{r}\,\mathrm{d}\theta \\
&\approx -\pi(\beta^+ + \beta^- - \zeta_0).
\end{aligned} \tag{5.9}$$

This finally gives us the envelope equations

$$\boxed{\frac{\partial \beta^{\pm}}{\partial T} \pm C_g' \frac{\partial \beta^{\pm}}{\partial X} = -\mathrm{i}\Omega_0 \left(\beta^{+} + \beta^{-} - \zeta_0\right)} \tag{5.10}$$

where Ω_0 was defined in (4.16). In physical terms, (5.10) reads

$$\frac{\partial \beta^{\pm}}{\partial t} \pm C_g \frac{\partial \beta^{\pm}}{\partial x} = -\mathrm{i}\frac{\omega a^2}{2d^2} \left(\beta^{+} + \beta^{-} - \frac{\zeta_0}{A}.\right) \tag{5.11}$$

The pair of equations in (5.10) expresses the coupling between the amplitude of the right- and left-going waves and the unknown buoy motion.

We must now examine the buoy displacement induced by the waves, in order to relate ζ_0 to $\alpha^{\pm}$ and $\beta^{\pm}$ in (5.10).

6. Buoy dynamics

The forcing term ζ' in the long-scale equation for the radiation problem are related to the scattering and radiation potentials, given respectively by (4.6) and (5.6). By Froude-Krylov approximation, the leading order vertical forces on a buoy by the scattering and radiation potentials are given respectively by

$$f_m' = \mathrm{i}\omega' \iint_{S_B^{(m)}} \phi_0 \, \mathrm{d}S = \mathrm{i}\pi\mu^2\omega' \frac{(-1)^m}{\mathrm{i}\omega'} \left(\alpha^{+} + \alpha^{-}\right), \tag{6.1}$$

$$F_m' = \mathrm{i}\omega' \iint_{S_B^{(m)}} \Phi_0 \, \mathrm{d}S = \mathrm{i}\pi\mu^2\omega' \frac{(-1)^m}{\mathrm{i}\omega'} \left(\beta^{+} + \beta^{-}\right), \tag{6.2}$$

where the forces are normalised according to

$$(f_m, F_m) = \rho g h^2 A (f_m', F_m'). \tag{6.3}$$

Use is made of the fact that the neighboring buoys move in opposite phases.

Let us assume that the energy extraction device exerts a force

$$\lambda_g' \frac{\partial \zeta'^{(m)}}{\partial t'} \tag{6.4}$$

on the m^{th} buoy. Applying Newton's law to the m^{th} buoy, we get

$$M\omega^2 \zeta'^{(m)} = f_m + F_m + \mathrm{i}\omega\lambda_g' \zeta'^{(m)} - \pi\rho a^2 \zeta'^{(m)}, \tag{6.5}$$

which gives us in dimensionless form

$$\begin{aligned} -\omega'^2\mu^3\pi\bar{H}\zeta_0(-1)^m = \pi\mu^2 \left(\alpha^{+} + \alpha^{-}\right)(-1)^m + \pi\mu^2 \left(\beta^{+} + \beta^{-}\right)(-1)^m \\ + \mathrm{i}\pi\omega'\mu^2\zeta_0(-1)^m\lambda_g' - \pi\mu^2\zeta_0(-1)^m. \end{aligned} \tag{6.6}$$

Again the mass of the buoy can be ignored so that

$$\zeta_0 = \mathcal{G}\left[\left(\alpha^{+} + \alpha^{-}\right) + \left(\beta^{+} + \beta^{-}\right)\right], \tag{6.7}$$

where for brevity we denote

$$\mathcal{G} \equiv \frac{1}{1 - \mathrm{i}\omega' \lambda_g'} + O(\mu), \tag{6.8}$$

which is the same as (3.9). Using (6.7) we can rewrite (5.10) as:

$$\boxed{\frac{\partial \beta^\pm}{\partial T} \pm C_g' \frac{\partial \beta^\pm}{\partial X} = -\mathrm{i}\Omega_0 \left[(1-\mathcal{G})(\beta^+ + \beta^-) - \mathcal{G}(\alpha^+ + \alpha^-)\right]} \tag{6.9}$$

which couple the radiation and scattering components. The scattering amplitudes $\alpha^\pm$ are already found in the previous section and serve as forcing terms here.

We can now study the envelopes due to waves interacting with a finite array of energy-extracting buoys.

7. Radiation by a finite array of energy-extracting buoys

(*a*) *Solution for radiation amplitudes*

Consider again an array of finite but large width $(L/\mu^2)h$ forced to oscillate by the incident and scattered waves and subjected to the reactive forces from the extractors and from the buoy motion. The potential outside the array must satisfy the radiation condition, hence

$$\Phi_0 = C^- Z(z') \mathrm{e}^{-\mathrm{i}k'x'} \mathrm{e}^{-\mathrm{i}KX} \mathrm{e}^{-\mathrm{i}\Omega T}, \qquad X < 0, \tag{7.1a}$$

$$\Phi_0 = C^+ Z(z') \mathrm{e}^{\mathrm{i}k'x'} \mathrm{e}^{\mathrm{i}K(X-L)} \mathrm{e}^{-\mathrm{i}\Omega T}, \qquad X > L, \tag{7.1b}$$

where the complex coefficients C^+ and C^- are yet unknown. As in the scattering problem, we introduce $\widehat{\beta}^\pm$ by defining $\beta^\pm = \widehat{\beta}^\pm e^{-\mathrm{i}\Omega T}$ and assume the solution inside the array to be of the form

$$\Phi_0 = \left(\widehat{\beta}^+(X)\mathrm{e}^{\mathrm{i}k'x'} + \widehat{\beta}^-(X)\mathrm{e}^{-\mathrm{i}k'x'}\right) Z(z') \mathrm{e}^{-\mathrm{i}\Omega T}. \tag{7.2}$$

Continuity of pressure and horizontal velocity are required at the edges of the array

$$\Phi_0\big|_{X=0^-} = \Phi_0\big|_{X=0^+}, \qquad \Phi_0\big|_{X=L^-} = \Phi_0\big|_{X=L^+}, \tag{7.3a}$$

$$\left.\frac{\partial \Phi_0}{\partial x'}\right|_{X=0^-} = \left.\frac{\partial \Phi_0}{\partial x'}\right|_{X=0^+}, \qquad \left.\frac{\partial \Phi_0}{\partial x'}\right|_{X=L^-} = \left.\frac{\partial \Phi_0}{\partial x'}\right|_{X=L^+}. \tag{7.3b}$$

Using (7.1) and (7.2), we get

$$C^- = \widehat{\beta}^+(0) + \widehat{\beta}^-(0), \qquad C^+ = \widehat{\beta}^+(L) + \widehat{\beta}^-(L), \tag{7.4a}$$

$$-C^- = \widehat{\beta}^+(0) - \widehat{\beta}^-(0), \qquad C^+ = \widehat{\beta}^+(L) - \widehat{\beta}^-(L). \tag{7.4b}$$

It follows that

$$\widehat{\beta}^-(L) = 0, \qquad \widehat{\beta}^+(0) = 0, \tag{7.5a}$$

which imply $C^- = \widehat{\beta}^-(0)$ and $C^+ = \widehat{\beta}^+(L)$. To complete the analytical solution, let us write the forcing term in (6.9) as

$$\widehat{\alpha}^+ + \widehat{\alpha}^- = C_g\left(\mathcal{A}^+ \mathrm{e}^{\mathrm{i}K_S x} + \mathcal{A}^- \mathrm{e}^{-\mathrm{i}K_S x}\right). \tag{7.6}$$

Using (4.30), we find

$$\mathcal{A}^+ = \mathrm{e}^{-\mathrm{i}K_S L}\frac{K_S + K}{2\left(-\mathrm{i}(\Omega-\Omega_0)\sin(K_S L) + \Omega_1\cos(K_S L)\right)}, \tag{7.7a}$$

$$\mathcal{A}^- = \mathrm{e}^{\mathrm{i}K_S L}\frac{K_S - K}{2\left(-\mathrm{i}(\Omega-\Omega_0)\sin(K_S L) + \Omega_1\cos(K_S L)\right)}. \tag{7.7b}$$

To find $\widehat{\beta}^\pm$, it is simpler to use linearity and just to solve first for one of the complex exponentials on the right-hand side of (6.9). Let the response to the forcing $(\mathrm{e}^{\mathrm{i}K_S X}, \mathrm{e}^{-\mathrm{i}K_S X})$ be denoted by $(\widehat{\beta}_1^\pm, \widehat{\beta}_2^\pm)$ respectively, then

$$-\mathrm{i}K\widehat{\beta}_1^+ + \frac{\partial\widehat{\beta}_1^+}{\partial X} = -\mathrm{i}K_0(1-\mathcal{G})(\widehat{\beta}_1^+ + \widehat{\beta}_1^-) + \mathrm{e}^{\mathrm{i}K_S X}, \tag{7.8a}$$

$$-\mathrm{i}K\widehat{\beta}_1^- - \frac{\partial\widehat{\beta}_1^-}{\partial X} = -\mathrm{i}K_0(1-\mathcal{G})(\widehat{\beta}_1^+ + \widehat{\beta}_1^-) + \mathrm{e}^{\mathrm{i}K_S X}, \tag{7.8b}$$

where $K \equiv \Omega/C_g'$ and $K_0 \equiv \Omega_0/C_g'$. By cross-differentiation these two equations can be decoupled to give

$$\frac{\mathrm{d}^2\widehat{\beta}_1^+}{\mathrm{d}X^2} + K_R^2\widehat{\beta}_1^+ = \mathrm{i}\,(K+K_S)\,\mathrm{e}^{\mathrm{i}K_S X}, \tag{7.9a}$$

$$\frac{\mathrm{d}^2\widehat{\beta}_1^-}{\mathrm{d}X^2} + K_R^2\widehat{\beta}_1^- = \mathrm{i}\,(K-K_S)\,\mathrm{e}^{\mathrm{i}K_S X}, \tag{7.9b}$$

with

$$K_R \equiv \sqrt{K\left[K - 2K_0(1-\mathcal{G})\right]} \tag{7.10}$$

being the natural wave number of the radiated wave. As $\mathcal{G}$ is complex for any non zero λ_g', K_R will be likewise, implying spatial attenuation. Since $K_R \neq K_S$, the spatial period of the forcing term differs from the natural period in (6.9) and no resonance is expected. Variations of the real and imaginary parts are shown in figure 7 as functions of the detuning frequency Ω and the extraction rate λ_g'. One sees that there is no band gap and that all radiated waves are spatially attenuated. The special case of free buoys without extractors will be treated later.

The general solutions of (7.9) are,

$$\widehat{\beta}_1^+ = C_1^{(1)}\mathrm{e}^{\mathrm{i}K_R X} + C_1^{(2)}\mathrm{e}^{-\mathrm{i}K_R X} + \mathrm{i}\frac{K+K_S}{K_R^2 - K_S^2}\mathrm{e}^{\mathrm{i}K_S X}, \tag{7.11a}$$

$$\widehat{\beta}_1^- = C_1^{(3)}\mathrm{e}^{\mathrm{i}K_R X} + C_1^{(4)}\mathrm{e}^{-\mathrm{i}K_R X} + \mathrm{i}\frac{K-K_S}{K_R^2 - K_S^2}\mathrm{e}^{\mathrm{i}K_S X}. \tag{7.11b}$$

For brevity we let

$$B_1^+ = \frac{K+K_S}{K_R^2 - K_S^2}, \qquad B_1^- = \frac{K-K_S}{K_R^2 - K_S^2}.$$

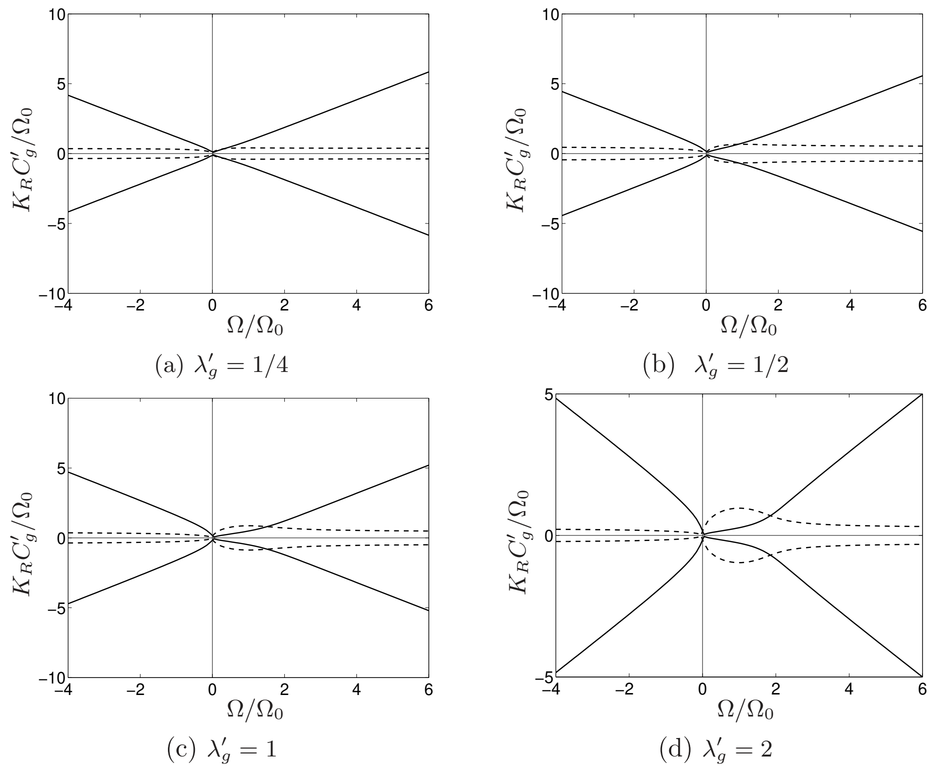


Figure 7. Dispersion relation for the radiation problem. The real part of $K_R C'_g/\Omega_0$ is represented in solid line, the imaginary part in dashed line. $d' = 1$.

Because of the coupling by (7.8), only two of the four coefficients $C_1^{(i)},\ i = 1, 2, 3, 4$ are independent. Two relations among them can be found by invoking (7.8) at any $X \in [0, L]$, say $X = 0$,

$$-\mathrm{i}K\widehat{\beta}_1^+(0) + \frac{\partial \widehat{\beta}_1^+}{\partial X}(0) = -\mathrm{i}K_0(1-\mathcal{G})(\widehat{\beta}_1^+(0) + \widehat{\beta}_1^-(0)) + 1, \tag{7.12}$$

$$-\mathrm{i}K\widehat{\beta}_1^-(0) - \frac{\partial \widehat{\beta}_1^-}{\partial X}(0) = -\mathrm{i}K_0(1-\mathcal{G})(\widehat{\beta}_1^+(0) + \widehat{\beta}_1^-(0)) + 1, \tag{7.13}$$

which give

$$-\mathrm{i}(K - K_R - K_{\mathcal{G}})C_1^{(1)} - \mathrm{i}(K + K_R - K_{\mathcal{G}})C_1^{(2)} + \mathrm{i}K_{\mathcal{G}}(C_1^{(3)} + C_1^{(4)}) = 1 + (-K + K_S + K_{\mathcal{G}})B_1^+ + K_{\mathcal{G}}B_1^-, \tag{7.14a}$$

$$\mathrm{i}K_{\mathcal{G}}(C_1^{(1)} + C_1^{(2)}) - \mathrm{i}(K + K_R - K_{\mathcal{G}})C_1^{(3)} - \mathrm{i}(K - K_R - K_{\mathcal{G}})C_1^{(4)} = 1 + K_{\mathcal{G}}B_1^+ + (-K - K_S + K_{\mathcal{G}})B_1^-, \tag{7.14b}$$

with $K_{\mathcal{G}} \equiv K_0(1-\mathcal{G})$. From the two boundary conditions (7.5) , we get

$$C_1^{(1)} + C_1^{(2)} = -\mathrm{i}B_1^+, \tag{7.14c}$$

$$C_1^{(3)}\mathrm{e}^{\mathrm{i}K_R L} + C_1^{(4)}\mathrm{e}^{-\mathrm{i}K_R L} = -\mathrm{i}B_1^-\mathrm{e}^{\mathrm{i}K_S L}. \tag{7.14d}$$

We can now solve for $C_1^{(i)}$ for all $i = 1, 2, 3, 4$:

$$C_1^{(1)} = -\mathrm{i}\left[K_R\left(B_1^+(K_R+K_S) + B_1^- K_{\mathcal{G}} + 1\right)\cos(K_R L) - A_1^- K_{\mathcal{G}} K_R \mathrm{e}^{\mathrm{i}K_S L}\right.$$
$$\left. +\mathrm{i}\left(-B_1^+(K-K_{\mathcal{G}})(K_R+K_S) + A_1^- K_{\mathcal{G}} K_S - K\right)\sin(K_R L)\right]$$
$$\left[2K_R\left(K_R\cos(K_R L) - \mathrm{i}(K-K_{\mathcal{G}})\sin(K_R L)\right)\right]^{-1}, \tag{7.15a}$$

$$C_1^{(2)} = \mathrm{i}\left[K_R\left(B_1^+(K_S-K_R) + B_1^- K_{\mathcal{G}} + 1\right)\cos(K_R L) - B_1^- K_{\mathcal{G}} K_R \mathrm{e}^{\mathrm{i}K_S L}\right.$$
$$\left. +\mathrm{i}\left(-B_1^+(K-K_{\mathcal{G}})(-K_R+K_S) + B_1^- K_{\mathcal{G}} K_S - K\right)\sin(K_R L)\right]$$
$$\left[2K_R\left(K_R\cos(K_R L) - \mathrm{i}(K-K_{\mathcal{G}})\sin(K_R L)\right)\right]^{-1}, \tag{7.15b}$$

$$C_1^{(3)} = -\mathrm{i}\frac{B_1^-\left((K_R-K+K_{\mathcal{G}})\mathrm{e}^{\mathrm{i}K_S L} + (K_S+K-K_{\mathcal{G}})\mathrm{e}^{-\mathrm{i}K_R L}\right) - \mathrm{e}^{-\mathrm{i}K_R L}}{2\left(K_R\cos(K_R L) - \mathrm{i}(K-K_{\mathcal{G}})\sin(K_R L)\right)}, \tag{7.15c}$$

$$C_1^{(4)} = \mathrm{i}\frac{-B_1^-\left((K_R+K-K_{\mathcal{G}})\mathrm{e}^{\mathrm{i}K_S L} - (K_S+K-K_{\mathcal{G}})\mathrm{e}^{\mathrm{i}K_R L}\right) - \mathrm{e}^{\mathrm{i}K_R L}}{2\left(K_R\cos(K_R L) - \mathrm{i}(K-K_{\mathcal{G}})\sin(K_R L)\right)}. \tag{7.15d}$$

Corresponding to the forcing $\exp(-\mathrm{i}K_S x)$, the responses $\widehat{\beta}_2^{\pm}$ can be treated similarly. Let the solution be of the form

$$\widehat{\beta}_2^+ = C_2^{(1)}\mathrm{e}^{\mathrm{i}K_R X} + C_2^{(2)}\mathrm{e}^{-\mathrm{i}K_R X} + \mathrm{i}\frac{K-K_S}{K_R^2-K_S^2}\mathrm{e}^{\mathrm{i}K_S X}, \tag{7.16a}$$

$$\widehat{\beta}_2^- = C_2^{(3)}\mathrm{e}^{\mathrm{i}K_R X} + C_2^{(4)}\mathrm{e}^{-\mathrm{i}K_R X} + \mathrm{i}\frac{K+K_S}{K_R^2-K_S^2}\mathrm{e}^{\mathrm{i}K_S X}. \tag{7.16b}$$

The coefficients $C_2^{(j)}$, $j = 1, 2, 3, 4$ can be obtained by replacing K_S by $-K_S$ in C_1^j, i,e.,

$$C_2^{(j)}(K_R, K_S) = C_1^{(j)}(K_R, -K_S), \qquad j = 1, 2, 3, 4. \tag{7.17}$$

It is now easy to see that

$$\widehat{\beta}^+ = \mathrm{i}\Omega_0\mathcal{G}\left(\mathcal{A}^+\widehat{\beta}_1^+ + \mathcal{A}^-\widehat{\beta}_2^+\right), \tag{7.18a}$$

$$\widehat{\beta}^- = \mathrm{i}\Omega_0\mathcal{G}\left(\mathcal{A}^+\widehat{\beta}_1^- + \mathcal{A}^-\widehat{\beta}_2^-\right), \tag{7.18b}$$

are the solutions of the radiation problem, satisfying (6.9) as well as the boundary conditions (7.5). We have also confirmed these formulas by direct numerical solution by the Finite Volume method.

(*b*) *Freely floating buoys*

In the limiting case of freely floating buoys, $\lambda_g' = 0$ so that $\mathcal{G} = 1$. The evolution equations for $\beta^{\pm}$ are no longer coupled. After omitting the factor $\mathrm{e}^{-\mathrm{i}\Omega T}$, we get simply

$$-\mathrm{i}K\widehat{\beta}^+ + \frac{\partial\widehat{\beta}^+}{\partial X} = -\mathrm{i}K_0(\widehat{\alpha}^+ + \widehat{\alpha}^-), \tag{7.19}$$

$$-\mathrm{i}K\widehat{\beta}^- - \frac{\partial\widehat{\beta}^-}{\partial X} = -\mathrm{i}K_0(\widehat{\alpha}^+ + \widehat{\alpha}^-). \tag{7.20}$$

These decoupled first order ordinary differential equations have solutions of the form

$$\widehat{\beta}^{+} = C_1 \mathrm{e}^{\mathrm{i}KX} - \frac{K_0\mathcal{A}^{+}}{K-K_S}\mathrm{e}^{\mathrm{i}K_S X} - \frac{K_0\mathcal{A}^{-}}{K+K_S}\mathrm{e}^{-\mathrm{i}K_S X}, \tag{7.21}$$

$$\widehat{\beta}^{-} = C_2 \mathrm{e}^{-\mathrm{i}KX} - \frac{K_0\mathcal{A}^{+}}{K+K_S}\mathrm{e}^{\mathrm{i}K_S X} - \frac{K_0\mathcal{A}^{-}}{K-K_S}\mathrm{e}^{-\mathrm{i}K_S X}, \tag{7.22}$$

with constants C_1 and C_2 to be determined. Note that

$$\frac{K_0}{K-K_S} + \frac{K_0}{K+K_S} = 2\frac{K_0 K}{K^2 - K_S^2} = 1 \tag{7.23}$$

after using the fact that the scattering wave number is given by $K_S = \sqrt{K^2 - 2KK_0}$, so that

$$\Phi_0 = \left(\widehat{\beta}^{+} + \widehat{\beta}^{-}\right)\mathrm{e}^{-\mathrm{i}\Omega T} = (C_1 - \widehat{\alpha}^{+})\mathrm{e}^{\mathrm{i}(KX-\Omega T)} + (C_2 - \widehat{\alpha}^{-})\mathrm{e}^{-\mathrm{i}(KX+\Omega T)}.$$

This gives the total potential

$$\begin{aligned}\phi_0 + \Phi_0 = Z(z')&\left(\widehat{\alpha}^{+} + \widehat{\alpha}^{-} + \widehat{\beta}^{+} + \widehat{\beta}^{-}\right)\mathrm{e}^{-\mathrm{i}\Omega T}\\ &= Z(z')\left(C_1\mathrm{e}^{\mathrm{i}(KX-\Omega T)} + C_2\mathrm{e}^{-\mathrm{i}(KX+\Omega T)}\right),\end{aligned} \tag{7.24a}$$

and thus

$$\zeta_0 = \eta_0 = C_1\mathrm{e}^{\mathrm{i}(KX-\Omega T)} + C_2\mathrm{e}^{-\mathrm{i}(KX+\Omega T)}, \tag{7.24b}$$

where η_0 is defined in a similar way as ζ_0. Matching directly the total potential $\phi_0 + \Phi_0$ with the potential outside the array, we find

$$C_1 = 1, \qquad\qquad C_2 = 0, \tag{7.25}$$

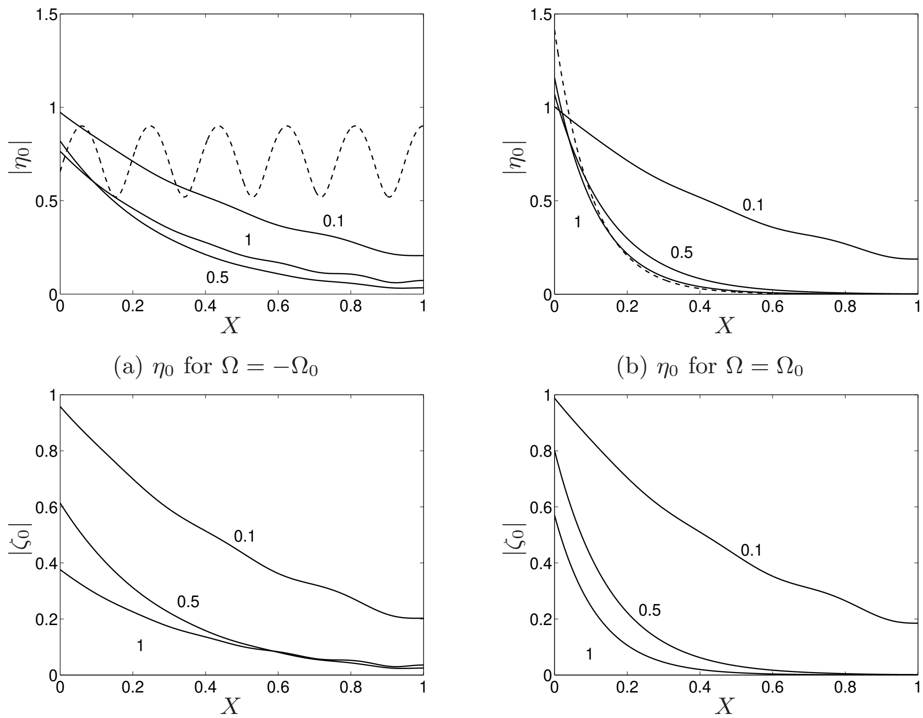
so

$$\eta_0 = \zeta_0 = \mathrm{e}^{\mathrm{i}(KX-\Omega T)}. \tag{7.26}$$

Hence the buoys move with the same amplitude and phase as the surrounding free surface and to the leading order a sparse array of small freely floating buoys does not affect the incoming wave. This result is expected as the consequence of negligible buoy inertia.

8. The combined effects of scattering, radiation and extraction

The combined effect of scattering, radiation and energy extraction on the free surface and buoy displacements are presented in figures 8 and 9. Two representative values of the detuning are chosen here: one outside the band gap of the scattering problem ($\Omega = -\Omega_0$) and one inside ($\Omega = \Omega_0$). One sees in figure 9 that the effect of the length is minor. In contrast, the effect of the extraction force shown in figure 8 has much stronger influence, as shown for the the two limits of fixed and free buoys. The displacements of the free surface and the buoys decrease faster through



(a) η_0 for $\Omega = -\Omega_0$ (b) η_0 for $\Omega = \Omega_0$

(c) ζ_0 for $\Omega = -\Omega_0$ (d) ζ_0 for $\Omega = \Omega_0$

Figure 8. Free surface elevation η_0 and buoy displacement ζ_0 in the array. $L = 1$, $d' = 1$ and $\lambda'_g = 0.1, 0.5, 1$ as marked next to the curves. In figures (a) and (b), the dashed curve represent the limiting case of fixed buoys, i.e. $\lambda'_g \to \infty$. As shown in §7(b), the free surface amplitude is constant and equal to 1 in the limiting case of free buoys $\lambda'_g = 0$.

the array for frequencies inside the band gap, and as the extraction rate increases. Understandably the displacement of the buoy is always smaller than that of the water surface. Note that there is no resonance.

In order to characterize the waves outside of the array, let us introduce transmission and reflection coefficients for the complete problem, denoted respectively by $\mathcal{T}$ and $\mathcal{R}$ – such that the free surface in the open water regions is given by

$$\eta_0 = \mathrm{e}^{\mathrm{i}(KX-\Omega T)} + \mathcal{R}\mathrm{e}^{-\mathrm{i}(KX-\Omega T)}, \qquad X < 0, \tag{8.1a}$$

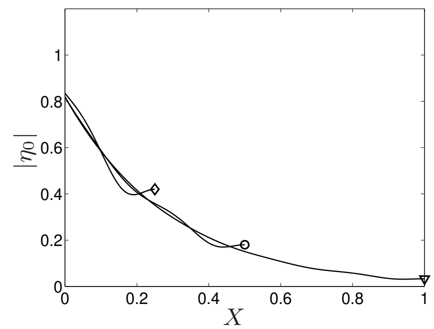
$$\eta_0 = \mathcal{T}\mathrm{e}^{\mathrm{i}(KX-\Omega T)}, \qquad X > L. \tag{8.1b}$$

In terms of the scattering and radiation problems studied previously, the coefficients are given by

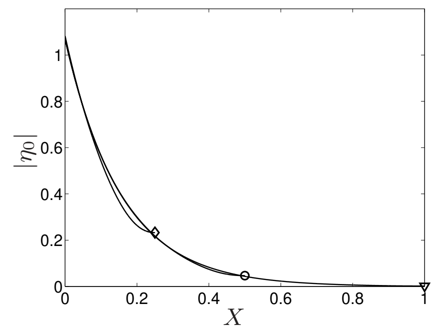
$$\mathcal{T} = \alpha^-(0) + \beta^-(0) = C_T + C^+, \tag{8.2a}$$

$$\mathcal{R} = \alpha^+(L) + \beta^+(L) = C_R + C^-. \tag{8.2b}$$

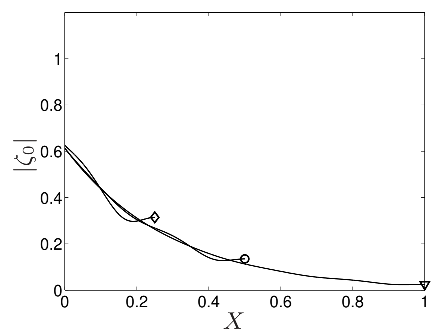
Analytical formulas can be obtained from the results in §4 and §7. While scattering is negligible for freely floating buoys when there is no power extracted as shown in §7(b), for moving and energy-extracting buoys $|\mathcal{R}|$ is close to 1 when the detuning frequency is within the band gap $0 \leq \Omega \leq 2\Omega_0$, as shown in figures 10 and 11.



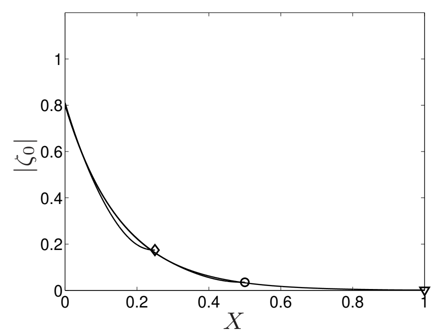
(a) η_0 for $\Omega = -\Omega_0$



(b) η_0 for $\Omega = \Omega_0$



(c) ζ_0 for $\Omega = -\Omega_0$



(d) ζ_0 for $\Omega = \Omega_0$

Figure 9. Effects of detuning frequencies and array length on free surface elevation η_0 and buoy displacement ζ_0 for $\lambda_g' = 0.5$ and $d' = 1$. $L = 0.25$ (diamond), $L = 0.5$ (square) and $L = 1$ (triangle).

In particular reflection increases with the energy extraction rate λ_g' and with the length of the array, and is of course the greatest for fixed buoys, which is equivalent to $\lambda_g' \to \infty$. In comparison with fixed buoys, the detuning corresponding to the maximum reflection is shifted slightly from $\Omega = \Omega_0$ towards $\Omega = 0$.

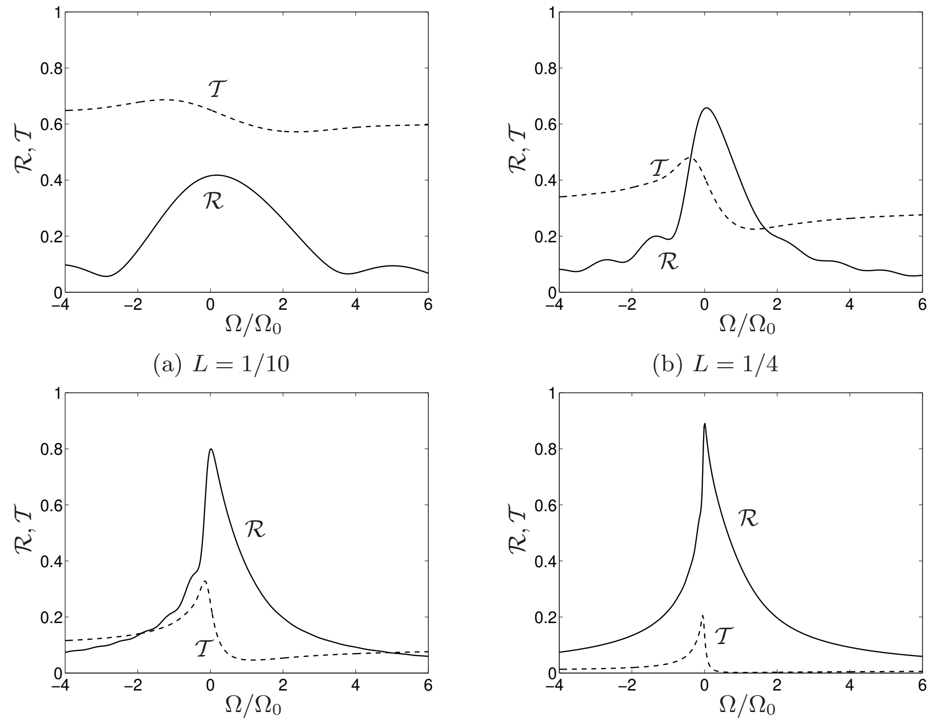
9. Energy extraction

Now the flow and buoy displacements are known. The period-averaged rate of energy extracted by the j^{th} buoy is given, in physical form, by:

$$E_j \equiv \frac{1}{2}\omega^2 \lambda_g |\zeta^{(j)}|^2. \tag{9.1}$$

The total energy extracted by N buoys is therefore

$$E = \sum_{j=1}^{N} E_j = \frac{1}{d}\sum_{j=1}^{N} E_j d. \tag{9.2}$$



(a) $L = 1/10$ (b) $L = 1/4$ (c) $L = 1/2$ (d) $L = 1$

Figure 10. Reflection and transmission coefficients for different L. The energy extraction rate is set at $\lambda'_g = 1/2$ and $d' = 1$.

Since $N \gg 1$, the above series can be approximated by an integral:

$$E \approx \frac{1}{2d}\omega^2\lambda_g \int_0^L |\zeta|^2 \,\mathrm{d}x$$
$$= \left(\frac{1}{2}\rho\sqrt{gh}^3 A^2\right) \frac{\pi}{d'}{\omega'}^2\lambda'_g \int_0^L |\zeta_0|^2 \,\mathrm{d}x'. \tag{9.3}$$

Dividing by the energy flux rate of the incident wave across the width $d : \frac{1}{2}\rho\sqrt{gh}^3 A^2 C'_g d'$, the efficiency of power absorption is found in terms of the buoy displacement:

$$\mathcal{E} \equiv \frac{\pi}{d'^2 C'_g}{\omega'}^2\lambda'_g \int_0^L |\zeta_0|^2 \,\mathrm{d}x. \tag{9.4}$$

As a check, the absorbed energy can also be calculated from the difference of the incoming and outgoing energy flux rates at $x \sim \pm\infty$. From the radiation and scattering coefficients, we can find

$$\mathcal{E} \equiv 1 - \left(\left| \widehat{\alpha}^-(0) + \widehat{\beta}^-(0) \right|^2 + \left| \widehat{\alpha}^+(L) + \widehat{\beta}^+(L) \right|^2 \right). \tag{9.5}$$

It is shown the appendix that (9.4) and (9.5) are the same.

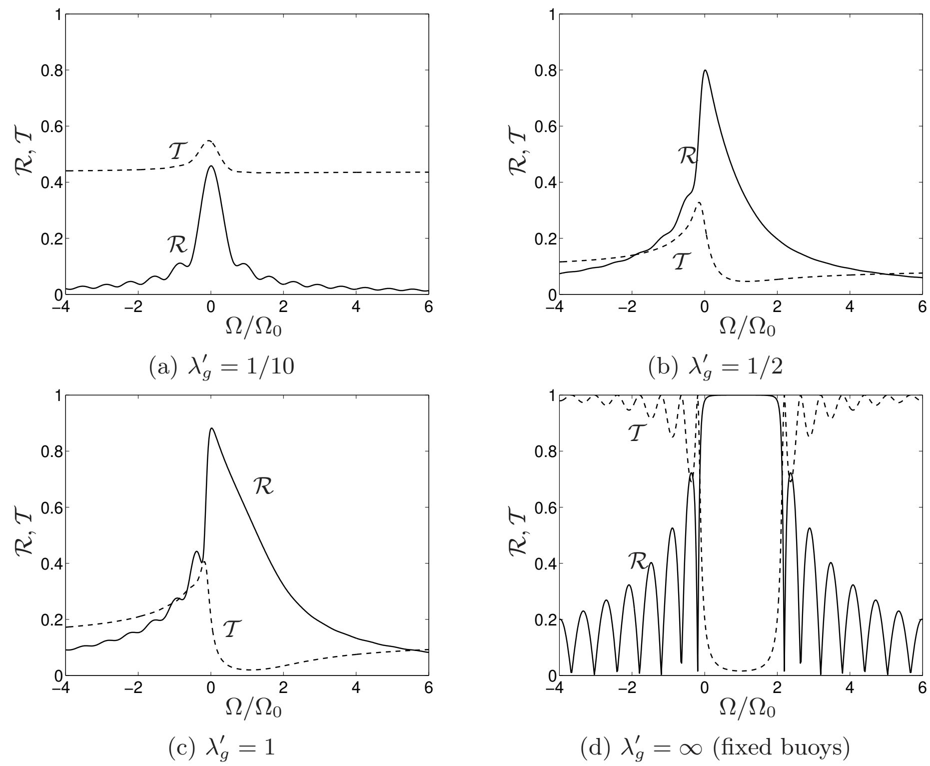


Figure 11. Reflection and transmission coefficients for different λ'_g. The length of the array is $L = 1/2$ and $d' = 1$.

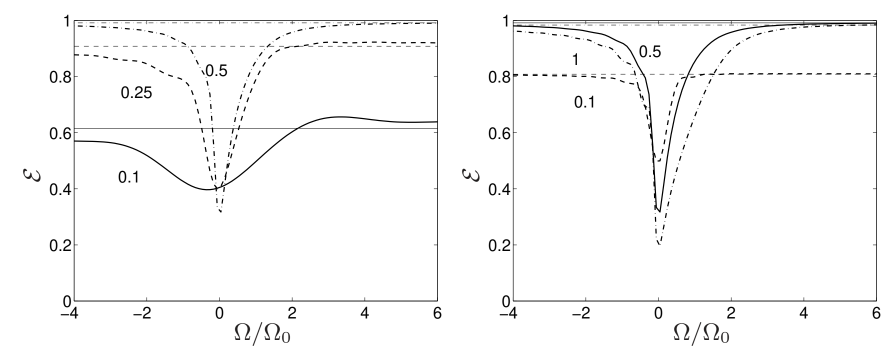


Figure 12. (a) Efficiency over a range of Ω/Ω_0 for $d' = 1$. (a): $\lambda'_g = 1/2$ and $L = 0.1, 0.25, 0.5$. (b) : $L = 1/2$ and $\lambda'_g = 0.1, 0.5, 1.0$. The horizontal lines give the results obtained by (3.12).

Figure 12 shows how the extracted energy varies with the detuning, the length of the array and the energy extraction rate. Note first that when the detuning frequency is outside the band gap of the pure scattering problem ($0 < \Omega/\Omega_0 < 2$), the energy extraction efficiency $\mathcal{E}$ tends to the same constant value predicted by the approximate value $\mathcal{E}_0$ given by (3.12) based on Froude-Krylov approximation.

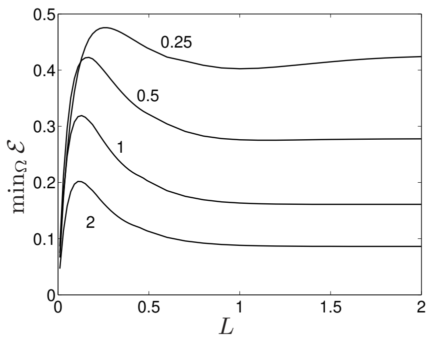


Figure 13. Influence of L and λ'_g on the the minimum energy extracted due to Bragg resonance. $d' = 1$ and $\lambda'_g = 0.25, 0.5, 1, 2$.

The good agreement shows the robustness of the asymptotic theory here despite its intended realm in the small neighbourhood of Bragg resonance.

It is clear that within the band gap, the extraction efficiency is substantially reduced by Bragg resonance. The reduction is the greatest around $\Omega = 0$ (perfect tuning) and is larger for wider arrays (see figure 12(a)) and greater extraction rate (see figure 12(b)).

Although the maximum efficiency over most of the detuning frequencies is approximately equal to $\mathcal{E}_0$ which increases with the length of the array, it is interesting that shorter arrays can yield more energy for small detuning, as shown in figure 13.

10. Conclusions

We have developed an asymptotic theory for wave interaction with a periodic array of small buoys in order to gain a better understanding of its potential as a wave-power farm.

We showed that fixed cylindrical buoys of small dimensions produce scattering effect of the same order of magnitude as vertical piles of the same radius but extending across the entire depth. It can easily be shown that the results derived here can be extended to small buoys of arbitrary geometry. In particular, we have shown that hemi-spherical buoys have the same influence as circular cylindrical buoys. We have extended the work of Li and Mei (2007*a*) and deduced analytically the frequency band gap within which one dimensional wave propagation is inhibited.

By multiple-scale analysis we have further solved the radiation problem of an array of movable buoys partially constrained by energy absorbing devices. The absorber is modeled as a damping force proportional to the velocity of the buoy. We have shown that Bragg resonance reduces the potential for energy extraction, somewhat similar to viscous damping in wall boundary layers studied recently by Tabaei and Mei (2009). While the present theory is designed only for the immediate vicinity of Bragg resonance, it agrees with the approximate theory valid far outside the band gap. Therefore it may be a practical tool for analyzing random incident waves with a broad frequency band. Modifications for shorter waves satisfying the Bragg condition $kd = n\pi$ appears straightforward. For practical design of arrays of energy converters with complex phase control of the power-takeoff system, numerical techniques would be necessary.

Future extensions may include oblique incidence on a wide array. In this case, the wave physics of multiple scattering is two dimensional. Approximation used here can be extended as in Li and Mei (2007*a*); Tabaei and Mei (2009) for piles.

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Appendix A. Energy extraction from a sparse array

To prove that the two expressions of the energy extracted given in §9 are equal, let us recall the governing equations of the scattering and radiation amplitudes:

$$C_g' \frac{\partial \widehat{\alpha}^+}{\partial X} = -\mathrm{i}\Omega_0 \left(\widehat{\alpha}^+ + \widehat{\alpha}^-\right) + \mathrm{i}\Omega \widehat{\alpha}^+, \tag{A 1a}$$

$$-C_g' \frac{\partial \widehat{\alpha}^-}{\partial X} = -\mathrm{i}\Omega_0 \left(\widehat{\alpha}^+ + \widehat{\alpha}^-\right) + \mathrm{i}\Omega \widehat{\alpha}^-, \tag{A 1b}$$

$$C_g' \frac{\partial \widehat{\beta}^+}{\partial X} = -\mathrm{i}\Omega_0 \left[(1-\mathcal{G})(\widehat{\beta}^+ + \widehat{\beta}^-) - \mathcal{G}(\widehat{\alpha}^+ + \widehat{\alpha}^-)\right] + \mathrm{i}\Omega \widehat{\beta}^+, \tag{A 1c}$$

$$-C_g' \frac{\partial \widehat{\beta}^-}{\partial X} = -\mathrm{i}\Omega_0 \left[(1-\mathcal{G})(\widehat{\beta}^+ + \widehat{\beta}^-) - \mathcal{G}(\widehat{\alpha}^+ + \widehat{\alpha}^-)\right] + \mathrm{i}\Omega \widehat{\beta}^-. \tag{A 1d}$$

The corresponding buoy displacement is given by:

$$\widehat{\zeta}_0 = \mathcal{G}\left[\left(\widehat{\alpha}^+ + \widehat{\alpha}^-\right) + \left(\widehat{\beta}^+ + \widehat{\beta}^-\right)\right]. \tag{A 2}$$

Let us introduce

$$\widehat{\xi} = \mathcal{G}\left[\left(\widehat{\alpha}^+ + \widehat{\beta}^+\right) - \left(\widehat{\alpha}^- + \widehat{\beta}^-\right)\right]. \tag{A 3}$$

Taking (A 1a) − (A 1b) + (A 1c) − (A 1d), we find that

$$C_g' \frac{\mathrm{d}\widehat{\zeta}_0}{\mathrm{d}X} = \mathrm{i}\Omega \widehat{\xi}.$$

Similarly, by taking (A 1a) + (A 1b) + (A 1c) + (A 1d), we find

$$C_g' \frac{\mathrm{d}\widehat{\xi}}{\mathrm{d}X} = -\mathrm{i}\left(2\Omega_0\left(1-\mathcal{G}\right) - \Omega\right)\widehat{\zeta}_0.$$

Denoting their complex conjugates of $\widehat{\zeta}_0, \widehat{\xi}$ by $\widehat{\zeta}_0^\dagger, \widehat{\xi}^\dagger$ respectively, we get

$$C_g' \frac{\mathrm{d}\widehat{\zeta}_0^\dagger \widehat{\xi}}{\mathrm{d}X} = -\mathrm{i}\left(2\Omega_0\left(1-\mathcal{G}\right) - \Omega\right)|\widehat{\zeta}_0|^2 - \mathrm{i}\Omega|\widehat{\xi}^2|,$$

and

$$C_g' \frac{\mathrm{d}\widehat{\zeta}_0 \widehat{\xi}^\dagger}{\mathrm{d}X} = \mathrm{i}\left(2\Omega_0\left(1-\mathcal{G}^\dagger\right) - \Omega\right)|\widehat{\zeta}_0|^2 + \mathrm{i}\Omega|\widehat{\xi}^2|,$$

which can be summed up to give

$$C_g' \frac{\mathrm{d}\left(\widehat{\zeta}_0^\dagger \widehat{\xi} + \widehat{\zeta}_0 \widehat{\xi}^\dagger\right)}{\mathrm{d}X} = 2\mathrm{i}\Omega_0\left(\mathcal{G} - \mathcal{G}^\dagger\right)|\widehat{\zeta}_0|^2.$$

Recall that

$$\mathcal{G} = \frac{1}{1 - \mathrm{i}\omega'\lambda_g'},$$

so that

$$\mathcal{G} - \mathcal{G}^\dagger = \frac{2\mathrm{i}\omega'\lambda_g'}{1 + \omega'^2\lambda_g'^2}.$$

Since

$$\widehat{\zeta}_0^\dagger\widehat{\xi} + \widehat{\zeta}_0\widehat{\xi}^\dagger = -2|\mathcal{G}|^2 \left(\left|\widehat{\alpha}^- + \widehat{\beta}^-\right|^2 - \left|\widehat{\alpha}^+ + \widehat{\beta}^+\right|^2 \right),$$

we finally get

$$\frac{\mathrm{d}}{\mathrm{d}X}\left[\left|\widehat{\alpha}^- + \widehat{\beta}^-\right|^2 - \left|\widehat{\alpha}^+ + \widehat{\beta}^+\right|^2 \right] = 2\Omega_0\lambda_g'\omega'|\widehat{\zeta}_0|^2.$$

and, after integration,

$$\frac{2\Omega_0\lambda_g'\omega'}{C_g'}\int_0^L |\widehat{\zeta}_0|^2\,\mathrm{d}X = \Omega_0\left[\left|\widehat{\alpha}^- + \widehat{\beta}^-\right|^2 - \left|\widehat{\alpha}^+ + \widehat{\beta}^+\right|^2 \right]_0^L. \tag{A 4}$$

Recall that the boundary conditions (4.29) and (7.5) give

$$\widehat{\alpha}^+(0) = 1, \quad \widehat{\alpha}^-(L) = \widehat{\beta}^+(0) = \widehat{\beta}^-(L) = 0,$$

so (A 4) reduces to

$$\begin{aligned}\frac{2\lambda_g'\omega'}{C_g'}\int_0^L |\widehat{\zeta}_0|^2\,\mathrm{d}X &= \left|\widehat{\alpha}^-(L) + \widehat{\beta}^-(L)\right|^2 - \left|\widehat{\alpha}^-(0) + \widehat{\beta}^-(0)\right|^2 - \\ &\quad \left|\widehat{\alpha}^+(L) + \widehat{\beta}^+(L)\right|^2 + \left|\widehat{\alpha}^+(0) + \widehat{\beta}^+(0)\right|^2 \\ &= 1 - \left(\left|\widehat{\alpha}^-(0) + \widehat{\beta}^-(0)\right|^2 + \left|\widehat{\alpha}^+(L) + \widehat{\beta}^+(L)\right|^2 \right).\end{aligned} \tag{A 5}$$

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