

# BRAID GROUP ACTIONS ON DERIVED CATEGORIES OF COHERENT SHEAVES

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## Abstract

*This paper gives a construction of braid group actions on the derived category of coherent sheaves on a variety  $X$ . The motivation for this is M. Kontsevich's homological mirror conjecture, together with the occurrence of certain braid group actions in symplectic geometry. One of the main results is that when  $\dim X \geq 2$ , our braid group actions are always faithful.*

*We describe conjectural mirror symmetries between smoothings and resolutions of singularities which lead us to find examples of braid group actions arising from crepant resolutions of various singularities. Relations with the McKay correspondence and with exceptional sheaves on Fano manifolds are given. Moreover, the case of an elliptic curve is worked out in some detail.*

## 1. Introduction

### 1.1. Derived categories of coherent sheaves

Let  $X$  be a smooth complex projective variety, and let  $D^b(X)$  be the bounded derived category of coherent sheaves. It is an interesting question how much information about  $X$  is contained in  $D^b(X)$ .

Certain invariants of  $X$  can be shown to depend only on  $D^b(X)$ . This is obviously true for  $K(X)$ , the Grothendieck group of both the abelian category  $\text{Coh}(X)$  of coherent sheaves and of  $D^b(X)$ . A deep result of D. Orlov [40] implies that the topological  $K$ -theory  $K_{\text{top}}^*(X)$  is also an invariant of  $D^b(X)$ ; hence, so are the sums of its even and odd Betti numbers. Because of the uniqueness of Serre functors (see [2]), the dimension of  $X$ , and whether it is Calabi-Yau ( $\omega_X \cong \mathcal{O}_X$ ) or not, can be read off from  $D^b(X)$ . Using Orlov's theorem quoted above, one can prove that the Hochschild cohomology of  $X$ ,  $HH^*(X) = \text{Ext}_{X \times X}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ , depends only on  $D^b(X)$ . As pointed out by Kontsevich [30, p. 131], it is implicit in the work of M. Gerstenhaber and

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S. Schack [15] that

$$HH^r(X) \cong \bigoplus_{p+q=r} H^p(X, \Lambda^q TX).$$

Thus, for Calabi-Yau varieties,  $\dim HH^r(X) = \sum_{p+q=r} h^{p,n-q}(X)$ ; in mirror symmetry these are the Betti numbers of the mirror manifold. Finally, a theorem of A. Bondal and Orlov [5] says that if the canonical sheaf  $\omega_X$  or its inverse is ample,  $X$  can be entirely reconstructed from  $D^b(X)$ . Contrary to what this list of results might suggest, there are in fact nonisomorphic varieties with equivalent derived categories. The first examples are due to S. Mukai: abelian varieties in [37] and  $K3$  surfaces in [38]. Examples with nontrivial  $\omega_X$  have been found by Bondal and Orlov [4].

This paper is concerned with a closely related object, the self-equivalence group  $\text{Auteq}(D^b(X))$ . Recall that an exact functor between two triangulated categories  $\mathfrak{C}, \mathfrak{D}$  is a pair  $(F, \nu_F)$  consisting of a functor  $F : \mathfrak{C} \rightarrow \mathfrak{D}$  and a natural isomorphism  $\nu_F : F \circ [1]_{\mathfrak{C}} \cong [1]_{\mathfrak{D}} \circ F$  (here  $[1]_{\mathfrak{C}}, [1]_{\mathfrak{D}}$  are the translation functors) with the property that exact triangles in  $\mathfrak{C}$  are mapped to exact triangles in  $\mathfrak{D}$ . The appropriate equivalence relation between such functors is *graded natural isomorphism*, which means natural isomorphism compatible with the maps  $\nu_F$  (see [5, Section 1]). Ignoring set-theoretic difficulties, which are irrelevant for  $\mathfrak{C} = D^b(X)$ , the equivalence classes of exact functors from  $\mathfrak{C}$  to itself form a monoid.  $\text{Auteq}(\mathfrak{C})$  is defined as the group of invertible elements in this monoid. Known results about  $\text{Auteq}(D^b(X))$  parallel those for  $D^b(X)$  itself. It always contains a subgroup  $A(X) \cong (\text{Aut}(X) \ltimes \text{Pic}(X)) \times \mathbb{Z}$  generated by the automorphisms of  $X$ , the functors of tensoring with an invertible sheaf, and the translation. Bondal and Orlov [5] have shown that if  $\omega_X$  or  $\omega_X^{-1}$  is ample, then  $\text{Auteq}(D^b(X)) = A(X)$ . Mukai's arguments in [37] imply that  $\text{Auteq}(D^b(X))$  is bigger than  $A(X)$  for all abelian varieties. (Recent work of Orlov [41] describes  $\text{Auteq}(D^b(X))$  completely in this case.)

Our own interest in self-equivalence groups comes from Kontsevich's homological mirror conjecture in [30]. One consequence of this conjecture is that, for Calabi-Yau varieties to which mirror symmetry applies, the group  $\text{Auteq}(D^b(X))$  should be related to the symplectic automorphisms of the mirror manifold. This conjectural relationship is rather abstract and difficult to spell out in concrete examples. Nevertheless, as a first and rather naive check, one can look at some special symplectic automorphisms of the mirror and try to guess the corresponding self-equivalences of  $D^b(X)$ . Having made this guess in a sufficiently plausible way (which means that the two objects show similar behaviour), the next step might be to take some unsolved questions about symplectic automorphisms and translate them into one about  $\text{Auteq}(D^b(X))$ . Using the smoother machinery of sheaf theory, one stands a good chance of solving this analogue; this in turn provides a conjectural answer, or “mirror symmetry prediction,” for the original problem. The present paper is an experiment

in this mode of thinking. We now state the main results independently of their motivation; the discussion of mirror symmetry is taken up again in Section 1.2.

Let  $X, Y$  be two (as before, smooth complex projective) varieties. The Fourier-Mukai transform (FMT) by an object  $\mathcal{P} \in D^b(X \times Y)$  is the exact functor

$$\Phi_{\mathcal{P}} : D^b(X) \longrightarrow D^b(Y), \quad \Phi_{\mathcal{P}}(\mathcal{G}) = \mathbf{R}\pi_{2*} \left( \pi_1^* \mathcal{G} \otimes^{\mathbf{L}} \mathcal{P} \right),$$

where  $\pi_1 : X \times Y \rightarrow X, \pi_2 : X \times Y \rightarrow Y$  are the projections. This is a very practical way of defining functors. Orlov [40] has proved that any equivalence  $D^b(X) \rightarrow D^b(Y)$  can be written as an FMT. Earlier work of A. Maciocia [32] shows that if  $\Phi_{\mathcal{P}}$  is an equivalence, then  $\mathcal{P}$  must satisfy a partial Calabi-Yau condition,  $\mathcal{P} \otimes \pi_1^* \omega_X \otimes \pi_2^* \omega_Y^{-1} \cong \mathcal{P}$ . T. Bridgeland [7] provides a partial converse to this.

Now take an object  $\mathcal{E} \in D^b(X)$  which is a complex of locally free sheaves. We define the *twist functor*  $T_{\mathcal{E}} : D^b(X) \rightarrow D^b(X)$  as the FMT with

$$\mathcal{P} = \text{Cone} \left( \eta : \mathcal{E}^{\vee} \boxtimes \mathcal{E} \longrightarrow \mathcal{O}_{\Delta} \right), \quad (1.1)$$

where  $\mathcal{E}^{\vee}$  is the dual complex,  $\boxtimes$  is the exterior tensor product,  $\Delta \subset X \times X$  is the diagonal, and  $\eta$  is the canonical pairing. Since quasi-isomorphic  $\mathcal{E}$  give rise to isomorphic functors  $T_{\mathcal{E}}$ , one can use locally free resolutions to extend the definition to arbitrary objects of  $D^b(X)$ .

#### Definition 1.1

- (a)  $\mathcal{E} \in D^b(X)$  is called *spherical* if  $\text{Hom}_{D^b(X)}^r(\mathcal{E}, \mathcal{E})$  is equal to  $\mathbb{C}$  for  $r = 0, \dim X$  and zero in all other degrees, and if in addition  $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ .
- (b) An  $(A_m)$ -configuration,  $m \geq 1$ , in  $D^b(X)$  is a collection of  $m$  spherical objects  $\mathcal{E}_1, \dots, \mathcal{E}_m$  such that

$$\dim_{\mathbb{C}} \text{Hom}_{D^b(X)}^*(\mathcal{E}_i, \mathcal{E}_j) = \begin{cases} 1 & |i - j| = 1, \\ 0 & |i - j| \geq 2. \end{cases}$$

Here, as elsewhere in the paper,  $\text{Hom}^r(\mathcal{E}, \mathcal{F})$  stands for  $\text{Hom}(\mathcal{E}, \mathcal{F}[r])$ , and  $\text{Hom}^*(\mathcal{E}, \mathcal{F})$  is the total space  $\bigoplus_{r \in \mathbb{Z}} \text{Hom}^r(\mathcal{E}, \mathcal{F})$ .

#### THEOREM 1.2

*The twist  $T_{\mathcal{E}}$  along any spherical object  $\mathcal{E}$  is an exact self-equivalence of  $D^b(X)$ . Moreover, if  $\mathcal{E}_1, \dots, \mathcal{E}_m$  is an  $(A_m)$ -configuration, the twists  $T_{\mathcal{E}_i}$  satisfy the braid relations up to graded natural isomorphism:*

$$\begin{aligned} T_{\mathcal{E}_i} T_{\mathcal{E}_{i+1}} T_{\mathcal{E}_i} &\cong T_{\mathcal{E}_{i+1}} T_{\mathcal{E}_i} T_{\mathcal{E}_{i+1}} && \text{for } i = 1, \dots, m-1, \\ T_{\mathcal{E}_i} T_{\mathcal{E}_j} &\cong T_{\mathcal{E}_j} T_{\mathcal{E}_i} && \text{for } |i - j| \geq 2. \end{aligned}$$

We should point out that the first part, the invertibility of  $T_{\mathcal{E}}$ , was also known to Kontsevich, Bridgeland, and Maciocia. Let  $\rho$  be the homomorphism from the braid group  $B_{m+1}$  to  $\text{Auteq}(D^b(X))$  defined by sending the standard generators  $g_1, \dots, g_m \in B_{m+1}$  to  $T_{\mathcal{E}_1}, \dots, T_{\mathcal{E}_m}$ . We call this a weak braid group action on  $D^b(X)$ . (There is a better notion of a group action on a category which requires the presence of certain additional natural transformations (see [11]); we have not checked whether these exist in our case.) There is an induced representation  $\rho_*$  of  $B_{m+1}$  on  $K(X)$ . Concretely, the twist along an arbitrary  $\mathcal{E} \in D^b(X)$  acts on  $K(X)$  by

$$(T_{\mathcal{E}})_*(y) = y - \langle [\mathcal{E}], y \rangle [\mathcal{E}], \quad (1.2)$$

where  $\langle [\mathcal{F}], [\mathcal{G}] \rangle = \sum_i (-1)^i \dim \text{Hom}^i(\mathcal{F}, \mathcal{G})$  is the Mukai pairing (see [38]) or “Euler form.” If  $\dim X$  is even, then  $\rho_*$  factors through the symmetric group  $S_{m+1}$ . The odd-dimensional case is slightly more complicated, but still  $\rho_*$  is far from being faithful, at least if  $m$  is large.

For  $\rho$  itself we have the following contrasting result.

#### THEOREM 1.3

*Assume that  $\dim X \geq 2$ . Then the homomorphism  $\rho$  generated by the twists in any  $(A_m)$ -configuration is injective.*

The assumption  $\dim X \neq 1$  cannot be removed; indeed, there is a  $B_4$ -action on the derived category of an elliptic curve which is not faithful (see Section 3.4).

#### 1.2. Homological mirror symmetry and self-equivalences

We begin by recalling Kontsevich’s homological mirror conjecture in [30]. On the one hand, one takes Calabi-Yau varieties  $X$  and their derived categories  $D^b(X)$ . On the other hand, using entirely different techniques, it is thought that one can attach to any compact symplectic manifold  $(M, \beta)$ , with zero first Chern class a triangulated category, the derived Fukaya category  $D^b \text{Fuk}(M, \beta)$ . (Despite the notation, this is not constructed as the derived category of an abelian category.) Kontsevich’s conjecture is that *whenever  $X$  and  $(M, \beta)$  form a mirror pair, there is a (noncanonical) exact equivalence*

$$D^b(X) \cong D^b \text{Fuk}(M, \beta). \quad (1.3)$$

A more prudent formulation would be to say that (1.3) should hold for the generally accepted constructions of mirror manifolds. Before discussing this conjecture further, we need to explain what  $D^b \text{Fuk}(M, \beta)$  looks like. This is necessarily a tentative description since a rigorous definition does not exist yet. Moreover, for simplicity we have omitted some of the more technical aspects.

Let  $(M, \beta)$  be as before, of real dimension  $2n$ . To simplify things we assume that  $\pi_1(M)$  is trivial; this excludes the case of the 2-torus, so that  $n \geq 2$ . Recall that a submanifold  $L^n \subset M$  is called Lagrangian if  $\beta|_L \in \Omega^2(L)$  is zero. Following Kontsevich [30, p. 134], one considers objects, denoted by  $\tilde{L}$ , which are Lagrangian submanifolds with some extra structure. We call such objects *graded Lagrangian submanifolds* and the extra structure the *grading*. This grading amounts approximately to an integer choice. In fact, there is a free  $\mathbb{Z}$ -action, denoted by  $\tilde{L} \mapsto \tilde{L}[j]$  for  $j \in \mathbb{Z}$ , on the set of graded Lagrangian submanifolds; and if  $L$  is a connected Lagrangian submanifold, all its possible gradings (assuming that there are any) form a single orbit of this action. For details we refer to [49]. For any pair  $(\tilde{L}_1, \tilde{L}_2)$  of graded Lagrangian submanifolds, one expects to have a Floer cohomology group  $HF^*(\tilde{L}_1, \tilde{L}_2)$ , which is a finite-dimensional graded  $\mathbb{R}$ -vector space satisfying  $HF^*(\tilde{L}_1, \tilde{L}_2[j]) = HF^*(\tilde{L}_1[-j], \tilde{L}_2) = HF^{*+j}(\tilde{L}_1, \tilde{L}_2)$ . Defining this is a difficult problem; a fairly general solution has been announced recently by K. Fukaya, Kontsevich, Y.-G. Oh, H. Ohta, and K. Ono.

The most essential property of  $D^b \text{Fuk}(M, \beta)$  is that *any graded Lagrangian submanifold  $\tilde{L}$  defines an object in this category*. The translation functor (which is part of the structure of  $D^b \text{Fuk}(M, \beta)$  as a triangulated category) acts on such objects by  $\tilde{L} \mapsto \tilde{L}[1]$ . The morphisms between two objects of this kind are given by the degree zero Floer cohomology with complex coefficients:

$$\text{Hom}_{D^b \text{Fuk}(M, \beta)}(\tilde{L}_1, \tilde{L}_2) = HF^0(\tilde{L}_1, \tilde{L}_2) \otimes_{\mathbb{R}} \mathbb{C}.$$

(Floer groups in other degrees can be recovered by changing  $\tilde{L}_2$  to  $\tilde{L}_2[j]$ .) Composition of such morphisms is given by certain products on Floer cohomology, which were first introduced by Donaldson. There is also a slight generalisation of this: any pair  $(\tilde{L}, E)$  consisting of a graded Lagrangian submanifold, together with a flat unitary vector bundle  $E$  on the underlying Lagrangian submanifold, defines an object of  $D^b \text{Fuk}(M, \beta)$ . The morphisms between such objects are a twisted version of Floer cohomology. It is important to keep in mind that  $D^b \text{Fuk}(M, \beta)$  *contains many objects other than those that we have described*. This is necessarily so because it is triangulated; there must be enough objects to complete each morphism to an exact triangle, and these objects do not usually have a direct geometric meaning. However, it is expected that the objects of the form  $(\tilde{L}, E)$  generate the category  $D^b \text{Fuk}(M, \beta)$  in some sense.

*Remark 1.4*

In the traditional picture of mirror symmetry,  $M$  carries a  $\mathbb{C}$ -valued closed 2-form  $\beta_{\mathbb{C}}$  with real part  $\beta$ . What we have said concerns the Fukaya category for  $\text{im}(\beta_{\mathbb{C}}) = 0$ . Apparently, the natural generalisation to  $\text{im}(\beta_{\mathbb{C}}) \neq 0$  would be to take objects

$(\tilde{L}, E, A)$  consisting of a graded Lagrangian submanifold  $\tilde{L}$ , a complex vector bundle  $E$  on the underlying Lagrangian submanifold  $L$ , and a unitary connection  $A$  on  $E$  with curvature  $F_A = -\beta_{\mathbb{C}} \mid L \otimes \text{id}_E$ . The point is that to any map  $w : (D^2, \partial D^2) \rightarrow (M, L)$  one can associate a complex number

$$\frac{\text{trace}(\text{monodromy of } A \text{ around } w \mid \partial D^2)}{\text{rank}(E)} \exp\left(-\int_{D^2} w^* \beta_{\mathbb{C}}\right),$$

which is invariant under deformations of  $w$ . These numbers, as well as certain variations of them, would be used as weights in the counting procedure that underlies the definition of Floer cohomology. For simplicity, we stick to the case  $\text{im}(\beta_{\mathbb{C}}) = 0$  in our discussion.

In parallel with graded Lagrangian submanifolds, there is also a notion of graded symplectic automorphisms; in fact, these are just a special kind of graded Lagrangian submanifolds on  $(M, -\beta) \times (M, \beta)$ . The graded symplectic automorphisms form a topological group  $\text{Symp}^{\text{gr}}(M, \beta)$  that is a central extension of the usual symplectic automorphism group  $\text{Symp}(M, \beta)$  by  $\mathbb{Z}$ .  $\text{Symp}^{\text{gr}}(M, \beta)$  acts naturally on the set of graded Lagrangian submanifolds. Moreover, the central subgroup  $\mathbb{Z}$  is generated by a graded symplectic automorphism denoted by  $[1]$ , which maps each graded Lagrangian submanifold  $\tilde{L}$  to  $\tilde{L}[1]$ ; we refer again to [49] for details. Because  $D^b \text{Fuk}(M, \beta)$  is defined in what are essentially symplectic terms, *every graded symplectic automorphism of  $M$  induces an exact self-equivalence of it*. Moreover, an isotopy of graded symplectic automorphisms gives rise to an equivalence between the induced functors. Thus one has a canonical map

$$\pi_0(\text{Symp}^{\text{gr}}(M, \beta)) \longrightarrow \text{Auteq}(D^b \text{Fuk}(M, \beta)).$$

Now we return to Kontsevich's conjecture. Assume that  $(M, \beta)$  has a mirror partner  $X$  such that (1.3) holds. Then there is an isomorphism between  $\text{Auteq}(D^b \text{Fuk}(M, \beta))$  and  $\text{Auteq}(D^b(X))$ . Combining this with the canonical map above yields a homomorphism

$$\mu : \pi_0(\text{Symp}^{\text{gr}}(M, \beta)) \longrightarrow \text{Auteq}(D^b(X)). \quad (1.4)$$

Somewhat oversimplifying, and ignoring the conjectural nature of the whole discussion, one can say that *symplectic automorphisms of  $M$  induce self-equivalences of the derived category of coherent sheaves on its mirror partner*. Note that the map  $\mu$  depends on the choice of equivalence (1.3) and hence is not canonical.

*Remark 1.5*

One can see rather easily that the central element  $[1] \in \text{Symp}^{\text{gr}}(M, \beta)$  induces the

translation functor on  $D^b \text{Fuk}(M, \beta)$  and hence on  $D^b(X)$ . Passing to the quotient yields a map

$$\bar{\mu} : \pi_0(\text{Symp}(M, \beta)) \longrightarrow \text{Auteq}(D^b(X))/(\text{translations}).$$

This simplified version may be more convenient for those readers who are unfamiliar with the “graded symplectic” machinery.

### 1.3. Dehn twists and mirror symmetry

A Lagrangian sphere in  $(M, \beta)$  is a Lagrangian submanifold  $S \subset M$  which is diffeomorphic to  $S^n$ . One can associate to any Lagrangian sphere a symplectic automorphism  $\tau_S$  called the generalised Dehn twist along  $S$ , which is defined by a local construction in a neighbourhood of  $S$ . (See [48] or [49] for details; strictly speaking,  $\tau_S$  depends on various choices, but since the induced functor on  $D^b \text{Fuk}(M, \beta)$  is expected to be independent of these choices, we ignore them in our discussion.) These maps are symplectic versions of the classical Picard-Lefschetz transformations. In particular, their action on  $H_*(M)$  is given by

$$(\tau_S)_*(x) = \begin{cases} x - ([S] \cdot x)[S] & \text{if } \dim(x) = n, \\ x & \text{otherwise,} \end{cases} \quad (1.5)$$

where  $\cdot$  is the intersection pairing twisted by a dimension-dependent sign. As explained in [49, Section 5b],  $\tau_S$  has a preferred lift  $\tilde{\tau}_S \in \text{Symp}^{\text{gr}}(M, \beta)$  to the graded symplectic automorphism group. Suppose that  $(M, \beta)$  has a mirror partner  $X$  such that Kontsevich’s conjecture (1.3) holds. Choose some lift  $\tilde{S}$  of  $S$  to a graded Lagrangian submanifold, and let  $\mathcal{E} \in D^b(X)$  be the object that corresponds to  $\tilde{S}$ . Then

$$\text{Hom}_{D^b(X)}^*(\mathcal{E}, \mathcal{E}) \cong \text{Hom}_{D^b \text{Fuk}(M, \beta)}^*(\tilde{S}, \tilde{S}) = HF^*(\tilde{S}, \tilde{S}) \otimes_{\mathbb{R}} \mathbb{C}. \quad (1.6)$$

The Floer cohomology group  $HF^*(\tilde{S}, \tilde{S})$  is isomorphic to the ordinary cohomology  $H^*(S; \mathbb{R})$ ; this is not true for general Lagrangian submanifolds, but it holds for spheres. Therefore  $\mathcal{E}$  must be a spherical object. (This motivated our use of the word spherical.) A natural conjecture about the homomorphism  $\mu$  introduced in Section 1.2 is that

$$\mu([\tilde{\tau}_S]) = [T_{\mathcal{E}}], \quad (1.7)$$

where  $T_{\mathcal{E}}$  is the twist functor as defined in Section 1.1. Roughly speaking, the idea is that *twist functors and generalised Dehn twists correspond to each other under mirror symmetry*. At present this is merely a guess, which can be motivated, for example, by comparing (1.2) with (1.5). But supposing that one wanted to actually prove this claim, how should one go about it? The first step would be to observe that for any

$\mathcal{F} \in D^b(X)$  there is an exact triangle

$$\begin{array}{ccccc} \mathrm{Hom}^*(\mathcal{E}, \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & T_{\mathcal{E}}(\mathcal{F}) \\ & & \searrow & \swarrow & \\ & & & & [1] \end{array}$$

Here  $\mathrm{Hom}^*(\mathcal{E}, \mathcal{F})$  is the graded group of homs in  $D^b(X)$ ,  $\mathrm{Hom}^*(\mathcal{E}, \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{E}$  is the corresponding direct sum of shifted copies of  $\mathcal{E}$ , and the first arrow is the evaluation map. This exact triangle determines  $T_{\mathcal{E}}(\mathcal{F})$  up to isomorphism; moreover, it does so in purely abstract terms, which involves only the triangulated structure of the category  $D^b(X)$ . Hence, *if there was an analogous abstract description of the action of  $\tilde{\tau}_S$  on  $D^b \mathrm{Fuk}(M, \beta)$ , one could indeed prove (1.7).* (This is slightly imprecise since it ignores a technical problem about nonfunctoriality of cones in triangulated categories.) The first step towards such a description will be provided in [50]. Note that here, for the first time in our discussion of mirror symmetry, we have made essential use of the triangulated structure of the categories.

Now define an  $(A_m)$ -configuration of Lagrangian spheres in  $(M, \beta)$  to be a collection of  $m \geq 1$  pairwise transverse Lagrangian spheres  $S_1, \dots, S_m \subset M$  such that

$$|S_i \cap S_j| = \begin{cases} 1 & |i - j| = 1, \\ 0 & |i - j| \geq 2. \end{cases} \quad (1.8)$$

Such configurations occur in Kähler manifolds that can be degenerated into a manifold with a singular point of type  $(A_m)$  (see [48] or [28]). The generalised Dehn twists  $\tilde{\tau}_{S_1}, \dots, \tilde{\tau}_{S_m}$  along such spheres satisfy the braid relations up to isotopy inside  $\mathrm{Symp}^{\mathrm{gr}}(M, \beta)$ . For  $n = 2$ , and ignoring the issue of gradings, this was proved in [48, appendix]; the argument given there can be adapted to yield the slightly sharper and more general statement that we are using here. Thus, by mapping the standard generators of the braid group to the classes  $[\tilde{\tau}_{S_i}]$ , one obtains a homomorphism from  $B_{m+1}$  to  $\pi_0(\mathrm{Symp}^{\mathrm{gr}}(M, \beta))$ . It is a difficult open question in symplectic geometry whether this homomorphism, which we denote by  $\rho'$ , is injective (see [28] for a partial result). We now see what mirror symmetry has to say about this.

Assume, as before, that Kontsevich's conjecture holds, and let  $\mathcal{E}_1, \dots, \mathcal{E}_m \in D^b(X)$  be the objects corresponding to some choice of gradings  $\tilde{S}_1, \dots, \tilde{S}_m$  for the  $S_j$ . We already know that each  $\mathcal{E}_i$  is a spherical object. An argument similar to (1.6) but based on (1.8) shows that  $\mathcal{E}_1, \dots, \mathcal{E}_m$  is an  $(A_m)$ -configuration in  $D^b(X)$  in the sense of Definition 1.1. Hence the twist functors  $T_{\mathcal{E}_i}$  satisfy the braid relations (see Theorem 1.2) and generate a homomorphism  $\rho$  from  $B_{m+1}$  to  $\mathrm{Auteq}(D^b(X))$ . Assuming that our claim (1.7) is true, one would have a commutative diagram

$$\begin{array}{ccc}
 B_{m+1} & \xrightarrow{\rho'} & \pi_0(\mathrm{Symp}^{\mathrm{gr}}(M, \beta)) \\
 & \searrow \rho & \downarrow \mu \\
 & & \mathrm{Auteq}(D^b(X))
 \end{array}$$

Since  $\dim_{\mathbb{C}} X = n \geq 2$ , we have Theorem 1.3, which says that  $\rho$  is injective. In the diagram above this would clearly imply that  $\rho'$  is injective. Thus we are led to a conjectural answer “based on mirror symmetry” to a question of symplectic geometry.

#### CONJECTURE 1.6

*Let  $(M, \beta)$  be a compact symplectic manifold with  $\pi_1(M)$  trivial and  $c_1(M, \beta) = 0$ , and let  $(S_1, \dots, S_m)$  be an  $(A_m)$ -configuration of Lagrangian spheres in  $M$  for some  $m \geq 1$ . Then the map  $\rho' : B_{m+1} \rightarrow \pi_0(\mathrm{Symp}^{\mathrm{gr}}(M, \beta))$  generated by the generalised Dehn twists  $\tilde{\tau}_{S_1}, \dots, \tilde{\tau}_{S_m}$  is injective.*

#### 1.4. A survey of the paper

Section 2 introduces spherical objects and twist functors for derived categories of fairly general abelian categories. The main result is the construction of braid group actions, Theorem 2.17.

Section 3.1 explains how the abstract framework specializes in the case of coherent sheaves; this recovers the definitions presented in Section 1.1 and, in particular, in Theorem 1.2. More generally, in Section 3.2 we consider singular and quasi-projective varieties, as well as equivariant sheaves on varieties with a finite group action; the latter give rise to what are probably the simplest examples of our theory. In Section 3.3 we present a more systematic way of producing spherical objects, which exploits their relations with the (much studied) exceptional objects on Fano varieties. Elliptic curves provide the only example where both sides of the homological mirror conjecture are completely understood; in Section 3.4 the group of symplectic automorphisms and the group of autoequivalences of the derived category are compared in an explicit way. Section 3.5 gives more explicit examples on  $K3$  surfaces. Finally, Section 3.6 puts our results in the framework of mirror symmetry for singularities; this was the underlying motivation for much of this work.

Section 4 contains the proof of the faithfulness result, Theorem 2.18. For the benefit of the reader, we provide here an outline of the argument, in the more concrete situation stated as Theorem 1.3; the general case does not differ greatly from this. Let  $\mathcal{E}_1, \dots, \mathcal{E}_m$  be a collection of spherical objects in  $D^b(X)$ , and set  $\mathcal{E} = \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_m$ . For a fixed  $m$  and dimension  $n$  of the variety, the endomorphism algebra

$$\mathrm{End}^*(\mathcal{E}) = \bigoplus_{i,j} \mathrm{Hom}^*(\mathcal{E}_i, \mathcal{E}_j)$$

is essentially the same for all  $(\mathcal{E}_1, \dots, \mathcal{E}_m)$ . More precisely, after possibly shifting each  $\mathcal{E}_i$  by some amount, one can achieve that  $\text{End}^*(\mathcal{E})$  is equal to a specific graded algebra  $A_{m,n}$  depending only on  $m, n$ . Moreover, one can define a functor  $\Psi^{\text{naive}} : D^b(X) \rightarrow A_{m,n}\text{-mod}$  into the category of graded modules over  $A_{m,n}$  by mapping  $\mathcal{F}$  to  $\text{Hom}^*(\mathcal{E}, \mathcal{F})$ . By a result of [28], the derived category  $D^b(A_{m,n}\text{-mod})$  carries a weak action of  $B_{m+1}$ , and one might hope that  $\Psi^{\text{naive}}$  would be compatible with these two actions. A little thought shows that this cannot possibly be true;  $A_{m,n}\text{-mod}$  can be embedded into  $D^b(A_{m,n}\text{-mod})$  as the subcategory of complexes of length 1, but the braid group action on  $D^b(A_{m,n}\text{-mod})$  does not preserve this subcategory. Nevertheless, the basic idea can be saved, at the cost of introducing some more homological algebra.

Take resolutions  $\mathcal{E}'_i$  of  $\mathcal{E}_i$  by bounded below complexes of injective quasi-coherent sheaves. Then one can define a differential graded algebra  $\text{end}(\mathcal{E}')$  whose cohomology is  $\text{End}^*(\mathcal{E})$ . The quasi-isomorphism type of  $\text{end}(\mathcal{E}')$  is independent of the choice of resolutions, so it is an invariant of the  $(A_m)$ -configuration  $\mathcal{E}_1, \dots, \mathcal{E}_m$ . As before, there is an exact functor  $\text{hom}(\mathcal{E}', -) : D^b(X) \rightarrow D(\text{end}(\mathcal{E}'))$  to the derived category of differential graded modules over  $\text{end}(\mathcal{E}')$ . Now assume that  $\text{end}(\mathcal{E}')$  is formal, that is to say, quasi-isomorphic to the differential graded algebra  $\mathcal{A}_{m,n} = (A_{m,n}, 0)$  with zero differential. Quasi-isomorphic differential graded algebras have equivalent derived categories, so what one obtains is an exact functor

$$\Psi : D^b(X) \longrightarrow D(\mathcal{A}_{m,n}),$$

which replaces the earlier  $\Psi^{\text{naive}}$ . A slight modification of the arguments of [28] shows that there is a weak braid group action on  $D(\mathcal{A}_{m,n})$ ; moreover, in contrast to the situation above, the functor  $\Psi$  now relates the two braid group actions. Still borrowing from [28], one can interpret the braid group action on  $D(\mathcal{A}_{m,n})$  in terms of low-dimensional topology and, more precisely, geometric intersection numbers of curves on a punctured disc. This leads to a strong faithfulness result for it, which through the functor  $\Psi$  implies the faithfulness of the original braid group action on  $D^b(X)$ .

This argument by reduction to the known case of  $D(\mathcal{A}_{m,n})$  hinges on the formality of  $\text{end}(\mathcal{E}')$ . We prove that this assumption is always satisfied when  $n \geq 2$ . This has nothing to do with the geometric origin of  $\text{end}(\mathcal{E}')$ ; in fact, what we show is that  $A_{m,n}$  is intrinsically formal for  $n \geq 2$ , which means that all differential graded algebras with this cohomology are formal. There is a general theory of intrinsically formal algebras, which goes back to the work of S. Halperin and J. Stasheff [19] in the commutative case; the Hochschild cohomology computation necessary to apply this theory to  $A_{m,n}$  is the final step in the proof of Theorem 2.18.

## 2. Braid group actions

### 2.1. Generalities

Fix a field  $k$ ; all categories are assumed to be  $k$ -linear. If  $\mathfrak{S}$  is an abelian category,  $\text{Ch}(\mathfrak{S})$  is the category of cochain complexes in  $\mathfrak{S}$  and cochain maps,  $K(\mathfrak{S})$  is the corresponding homotopy category (morphisms are homotopy classes of cochain maps), and  $D(\mathfrak{S})$  is the derived category. The variants involving bounded (below, above, or on both sides) complexes are denoted by  $\text{Ch}^+(\mathfrak{S})$ ,  $\text{Ch}^-(\mathfrak{S})$ ,  $\text{Ch}^b(\mathfrak{S})$ , and so on. Let  $(C_j, \delta_j)_{j \in \mathbb{Z}}$  be a cochain complex of objects and morphisms in  $\text{Ch}(\mathfrak{S})$ , that is to say,  $C_j \in \text{Ch}(\mathfrak{S})$  and  $\delta_j \in \text{Hom}_{\text{Ch}(\mathfrak{S})}(C_j, C_{j+1})$  satisfying  $\delta_{j+1}\delta_j = 0$ . Such a complex is exactly the same as a bicomplex in  $\mathfrak{S}$ . In this case we write  $\{\cdots \rightarrow C_{-1} \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots\}$  for the associated total complex, obtained by collapsing the bigrading; this is a single object in  $\text{Ch}(\mathfrak{S})$ . The same notation is applied to bicomplexes of objects of  $\text{Ch}(\mathfrak{S})$  (which are triple complexes in  $\mathfrak{S}$ ).

For  $C, D \in \text{Ch}(\mathfrak{S})$ , let  $\text{hom}(C, D)$  be the standard cochain complex of  $k$ -vector spaces whose cohomology is  $H^i \text{hom}(C, D) = \text{Hom}_{K(\mathfrak{S})}^i(C, D)$ ; that is,  $\text{hom}^i(C, D) = \prod_{j \in \mathbb{Z}} \text{Hom}_{\mathfrak{S}}(C^j, D^{j+i})$  with  $d_{\text{hom}(C, D)}^i(\phi) = d_D \phi - (-1)^i \phi d_C$ . Now suppose that  $\mathfrak{S}$  contains infinite direct sums and products. Given an object  $C \in \text{Ch}(\mathfrak{S})$  and a cochain complex  $b$  of  $k$ -vector spaces, one can form the tensor product  $b \otimes C$  and the complex of linear maps  $\text{lin}(b, C)$ , both of which are again objects of  $\text{Ch}(\mathfrak{S})$ . They are defined by choosing a basis of  $b$  and taking a corresponding direct sum (for  $b \otimes C$ ) or product (for  $\text{lin}(b, C)$ ) of shifted copies of  $C$ , with a differential that combines  $d_b$  and  $d_C$ . The outcome is independent of the chosen basis up to canonical isomorphism. The definition of  $b \otimes C$  is clear, but for  $\text{lin}(b, C)$  there are two possible choices of signs. Ours is fixed to fit in with an evaluation map  $b \otimes \text{lin}(b, C) \rightarrow C$ . To clarify the issue, we now spell out the definition. Take a homogeneous basis  $(x_i)_{i \in I}$  of the total space  $b$ , and write  $d_b(x_i) = \sum_j z_{ji} x_j$ . Then  $\text{lin}^q(b, C) = \prod_{i \in I} C_i^q$ , where  $C_i$  is a copy of  $C$  shifted by  $\deg(x_i)$ . The differential  $d^q : \text{lin}^q(b, C) \rightarrow \text{lin}^{q+1}(b, C)$  has components  $d_{ji}^q : C_i^q \rightarrow C_j^{q+1}$ , which are given by

$$d_{ji}^q = \begin{cases} (-1)^{\deg(x_i)} d_C & i = j, \\ (-1)^{\deg(x_i)} z_{ij} \cdot \text{id}_C & \deg(x_i) = \deg(x_j) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

One can verify that the map  $b \otimes \text{lin}(b, C) \rightarrow C$ ,  $x_j \otimes (c_i)_{i \in I} \mapsto c_j$ , is indeed a morphism in  $\text{Ch}(\mathfrak{S})$ . Moreover, there are canonical monomorphic cochain maps

$$\begin{aligned}
b \otimes \mathrm{hom}(D, C) &\longrightarrow \mathrm{hom}(D, b \otimes C), \\
\mathrm{hom}(D, C) \otimes b &\longrightarrow \mathrm{hom}(\mathrm{lin}(b, D), C), \\
\mathrm{hom}(B, \mathrm{lin}(b, C)) \otimes D &\longrightarrow \mathrm{lin}(b, \mathrm{hom}(B, C) \otimes D),
\end{aligned} \tag{2.1}$$

where  $b$  is as before and  $B, C, D \in \mathrm{Ch}(\mathfrak{S})$ . These maps are isomorphisms if  $b$  is finite-dimensional, and they are quasi-isomorphisms if  $b$  has finite-dimensional cohomology.

From now on  $\mathfrak{S}$  is an abelian category and  $\mathfrak{S}' \subset \mathfrak{S}$  a full subcategory such that the following conditions hold:

- (C1)  $\mathfrak{S}'$  is a Serre subcategory of  $\mathfrak{S}$  (this means that any subobject and quotient object of an object in  $\mathfrak{S}'$  lies again in  $\mathfrak{S}'$  and that  $\mathfrak{S}'$  is closed under extension);
- (C2)  $\mathfrak{S}$  contains infinite direct sums and products;
- (C3)  $\mathfrak{S}$  has enough injectives, and any direct sum of injectives is again injective (this is *not* a trivial consequence of the definition of an injective object);
- (C4) for any epimorphism  $f : A \twoheadrightarrow A'$  with  $A \in \mathfrak{S}$  and  $A' \in \mathfrak{S}'$ , there is a  $B' \in \mathfrak{S}'$  and a  $g : B' \rightarrow A$  such that  $fg$  is an epimorphism (because  $\mathfrak{S}'$  is a Serre subcategory,  $g$  may be taken to be mono):

$$\begin{array}{ccc}
& & A \\
& \nearrow g & \downarrow f \\
B' & \dashrightarrow & A'
\end{array}$$

#### LEMMA 2.1

Let  $X$  be a noetherian scheme over  $k$ , and let  $\mathfrak{S} = \mathrm{Qco}(X)$ ,  $\mathfrak{S}' = \mathrm{Coh}(X)$  be the categories of quasi-coherent, respectively, coherent sheaves. Then properties (C1)–(C4) are satisfied.

#### Proof

(C1) and (C2) are obvious.  $\mathfrak{S}$  has enough injectives by [20, Chapter II, Theorem 7.18]. Moreover,  $X$  is locally noetherian, which implies that direct sums of injectives are again injective (see [20, p. 121] and the references quoted therein). This proves (C3). Finally, we need to verify that a diagram as in (C4) with  $A$  quasi-coherent and  $A'$  coherent can be completed with a coherent sheaf  $B'$ . Such a  $B'$  certainly exists locally, and, replacing it by its image in  $A$  (which is also coherent), we may extend it to be a coherent subsheaf on all of  $X$  (see [17, Chapter I, Theorem 6.9.7]). Since  $X$  is quasi-compact, repeating this a finite number of times and taking the union yields a  $B'$  whose map to  $A'$  is globally surjective.  $\square$

As indicated by this example, our main interest is in  $D^b(\mathfrak{S}')$ . However, we find it

convenient to replace all complexes by injective resolutions. These resolutions may exist only in  $\mathfrak{S}$ , and they are not necessarily bounded. The precise category we want to work with is defined as follows.

*Definition 2.2*

$\mathfrak{K} \subset K^+(\mathfrak{S})$  is the full subcategory whose objects are those bounded below cochain complexes  $C$  of  $\mathfrak{S}$ -injectives which satisfy  $H^i(C) \in \mathfrak{S}'$  for all  $i$ , and  $H^i(C) = 0$  for  $i \gg 0$ .

We now prove, in several steps, that  $\mathfrak{K}$  is equivalent to  $D^b(\mathfrak{S}')$ . First, let  $\mathfrak{D} \subset D^+(\mathfrak{S})$  be the full subcategory of objects whose cohomology has the same properties as in Definition 2.2. The assumption that  $\mathfrak{S}$  has enough injectives implies that the obvious functor  $\mathfrak{K} \rightarrow \mathfrak{D}$  is an equivalence. Now let  $\text{Ch}_{\mathfrak{S}'}^b(\mathfrak{S})$  be the category of bounded cochain complexes in  $\mathfrak{S}$  whose cohomology objects lie in  $\mathfrak{S}'$ , and let  $D_{\mathfrak{S}'}^b(\mathfrak{S})$  be the corresponding full subcategory of  $D^b(\mathfrak{S})$ . It is a standard result (proved by truncating cochain complexes) that the obvious functor  $D_{\mathfrak{S}'}^b(\mathfrak{S}) \rightarrow \mathfrak{D}$  is an equivalence. The final step (and the only nontrivial one) is to relate  $D_{\mathfrak{S}'}^b(\mathfrak{S})$  and  $D^b(\mathfrak{S}')$ .

*LEMMA 2.3*

*For any  $C \in \text{Ch}_{\mathfrak{S}'}^b(\mathfrak{S})$ , there is an  $E \in \text{Ch}^b(\mathfrak{S}')$  and a monomorphic cochain map  $\iota : E \rightarrow C$  which is a quasi-isomorphism.*

*Proof*

Recall that, as an abelian category,  $\mathfrak{S}$  has fibre products. The fibre product of two maps  $f_1 : A_1 \rightarrow A$ ,  $f_2 : A_2 \rightarrow A$  is the kernel of  $f_1 \oplus 0 - 0 \oplus f_2 : A_1 \oplus A_2 \rightarrow A$ . If  $f_1$  is mono (thought of as an inclusion), we write  $f_2^{-1}(A_1)$  for the fibre product, and if both  $f_1$  and  $f_2$  are mono, we write  $A_1 \cap A_2$ . In the latter case, one can also define the sum  $A_1 + A_2$  as the image (kernel of the map to the cokernel) of  $f_1 \oplus f_2$ .

Let  $N$  be the largest integer such that  $C^N \neq 0$ . Set  $E^n = 0$  for all  $n > N$ . For  $n \leq N$ , define  $E^n \subset C^n$  (for brevity, we write the monomorphisms as inclusions) inductively as follows. By invoking (C4), one finds subobjects  $F^n, G^n \subset C^n$  which lie in  $\mathfrak{S}'$  and complete the diagrams

$$\begin{array}{ccc} & (d_C^n)^{-1}(E^{n+1}) & \\ \nearrow & \downarrow d_C^n & \nearrow \\ F^n & \dashrightarrow E^{n+1} \cap \text{im } d_C^n & \end{array} \quad \text{and} \quad \begin{array}{ccc} & \ker d_C^n & \\ \nearrow & \downarrow & \nearrow \\ G^n & \dashrightarrow H^n(C) & \end{array}$$

Set  $E^n = F^n + G^n$  (this is again in  $\mathfrak{S}'$ ), and define  $d_E^n = d_C^n \mid E^n$ . Since  $E^n$  is a subobject of  $C^n$  for any  $n$ ,  $E$  is a bounded complex. Consider the obvious map

$j^n : \ker d_E^n = E^n \cap \ker d_C^n \rightarrow H^n(C)$ . The definition of  $G^n$  implies that  $j^n$  is an epimorphism, and the definition of  $F^{n-1}$  yields  $\ker j^n = E^n \cap \operatorname{im} d_C^{n-1} = \operatorname{im} d_E^{n-1}$ . It follows that the inclusion induces an isomorphism  $H^*(E) \cong H^*(C)$ .  $\square$

From Lemma 2.3 it now follows by standard homological algebra (see [14, Proposition III.2.10]) that the obvious functor  $D^b(\mathfrak{S}') \rightarrow D_{\mathfrak{S}'}^b(\mathfrak{S})$  is an equivalence of categories. Combining this with the remarks made above, one gets the following proposition.

**PROPOSITION 2.4**

*There is an exact equivalence (canonical up to natural isomorphism)  $D^b(\mathfrak{S}') \cong \mathfrak{K}$ .*

## 2.2. Twist functors and spherical objects

### Definition 2.5

Let  $E \in \mathfrak{K}$  be an object satisfying the following finiteness conditions:

- (K1)  $E$  is a bounded complex;
- (K2) for any  $F \in \mathfrak{K}$ , both  $\operatorname{Hom}_{\mathfrak{K}}^*(E, F)$  and  $\operatorname{Hom}_{\mathfrak{K}}^*(F, E)$  have finite (total) dimension over  $k$ .

Then we define the *twist functor*  $T_E : \mathfrak{K} \rightarrow \mathfrak{K}$  by

$$T_E(F) = \{ \operatorname{hom}(E, F) \otimes E \xrightarrow{\operatorname{ev}} F \}. \quad (2.2)$$

This expression requires some explanation;  $\operatorname{ev}$  is the obvious evaluation map. The grading is such that if one ignores the differential,  $T_E(F) = F \oplus (\operatorname{hom}(E, F) \otimes E)[1]$ . In other words,  $T_E(F)$  is the cone of  $\operatorname{ev}$ . Since  $E$  is bounded and  $F$  is bounded below,  $\operatorname{hom}(E, F)$  is again bounded below. Hence  $\operatorname{hom}(E, F) \otimes E$  is a bounded below complex of injectives in  $\mathfrak{S}$ . (This uses property (C3) of  $\mathfrak{S}$ .) Its cohomology  $H^*(\operatorname{hom}(E, F) \otimes E)$  is isomorphic to  $\operatorname{Hom}_{\mathfrak{K}}^*(E, F) \otimes H^*(E)$  (because  $\operatorname{hom}(E, F)$  is quasi-isomorphic to  $\operatorname{Hom}_{\mathfrak{K}}^*(E, F)$ , which is finite-dimensional) and so is bounded, and the finiteness conditions imply that each cohomology group lies in  $\mathfrak{S}'$ . Therefore  $\operatorname{hom}(E, F) \otimes E$  lies in  $\mathfrak{K}$ , and the same holds for  $T_E(F)$ . The functoriality of  $T_E$  is obvious, and one sees easily that it is an exact functor. Actually, for any  $F, G \in \mathfrak{K}$  there is a canonical map of complexes  $(T_E)_* : \operatorname{hom}(F, G) \rightarrow \operatorname{hom}(T_E(F), T_E(G))$ . In fancy language, this means that  $T_E$  is functorial on the differential graded category that underlies  $\mathfrak{K}$ .

**PROPOSITION 2.6**

*If two objects  $E_1, E_2 \in \mathfrak{K}$  satisfying (K1), (K2) are isomorphic, the corresponding functors  $T_{E_1}, T_{E_2}$  are isomorphic.*

*Proof*

Take cones of the rows of the following commutative diagram:

$$\begin{array}{ccc}
 \mathrm{hom}(E_1, F) \otimes E_1 & \longrightarrow & F \\
 \uparrow & & \parallel \\
 \mathrm{hom}(E_2, F) \otimes E_1 & \longrightarrow & F \\
 \downarrow & & \parallel \\
 \mathrm{hom}(E_2, F) \otimes E_2 & \longrightarrow & F
 \end{array}$$

Here the vertical arrows are induced by a quasi-isomorphism of complexes  $E_1 \rightarrow E_2$ .  $\square$

Note also that  $T_{E[j]}$  is isomorphic to  $T_E$  for any  $j \in \mathbb{Z}$ .

*Definition 2.7*

For an object  $E$  as in Definition 2.5, we define the *dual twist* functor  $T'_E : \mathfrak{K} \rightarrow \mathfrak{K}$  by  $T'_E(F) = \{\mathrm{ev}' : F \rightarrow \mathrm{lin}(\mathrm{hom}(F, E), E)\}$ .

Here the grading is such that  $F$  lies in degree zero;  $\mathrm{ev}'$  is again some kind of evaluation map. To write it down explicitly, choose a homogeneous basis  $(\psi_i)$  of  $\mathrm{hom}(F, E)$ . Then  $\mathrm{lin}^q(\mathrm{hom}(F, E), E) = \prod_i E_i^q$ , where  $E_i$  is a copy of  $E[\mathrm{deg}(\psi_i)]$ , and the  $i$ th component of  $\mathrm{ev}'$  is simply  $\psi_i$  itself.  $T'_E$  is again an exact functor from  $\mathfrak{K}$  to itself.

LEMMA 2.8

$T'_E$  is left adjoint to  $T_E$ .

*Proof*

Using the maps from (2.1) and condition (K2), one constructs a chain of natural (in  $F, G \in \mathfrak{K}$ ) quasi-isomorphisms

$$\begin{aligned}
 \mathrm{hom}(F, T_E(G)) &= \{ \mathrm{hom}(F, \mathrm{hom}(E, G) \otimes E) \longrightarrow \mathrm{hom}(F, G) \} \\
 &\longleftarrow \{ \mathrm{hom}(E, G) \otimes \mathrm{hom}(F, E) \longrightarrow \mathrm{hom}(F, G) \} \\
 &\longrightarrow \{ \mathrm{hom}(\mathrm{lin}(\mathrm{hom}(F, E), E), G) \longrightarrow \mathrm{hom}(F, G) \} \\
 &= \mathrm{hom}(T'_E(F), G).
 \end{aligned}$$

Here the chain map  $\mathrm{hom}(E, G) \otimes \mathrm{hom}(F, E) \rightarrow \mathrm{hom}(F, G)$  is just composition. The reader may easily check that the required diagrams commute. Taking  $H^0$  on both sides yields a natural isomorphism  $\mathrm{Hom}_{\mathfrak{K}}(F, T_E(G)) \cong \mathrm{Hom}_{\mathfrak{K}}(T'_E(F), G)$ .  $\square$

*Definition 2.9*

An object  $E \in \mathfrak{K}$  is called *n-spherical* for some  $n > 0$  if it satisfies (K1), (K2) of Definition 2.5 and, in addition,

(K3)  $\text{Hom}_{\mathfrak{K}}^i(E, E)$  is equal to  $k$  for  $i = 0, n$  and zero in all other degrees;

(K4) the composition  $\text{Hom}_{\mathfrak{K}}^j(F, E) \times \text{Hom}_{\mathfrak{K}}^{n-j}(E, F) \rightarrow \text{Hom}_{\mathfrak{K}}^n(E, E) \cong k$  is a nondegenerate pairing for all  $F \in \mathfrak{K}$ ,  $j \in \mathbb{Z}$ .

One can also define zero-spherical objects; these are objects  $E$  for which  $\text{Hom}_{\mathfrak{K}}^*(E, E)$  is 2-dimensional and concentrated in degree zero, and such that the pairings  $\text{Hom}_{\mathfrak{K}}^j(E, F) \times \text{Hom}_{\mathfrak{K}}^{-j}(F, E) \rightarrow \text{Hom}_{\mathfrak{K}}^0(E, E)/k \cdot \text{id}_E$  are nondegenerate. (This means in particular that  $\text{Hom}_{\mathfrak{K}}^0(E, E)$  is isomorphic to  $k[t]/t^2$  as a  $k$ -algebra.) We do not pursue this further; the interested reader can easily verify that the proof of the next proposition extends to this case.

*PROPOSITION 2.10*

If  $E$  is *n-spherical* for some  $n > 0$ , both  $T'_E T_E$  and  $T_E T'_E$  are naturally isomorphic to the identity functor  $\text{Id}_{\mathfrak{K}}$ . In particular,  $T_E$  is an exact self-equivalence of  $\mathfrak{K}$ .

*Proof*<sup>1</sup>

$T_E T'_E(F)$  is a total complex

$$\left\{ \begin{array}{ccc} \text{hom}(E, F) \otimes E & \xrightarrow{\delta} & \text{hom}(E, \text{lin}(\text{hom}(F, E), E)) \otimes E \\ \alpha \downarrow & & \downarrow \gamma \\ F & \xrightarrow{\beta} & \text{lin}(\text{hom}(F, E), E) \end{array} \right\} \quad (2.3)$$

Here  $\alpha = \text{ev}$ ,  $\beta = \text{ev}'$ ,  $\gamma$  is a map induced by  $\text{ev}$ , and  $\delta$  is a map induced by  $\text{ev}'$ . We need to know a little more about  $\delta$ . By the very definition of  $\text{ev}'$  by duality,  $\delta$ 's induced map on cohomology

$$\text{Hom}_{\mathfrak{K}}^*(E, F) \otimes H^*(E) \longrightarrow \text{Hom}_{\mathfrak{K}}^*(F, E)^{\vee} \otimes \text{Hom}_{\mathfrak{K}}^*(E, E) \otimes H^*(E) \quad (2.4)$$

is dual to the composition  $\text{Hom}_{\mathfrak{K}}^*(F, E) \otimes \text{Hom}_{\mathfrak{K}}^*(E, F) \rightarrow \text{Hom}_{\mathfrak{K}}^*(E, E)$ , tensored with the identity map on  $H^*(E)$ . This second pairing is, by the conditions (K3) and (K4) on  $E$ , perfect when we divide  $\text{Hom}_{\mathfrak{K}}^*(E, E)$  by its degree zero piece ( $k \cdot \text{id}_E$ ). Thus the following modification of the map (2.4),

$$\text{Hom}_{\mathfrak{K}}^*(E, F) \otimes H^*(E) \longrightarrow \text{Hom}_{\mathfrak{K}}^*(F, E)^{\vee} \otimes \frac{\text{Hom}_{\mathfrak{K}}^*(E, E)}{k \cdot \text{id}_E} \otimes H^*(E), \quad (2.5)$$

is an isomorphism.

<sup>1</sup> We thank one of the referees for simplifying our original proof of this result.

We now enlarge slightly the object in the top right-hand corner of (2.3) to produce a new, quasi-isomorphic, complex  $Q_E(F)$ . The last equation in (2.1) gives a map  $\text{hom}(E, \text{lin}(\text{hom}(F, E), E)) \otimes E \hookrightarrow \text{lin}(\text{hom}(F, E), \text{hom}(E, E) \otimes E)$ . Since  $\text{hom}(F, E)$  has finite-dimensional cohomology, this is a quasi-isomorphism. The map  $\bar{\gamma} : \text{lin}(\text{hom}(F, E), \text{hom}(E, E) \otimes E) \rightarrow \text{lin}(\text{hom}(F, E), E)$  induced by  $\text{ev} : \text{hom}(E, E) \otimes E \rightarrow E$  naturally extends  $\gamma$ . In fact,  $\bar{\gamma}$  splits canonically. Define the map  $\phi : \text{lin}(\text{hom}(F, E), E) \rightarrow \text{lin}(\text{hom}(F, E), \text{hom}(E, E) \otimes E)$  induced by  $k \mapsto \text{hom}(E, E), 1 \mapsto \text{id}_E$ . From the definition of  $\bar{\gamma}$ , it follows that  $\bar{\gamma} \circ \phi = \text{id}$ . This splitting gives a way of embedding an acyclic complex  $\{\text{id} : \text{lin}(\text{hom}(F, E), E) \rightarrow \text{lin}(\text{hom}(F, E), E)\}$  into our enlarged complex  $Q_E(F)$ ; the cokernel is

$$\left\{ \text{hom}(E, F) \otimes E \xrightarrow{\delta \oplus \alpha} \text{lin} \left( \text{hom}(F, E), \frac{\text{hom}(E, E)}{k \cdot \text{id}_E} \otimes E \right) \oplus F \right\}.$$

There is an obvious map of  $F$  to this, and everything we have done is functorial in  $F$ . Thus to prove that  $T_E T'_E \cong \text{Id}_{\mathfrak{K}}$  we are left with showing that the cokernel

$$\left\{ \text{hom}(E, F) \otimes E \xrightarrow{\delta} \text{lin} \left( \text{hom}(F, E), \frac{\text{hom}(E, E)}{k \cdot \text{id}_E} \otimes E \right) \right\} \quad (2.6)$$

is acyclic; that is, the arrow induces an isomorphism on cohomology. But passing to cohomology yields (2.5), which we already noted was an isomorphism.

The proof that  $T'_E T_E \cong \text{Id}_{\mathfrak{K}}$  is similar; one passes from  $T'_E T_E(F)$  to a quasi-isomorphic but slightly smaller object, which then has a natural map to  $F$ . The details are almost the same as before, and we leave them to the reader.  $\square$

### 2.3. The braid relations

#### LEMMA 2.11

Let  $E_1, E_2 \in \mathfrak{K}$  be two objects such that  $E_1$  satisfies conditions (K1), (K2) of Definition 2.5 and  $E_2$  is  $n$ -spherical for some  $n > 0$ . Then  $T_{E_2}(E_1)$  also satisfies (K1), (K2), and  $T_{E_2} T_{E_1}$  is naturally isomorphic to  $T_{T_{E_2}(E_1)} T_{E_2}$ .

#### Proof

Since  $E_1$  and  $E_2$  are bounded complexes, so are  $\text{hom}(E_1, E_2)$  and  $T_{E_2}(E_1)$ . Lemma 2.8 says that  $\text{Hom}_{\mathfrak{K}}^*(F, T_{E_2}(E_1)) \cong \text{Hom}_{\mathfrak{K}}^*(T'_{E_2}(F), E_1)$ . By assumption on  $E_1$ , this implies that  $\text{Hom}_{\mathfrak{K}}^*(F, T_{E_2}(E_1))$  is always finite-dimensional. Similarly, the finite-dimensionality of  $\text{Hom}_{\mathfrak{K}}^*(T_{E_2}(E_1), F)$  follows from Proposition 2.10 since  $\text{Hom}_{\mathfrak{K}}^*(T_{E_2}(E_1), F) \cong \text{Hom}_{\mathfrak{K}}^*(E_1, T'_{E_2}(F))$ . We have now proved that  $T_{E_2}(E_1)$  satisfies (K1), (K2).  $T_{E_2} T_{E_1}(F)$  is a total complex

$$\left\{ \begin{array}{ccc} \mathrm{hom}(E_2, \mathrm{hom}(E_1, F) \otimes E_1) \otimes E_2 & \longrightarrow & \mathrm{hom}(E_2, F) \otimes E_2 \\ \downarrow & & \downarrow \\ \mathrm{hom}(E_1, F) \otimes E_1 & \longrightarrow & F \end{array} \right\}$$

where all arrows are evaluation maps or are induced by them. We argue as in the proof of Proposition 2.10. Using (2.1), one sees that the object in the top left-hand corner can be replaced by the smaller quasi-isomorphic  $\mathrm{hom}(E_1, F) \otimes \mathrm{hom}(E_2, E_1) \otimes E_2$ . More precisely, this modification defines another functor  $R_{E_1, E_2}$  on  $\mathfrak{K}$  which is naturally isomorphic to  $T_{E_2} T_{E_1}$ . One can rewrite the definition of this functor as

$$R_{E_1, E_2}(F) = \{ \mathrm{hom}(E_1, F) \otimes T_{E_2}(E_1) \longrightarrow T_{E_2}(F) \}. \quad (2.7)$$

The arrow in (2.7) is obtained by composing

$$\mathrm{hom}(E_1, F) \otimes T_{E_2}(E_1) \xrightarrow{(T_{E_2})_* \otimes \mathrm{id}} \mathrm{hom}(T_{E_2}(E_1), T_{E_2}(F)) \otimes T_{E_2}(E_1)$$

with the evaluation map  $\mathrm{ev} : \mathrm{hom}(T_{E_2}(E_1), T_{E_2}(F)) \otimes T_{E_2}(E_1) \rightarrow T_{E_2}(F)$ . This means that one has a natural map from  $R_{E_1, E_2}(F)$  to  $T_{T_{E_2}(E_1)} T_{E_2}(F)$ , given by  $(T_{E_2})_* \otimes \mathrm{id}$  on the first component and by the identity on the second one. Since  $(T_{E_2})_*$  is a quasi-isomorphism by Proposition 2.10, this natural transformation is an isomorphism.  $\square$

#### PROPOSITION 2.12

Let  $E_1, E_2$  be as before, and assume in addition that  $\mathrm{Hom}_{\mathfrak{K}}^i(E_2, E_1) = 0$  for all  $i$ . Then  $T_{E_1} T_{E_2} \cong T_{E_2} T_{E_1}$ .

##### *Proof*

The assumption implies that  $T_{E_2}(E_1)$  is isomorphic to  $E_1$ . Hence the result follows directly from Lemma 2.11 and Proposition 2.6. (One can also prove this by a direct computation, without using Lemma 2.11.)  $\square$

#### PROPOSITION 2.13

Let  $E_1, E_2 \in \mathfrak{K}$  be two  $n$ -spherical objects for some  $n > 0$ . Assume that the total dimension of  $\mathrm{Hom}_{\mathfrak{K}}^*(E_2, E_1)$  is 1. Then  $T_{E_1} T_{E_2} T_{E_1} \cong T_{E_2} T_{E_1} T_{E_2}$ .

##### *Proof*

Since the twists are not affected by shifting, we may assume that  $\mathrm{Hom}_{\mathfrak{K}}^i(E_2, E_1)$  is 1-dimensional for  $i = 0$  and zero in all other dimensions. A simple computation shows that

$$T_{E_2}(E_1) \cong \{E_2 \xrightarrow{g} E_1\}, \quad T'_{E_1}(E_2) \cong \{E_2 \xrightarrow{h} E_1\},$$

where  $g$  and  $h$  are nonzero maps. As  $\text{Hom}_{\mathfrak{K}}(E_2, E_1)$  is 1-dimensional, it follows that  $T_{E_2}(E_1)$  and  $T'_{E_1}(E_2)$  are isomorphic up to the shift  $[1]$ . By applying Lemma 2.11 and Proposition 2.6, one finds that

$$T_{E_1}T_{E_2}T_{E_1} \cong T_{E_1}T_{T_{E_2}(E_1)}T_{E_2} \cong T_{E_1}T_{T'_{E_1}(E_2)}T_{E_2}.$$

On the other hand, applying Lemma 2.11 to  $T'_{E_1}(E_2)$  and  $E_1$  and using Proposition 2.10 shows that  $T_{E_1}T_{T'_{E_1}(E_2)}T_{E_2} \cong T_{E_2}T_{E_1}T_{E_2}$ .  $\square$

We now carry over the results obtained so far to the derived category  $D^b(\mathfrak{S}')$ . Throughout the rest of this section,  $\text{Hom}$  always means  $\text{Hom}_{D^b(\mathfrak{S}')}$ .

*Definition 2.14*

An object  $E \in D^b(\mathfrak{S}')$  is called  $n$ -spherical for some  $n > 0$  if it has the following properties:

- (S1)  $E$  has a finite resolution by injective objects in  $\mathfrak{S}$ ;
- (S2)  $\text{Hom}^*(E, F), \text{Hom}^*(F, E)$  are finite-dimensional for any  $F \in D^b(\mathfrak{S}')$ ;
- (S3)  $\text{Hom}^i(E, E)$  is equal to  $k$  for  $i = 0, n$  and zero in all other dimensions;
- (S4) the composition map  $\text{Hom}^i(F, E) \times \text{Hom}^{n-i}(E, F) \rightarrow \text{Hom}^n(E, E) \cong k$  is a nondegenerate pairing for all  $F \in \mathfrak{K}$  and  $i \in \mathbb{Z}$ .

Clearly, if  $E$  is such an object, any finite resolution by  $\mathfrak{S}$ -injectives is an  $n$ -spherical object of  $\mathfrak{K}$  in the sense of Definition 2.9. Using such a resolution and the equivalence of categories from Proposition 2.4, one can associate to  $E$  a twist functor  $T_E$  which, by Proposition 2.10, is an exact self-equivalence of  $D^b(\mathfrak{S}')$ . This is independent of the choice of resolution up to isomorphism, thanks to Proposition 2.6.

LEMMA 2.15

*In the presence of (S2) and (S3), condition (S4) is equivalent to the following apparently weaker one.*

- (S4') *There is an isomorphism  $\text{Hom}(E, F) \cong \text{Hom}^n(F, E)^\vee$  which is natural in  $F \in D^b(\mathfrak{S}')$ .*

*Proof*

The proof is by a “general nonsense” argument. Take any natural isomorphism as in (S4'), and let  $q_F : \text{Hom}(E, F) \times \text{Hom}^n(F, E) \rightarrow k$  be the family of nondegenerate pairings induced by it. Because of the naturality, these pairings satisfy  $q_F(\phi, \psi) = q_F(\phi \circ \text{id}_E, \psi) = q_E(\text{id}_E, \phi \circ \psi)$ . Since the pairings are all nondegenerate,  $q_E(\text{id}_E, -) : \text{Hom}^n(E, E) \rightarrow k$  is nonzero and hence by (S3) an isomorphism. We have therefore shown that

$$\mathrm{Hom}(E, F) \times \mathrm{Hom}^n(F, E) \xrightarrow{\text{composition}} \mathrm{Hom}^n(E, E) \cong k$$

is a nondegenerate pairing for any  $F$ , which is the special case  $i = 0$  of (S4). The other cases follow by replacing  $F$  by  $F[i]$ .  $\square$

LEMMA 2.16

Let  $X$  be a noetherian scheme over  $k$ , and  $\mathfrak{S} = Qco(X)$ ,  $\mathfrak{S}' = \mathrm{Coh}(X)$ . Then condition (S4) or (S4') for an object of  $D^b(\mathfrak{S}')$  implies condition (S1).

*Proof*

Let  $\mathcal{E}$  be an object of  $D^b(\mathfrak{S}')$ , and let  $\mathcal{F} \in \mathfrak{S}'$  be a coherent sheaf. Since  $\mathcal{E}$  is bounded and  $\mathcal{F}$  has a bounded below resolution by  $\mathfrak{S}$ -injectives, one has  $\mathrm{Hom}^i(\mathcal{E}, \mathcal{F}) = 0$  for  $i \ll 0$ . Using (S4) or (S4'), it follows that  $\mathrm{Hom}^i(\mathcal{F}, \mathcal{E}) = 0$  for  $i \gg 0$ , and [20, Proposition II.7.20] completes the proof.  $\square$

Now define an  $(A_m)$ -configuration ( $m > 0$ ) of  $n$ -spherical objects in  $D^b(\mathfrak{S}')$  to be a collection  $(E_1, \dots, E_m)$  of such objects, satisfying

$$\dim_k \mathrm{Hom}_{D^b(\mathfrak{S}')}^*(E_i, E_j) = \begin{cases} 1 & |i - j| = 1, \\ 0 & |i - j| \geq 2. \end{cases} \quad (2.8)$$

THEOREM 2.17

Let  $(E_1, \dots, E_m)$  be an  $(A_m)$ -configuration of  $n$ -spherical objects in  $D^b(\mathfrak{S}')$ . Then the twists  $T_{E_1}, \dots, T_{E_m}$  satisfy the relations of the braid group  $B_{m+1}$  up to graded natural isomorphism. That is to say, they generate a homomorphism  $\rho : B_{m+1} \rightarrow \mathrm{Auteq}(D^b(\mathfrak{S}'))$ .

This follows immediately from the corresponding results for  $\mathfrak{K}$  (see Propositions 2.12 and 2.13). One minor point remains to be cleared up. Theorem 2.17 states that the braid relations hold up to graded natural isomorphism, whereas before we have only talked about ordinary natural isomorphism. But one can easily see that all the natural isomorphisms we have constructed are graded ones, essentially because everything commutes with the translation functors. We can now state the main result of this paper.

THEOREM 2.18

Suppose that  $n \geq 2$ . Then the homomorphism  $\rho$  defined in Theorem 2.17 is injective, and in fact the following stronger statement holds. If  $g \in B_{m+1}$  is not the identity element, then  $\rho(g)(E_i) \not\cong E_i$  for some  $i \in \{1, \dots, m\}$ .

### 3. Applications

#### 3.1. Smooth projective varieties

We now return to the concrete situation of derived categories of coherent sheaves. The main theme is the use of suitable duality theorems to simplify condition (S4') in the definition of spherical objects. Throughout, all varieties are over an algebraically closed field  $k$ .

For the moment we consider only smooth projective varieties  $X$ , of dimension  $n$ . Let us recall some facts about duality on such varieties. Serre duality says that for any  $\mathcal{G} \in D^b(X)$  the composition

$$\mathrm{Hom}^{n-*}(\mathcal{G}, \omega_X) \otimes \mathrm{Hom}^*(\mathcal{O}, \mathcal{G}) \longrightarrow \mathrm{Hom}^n(\mathcal{O}, \omega_X) = H^n(\omega_X) \cong k \quad (3.1)$$

is a nondegenerate pairing. (The classical form is for a single sheaf  $\mathcal{G}$ ; the general case can be derived from this by induction on the length of the complex, using the five lemma.) Now let  $\mathcal{E}$  be a bounded complex of locally free coherent sheaves on  $X$ . For all  $\mathcal{G}_1, \mathcal{G}_2 \in D^+(X)$  there is a natural isomorphism

$$\mathrm{Hom}^*(\mathcal{G}_1 \otimes \mathcal{E}, \mathcal{G}_2) \cong \mathrm{Hom}^*(\mathcal{G}_1, \mathcal{G}_2 \otimes \mathcal{E}^\vee). \quad (3.2)$$

This is proved using a resolution  $\mathcal{G}'_2$  of  $\mathcal{G}_2$  by injective quasi-coherent sheaves; the point is that  $\mathcal{G}'_2 \otimes \mathcal{E}^\vee$  is an injective resolution of  $\mathcal{G}_2 \otimes \mathcal{E}^\vee$  (see [20, Proposition 7.17]). Setting  $\mathcal{G} = \mathcal{F} \otimes \mathcal{E}^\vee$  in (3.1) for some  $\mathcal{F} \in D^b(X)$  and using (3.2) shows that there is an isomorphism, natural in  $\mathcal{F}$ ,

$$\mathrm{Hom}^*(\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}^{n-*}(\mathcal{F}, \mathcal{E} \otimes \omega_X)^\vee. \quad (3.3)$$

Again, by (3.2) and the standard finiteness theorems,  $\mathrm{Hom}^*(\mathcal{E}, \mathcal{F}) \cong \mathbb{H}^*(\mathcal{E}^\vee \otimes \mathcal{F})$  is of finite total dimension; hence so is  $\mathrm{Hom}^*(\mathcal{F}, \mathcal{E})$  by (3.3). Finally, because of the existence of finite locally free resolutions, everything we have said holds for an arbitrary  $\mathcal{E} \in D^b(X)$ .

#### LEMMA 3.1

*An object  $\mathcal{E} \in D^b(X)$  is spherical, in the sense of Definition 2.14, if and only if it satisfies the following two conditions:  $\mathrm{Hom}^j(\mathcal{E}, \mathcal{E})$  is 1-dimensional for  $j = 0, n$  and zero for all other  $j$ ; and  $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ .*

#### *Proof*

It follows from (3.3) and the previous discussion that the conditions are sufficient. Conversely, assume that  $\mathcal{E}$  is a spherical object. Then property (S4) and (3.3) imply that the functors  $\mathrm{Hom}(-, \mathcal{E} \otimes \omega_X)$  and  $\mathrm{Hom}(-, \mathcal{E})$  are isomorphic. By a general nonsense argument,  $\mathcal{E}$  must be isomorphic to  $\mathcal{E} \otimes \omega_X$ .  $\square$

This shows that the abstract definition of spherical objects specializes to the one in Section 1.1. We now prove the corresponding statement for twist functors.

LEMMA 3.2

Let  $\mathcal{E} \in D^b(X)$  be a bounded complex of locally free sheaves, which is a spherical object. Then the twist functor  $T_{\mathcal{E}}$  as defined in Section 2.3 is isomorphic to the FMT by  $\mathcal{P} = \text{Cone}(\eta : \mathcal{E}^\vee \boxtimes \mathcal{E} \rightarrow \mathcal{O}_\Delta)$ .

*Proof*

Let  $\mathcal{E}' \in \mathfrak{K}$  be a bounded resolution of  $\mathcal{E}$  by injective quasi-coherent sheaves. Let  $T : \mathfrak{K} \rightarrow D^+(X)$  be the functor that sends  $\mathcal{F}$  to  $\text{Cone}(\text{ev} : \text{hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F})$ . We show that the diagram

$$\begin{array}{ccc} \mathfrak{K} & \xrightarrow{T_{\mathcal{E}'}} & \mathfrak{K} \\ \downarrow & \searrow T & \downarrow \\ D^b(X) & \xrightarrow{\Phi_{\mathcal{P}}} & D^+(X) \end{array} \quad (3.4)$$

where the unlabeled arrows are the equivalence  $\mathfrak{K} \cong D^b(X)$  and its inclusion into  $D^+(X)$  commutes up to isomorphism. Since  $T_{\mathcal{E}}$  is defined using the twist functor  $T_{\mathcal{E}'}$  on  $\mathfrak{K}$  and  $\mathfrak{K} \cong D^b(X)$ , the commutativity of (3.4) implies that  $\Phi_{\mathcal{P}} \cong T_{\mathcal{E}}$ . Take an object  $\mathcal{F} \in D^b(X)$  and a resolution  $\mathcal{F}' \in \mathfrak{K}$ . Then

$$\begin{aligned} \Phi_{\mathcal{P}}(\mathcal{F}) &= \mathbf{R}\pi_{2*} \{ \pi_1^* \mathcal{F} \otimes \pi_1^* \mathcal{E}^\vee \otimes \pi_2^* \mathcal{E} \rightarrow \mathcal{O}_\Delta \otimes \pi_1^* \mathcal{F} \} \\ &\cong \mathbf{R}\pi_{2*} \{ \pi_1^* \mathcal{F}' \otimes \pi_1^* \mathcal{E}^\vee \otimes \pi_2^* \mathcal{E} \rightarrow \mathcal{O}_\Delta \otimes \pi_1^* \mathcal{F}' \} \\ &\cong \pi_{2*} \{ \pi_1^* \mathcal{H}om(\mathcal{E}, \mathcal{F}') \otimes \pi_2^* \mathcal{E} \rightarrow \mathcal{O}_\Delta \otimes \pi_1^* \mathcal{F}' \} \\ &\cong \{ \text{hom}(\mathcal{E}, \mathcal{F}') \otimes \mathcal{E} \rightarrow \mathcal{F}' \} = T(\mathcal{F}'), \end{aligned}$$

where the arrow in the last line is evaluation. This provides a natural isomorphism that makes the left lower triangle in (3.4) commute. To deal with the other triangle, set up a diagram as in the proof of Proposition 2.6.  $\square$

Example 3.3

Let  $X$  be a variety that is Calabi-Yau in the strict sense; that is to say,  $\omega_X \cong \mathcal{O}$  and  $H^i(X, \mathcal{O}) = 0$  for  $0 < i < n$ . Then any invertible sheaf on  $X$  is spherical. For the trivial sheaf, the twist  $T_{\mathcal{O}}$  is the FMT given by the object on  $X \times X$  which is the ideal sheaf of the diagonal shifted by [1]. This is what Mukai [38] calls the *reflection functor*.

## LEMMA 3.4

Let  $Y \subset X$  be a connected subscheme that is a local complete intersection, with (locally free) normal sheaf  $\nu = (\mathcal{I}_Y / \mathcal{I}_Y^2)^\vee$ . Assume that  $\omega_X|_Y$  is trivial, and assume that  $H^i(Y, \Lambda^j \nu) = 0$  for all  $0 < i + j < n$ . Then  $\mathcal{O}_Y \in D^b(X)$  is a spherical object.

## Proof

Denote by  $\iota$  the embedding of  $Y$  into  $X$ . The local Koszul resolution of  $\iota_* \mathcal{O}_Y$  gives the well-known formula for the sheaf Exts,  $\mathcal{E}xt^j(\iota_* \mathcal{O}_Y, \iota_* \mathcal{O}_Y) \cong \iota_*(\Lambda^j \nu)$ . The assumptions and the spectral sequence  $H^i(\mathcal{E}xt^j) \Rightarrow \text{Ext}^{i+j}$  (i.e., the hypercohomology spectral sequence of  $\mathbb{H}(\mathbf{R}\mathcal{H}om) = \text{Ext}$ ) give  $\text{Ext}^r(\iota_* \mathcal{O}_Y, \iota_* \mathcal{O}_Y) = 0$  for  $0 < r < n$ . We have  $\text{Hom}(\iota_* \mathcal{O}_Y, \iota_* \mathcal{O}_Y) \cong k$ , and hence  $\text{Ext}^n(\iota_* \mathcal{O}_Y, \iota_* \mathcal{O}_Y) \cong k$  by duality.  $\square$

## Example 3.5

Let  $X$  be a surface. Then any smooth rational curve  $C \subset X$  with  $C \cdot C = -2$  satisfies the conditions of Lemma 3.4. Now take a chain  $C_1, \dots, C_m$  of such curves such that  $C_i \cap C_j = \emptyset$  for  $|i - j| \geq 2$ , and  $C_i \cdot C_{i+1} = 1$  for  $i = 1, \dots, m - 1$ . Then  $(\mathcal{O}_{C_1}, \dots, \mathcal{O}_{C_m})$  is an  $(A_m)$ -configuration of spherical objects.

## Remark 3.6

As far as Lemma 3.1 is concerned, one could remove the assumption of smoothness and work with arbitrary projective varieties  $X$ . Serre duality must then be replaced by the general duality theorem (see [20, Theorem III.11.1]) applied to the projection  $\pi : X \rightarrow \text{Spec } k$ . This yields a natural isomorphism, for  $\mathcal{G} \in D^-(X)$ ,

$$\text{Ext}^{n-*}(\mathcal{G}, \omega_X) \cong \text{Ext}^*(\mathcal{O}_X, \mathcal{G})^\vee,$$

where now  $\omega_X = \pi^!(\mathcal{O}_{\text{Spec } k}) \in D^+(X)$  is the dualizing complex. With this replacing (3.1), one can essentially repeat the same discussion as in the smooth case, leading to an analogue of Lemma 3.1. The only difference is that the condition that  $\mathcal{E}$  has a finite locally free resolution must be included as an assumption. We do not pursue this further, for lack of a really relevant application.

## 3.2. Two generalisations

We now look at smooth quasi-projective varieties. Rather than aiming at a comprehensive characterisation of spherical objects, we just carry over Lemma 3.4, which provides one important source of examples.

Let  $X$  be a smooth quasi-projective variety of dimension  $n$ , and let  $Y \subset X$  be a complete subscheme of codimension  $c$ ;  $\iota$  denotes the embedding  $Y \hookrightarrow X$ . Complete  $X$  to a projective variety  $\bar{X}$ . Then  $Y \subset \bar{X}$  is closed, and  $\bar{X}$  is smooth, so  $\iota_* \mathcal{O}_Y$  has a finite locally free resolution; thus we may use Serre duality (see [20, Theorem III.11.1])

on  $\bar{X}$ , and the methods of (3.3), to conclude that

$$\mathrm{Hom}(\iota_* \mathcal{O}_Y, \mathcal{F}) \cong \mathrm{Hom}^n(\mathcal{F}, \iota_* \mathcal{O}_Y \otimes \omega_X)^\vee$$

on  $X$ . By continuing as in the projective case and using the same spectral sequence as in Lemma 3.4, one obtains the following result.<sup>2</sup>

LEMMA 3.7

*Assume that  $H^i(Y, \Lambda^j \nu) = 0$  for all  $0 < i + j < n$ , and assume that  $\iota^* \omega_X$  is trivial. Then  $\iota_* \mathcal{O}_Y$  is a spherical object in  $D^b(X)$ .*

One can now, for example, extend Example 3.5 to quasi-projective surfaces. For subschemes of codimension 1, we later provide a stronger result, Proposition 3.15, which can be used to construct more interesting spherical objects.

The other generalisation at which we want to look is technically much simpler. Let  $X$  be a smooth  $n$ -dimensional projective variety over  $k$  with an action of a finite group  $G$ . We assume that  $\mathrm{char}(k) = 0$ ; this implies the complete reducibility of  $G$ -representations, which we use in an essential way. Let  $Qco_G(X)$  be the category whose objects are  $G$ -equivariant quasi-coherent sheaves and whose morphisms are the  $G$ -equivariant sheaf homomorphisms. One can write

$$\mathrm{Hom}_{Qco_G(X)}(\mathcal{E}_1, \mathcal{E}_2) = \mathrm{Hom}_{Qco(X)}(\mathcal{E}_1, \mathcal{E}_2)^G \quad (3.5)$$

with respect to the obvious  $G$ -action on  $\mathrm{Hom}_{Qco(X)}(\mathcal{E}_1, \mathcal{E}_2)$ . Because taking the invariant part of a  $G$ -vector space is an exact functor, it follows that a  $G$ -sheaf is injective in  $Qco_G(X)$  if and only if it is injective in  $Qco(X)$ . This can be used to show that  $Qco_G(X)$  has enough injectives and also that  $\mathfrak{S} = Qco_G(X)$  and its Serre subcategory  $\mathfrak{S}' = \mathrm{Coh}_G(X)$  of coherent  $G$ -sheaves satisfy conditions (C1)–(C4) from Section 2.1. As a further application, one derives a formula similar to (3.5) for the derived category

$$\mathrm{Hom}_{D^+(Qco_G(X))}(\mathcal{F}_1, \mathcal{F}_2) = \mathrm{Hom}_{D^+(Qco(X))}(\mathcal{F}_1, \mathcal{F}_2)^G \quad (3.6)$$

for all  $\mathcal{F}_1, \mathcal{F}_2 \in D^+(Qco_G(X))$ . This allows one to carry over the usual finiteness results for coherent sheaf cohomology, as well as Serre duality, to the equivariant context. The same argument as in the nonequivariant case now leads to the following lemma.

LEMMA 3.8

*An object  $\mathcal{E}$  in the derived category  $D_G^b(X) = D^b(\mathrm{Coh}_G(X))$  of coherent equivariant sheaves is spherical if and only if the following two conditions are satisfied:*

<sup>2</sup> We thank one of the referees for simplifying our original version of this proof.

$\mathrm{Hom}_{D_G^b(X)}^j(\mathcal{E}, \mathcal{E})$  is 1-dimensional for  $j = 0, n$  and zero in other degrees; and  $\mathcal{E} \otimes \omega_X$  is equivariantly isomorphic to  $\mathcal{E}$ .

Finally, one can combine the two generalisations and obtain an equivariant version of Lemma 3.7. This is useful in examples that arise in connection with the McKay correspondence. We concentrate on the simplest of these examples, which also happens to be particularly relevant for our purpose. Consider the diagonal subgroup  $G \cong \mathbb{Z}/(m+1)$  of  $\mathrm{SL}_2(k)$ . Write  $R$  for its regular representation, and write  $V_1, \dots, V_m$  for its (nontrivial) irreducible representations. Let  $X$  be a smooth quasi-projective surface with a complex symplectic form, carrying an effective symplectic action of  $G$ . Choose a fixed point  $x \in X$ ; the tangent space  $T_x X$  must necessarily be isomorphic to  $R$  as a  $G$ -vector space. For  $i = 1, \dots, m$ , set  $\mathcal{E}_i = \mathcal{O}_x \otimes V_i \in \mathrm{Coh}_G(X)$ . The Koszul resolution of  $\mathcal{O}_x$ , together with (3.6), shows that

$$\mathrm{Hom}_{D_G^b(X)}^r(\mathcal{E}_i, \mathcal{E}_j) \cong (\Lambda^r R \otimes V_i^\vee \otimes V_j)^G.$$

This implies that each  $\mathcal{E}_i$  is a spherical object and that these objects form an  $(A_m)$ -configuration, so that we obtain a braid group action on  $D_G^b(X)$ .

### Example 3.9

In particular, we have a braid group action on the equivariant derived category of coherent sheaves over  $\mathbb{A}^2$ , with respect to the obvious linear action of  $G$ . (This is probably the simplest example of a braid group action on a category in the present paper.)

Let  $\pi : Z \rightarrow X/G$  be the minimal resolution. This is again a quasi-projective surface with a symplectic form; it can be constructed as a Hilbert scheme of  $G$ -clusters on  $X$ . The irreducible components of  $\pi^{-1}(x)$  are smooth rational curves  $C_1, \dots, C_m$  which are arranged as in Example 3.5, so that their structure sheaves generate a braid group action on  $D^b(Z)$ . A theorem of M. Kapranov and E. Vasserot [24] provides an equivalence of categories

$$D_G^b(X) \cong D^b(Z), \tag{3.7}$$

which takes  $\mathcal{E}_j$  to  $\mathcal{O}_{C_j}$  up to tensoring by a line bundle (see [24, p. 7]). This means that the braid group actions on the two categories essentially correspond to each other. Adding the trivial 1-dimensional representation  $V_0$ , and the corresponding equivariant sheaf  $\mathcal{E}_0 = \mathcal{O}_x = \mathcal{O}_x \otimes V_0$ , extends the action on  $D_G^b(X)$  to an action of the affine braid group, except for  $m = 1$ . Interestingly, the cyclic symmetry between  $V_0, V_1, \dots, V_m$  is not immediately visible on  $D^b(Z)$ ; the equivalence (3.7) takes  $\mathcal{E}_0$  to the structure sheaf of the whole exceptional divisor  $\pi^{-1}(x)$ . Finally, everything we have said carries

over to the other finite subgroups of  $SL(2, k)$  with the obvious modifications. The Dynkin diagram of type  $(A_m)$  which occurs implicitly several times in our discussion must be replaced by those of type D/E, and one obtains actions of the corresponding (affine) generalised braid groups.

A recent deep theorem of Bridgeland, A. King, and M. Reid [8] extends the equivalence (3.7) to certain higher-dimensional quotient singularities. We consider only one very concrete case.

*Example 3.10*

Let  $X$  be the Fermat quintic in  $\mathbb{P}^4$  with the diagonal action of  $G = (\mathbb{Z}/5)^3$  familiar from mirror symmetry. The fixed-point set  $X^H$  of the subgroup  $H = (\mathbb{Z}/5)^2 \times 1$  consists of a single  $G$ -orbit  $\Sigma$ , whose structure sheaf is a spherical object in  $D_G^b(X)$ . By considering other subgroups of the same kind, one finds a total of ten spherical objects with no Homs between any two of them. Now let  $\pi : Z \rightarrow X/G$  be the crepant resolution given by the Hilbert scheme of  $G$ -clusters. Then  $D_G^b(X) \cong D^b(Z)$  by [8], so that one gets corresponding spherical objects on  $Z$ . Because of the nature of the equivalence, the object corresponding to  $\mathcal{O}_\Sigma$  must be supported on the exceptional set  $p^{-1}(\Sigma)$  of the resolution. We have not determined its precise nature, but this is clearly related to Proposition 3.15 and Examples 3.20.

### 3.3. Spherical and exceptional objects

The reader familiar with the theory of exceptional sheaves (see [47]), or with certain aspects of tilting theory in representation theory, will have noticed a similarity between our twist functors and mutations of exceptional objects. (See also [6], and note that their “elliptical exceptional” objects are examples of 1-spherical objects.) The braid group also occurs in the mutation context, but there it acts on collections of exceptional objects in a triangulated category instead of on the category itself. The relation of the two kinds of braid group actions is not at all clear. Here we content ourselves with two observations, the first of which is motivated by examples in [31].

*Definition 3.11*

Let  $X, Y$  be smooth projective varieties, with  $\omega_X$  trivial. A morphism  $f : X \rightarrow Y$  (of codimension  $c = \dim X - \dim Y$ ) is called *simple* if there is an exact triangle

$$\mathcal{O}_Y \longrightarrow \mathbf{R}f_* \mathcal{O}_X \longrightarrow \omega_Y[-c].$$

In most applications,  $Y$  would be Fano because one could then use the wealth of known results about exceptional sheaves on such varieties. However, the general theory does not require this assumption on  $Y$ .

LEMMA 3.12

Suppose that  $c > 0$  and

$$R^i f_* \mathcal{O}_X \cong \begin{cases} \mathcal{O}_Y & \text{for } i = 0, \\ 0 & \text{for } 0 < i < c, \\ \omega_Y & \text{for } i = c. \end{cases}$$

Then  $f$  is simple.

*Proof*

$\mathbf{R}f_* \mathcal{O}_X$  is a complex of sheaves whose cohomology is nonzero only in two degrees; a general argument, valid in any derived category, shows that there is an exact triangle  $R^0 f_* \mathcal{O}_X \rightarrow \mathbf{R}f_* \mathcal{O}_X \rightarrow (R^c f_* \mathcal{O}_X)[-c]$ .  $\square$

PROPOSITION 3.13

Suppose that  $f$  is simple, and suppose that  $\mathcal{F} \in D^b(Y)$  is an exceptional object, in the sense that  $\text{Hom}(\mathcal{F}, \mathcal{F}) \cong k$  and  $\text{Hom}^i(\mathcal{F}, \mathcal{F}) = 0$  for all  $i \neq 0$ . Then  $\mathbf{L}f^* \mathcal{F} \in D^b(X)$  is a spherical object.

*Proof*

One can easily show, using, for example, a finite locally free resolution of  $\mathcal{F}$  and a finite injective quasi-coherent resolution of  $\mathcal{O}_X$ , that  $\mathbf{R}f_* \mathbf{L}f^* \mathcal{F} \cong \mathcal{F} \otimes^{\mathbf{L}} (\mathbf{R}f_* \mathcal{O}_X)$ . Hence, by tensoring the triangle in Definition 3.11 with  $\mathcal{F}$ , one obtains another exact triangle  $\mathcal{F} \rightarrow \mathbf{R}f_* \mathbf{L}f^* \mathcal{F} \rightarrow \mathcal{F} \otimes \omega_Y[-c]$ . This yields a long exact sequence

$$\cdots \longrightarrow \text{Hom}^*(\mathcal{F}, \mathcal{F}) \longrightarrow \text{Hom}^*(\mathcal{F}, \mathbf{R}f_* \mathbf{L}f^* \mathcal{F}) \longrightarrow \text{Hom}^{*-c}(\mathcal{F}, \mathcal{F} \otimes \omega_Y) \longrightarrow \cdots.$$

The second and third group are  $\text{Hom}^*(\mathbf{L}f^* \mathcal{F}, \mathbf{L}f^* \mathcal{F})$  and  $\text{Hom}^{\dim X - *}( \mathcal{F}, \mathcal{F})$  by, respectively, adjointness and Serre duality. From the assumption that  $\mathcal{F}$  is exceptional, one now immediately obtains the desired result.  $\square$

Examples 3.14

(a) (This assumes  $\text{char}(k) = 0$ .) Consider a Calabi-Yau  $X$  with a fibration  $f : X \rightarrow Y$  over a variety  $Y$  such that the generic fibres are elliptic curves or  $K3$  surfaces. Clearly  $f_* \mathcal{O}_X \cong \mathcal{O}_Y$ ; relative Serre duality shows that  $R^c f_* \mathcal{O}_X \cong \omega_Y$ ; and in the  $K3$ -fibred case one has also  $R^1 f_* \mathcal{O}_X = 0$ . Hence  $f$  is simple.

(b) (This assumes  $\text{char}(k) \neq 2$ .) Let  $f : X \rightarrow Y$  be a 2-fold covering branched over a double anticanonical divisor. One can use the  $\mathbb{Z}/2$ -action on  $X$  to split  $f_* \mathcal{O}_X$  into two direct summands, which are isomorphic to  $\mathcal{O}_Y$  and  $\omega_Y$ , respectively; this implies that  $f$  is simple. An example, already considered in [31], is a  $K3$  double covering of  $\mathbb{P}^2$  branched over a sextic. Another example, which is slightly degenerate but still

works, is the unbranched covering map from a  $K3$  surface to an Enriques surface.

(c) Examples with  $c = -1$  come from taking  $X$  to be a smooth anticanonical divisor in  $Y$  and taking  $f$  to be the embedding. Then  $\mathbf{R}f_*\mathcal{O}_X = f_*\mathcal{O}_X \cong \{\omega_Y \rightarrow \mathcal{O}_Y\}$  with the map given by the section of  $\omega_Y^{-1}$  defining  $X$ . Quartic surfaces in  $\mathbb{P}^3$  are an example considered in [31].

We now describe a second connection between spherical and exceptional objects, this time using pushforwards instead of pullbacks. The result applies to quasi-projective varieties as well, but it is limited to embeddings of divisors. Let  $X \subset \mathbb{P}^N$  be a smooth quasi-projective variety, and let  $\iota : Y \hookrightarrow X$  be an embedding of a complete connected hypersurface  $Y$ . As in the parallel argument in the previous section, we work on the projective completion  $\bar{X}$  of  $X$ , in which  $Y$  is closed. By the smoothness of  $X$ , given that  $\mathcal{F} \in D^b(Y)$ ,  $\iota_*\mathcal{F}$  has a finite locally free resolution, and Serre duality on  $\bar{X}$  (see [20, Theorem III.11.1]) yields

$$\mathrm{Hom}(\iota_*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}^{\dim X}(\mathcal{G}, \iota_*\mathcal{F} \otimes \omega_X)^\vee$$

on  $X$ .

**PROPOSITION 3.15**

*Assume that  $\iota^*\omega_X$  is trivial. If  $\mathcal{F} \in D^b(Y)$  is an exceptional object with a finite locally free resolution, then  $\iota_*\mathcal{F}$  is spherical in  $D^b(X)$ .*

*Proof*

In view of the previous discussion, what remains to be done is to compute  $\mathrm{Hom}^i(\iota_*\mathcal{F}, \iota_*\mathcal{F})$ , which, by [20, Theorem III.11.1] applied to  $\iota_*$ , is isomorphic to  $\mathrm{Hom}^{i-1}(\mathcal{F}, \mathbf{L}\iota^*\iota_*\mathcal{F} \otimes \omega_Y)$ . We need the following result (which, perhaps surprisingly, need not be true without the  $\iota_*$ s).  $\square$

**LEMMA 3.16**

*We have  $\iota_*\mathbf{L}\iota^*(\iota_*\mathcal{F}) \cong \iota_*(\mathcal{F} \otimes \omega_Y^{-1})[1] \oplus \iota_*\mathcal{F}$ .*

*Proof*

Replacing  $\iota_*\mathcal{F}$  by a quasi-isomorphic complex  $\mathcal{F}'$  of locally free sheaves, the left-hand side of the above equation is  $\iota_*(\mathcal{F}'|_Y) = \mathcal{F}' \otimes \mathcal{O}_Y$ , which is quasi-isomorphic to

$$\mathcal{F}' \otimes \{\mathcal{O}(-Y) \longrightarrow \mathcal{O}\} \simeq \iota_*\mathcal{F} \otimes \{\mathcal{O}(-Y) \longrightarrow \mathcal{O}\},$$

where the arrow is multiplication by the canonical section of  $\mathcal{O}(Y)$ . Since this vanishes on  $Y$ , which contains the support of  $\iota_*\mathcal{F}$ , we obtain  $\iota_*(\mathcal{F} \otimes \mathcal{O}(-Y)|_Y)[1] \oplus \iota_*\mathcal{F}$ , as required.  $\square$

By hypothesis we may assume that  $\mathcal{F}$  is a finite complex of locally free sheaves on  $Y$ , so that  $\mathcal{H}om(\mathcal{F}, \mathcal{F}) \cong \mathcal{F} \otimes \mathcal{F}^\vee$ . Thus, computing  $\mathrm{Hom}^i(\iota_* \mathcal{F}, \iota_* \mathcal{F})$  as the  $(i - 1)$ th (derived/hyper) sheaf cohomology of the complex of  $\mathcal{O}_Y$ -module sheaves  $\mathbf{L}t^* \iota_* \mathcal{F} \otimes \mathcal{F}^\vee \otimes \omega_Y$ , we may push forward to  $X$  and there use Lemma 3.16. That is, pick an injective resolution  $\mathcal{O}_Y \rightarrow I$  on  $Y$ , so that  $\mathrm{Hom}^i(\iota_* \mathcal{F}, \iota_* \mathcal{F})$  is the  $(i - 1)$ th cohomology of

$$\Gamma_Y(\mathbf{L}t^* \iota_* \mathcal{F} \otimes \mathcal{F}^\vee \otimes \omega_Y \otimes I),$$

where  $\Gamma$  is the global sections functor. Pushing forward to  $X$ , this is

$$\Gamma_X(\iota_*(\mathbf{L}t^* \iota_* \mathcal{F}) \otimes \iota_*((\mathcal{F}^\vee \otimes \omega_Y \otimes I)),$$

which by Lemma 3.16 is

$$\Gamma_X(\iota_*(\mathcal{F} \otimes \mathcal{F}^\vee \otimes I))[1] \oplus \Gamma_X(\iota_*(\mathcal{F} \otimes \mathcal{F}^\vee \otimes \omega_Y \otimes I)).$$

This may be brought back onto  $Y$  to give the  $(i - 1)$ th cohomology of  $\Gamma_Y(\mathcal{F} \otimes \mathcal{F}^\vee \otimes I)[1] \oplus \Gamma_Y(\mathcal{F} \otimes \mathcal{F}^\vee \otimes \omega_Y \otimes I)$ . This is  $\mathrm{Hom}^i(\mathcal{F}, \mathcal{F}) \oplus \mathrm{Hom}^{n-i}(\mathcal{F}, \mathcal{F})^\vee$ , where for the second term we have used Serre duality on  $Y$ . Since  $\mathcal{F}$  is exceptional, this completes the proof.  $\square$

### 3.4. Elliptic curves

The homological mirror conjecture for elliptic curves has been studied extensively by A. Polishchuk and E. Zaslow [45], [44]. (Unfortunately, their formulation of the conjecture differs somewhat from that in Section 1.2, so their results cannot be applied directly here.) Polishchuk [43] and Orlov [41], following earlier work of Mukai [37], have completely determined the automorphism group of the derived category of coherent sheaves. These are difficult results, to which we have little to add. Still, it is perhaps instructive to see how things work out in a well-understood case.

We begin with the symplectic side of the story. Let  $(M, \beta)$  be the torus  $M = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$  with its standard volume form  $\beta = ds_1 \wedge ds_2$ . Matters are slightly more complicated than in Section 1.2 because the fundamental group is nontrivial. In particular, the  $C^\infty$ -topology on  $\mathrm{Symp}(M, \beta)$  is no longer the correct one; this is due to the fact that Floer cohomology is not invariant under arbitrary isotopies, but only under Hamiltonian ones. There is a bi-invariant foliation  $\mathfrak{F}$  of codimension 2 on  $\mathrm{Symp}(M, \beta)$ , and the Hamiltonian isotopies are precisely those that are tangent to the leaves. To capture this idea one introduces a new topology, the Hamiltonian topology, on  $\mathrm{Symp}(M, \beta)$ . This is the topology generated by the leaves of  $\mathfrak{F} \mid U$ , where  $U \subset \mathrm{Symp}(M, \beta)$  runs over all  $C^\infty$ -open subsets. To avoid confusion, we write  $\mathrm{Symp}^h(M, \beta)$  whenever we have the Hamiltonian topology in mind, and we call this the Hamiltonian automorphism group; this differs from the terminology in most of the literature, where the name is reserved for what, in our terms, is the

connected component of the identity in  $\text{Symp}^h(M, \beta)$ . The difference between the two topologies becomes clear if one considers the group  $\text{Aff}(M) = M \rtimes \text{SL}(2, \mathbb{Z})$  of oriented affine diffeomorphisms of  $M$ . As a subgroup of  $\text{Symp}(M, \beta)$ , this has its Lie group topology, in which the translation subgroup  $M$  is connected. In contrast, as a subgroup of  $\text{Symp}^h(M, \beta)$  it has the discrete topology.

LEMMA 3.17

*The embedding of  $\text{Aff}(M)$  into  $\text{Symp}^h(M, \beta)$  as a discrete subgroup is a homotopy equivalence.*

The proof consists of combining the known topology of  $\text{Diff}^+(M)$ , Moser's theorem that  $\text{Symp}(M, \beta) \subset \text{Diff}^+(M)$  is a homotopy equivalence, and the flux homomorphism that describes the global structure of the foliation  $\mathfrak{F}$ . We omit the details.

Let  $\pi : \mathbb{R} \rightarrow \mathbb{RP}^1, s \mapsto [\cos(\pi s) : \sin(\pi s)]$  be the universal covering of  $\mathbb{RP}^1$ . Consider the subgroup  $\widetilde{\text{SL}}(2, \mathbb{R}) \subset \text{SL}(2, \mathbb{R}) \times \text{Diff}(\mathbb{R})$  of pairs  $(g, \tilde{g})$  such that  $\tilde{g}$  is a lift of the action of  $g$  on  $\mathbb{RP}^1$ .  $\widetilde{\text{SL}}(2, \mathbb{R})$  is a central extension of  $\text{SL}(2, \mathbb{R})$  by  $\mathbb{Z}$ . (Topologically, it consists of two copies of the universal cover.) We define a graded symplectic automorphism of  $(M, \beta)$  to be a pair

$$(\phi, \tilde{\phi}) \in \text{Symp}^h(M, \beta) \times C^\infty(M, \widetilde{\text{SL}}(2, \mathbb{R}))$$

such that  $\tilde{g}$  is a lift of  $Dg : M \rightarrow \text{SL}(2, \mathbb{R})$ ; here we have used the standard trivialisation of  $TM$ . The graded symplectic automorphisms form a group under the composition  $(\phi, \tilde{\phi})(\psi, \tilde{\psi}) = (\phi\psi, (\tilde{\phi} \circ \psi)\tilde{\psi})$ . We denote this group by  $\text{Symp}^{h, \text{gr}}(M, \beta)$ , and we equip it with the topology induced from  $\text{Symp}^h(M, \beta) \times C^\infty(M, \widetilde{\text{SL}}(2, \mathbb{R}))$ . It is a central extension of  $\text{Symp}^h(M, \beta)$  by  $\mathbb{Z}$ . One can easily verify that the definition is equivalent to that in [49], which in turn goes back to ideas of Kontsevich [30].

Even in this simplest example, the construction of the derived Fukaya category  $D^b \text{Fuk}(M, \beta)$  has not yet been carried out in detail, so we proceed on the basis of guesswork in the style of Section 1.2. The basic objects of  $D^b \text{Fuk}(M, \beta)$  are pairs  $(L, E)$  consisting of a Lagrangian submanifold and a flat unitary bundle on it. Thus, in addition to symplectic automorphisms, the category should admit another group of self-equivalences, which act on all objects  $(L, E)$  by tensoring  $E$  with some fixed flat unitary line bundle  $\xi \rightarrow M$ . The two kinds of self-equivalence should give a homomorphism

$$\gamma : G \stackrel{\text{def}}{=} M^\vee \rtimes \pi_0(\text{Symp}^{h, \text{gr}}(M, \beta)) \longrightarrow \text{Auteq}(D^b \text{Fuk}(M, \beta)), \quad (3.8)$$

where  $M^\vee = H^1(M; \mathbb{R}/\mathbb{Z})$  is the Jacobian, or a dual torus. In order to make the picture more concrete, we now write down the group  $G$  explicitly. Take the standard presentation of  $\text{SL}(2, \mathbb{Z})$  by generators  $g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ,  $g_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$  and relations

$g_1 g_2 g_1 = g_2 g_1 g_2$ ,  $(g_1 g_2)^6 = 1$ . Let  $\widetilde{\mathrm{SL}}(2, \mathbb{Z}) \subset \widetilde{\mathrm{SL}}(2, \mathbb{R})$  be the preimage of  $\mathrm{SL}(2, \mathbb{Z})$ . One can lift  $g_1, g_2$  to elements  $a_1 = (g_1, \tilde{g}_1)$  and  $a_2 = (g_2, \tilde{g}_2)$  in  $\widetilde{\mathrm{SL}}(2, \mathbb{Z})$  which satisfy  $\tilde{g}_1(1/2) = 1/4$  and  $\tilde{g}_2(1/4) = 0$ . Together with the central element  $t = (\mathrm{id}, s \mapsto s - 1)$ , these generate  $\widetilde{\mathrm{SL}}(2, \mathbb{Z})$ , and one can easily work out what the relations are:

$$\widetilde{\mathrm{SL}}(2, \mathbb{Z}) = \langle a_1, a_2, t \mid a_1 a_2 a_1 = a_2 a_1 a_2, (a_1 a_2)^6 = t^2, [a_1, t] = [a_2, t] = 1 \rangle.$$

Any element of  $(g, \tilde{g}) \in \widetilde{\mathrm{SL}}(2, \mathbb{Z})$  defines a graded symplectic automorphism of  $(M, \beta)$ ; one simply takes  $\phi = g$  and  $\tilde{\phi}$  to be the constant map with value  $\tilde{g}$ . Moreover, any translation of  $M$  has a canonical lift to a graded symplectic automorphism by taking  $\tilde{\phi}$  to be the constant map with value  $1 \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$ . These two observations together give a subgroup  $\widetilde{\mathrm{Aff}}(M) = M \rtimes \widetilde{\mathrm{SL}}(2, \mathbb{Z})$  of  $\mathrm{Symp}^{h, \mathrm{gr}}(M, \beta)$ , which fits into a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathrm{Aff}}(M) & \longrightarrow & \mathrm{Aff}(M) \longrightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Symp}^{h, \mathrm{gr}}(M, \beta) & \longrightarrow & \mathrm{Symp}^h(M, \beta) \longrightarrow 1 \end{array}$$

Hence, in view of Lemma 3.17,  $\pi_0(\mathrm{Symp}^{h, \mathrm{gr}}(M, \beta)) \cong \widetilde{\mathrm{Aff}}(M)$ . After spelling out everything, one finds that  $G$  is the semidirect product  $(\mathbb{R}/\mathbb{Z})^4 \rtimes \widetilde{\mathrm{SL}}(2, \mathbb{Z})$ , with respect to the action of  $\widetilde{\mathrm{SL}}(2, \mathbb{Z})$  on  $\mathbb{R}^4$  given by

$$a_1 \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad a_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad t \mapsto \mathrm{id}. \quad (3.9)$$

We now pass to the mirror dual side. Let  $X$  be a smooth elliptic curve over  $\mathbb{C}$ . We choose a point  $x_0 \in X$  which is the identity for the group law on  $x$ . The derived category  $D^b(X)$  has self-equivalences

$$T_{\mathcal{O}}, S \quad \text{and} \quad R_x, L_x, T_{\mathcal{O}_x} \quad (x \in X)$$

defined as follows.  $T_{\mathcal{O}}$  is the twist by  $\mathcal{O}$ , which is spherical for obvious reasons;  $S$  is the original example of an FMT,  $S = \Phi_{\mathcal{L}}$  with  $\mathcal{L} = \mathcal{O}(\Delta - \{x_0\} \times X - X \times \{x_0\})$  the Poincaré line bundle. It maps the structure sheaves of points  $\mathcal{O}_x$  to the line bundles  $\mathcal{O}(x - x_0)$ , and it was shown to be an equivalence by Mukai [37].  $R_x$  is the self-equivalence induced by the translation  $y \mapsto y + x$ ;  $L_x$  is the functor of tensoring with the degree zero line bundle  $\mathcal{O}(x - x_0)$ ; and  $T_{\mathcal{O}_x}$  is the twist along  $\mathcal{O}_x$  which is spherical by Lemma 3.4. These functors have the following properties:

$$[L_x, R_y] \cong \text{id} \quad \text{for all } x, y, \quad (3.10)$$

$$T_{\mathcal{O}_x} \text{ is isomorphic to } \mathcal{O}(x) \otimes -, \quad (3.11)$$

$$S^4 \cong [-2], \quad (3.12)$$

$$T_{\mathcal{O}_{x_0}} T_{\mathcal{O}} T_{\mathcal{O}_{x_0}} \cong T_{\mathcal{O}} T_{\mathcal{O}_{x_0}} T_{\mathcal{O}} \cong S^{-1}, \quad (3.13)$$

$$T_{\mathcal{O}_{x_0}} R_x T_{\mathcal{O}_{x_0}}^{-1} \cong R_x L_x^{-1}, \quad (3.14)$$

$$T_{\mathcal{O}_{x_0}} L_x T_{\mathcal{O}_{x_0}}^{-1} \cong L_x, \quad (3.15)$$

$$T_{\mathcal{O}} R_x T_{\mathcal{O}}^{-1} \cong R_x, \quad (3.16)$$

$$T_{\mathcal{O}} L_x T_{\mathcal{O}}^{-1} \cong R_x L_x. \quad (3.17)$$

Most of these isomorphisms are easy to prove; those that present any difficulties are (3.11), (3.12), and (3.13). The first and third of these are proved below, and the second one is a consequence of [37, Theorem 3.13(1)].

*Proof of (3.11)*

(This argument is valid for the structure sheaf of a point on any algebraic curve.) A simple computation shows that the dual in the derived sense is  $\mathcal{O}_x^\vee \cong \mathcal{O}_x[-1]$ . The formula for inverses of FMTs (for which, see, e.g., [7, Lemma 4.5]) shows that  $T_{\mathcal{O}_x}^{-1} \cong \Phi_{\mathcal{Q}}$  for some object  $\mathcal{Q}$  fitting into an exact triangle

$$\mathcal{Q} \longrightarrow \mathcal{O}_\Delta \xrightarrow{f} \mathcal{O}_{(x,x)}.$$

When following through the computation, it is not easy to keep track of the map  $f$ , but that is not really necessary. All we need to know is that  $f \neq 0$ , which is true because the converse would violate the fact that  $\Phi_{\mathcal{Q}}$  is an equivalence. Then, since any morphism  $\mathcal{O}_\Delta \rightarrow \mathcal{O}_{(x,x)}$  in the derived category is represented by a genuine map of sheaves,  $f$  must be some nonzero multiple of the obvious restriction map. It follows that  $\mathcal{Q}$  is isomorphic to the kernel of  $f$ , which is  $\mathcal{O}_\Delta \otimes \pi_1^* \mathcal{O}(-x)$ . This means that  $T_{\mathcal{O}_x}^{-1}$  is the functor of tensoring with  $\mathcal{O}(-x)$ . Passing to inverses yields the desired result.  $\square$

*Proof of (3.13)*

The equality between the first two terms follows from Theorem 2.17 because  $\mathcal{O}_{x_0}, \mathcal{O}$  form an  $(A_2)$ -configuration of spherical objects. By the standard formula for the adjoints of an FMT, the inverse of  $S$  is the FMT with  $\mathcal{L}^\vee[1]$ . By definition,  $T_{\mathcal{O}}$  is the FMT with  $\mathcal{O}(-\Delta)[1]$ . Using (3.11), it follows that  $T_{\mathcal{O}_{x_0}} T_{\mathcal{O}} T_{\mathcal{O}_{x_0}}$  is the FMT with  $\pi_1^* \mathcal{O}(x_0) \otimes \mathcal{O}(-\Delta)[1] \otimes \pi_2^* \mathcal{O}(x_0) \cong \mathcal{L}^\vee[1]$ .  $\square$

Equations (3.12) and (3.13) show that  $(T_{\mathcal{O}} T_{\mathcal{O}_{x_0}})^6 \cong [2]$ . Therefore one can define a

homomorphism

$$\widetilde{\mathrm{SL}}(2, \mathbb{Z}) \longrightarrow \mathrm{Auteq}(D^b(X))$$

by mapping the generators  $a_1, a_2, t$  to  $T_\theta, T_{\theta_{x_0}}$ , and the translation [1]; this already occurs in Mukai's paper [37], slightly disguised by the fact that he uses a different presentation of  $\mathrm{SL}(2, \mathbb{Z})$ . The functors  $L_x, R_x$  yield another homomorphism  $X \times X \rightarrow \mathrm{Auteq}(D^b(X))$ ; and one can combine the two constructions into a map

$$\gamma' : G' \stackrel{\mathrm{def}}{=} (X \times X) \rtimes \widetilde{\mathrm{SL}}(2, \mathbb{Z}) \longrightarrow \mathrm{Auteq}(D^b(X)). \quad (3.18)$$

Here the semidirect product is taken with respect to the  $\widetilde{\mathrm{SL}}(2, \mathbb{Z})$ -action on  $X \times X$  indicated by (3.14)–(3.17); explicitly, it is given by the matrices

$$a_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad a_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad t \mapsto \mathrm{id}. \quad (3.19)$$

LEMMA 3.18

*The group  $G$  in (3.8) is isomorphic to the group  $G'$  in (3.18).*

*Proof*

Introduce complex coordinates  $z_1 = r_1 + ir_4, z_2 = r_2 - ir_3$  on  $\mathbb{R}^4/\mathbb{Z}^4$ . Then the action of  $\widetilde{\mathrm{SL}}(2, \mathbb{Z})$  described in (3.9) becomes  $\mathbb{C}$ -linear and is given by the same matrices as in (3.19). This is sufficient to identify the two semidirect products that define  $G$  and  $G'$ . We should point out that although the argument is straightforward, the change of coordinates is by no means obvious from the geometric point of view; a look back at the definition of  $G$  shows that  $z_1, z_2$  mix genuine symplectic automorphisms with the extra symmetries of  $D^b \mathrm{Fuk}(M, \beta)$  which come from tensoring with flat line bundles.  $\square$

The way in which this fits into the general philosophy is that one expects to have a commutative diagram, with the right vertical arrow given by Kontsevich's conjecture:

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & \mathrm{Auteq}(D^b \mathrm{Fuk}(M, \beta)) \\ \cong \downarrow & & \cong \downarrow \\ G' & \xrightarrow{\gamma'} & \mathrm{Auteq}(D^b(X)) \end{array} \quad (3.20)$$

To be accurate, one should adjust the modular parameter of  $X$  and the volume of  $(M, \beta)$ , eventually introducing a complex part  $\beta_{\mathbb{C}}$  as in Remark 1.4, so that they are indeed mirror dual. This has not played any role up to now since the groups  $G$  and  $G'$  are independent of the parameters, but it would become important in further study. A theorem of Orlov [41] says that  $\gamma'$  is always injective and that it is an

isomorphism if and only if  $X$  has no complex multiplication. Only the easy part of the theorem is important for us here. If  $X$  has complex multiplication, then its symmetries induce additional automorphisms of  $D^b(X)$ , which are not contained in the image of  $\gamma'$ . Therefore, if the picture (3.20) is correct, the derived Fukaya category for the corresponding values of  $\beta_{\mathbb{C}}$  must admit exotic automorphisms that do not come from symplectic geometry or from flat line bundles. It would be interesting to check this claim, especially because similar phenomena may be expected to occur in higher dimensions.

We now apply the intuition provided by the general discussion to the specific topic of braid group actions. To a simple closed curve  $S$  on  $(M, \beta)$ , one can associate a Dehn twist  $\tau_S \in \text{Symp}^h(M, \beta)$  which is unique up to Hamiltonian isotopy. This is defined by taking a symplectic embedding  $\iota$  of  $(U, \theta) = ([-\epsilon; \epsilon] \times \mathbb{R}/\mathbb{Z}, ds_1 \wedge ds_2)$  into  $M$  for some  $\epsilon > 0$ , with  $\iota(\{0\} \times \mathbb{R}/\mathbb{Z}) = S$ , and by using a local model

$$\tau : U \longrightarrow U, \quad \tau(s_1, s_2) = (s_1, s_2 - h(s_1)),$$

where  $h \in C^\infty(\mathbb{R}, \mathbb{R})$  is some function with  $h(s) = 0$  for  $s \leq -\epsilon/2$ ,  $h(s) = 1$  for  $s \geq \epsilon/2$ , and  $h(s) + h(-s) = 1$  for all  $s$ . The interesting fact is that the Dehn twists along two parallel geodesic lines are not Hamiltonian isotopic; they differ by a translation that depends on the area lying between the two lines. Now take

$$S_1 = \mathbb{R}/\mathbb{Z} \times \{0\}, \quad S_2 = \{0\} \times \mathbb{R}/\mathbb{Z}, \quad S_3 = \mathbb{R}/\mathbb{Z} \times \{1/2\}.$$

This is an  $(A_3)$ -configuration of circles. Hence the Dehn twists  $\tau_{S_1}, \tau_{S_2}, \tau_{S_3}$  define a homomorphism from the braid group  $B_4$  to  $\pi_0(\text{Symp}^h(M, \beta))$ . However, this is not injective;  $\tau_{S_3}^{-1}\tau_{S_1}$  is Hamiltonian isotopic to a translation that has order 2, so the nontrivial braid  $(g_3^{-1}g_1)^2 \in B_4$  gets mapped to the identity element. The natural lift of this homomorphism to  $\text{Symp}^{h, \text{gr}}(M, \beta)$  has the same noninjectivity property. Guided by mirror symmetry, one translates this example into algebraic geometry as follows:  $\mathcal{E}_1 = \mathcal{O}_{x_0}, \mathcal{E}_2 = \mathcal{O}, \mathcal{E}_3 = \mathcal{O}_x \in D^b(X)$ , where  $x \neq x_0$  is a point of order 2 on  $X$ , form an  $(A_3)$ -configuration of spherical objects. Hence their twist functors generate a weak action of  $B_4$  on  $D^b(X)$ . By (3.11),  $T_{\mathcal{E}_3}^{-1}T_{\mathcal{E}_1}$  is the functor of tensoring with  $\mathcal{O}(x - x_0)$ . Since the square of this is the identity functor, we have the same relation as in the symplectic case, so that the action is not faithful.

### 3.5. $K3$ surfaces

Let  $X$  be a smooth complex  $K3$  surface. Consider, as in Example 3.5, a chain of embeddings  $\iota_1, \dots, \iota_m : \mathbb{P}^1 \rightarrow X$  whose images  $C_i$  satisfy  $C_i \cdot C_j = 1$  for  $|i - j| = 1$  and  $C_i \cap C_j = \emptyset$  for  $|i - j| \geq 2$ . One can then use the structure sheaves  $\mathcal{O}_{C_i}$  to define a braid group action on  $D^b(X)$ . However, this is not the only way.

## PROPOSITION 3.19

For each  $i = 1, \dots, m$ , choose  $\mathcal{E}_i$  to be either  $\mathcal{O}(-C_i)$  or  $\mathcal{O}_{C_i}(-1) := (\iota_i)_* \mathcal{O}_{\mathbb{P}^1}(-1)$ . Then the  $\mathcal{E}_i$  form an  $(A_m)$ -configuration of spherical objects in  $D^b(X)$  and hence generate a weak braid group action on that category.

The choice can be made for each  $\mathcal{E}_i$  independently. These multiple possibilities are relevant from the mirror symmetry point of view. This is explained in [53], so we only summarize the discussion here.

Suppose that  $X$  is elliptically fibred with a section  $S$ . Its mirror should be the symplectic 4-manifold  $(M, \beta)$  with  $M = X$  and where  $\beta$  is the real part of some holomorphic 2-form on  $X$  (hyper-Kähler rotation). The smooth holomorphic curves in  $M$  are precisely the Lagrangian submanifolds in  $(M, \beta)$  which are special (with respect to the calibration given by the Kähler form of a Ricci-flat metric on  $X$ ). In particular, the curves  $C_i$  turn into an  $(A_m)$ -configuration of Lagrangian 2-spheres; hence the generalised Dehn twists along them generate a homomorphism  $B_{m+1} \rightarrow \pi_0(\text{Symp}^{\text{gr}}(M, \beta))$ . One can wonder what the corresponding braid group action on  $D^b(X)$  should be. This question is not really meaningful without a *distinguished* equivalence between the derived Fukaya category of  $(M, \beta)$  and that of coherent sheaves on  $X$ , which is not what is predicted by Kontsevich's conjecture. But if we adopt the Strominger-Yau-Zaslow [51] picture of mirror symmetry, then conjecturally there should be a distinguished full and faithful embedding of triangulated categories

$$D^b \text{Fuk}(M, \beta) \hookrightarrow D^b(X)$$

induced by the particular special Lagrangian torus fibration of  $M$  which comes from the elliptic fibration of  $X$ . (This fibration may, of course, not be distinguished.) That this is an embedding, and not an equivalence, is a feature of even-dimensional mirror symmetry. This embedding should be an extension of the FMT which takes special Lagrangian submanifolds of  $M$  (algebraic curves in  $X$ ) to coherent sheaves on  $X$  using the relative Poincaré sheaf on  $X \times_{\mathbb{P}^1} X$  that comes from considering the elliptic fibres to be self-dual using the section (see, e.g., [53]).

Assuming this, it now makes sense to ask what spherical objects of  $D^b(X)$  correspond to the special Lagrangian spheres  $C_1, \dots, C_m$ . The FMT takes any special Lagrangian submanifold  $C$  that is a section of the elliptic fibration to the invertible sheaf  $\mathcal{O}(S - C)$ ; if  $C$  lies in a fibre of the fibration, it goes to the structure sheaf  $\mathcal{O}_C$ . If we assume that all curves  $C_i$  fall into one of these two categories and that  $S$  intersects all those that lie in one fibre, then the FMT takes the special Lagrangian submanifolds  $C_1, \dots, C_m$  in  $(M, \beta)$  to sheaves  $\mathcal{E}_1, \dots, \mathcal{E}_m$  as in Proposition 3.19, tensored with  $\mathcal{O}(S)$ . Then, up to the minor difference of tensoring by  $\mathcal{O}(S)$ , one of the braid group actions mentioned in that proposition would be the correct mirror dual of the symplectic one.

As mentioned in Section 3.2, such configurations of curves  $C_i$  are the exceptional loci in the resolution of any algebraic surface with an  $(A_m)$ -singularity. Now,  $(A_m)$ -configurations of Lagrangian 2-spheres occur as vanishing cycles in the smoothing of the same singularity. Thus, in a sense, mirror symmetry interchanges smoothings and resolutions. A more striking, though maybe less well-understood, instance of this phenomenon is H. Pinkham's interpretation of Arnold's strange duality (see, e.g., [42]), which has been interpreted as a manifestation of mirror symmetry by a number of people (P. Aspinwall and D. Morrison, M. Kobayashi, I. Dolgachev, W. Ebeling, and so on). Each of the 14 singular affine surfaces  $S(c_1, c_2, c_3)$  on Arnold's list has a natural compactification  $\bar{S}(c_1, c_2, c_3)$  which has four singular points. One of these points is the original singularity at the origin; the other three are quotient singularities lying on the divisor at infinity, which is a  $\mathbb{P}^1$ . One can smooth the singular point at the origin; the intersection form of the vanishing cycles obtained in this way is  $T(c_1, c_2, c_3) \oplus H$ , where  $T(c_1, c_2, c_3)$  is the matrix associated to the Dynkin-type diagram (see Figure 1) and  $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . On the other hand, one can resolve the three

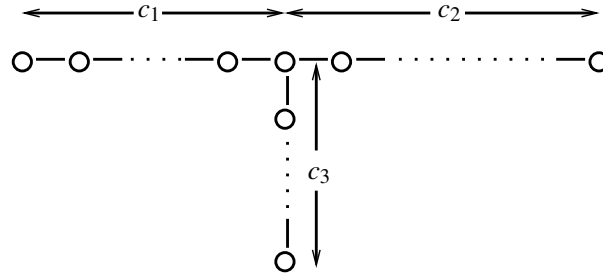


Figure 1

singular points at infinity. Inside the resolution this yields a configuration of smooth rational curves of the form  $T(b_1, b_2, b_3)$  for certain other numbers  $(b_1, b_2, b_3)$ . One can also do the two things together; this removes all singularities, yielding a smooth  $K3$  surface  $X(c_1, c_2, c_3)$  with a splitting of its intersection form as

$$T(c_1, c_2, c_3) \oplus H \oplus T(b_1, b_2, b_3).$$

Strange duality is the observation that the numbers  $(b_1, b_2, b_3)$  associated to one singularity on the list occur as  $(c_1, c_2, c_3)$  for another singularity, and vice versa. Kobayashi [29] (extended by Ebeling [12] to more general singularities) explains this by showing that the  $K3$ 's,  $X(c_1, c_2, c_3)$  and  $X(b_1, b_2, b_3)$ , belong to mirror dual families. The associated map on homology interchanges the  $T(c_1, c_2, c_3)$  and  $T(b_1, b_2, b_3)$  summands of the intersection form. (The extra hyperbolic of vanishing

cycles goes to the  $H^0 \oplus H^4$  of the other  $K3$  surface.) Thus the smoothing of the singular point at the origin in  $\bar{S}(c_1, c_2, c_3)$  corresponds, in a slightly vague sense, to the resolution of the divisor at infinity of  $\bar{S}(b_1, b_2, b_3)$ . From our point of view, since the rational curves at infinity in  $X(b_1, b_2, b_3)$  can be used to define a braid group action on its derived category, one would like to have a similar configuration of Lagrangian 2-spheres (vanishing cycles) in the finite part of  $X(c_1, c_2, c_3)$ . On the level of homology, such a configuration exists of course, but it is apparently unknown whether it can be realized geometrically. (Recall that Lagrangian submanifolds can have many more nonremovable intersection points than their intersection number suggests.)

### 3.6. Singularities of 3-folds

Throughout the following discussion, all varieties are smooth projective 3-folds that are Calabi-Yau in the strict sense. (Some singular 3-folds also occur, but they are specifically designated as such.) Let  $X$  be such a variety.

#### Examples 3.20

Any invertible sheaf on  $X$  is a spherical object in  $D^b(X)$ . If  $S$  is a smooth connected surface in  $X$  with  $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$  (e.g., a rational surface or Enriques surface), the structure sheaf  $\mathcal{O}_S$  is a spherical object, by Lemma 3.4. Similarly, for  $C$  a smooth rational curve in  $X$  with normal bundle  $\nu_C \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  (usually referred to as a  $(-1, -1)$ -curve),  $\mathcal{O}_C$  is spherical. The ideal sheaf  $\mathcal{I}_C$  of such a curve is also a spherical object; this follows from  $\mathcal{I}_C[1] \cong T_{\mathcal{O}}(\mathcal{O}_C)$ .

Supposing the ground field to be  $k = \mathbb{C}$ , we now return to the conjectural duality between smoothings and resolutions that played a role in Section 3.5 and that has been considered by many physicists. (Of course, our interest in this is in trying to mirror Dehn twists on smoothings, which arise as monodromy transformations around a degeneration of the smoothing collapsing the appropriate spherical vanishing cycle, by twists on the derived categories of the resolutions.) To explain the approach of physicists (as described in [36], for instance), it is better to adopt the traditional point of view in which mirror symmetry relates the combined complex and (complexified) Kähler moduli spaces of two varieties, rather than Kontsevich's conjecture, which considers a fixed value of the moduli variables. Then the idea can be phrased as follows. Moving towards the discriminant locus in the complex moduli space of a variety  $X$ , which means degenerating it to a singular variety  $Y$ , should be mirror dual to going to a “boundary wall” of the complexified Kähler cone of the mirror  $\widehat{X}$  (the annihilator of a face of the Mori cone), thus inducing an extremal contraction  $\widehat{X} \rightarrow \widehat{Y}$ . A second application of the same idea, with the roles of the mirrors reversed, shows

that an arbitrary crepant resolution  $Z \rightarrow Y$  should be mirror dual to a smoothing  $\widehat{Z}$  of  $\widehat{Y}$ . A case that is reasonably well understood is that of the ordinary double point (ODP:  $x^2 + y^2 + z^2 + t^2 = 0$  in local analytic coordinates) singularity. ODPs should be self-dual, in the sense that if  $Y$  has  $d$  distinct ODPs, then so does  $\widehat{Y}$ . (This can be checked for Calabi-Yau hypersurfaces in toric varieties, for instance.) We now review briefly C. Clemens's work in [9] on the homology of smoothings and resolutions of such singularities.

A degeneration of  $X$  to a variety  $Y$  with  $d$  ODPs determines  $d$  vanishing cycles in  $X$  and hence a map  $v : \mathbb{Z}^d \rightarrow H_3(X)$ . Let  $v^\vee : H_3(X) \rightarrow \mathbb{Z}^d$  be the Poincaré dual of  $v$ . Suppose that  $Y$  has a crepant resolution  $Z$  that, in local analytic coordinates near each ODP, looks like the standard small resolution. This means that the exceptional set in  $Z$  consists of  $d$  disjoint  $(-1, -1)$ -curves. By [9] and [16], one has

$$H_3(Z) \cong \frac{\ker(v^\vee)}{\text{im}(v)}$$

and exact sequences

$$\begin{aligned} H_3(X) &\xrightarrow{v^\vee} \mathbb{Z}^d \rightarrow H_2(Z) \rightarrow H_2(X) \rightarrow 0, \\ 0 &\rightarrow H_4(X) \rightarrow H_4(Z) \rightarrow \mathbb{Z}^d \xrightarrow{v} H_3(X). \end{aligned}$$

Thus, if there are  $r$  relations between the vanishing cycles (the image of  $v$  is of rank  $d - r$ ), the Betti numbers are

$$b_2(Z) = b_2(X) + r, \quad b_3(Z) = b_3(X) - 2(d - r), \quad b_4(Z) = b_4(X) + r. \quad (3.21)$$

Topologically,  $Z$  arises from  $X$  through codimension 3 surgery along the vanishing cycles, and the statements above can be proved, for example, by considering the standard cobordism between them. More intuitively, one can explain matters as follows. Going from  $X$  to  $Y$  shrinks the vanishing cycles to points; at the same time, the relations between vanishing cycles are given by 4-dimensional chains that become cycles in the limit  $Y$  because their boundaries shrink to points. This means that we lose  $d - r$  generators of  $H_3$  and get  $r$  new generators of  $H_4$ . In  $Z$  there are  $d - r$  relations between the homology classes of the exceptional  $\mathbb{P}^1$ 's; these relations are pullbacks of closed 3-dimensional cycles on  $Y$  which do not lift to cycles on  $Z$ , so going from  $Y$  to  $Z$  adds  $r$  new generators to  $H_2$  while removing another  $d - r$  generators from  $H_3$ . Finally,  $H_4(Z) = H_4(Y)$  for codimension reasons. Mirror symmetry exchanges odd- and even-dimensional homology, so if  $X$  and  $Z$  have mirrors  $\widehat{X}$  and  $\widehat{Z}$ , then

$$b_2(\widehat{Z}) = b_2(\widehat{X}) - (d - r), \quad b_3(\widehat{Z}) = b_2(\widehat{X}) + 2r, \quad b_4(\widehat{Z}) = b_4(\widehat{X}) - (d - r).$$

The suggested explanation, in the general framework explained above, is that  $\widehat{Z}$  should contain  $d$  vanishing cycles with  $d - r$  relations between them, obtained from a degeneration to a variety  $\widehat{Y}$  with  $d$  ODP, and that  $\widehat{X}$  should be a crepant resolution of  $\widehat{Y}$ . Thus mirror symmetry exchanges ODPs with the opposite number of relations between their vanishing cycles. Moreover, to the  $d$  vanishing cycles in the original variety  $X$  correspond  $d$  rational  $(-1, -1)$ -curves in its mirror  $\widehat{X}$ . It seems plausible to think that the structure sheaves or ideal sheaves of these curves (possibly twisted by some line bundle) should be mirror dual to the Lagrangian spheres representing the vanishing cycles in  $X$ ; however, as in the  $K3$  case, such a statement is not really meaningful unless one has chosen some specific equivalence  $D^b(X) \cong D^b \text{Fuk}(\widehat{X})$ .

*Remark 3.21*

When  $r = 0$ ,  $H_2(Y) \cong H_2(X)$ , so the exceptional cycles in  $Y$  are homologous to zero. This is not possible if the resolution is algebraic, so we exclude this case and also the case  $d = r$  to avoid the same problem on the mirror.

Going a bit beyond this, we now propose a possible mirror dual to the  $(A_{2d-1})$ -singularity. Let  $X$  be a variety that can be degenerated to some  $Y$  with an  $(A_{2d-1})$ -singularity, and let  $v_1, \dots, v_{2d-1} \in H_n(X)$  be the corresponding vanishing cycles. The signs are fixed in such a way that  $v_i \cdot v_{i+1} = 1$  for all  $i$ . We impose two additional conditions. One is that  $Y$  should have a partial smoothing  $Y'$  (equivalently,  $X$  a partial degeneration) having  $d$  ODPs, built according to the local model

$$x^2 + y^2 + z^2 + \prod_{i=1}^d (t - \epsilon_i)^2 = 0$$

with the  $\epsilon_i$ 's distinct and small. This means that in the  $(A_{2d-1})$ -configuration of vanishing cycles in  $X$ , one can degenerate the 1st, 3rd,  $\dots$ ,  $(2d-1)$ th to ODPs. The second additional condition is that  $Y'$  should admit a resolution  $Z'$  of the standard kind considered above. Then, according to Remark 3.21, there is at least one relation between  $v_1, v_3, \dots, v_{2d-1}$ . In fact, since the intersection matrix of all  $v_i$  has only a 1-dimensional nullspace, there must be precisely one relation.

*Remark 3.22*

This relation is in fact  $v_1 + v_3 + \dots + v_{2d-1} = 0$ . The corresponding situation on  $Z'$  is that all the exceptional  $\mathbb{P}^1$ 's are homologous. This should not be too surprising; they can be moved back together to give the  $d$ -times thickened  $\mathbb{P}^1$  in the resolution of the original  $(A_{2d-1})$ -singularity that one gets by taking the  $d$ -fold branch cover  $t \mapsto t^d$  of the resolution of the ODP  $x^2 + y^2 + z^2 + t^2 = 0$ . We note in passing that out of the  $2^d$  possible ways of resolving the ODPs in  $Y'$  (differing by flops), at most two

can lead to an algebraic manifold since an exceptional  $\mathbb{P}^1$  cannot be homologous to minus another one.

In view of our previous discussion, we expect that the mirror  $\widehat{X}$  of  $X$  admits a contraction  $\widehat{X} \rightarrow \widehat{Y}'$  to a variety with  $d$  ODPs; any smoothing  $\widehat{Z}'$  of  $\widehat{Y}'$  should contain  $d$  vanishing cycles with  $(d - 1)$  relations between them. By (3.21), these give rise to a  $(d - 1)$ -dimensional subspace of  $H_4(\widehat{X}; \mathbb{C}) \cong H^{1,1}(\widehat{X})$ . There is a natural basis for the relations between the exceptional  $\mathbb{P}^1$ 's in  $Z'$ , which comes from the even-numbered vanishing cycles  $v_{2i}$ . The corresponding basis of the subspace of  $H^{1,1}(\widehat{X})$  can be represented by divisors  $S_2, S_4, \dots, S_{2d-2}$  such that  $S_{2i}$  intersects only the  $(i - 1)$ th and  $i$ th exceptional  $\mathbb{P}^1$ . Based on these considerations and others described below, we make a concrete guess as to what  $\widehat{X}$  looks like.

*Definition 3.23*

An  $(\widehat{A}_{2d-1})$ -configuration of subvarieties inside a smooth 3-fold consists of embedded smooth surfaces  $S_2, S_4, \dots, S_{2d-2}$  and curves  $C_1, C_3, \dots, C_{2d-1}$  such that

- (1) the canonical sheaf of the 3-fold is trivial along each  $S_{2i}$ ;
- (2) each  $S_{2i}$  is isomorphic to  $\mathbb{P}^2$  with two points blown up;
- (3)  $S_{2i} \cap S_{2j} = \emptyset$  for  $|i - j| \geq 2$ ;
- (4)  $S_{2i-2}, S_{2i}$  are transverse and intersect in  $C_{2i-1}$ , which is a rational curve and exceptional (i.e., has self-intersection  $-1$ ) both in  $S_{2i-2}$  and  $S_{2i}$ .

Note that the last condition implies that  $C_{2i-1}$  is a  $(-1, -1)$ -curve in the 3-fold. What we postulate is that the mirror  $\widehat{X}$  contains such a configuration of subvarieties, with the  $C_{2i-1}$  being the exceptional set of the contraction  $\widehat{X} \rightarrow \widehat{Y}'$ . Apart from being compatible with the informal discussion above, there are some more feasibility arguments in favour of this proposal. First, such configurations exist as exceptional loci in crepant resolutions of singularities; Figure 2 represents a toric 3-fold with trivial canonical bundle containing such a configuration. The thick lines represent the  $C_{2i-1}$ 's joining consecutive surfaces  $S_{2i-2}, S_{2i}$ , which are themselves represented by the nodes. Removing these nodes and lines gives the singularity of which it is a resolution by collapsing the whole chain of surfaces and lines; this singularity we think of as the dual of the  $(A_{2d-1})$ -singularity.

We could have deformed the  $(A_{2d-1})$ -singularity in  $X$  differently, for instance, by degenerating an even-numbered vanishing cycle  $v_{2i}$  to an ODP. This should correspond to contracting a  $\mathbb{P}^1$  in  $\widehat{X}$ . Assuming that our guess is right, so that  $\widehat{X}$  contains an  $(\widehat{A}_{2d-1})$ -configuration, this should be the other exceptional curve in the  $S_{2i}$  besides  $C_{2i-1}$  and  $C_{2i+1}$  (i.e., the line that we call  $C_{2i} \cong \mathbb{P}^1$  joining  $C_{2i-1}$  and  $C_{2i+1}$ ; in Figure 2 these are represented by the vertical lines). Contracting these curves while *not*

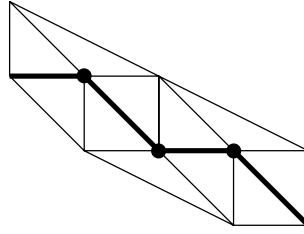


Figure 2

contracting  $C_{2i-1}, C_{2i+1}$  turns  $S_{2i}$  into a  $\mathbb{P}^1 \times \mathbb{P}^1$ . The whole 4-cycle  $S_{2i}$  contracts to a lower dimensional cycle only when we contract another of the  $\mathbb{P}^1$ 's in it, leaving the final  $\mathbb{P}^1$  (over which the surface fibres) still uncontracted (on  $X$ , this corresponds to degenerating two consecutive vanishing cycles while leaving the others finite). Thus there are contractions of  $\widehat{X}$  mirroring various possible partial degenerations of  $X$ .

A final argument in favour of our proposal, and much of the motivation for it, is that it leads to braid group actions on derived categories of coherent sheaves. These are of interest in themselves, independent of whether or not they can be considered to be mirror dual to the braid groups of Dehn twist symplectomorphisms on smoothings of  $(A_{2d-1})$ -singularities.

**PROPOSITION 3.24**

*Let  $X$  be a smooth quasi-projective 3-fold, and let  $S_2, S_4, \dots, S_{2d-2}, C_1, C_3, \dots, C_{2d-1}$  be an  $(\widehat{A}_{2d-1})$ -configuration of subvarieties in  $X$ . Then taking  $\mathcal{E}_i = \mathcal{O}_{C_i}$  if  $i$  is odd, or  $\mathcal{O}_{S_i}$  if  $i$  is even, gives an  $(A_{2d-1})$ -configuration  $(\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_{2d-1})$  of spherical objects in  $D^b(X)$ .*

The assumption that the  $S_i$  are  $\mathbb{P}^2$ 's with two points blown up can be weakened considerably for this result to hold; any other rational surface will do. Proposition 3.24 is a 3-dimensional analogue of Example 3.5 and hence, as a comparison with our discussion of  $K3$  surfaces shows, possibly too naive from the mirror symmetry point of view. There is an alternative way of constructing spherical objects, closer to Proposition 3.19.

**PROPOSITION 3.25**

*Let  $X$  be a smooth projective 3-fold that is Calabi-Yau in the strict sense, containing an  $(\widehat{A}_{2d-1})$ -configuration as in Proposition 3.24. Take rational curves  $L_{2i}$  in  $S_{2i}$  such that  $L_{2i} \cap C_{2j+1} = \emptyset$  for all  $i, j$ . (The inverse image of the generic line in  $\mathbb{P}^2$  is such a rational curve in the blow-up of  $\mathbb{P}^2$ .) Choose*

$$\mathcal{E}_i = \begin{cases} \mathcal{O}_{C_i}(-1) \text{ or } \mathcal{I}_{C_i} & \text{if } i \text{ is odd,} \\ \mathcal{O}_{S_i}(-L_i) \text{ or } \mathcal{O}_X(-S_i) & \text{if } i \text{ is even.} \end{cases}$$

Then the  $\mathcal{E}_i$ ,  $i = 1, 2, \dots, 2d-1$ , form an  $(A_{2d-1})$ -configuration of spherical objects in  $D^b(X)$ .

Here  $\mathcal{O}_{C_i}(-1)$  is shorthand for  $\iota_*(\mathcal{O}_{\mathbb{P}^1}(-1))$ , where  $\iota : \mathbb{P}^1 \rightarrow X$  is some embedding with image  $C_i$ , and  $\mathcal{O}_{S_i}(-L_i)$  should be interpreted in the same way. As in Lemma 3.19, the choice of  $\mathcal{E}_i$  can be made independently for each  $i$ .

There are many other interesting configurations of spherical objects which arise in connection with 3-fold singularities. Their mirror symmetry interpretations are mostly unclear. For instance, a slight variation of the situation above yields braid group actions built only from structure sheaves of surfaces.

**PROPOSITION 3.26**

*Let  $X$  be a smooth quasi-projective 3-fold, and let  $S_1, S_2, \dots, S_m$  be a chain of smooth embedded rational surfaces in  $X$  with the following properties:  $S_i \cap S_{i+1}$  is transverse and consists of one rational curve whose self-intersection in  $S_i$  and  $S_{i+1}$  is either zero or  $-2$ ;  $S_i \cap S_j = \emptyset$  for  $|i - j| \geq 2$ ; and  $\omega_X|_{S_i}$  is trivial. Then  $\mathcal{E}_i = \mathcal{O}_{S_i}$  is an  $(A_m)$ -configuration of spherical objects in  $D^b(X)$ .*

The conditions actually imply that every intersection  $S_i \cap S_{i+1}$  is a rational curve with normal bundle  $\cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  in  $X$ . Note also that the presence of rational curves with self-intersection zero forces at least every second of the surfaces  $S_i$  to be fibred over  $\mathbb{P}^1$ . Configurations of this kind are the exceptional loci of crepant resolutions of suitable toric singularities.

In a different direction, I. Nakamura's resolutions of abelian quotient singularities using Hilbert schemes of clusters, with their toric representations as tessellations of hexagons (see [39], [10]), lead to situations similar to Proposition 3.24. The nodes of the hexagons in Figure 3 represent surfaces that are the blow-ups of  $\mathbb{P}^1 \times \mathbb{P}^1$  in two distinct points; the six lines emanating from a node represent the six exceptional  $\mathbb{P}^1$ 's in the surface, in which it intersects the other surfaces represented by the other nodes that the lines join. The structure sheaves of these curves and surfaces give rise to twists on the derived category satisfying braid relations according to the Dynkin-type diagram obtained by adding a vertex in the middle of each edge. (These added vertices represent the structure sheaves of the curves; see Figure 3.) The McKay correspondence (see Section 3.2) translates this into a group of twists on the equivariant derived category of the 3-fold on which the finite group acted.

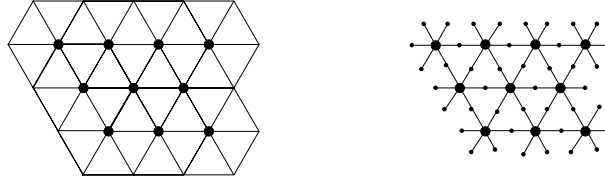


Figure 3

## 4. Faithfulness

### 4.1. Differential graded algebras and modules

The notions discussed in this section are, for the most part, familiar ones; we collect them here to set up the terminology and also for the reader's convenience. A detailed exposition of the general theory of differential graded modules can be found in [1, Section 10].

Fix a field  $k$ , and fix an integer  $m \geq 1$ . Take the semisimple  $k$ -algebra  $R = k^m$  with generators  $e_1, \dots, e_m$  and relations  $e_i^2 = e_i$  for all  $i$ ,  $e_i e_j = 0$  for  $i \neq j$  (so  $1 = e_1 + \dots + e_m$  is the unit element).  $R$  plays the role of ground ring in the following considerations. In particular, by a graded algebra we always mean a  $\mathbb{Z}$ -graded unital associative  $k$ -algebra  $A$ , together with a homomorphism (of algebras, and unital)  $\iota_A : R \rightarrow A^0$ . This equips  $A$  with the structure of a graded  $R$ -bimodule, and the multiplication becomes a bimodule map. For the sake of brevity, we denote the bimodule structure by  $e_i a$  and  $a e_i$  ( $a \in A$ ) instead of  $\iota_A(e_i)a$ , respectively,  $a \iota_A(e_i)$ . All homomorphisms  $A \rightarrow B$  between graded algebras are required to commute with the maps  $\iota_A, \iota_B$ . A differential graded algebra (dga)  $\mathcal{A} = (A, d_A)$  is a graded algebra  $A$  together with a derivation  $d_A$  of degree 1, which satisfies  $d_A^2 = 0$  and  $d_A \circ \iota_A = 0$ . The cohomology  $H(\mathcal{A})$  of a dga is a graded algebra. A homomorphism of dgas is called a quasi-isomorphism if it induces an isomorphism on cohomology. Two dgas  $\mathcal{A}, \mathcal{B}$  are called quasi-isomorphic if there is a chain of dgas and quasi-isomorphisms  $\mathcal{A} \leftarrow \mathcal{C}_1 \rightarrow \dots \leftarrow \mathcal{C}_k \rightarrow \mathcal{B}$  connecting them. (In fact, it is sufficient to allow  $k = 1$  since the category of dgas admits a calculus of fractions; see [26, Lemma 3.2].)

A dga  $\mathcal{A}$  is called formal if it is quasi-isomorphic to its own cohomology algebra  $H(\mathcal{A})$ , thought of as a dga with zero differential.

By a graded module over a graded algebra  $A$ , we always mean a graded *right*  $A$ -module. Through the map  $\iota_A$ , any such module  $M$  becomes a right  $R$ -module; again, we write  $x e_i$  ( $x \in M$ ) instead of  $x \iota_A(e_i)$ . A differential graded module (dgm) over a dga  $\mathcal{A} = (A, d_A)$  is a pair  $\mathcal{M} = (M, d_M)$  consisting of a graded  $A$ -module

$M$  and a  $k$ -linear map  $d_M : M \rightarrow M$  of degree 1 such that  $d_M^2 = 0$  and  $d_M(xa) = (d_M x)a + (-1)^{\deg(x)} x(d_A a)$  for  $a \in A$ . The cohomology  $H(\mathcal{M})$  is a graded module over  $H(\mathcal{A})$ . For instance,  $\mathcal{A}$  is a dgm over itself, and as such it splits into a direct sum of dgms

$$\mathcal{P}_i = (e_i A, d_A|_{e_i A}), \quad 1 \leq i \leq m. \quad (4.1)$$

By definition, a dgm homomorphism  $M \rightarrow N$  is a homomorphism of graded modules which is at the same time a homomorphism of chain complexes. Dgms over  $\mathcal{A}$  and their homomorphisms form an abelian category  $\text{Dgm}(\mathcal{A})$ . One can also define chain homotopies between dgm homomorphisms. The category  $K(\mathcal{A})$ , with the same objects as  $\text{Dgm}(\mathcal{A})$  and with the homotopy classes of dgm homomorphisms as morphisms, is triangulated. The translation functor in it takes  $\mathcal{M} = (M, d_M)$  to  $\mathcal{M}[1] = (M[1], -d_M)$ , with no change of sign in the module structure. Exact triangles are all those isomorphic to one of the standard triangles involving a dgm homomorphism and its cone.

Having mentioned cones, we use the opportunity to introduce a slight generalisation, which is used later on. Assume that one has a chain complex in  $\text{Dgm}(\mathcal{A})$ , namely, dgms  $\mathcal{C}_i$ ,  $i \in \mathbb{Z}$ , and dgm homomorphisms  $\delta_i : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}$  such that  $\delta_{i+1}\delta_i = 0$ . Then one can form a new dgm  $\mathcal{C}$  by setting  $C = \bigoplus_{i \in \mathbb{Z}} C_i[i]$  and

$$d_C = \begin{pmatrix} \cdots & & & \\ \delta_{i-1} & (-1)^i d_{C_i} & & \\ & \delta_i & (-1)^{i+1} d_{C_{i+1}} & \\ & & \delta_{i+1} & \cdots \end{pmatrix}.$$

We refer to this as collapsing the chain complex (it can also be viewed as a special case of a twisted complex; see, e.g., [3]), and we write  $\mathcal{C} = \{\cdots \rightarrow \mathcal{C}_i \rightarrow \mathcal{C}_{i+1} \rightarrow \cdots\}$ ; for complexes of length 2, it specializes to the cone of a dgm homomorphism.

Inverting the dgm quasi-isomorphisms in  $K(\mathcal{A})$  yields another triangulated category  $D(\mathcal{A})$ , in which any short exact sequence of dgms can be completed to an exact triangle. As usual,  $D(\mathcal{A})$  can also be defined by inverting the quasi-isomorphisms directly in  $\text{Dgm}(\mathcal{A})$ , but then the triangulated structure is more difficult to see. We call  $D(\mathcal{A})$  the derived category of dgms over  $\mathcal{A}$ .

### Warning

Even though we use the same notation as in ordinary homological algebra, the expressions  $K(\mathcal{A})$  and  $D(\mathcal{A})$  have a different meaning here. In particular,  $D(\mathcal{A})$  is not the derived category, in the usual sense, of  $\text{Dgm}(\mathcal{A})$ .

For any dga homomorphism  $f : \mathcal{A} \rightarrow \mathcal{B}$  there is a “restriction of scalars” functor  $\text{Dgm}(\mathcal{B}) \rightarrow \text{Dgm}(\mathcal{A})$ . This preserves homotopy classes of homomorphisms, takes

cones to cones, and commutes with the shift functors. Hence it descends to an exact functor  $K(\mathcal{B}) \rightarrow K(\mathcal{A})$ . Moreover, it obviously preserves quasi-isomorphisms, so that it also descends to an exact functor  $D(\mathcal{B}) \rightarrow D(\mathcal{A})$ ; we denote any of these functors by  $f^*$ . The next result, taken from [1, Theorem 10.12.5.1], shows that two quasi-isomorphic dgas have equivalent derived categories.

**THEOREM 4.1**

*If  $f$  is a quasi-isomorphism,  $f^* : D(\mathcal{B}) \rightarrow D(\mathcal{A})$  is an exact equivalence.*

Let  $\mathcal{A}$  be a dga. The *standard twist functors*  $t_1, \dots, t_m$  from  $\text{Dgm}(\mathcal{A})$  to itself are defined by

$$t_i(\mathcal{M}) = \{ \mathcal{M} e_i \otimes_k \mathcal{P}_i \longrightarrow \mathcal{M} \}.$$

The tensor product of  $\mathcal{M} e_i = (M e_i, d_M|_{M e_i})$  with the dgm  $\mathcal{P}_i$  of (4.1) is one of complexes of  $k$ -vector spaces; it becomes a dgm with the module structure inherited from  $\mathcal{P}_i$ . The arrow is the multiplication map  $M e_i \otimes_k e_i A \rightarrow M$ , which is a homomorphism of dgms, and we are taking its cone;  $t_i$  descends to exact functors  $K(\mathcal{A}) \rightarrow K(\mathcal{A})$  and  $D(\mathcal{A}) \rightarrow D(\mathcal{A})$ , for which we use the same notation. This is straightforward for  $K(\mathcal{A})$ . As for  $D(\mathcal{A})$ , one needs to show that  $t_i$  preserves quasi-isomorphisms; this follows from looking at the long exact sequence

$$\dots \longrightarrow H(\mathcal{M}) e_i \otimes_k e_i H(\mathcal{A}) \longrightarrow H(\mathcal{M}) \longrightarrow H(t_i(\mathcal{M})) \longrightarrow \dots.$$

**LEMMA 4.2**

*Let  $f : \mathcal{A} \rightarrow \mathcal{B}$  be a quasi-isomorphism of dgas. Then the following diagram commutes up to isomorphism, for each  $1 \leq i \leq m$ :*

$$\begin{array}{ccc} D(\mathcal{B}) & \xrightarrow{t_i} & D(\mathcal{B}) \\ f^* \downarrow & & \downarrow f^* \\ D(\mathcal{A}) & \xrightarrow{t_i} & D(\mathcal{A}) \end{array}$$

*Proof*

Let  $\mathcal{M} = (M, d_M)$  be a dgm over  $\mathcal{B}$ . Consider the commutative diagram of dgms over  $\mathcal{A}$ :

$$\begin{array}{ccc} \mathcal{M} e_i \otimes_k e_i \mathcal{A} & \longrightarrow & f^* \mathcal{M} \\ \text{id} \otimes (f|_{e_i \mathcal{A}}) \downarrow & & \downarrow \text{id} \\ \mathcal{M} e_i \otimes_k f^*(e_i \mathcal{B}) & \longrightarrow & f^* \mathcal{M} \end{array}$$

The upper horizontal arrow is  $m \otimes a \mapsto m f(a)$ , and the lower one is multiplication.

The cone of the upper row is  $t_i(f^*(\mathcal{M}))$ , while that of the lower one is  $f^*(t_i(\mathcal{M}))$ . The two vertical arrows combine to give a quasi-isomorphism between these cones.  $\square$

Now let  $\mathfrak{S}' \subset \mathfrak{S}$  be as in Section 2.1, and let  $\mathfrak{K}$  be the category from Definition 2.2. Let  $E_1, \dots, E_m$  be objects of  $\mathfrak{K}$ , and let  $E$  be their direct sum. The chain complex of endomorphisms

$$\text{end}(E) := \text{hom}(E, E) = \bigoplus_{1 \leq i, j \leq m} \text{hom}(E_i, E_j)$$

has a natural structure of a dga. Multiplication is given by composition of homomorphisms;  $\iota_{\text{end}(E)}$  maps  $e_i \in R$  to  $\text{id}_{E_i} \in \text{hom}(E_i, E_i)$ , so left multiplication with  $e_i$  is the projection to  $\text{hom}(E, E_i)$ , while right multiplication is the projection to  $\text{hom}(E_i, E)$ . In the same way, for any  $F \in \mathfrak{K}$ , the complex  $\text{hom}(E, F)$  is a dgm over  $\text{end}(E)$ . The functor  $\text{hom}(E, -) : \mathfrak{K} \rightarrow K(\text{end}(E))$  defined in this way is exact because it carries cones to cones. The objects  $E_i$  get mapped to the dgms  $\text{hom}(E, E_i) = e_i \text{end}(E)$ , which are precisely the  $\mathcal{P}_i$  from (4.1). We define a functor  $\Psi_E$  to be the composition

$$\mathfrak{K} \xrightarrow{\text{hom}(E, -)} K(\text{end}(E)) \xrightarrow{\text{quotient functor}} D(\text{end}(E)).$$

LEMMA 4.3

Assume that  $E_1, \dots, E_m$  satisfy the conditions from Definition 2.5, so that the twist functors  $T_{E_i}$  are defined. Then the following diagram is commutative up to isomorphism, for each  $1 \leq i \leq m$ :

$$\begin{array}{ccc} \mathfrak{K} & \xrightarrow{T_{E_i}} & \mathfrak{K} \\ \Psi_E \downarrow & & \downarrow \Psi_E \\ D(\text{end}(E)) & \xrightarrow{t_i} & D(\text{end}(E)) \end{array}$$

*Proof*

For  $F \in \mathfrak{K}$ , consider the commutative diagram of dgms over  $\text{end}(E)$ ,

$$\begin{array}{ccc} \text{hom}(E_i, F) \otimes_k \text{hom}(E, E_i) & \longrightarrow & \text{hom}(E, F) \\ \downarrow & & \downarrow \text{id} \\ \text{hom}(E, \text{hom}(E_i, F) \otimes_k E_i) & \longrightarrow & \text{hom}(E, F) \end{array}$$

with the following maps: the horizontal arrow in the first row is the composition; that in the second row is induced by the evaluation map  $\text{hom}(E_i, F) \otimes_k E_i \rightarrow F$ . The left-hand vertical arrow is the first of the canonical maps from (2.1), which is a quasi-isomorphism since  $\text{hom}(E_i, F)$  has finite-dimensional cohomology. The cone

of the first row is  $t_i(\Psi_E(F))$ , while that of the second row is  $\Psi_E(T_{E_i}(F))$ . The vertical arrows combine to give a natural quasi-isomorphism between these cones.  $\square$

Later on, in our application, the  $E_i$  occur as resolutions of objects in  $D^b(\mathfrak{S}')$ . The next two lemmas address the question of how the choice of resolutions affects the construction. This is not strictly necessary for our purpose, but it rounds off the picture.

LEMMA 4.4

Let  $E_i, E'_i$  ( $1 \leq i \leq m$ ) be objects in  $\mathfrak{K}$  such that  $E_i \cong E'_i$  for all  $i$ . Then the dgas  $\text{end}(E)$  and  $\text{end}(E')$  are quasi-isomorphic.

*Proof*

Choose for each  $i$  a map  $g_i : E_i \rightarrow E'_i$  which is a chain homotopy equivalence. Set  $C_i = \text{Cone}(g_i)$ , and let  $C$  be the direct sum of these cones; this is the same as the cone of  $g = g_1 \oplus \cdots \oplus g_m$ . Let  $\text{end}(C)$  be the endomorphism dga of  $C_1, \dots, C_m$ . An element of  $\text{end}(C)$  of degree  $r$  is a matrix

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

with  $\phi_{11} \in \text{hom}^r(E, E)$ ,  $\phi_{21} \in \text{hom}^{r-1}(E, E')$ ,  $\phi_{12} \in \text{hom}^{r+1}(E', E)$ ,  $\phi_{22} \in \text{hom}^r(E', E')$ . The differential in  $\text{end}(C)$  maps  $\phi$  to

$$\begin{pmatrix} -d_E \phi_{11} + (-1)^r \phi_{11} d_E - (-1)^r \phi_{12} g & -d_E \phi_{12} - (-1)^r \phi_{12} d_E \\ g \phi_{11} - (-1)^r \phi_{22} g + d_{E'} \phi_{21} + (-1)^r \phi_{21} d_E & g \phi_{12} + d_{E'} \phi_{22} - (-1)^r \phi_{22} d_{E'} \end{pmatrix}.$$

Let  $\mathcal{C} \subset \text{end}(C)$  be the subalgebra of matrices that are lower-triangular ( $\phi_{12} = 0$ ). The formula above shows that this is closed under the differential and hence again a dga. The projection  $\pi_2 : \mathcal{C} \rightarrow \text{end}(E')$ ,  $\pi_2(\phi) = \phi_{22}$ , is a homomorphism of dgas. Its kernel is isomorphic (as a complex of  $k$ -vector spaces, and up to a shift) to the cone of the composition with  $g$  map

$$g \circ - : \text{hom}(E, E) \longrightarrow \text{hom}(E, E').$$

Since  $g$  is a homotopy equivalence, this cone is acyclic, so that  $\pi_2$  is a quasi-isomorphism of dgas. A similar argument shows that the projection  $\pi_1 : \mathcal{C} \rightarrow \text{end}(E)$ ,  $\pi_1(\phi) = (-1)^{\deg(\phi)} \phi_{11}$ , is a quasi-isomorphism of dgas. The two maps together prove that  $\text{end}(E)$  and  $\text{end}(E')$  are quasi-isomorphic.  $\square$

As a consequence of this and Theorem 4.1, the categories  $D(\text{end}(E))$  and  $D(\text{end}(E'))$  are equivalent. Actually, we have shown a more precise statement: any choice of

$g_i : E_i \rightarrow E'_i$  yields, up to isomorphism of functors, an exact equivalence  $(\pi_2^*)^{-1}\pi_1^* : D(\text{end}(E)) \rightarrow D(\text{end}(E'))$ . We now see that this equivalence is compatible with the functors  $\Psi_E, \Psi_{E'}$ .

LEMMA 4.5

In the situation of Lemma 4.4,  $(\pi_2^*)^{-1}\pi_1^* \circ \Psi_E \cong \Psi_{E'}$ .

*Proof*

The obvious short exact sequence  $0 \rightarrow E' \rightarrow C \rightarrow E[1] \rightarrow 0$  induces, for any  $F \in \mathfrak{K}$ , a short exact sequence of dgms over  $\mathcal{C}$ :

$$0 \longrightarrow \pi_1^* \text{hom}(E, F)[-1] \longrightarrow \text{hom}(C, F) \longrightarrow \pi_2^* \text{hom}(E', F) \longrightarrow 0.$$

In the derived category  $D(\mathcal{C})$ , this short exact sequence can be completed to an exact triangle by a morphism

$$\pi_2^* \text{hom}(E', F) \longrightarrow \pi_1^* \text{hom}(E, F). \quad (4.2)$$

One can define such a morphism explicitly by replacing the given sequence with a (canonically constructed) quasi-isomorphic one, for which the corresponding morphism can be realized by an actual homomorphism of dgms (cf. [14, Proposition III.3.5]). The advantage of this explicit construction is that (4.2) is now natural in  $F$ . Since  $C$  is a contractible complex,  $\text{hom}(C, F)$  is acyclic, which implies that (4.2) is an isomorphism in  $D(\mathcal{C})$  for any  $F$ . This shows that the diagram

$$\begin{array}{ccc} & \mathfrak{K} & \\ \Psi_E \swarrow & & \searrow \Psi_{E'} \\ D(\text{end}(E)) & \xrightarrow{\pi_1^*} D(\mathcal{C}) \xleftarrow{\pi_2^*} & D(\text{end}(E')) \end{array}$$

commutes up to isomorphism, as desired.  $\square$

#### 4.2. Intrinsic formality

Applications of dga methods to homological algebra often hinge on constructing a chain of quasi-isomorphisms connecting two given dgas. For instance, in the situation explained in Section 4.1, one can try to use the dga  $\text{end}(E)$  to study the twists  $T_{E_i}$  via the functor  $\Psi_E$ . What really matters for this purpose is only the quasi-isomorphism type of  $\text{end}(E)$ . In general, quasi-isomorphism type is a rather subtle invariant. However, there are some cases where the cohomology already determines the quasi-isomorphism type.

*Definition 4.6*

A graded algebra  $A$  is called *intrinsically formal* if any two dgas with cohomology  $A$  are quasi-isomorphic or, equivalently, if any dga  $\mathcal{B}$  with  $H(\mathcal{B}) \cong A$  is formal.

For instance, one can show easily that any graded algebra  $A$  concentrated in degree zero is intrinsically formal. (This particular example can be viewed as the starting point for J. Rickard's theory of derived Morita equivalences (see [46]), as recast in dga language by B. Keller [25].) However, our intended application is to algebras of a rather different kind.

An augmented graded algebra is a graded algebra  $A$  together with a graded algebra homomorphism  $\epsilon_A : A \rightarrow R$  which satisfies  $\epsilon_A \circ \iota_A = \text{id}_R$ . Its kernel is a 2-sided ideal, called the augmentation ideal; we write it as  $A^+ \subset A$ . A special case is when  $A$  is connected, which means  $A^i = 0$  for  $i < 0$  and  $\iota_A : R \rightarrow A^0$  is an isomorphism; then there is of course a unique augmentation map, and  $A^+$  is the subspace of elements of positive degree.

*THEOREM 4.7*

*Let  $A$  be an augmented graded algebra. If  $HH^q(A, A[2 - q]) = 0$  for all  $q > 2$ , then  $A$  is intrinsically formal.*

We remind the reader that the Hochschild cohomology  $HH^*(A, M)$  of a graded  $A$ -bimodule  $M$  is the cohomology of the cochain complex

$$\begin{aligned} C^q(A, M) &= \text{Hom}_{R-R} \left( \overbrace{A^+ \otimes_R \cdots \otimes_R A^+}^q, M \right), \\ (\partial^q \phi)(a_1, \dots, a_{q+1}) &= (-1)^\epsilon a_1 \phi(a_2, \dots, a_{q+1}) \\ &\quad + \sum_{i=1}^q (-1)^{\epsilon_i} \phi(a_1, \dots, a_i a_{i+1}, \dots, a_{q+1}) \\ &\quad - (-1)^{\epsilon_q} \phi(a_1, \dots, a_q) a_{q+1}, \end{aligned}$$

where  $\text{Hom}_{R-R}$  denotes homomorphisms of graded  $R$ -bimodules. (By definition, these are homomorphisms of degree zero.) The signs are  $\epsilon = q \deg(a_1)$ ,  $\epsilon_i = \deg(a_1) + \cdots + \deg(a_i) - i$ . The bimodules relevant for our application are  $M = A[s]$  with the left multiplication twisted by a sign:  $a \cdot x \cdot a' = (-1)^{s \deg(a)} a x a'$  for  $a, a' \in A$  and  $x \in M$ . Note that the chain complex  $C^*(A, A[s])$  depends on  $s$ , so the cohomology groups that occur in Theorem 4.7 belong to different complexes.

We give a proof of Theorem 4.7 for lack of an accessible reference, and also because our framework (in which dgas may be nonzero in positive and negative

degrees) differs slightly from the usual one. However, the result is by no means new. Originally, the phenomenon of intrinsic formality was discovered by Halperin and Stasheff [19] in the framework of commutative dgas. They constructed a series of obstruction groups whose vanishing implies intrinsic formality. Later D. Tanré [52] identified these obstruction groups as Harrison cohomology groups. To the best of our knowledge, the noncommutative version, in which Hochschild cohomology replaces Harrison cohomology, is due to T. Kadeishvili [23], who also realized the importance of  $A_\infty$ -algebras in this context. A general survey of  $A_\infty$ -algebras and applications is [27]. It is difficult to find a concrete counterexample, but apparently Theorem 4.7 is not true without the augmentedness assumption. This is related to the problem that a general  $A_\infty$ -algebra with unit might not be quasi-isomorphic to a dga with unit. (We do not know if this question has been settled, but the construction of  $\mathcal{X}$  below would not work.) Let  $A$  be an augmented graded algebra, and let  $\mathcal{B} = (B, d_B)$  be a dga. An  $A_\infty$ -morphism  $\gamma : A \rightarrow \mathcal{B}$  is a sequence of maps of graded  $R$ -bimodules  $\gamma_q \in \text{Hom}_{R-R}((A^+)^{\otimes R^q}, B[1-q])$ ,  $q \geq 1$ , satisfying the equations

$$\begin{aligned} d_B \gamma_q(a_1, \dots, a_q) &= \sum_{i=1}^{q-1} (-1)^{\epsilon_i} (\gamma_{q-1}(a_1, \dots, a_i a_{i+1}, \dots, a_q) \\ &\quad - \gamma_i(a_1, \dots, a_i) \gamma_{q-i}(a_{i+1}, \dots, a_q)). \end{aligned} \quad (E_q)$$

The  $\epsilon_i$  are as in the definition of  $HH^*(A, M)$  above. The first two of these equations are

$$d_B \gamma_1(a_1) = 0, \quad (E_1)$$

$$d_B \gamma_2(a_1, a_2) = (-1)^{\deg(a_1)-1} (\gamma_1(a_1 a_2) - \gamma_1(a_1) \gamma_1(a_2)). \quad (E_2)$$

This means that  $\gamma_1$ , which need not be a homomorphism of algebras, nevertheless induces a multiplicative map  $(\gamma_1)_* : A^+ \rightarrow H(\mathcal{B})$ . In a sense, the nonmultiplicativity of  $\gamma_1$  is corrected by the higher order maps  $\gamma_q$ , so that  $A_\infty$ -morphisms are “approximately multiplicative maps.”

From a more classical point of view, one can see  $A_\infty$ -morphisms simply as a convenient way of encoding dga homomorphisms from a certain large dga canonically associated to  $A$ , a kind of “thickening of  $A$ .” Consider  $V = A^+[1]$  as a graded  $R$ -bimodule, and let  $T^+V = \bigoplus_{q \geq 1} V^{\otimes R^q}$  be its tensor algebra, without unit. We write  $\langle a_1, \dots, a_q \rangle \in T^+V$  instead of  $a_1 \otimes \dots \otimes a_q$ . Now consider  $W = T^+V[-1]$  as a graded  $R$ -bimodule in its own right, and form its tensor algebra with unit  $TW = R \oplus \bigoplus_{r \geq 1} V^{\otimes R^r}$ . The elements of  $TW$  (apart from  $R \subset TW$ ) are linear combinations of expressions of the form

$$x = \langle a_{11}, a_{12}, \dots, a_{1,q_1} \rangle \otimes \dots \otimes \langle a_{r1}, \dots, a_{r,q_r} \rangle$$

with  $r > 0$ ,  $q_1, \dots, q_r > 0$ , and  $a_{ij} \in A^+$ . The degree of such an expression is

$\deg_{TW}(x) = \sum_{ij} \deg_A(a_{ij}) - \sum_i q_i + r$ . One defines a dga  $\mathcal{X} = (X, d_X)$  by taking  $X = TW$  with the tensor multiplication and taking  $d_X$  to be the derivation that acts on elements of  $W$  as follows:

$$d_X \langle a_1, \dots, a_q \rangle = \sum_{i=1}^{q-1} (-1)^{\epsilon_i} (\langle a_1, \dots, a_i a_{i+1}, \dots, a_q \rangle - \langle a_1, \dots, a_i \rangle \otimes \langle a_{i+1}, \dots, a_q \rangle).$$

The passage from  $A$  to  $\mathcal{X}$  is usually written as a composition of the bar and cobar functors, which go from augmented dg algebras to dg coalgebras and back (see, e.g., [35]). We can now make the above-mentioned connection with  $A_\infty$ -morphisms.

LEMMA 4.8

*For any  $A_\infty$ -morphism  $\gamma : A \rightarrow \mathcal{B}$ , one can define a dga homomorphism  $\Gamma : \mathcal{X} \rightarrow \mathcal{B}$  by setting  $\Gamma|_R$  to be the unit map  $\iota_B$ , and  $\Gamma(\langle a_1, \dots, a_q \rangle) = \gamma_q(a_1, \dots, a_q)$ .  $\Gamma$  is a quasi-isomorphism if and only if  $\iota_B \oplus \gamma_1$  induces an isomorphism between  $R \oplus A^+ \cong A$  and  $H(\mathcal{B})$ .*

*Proof*

The first part follows immediately from comparing the equations  $(E_q)$  with the definition of the differential  $d_X$ . As for the second part, a classical computation due to J. Moore [34, Théorème 6.2], [35] shows that the inclusion  $R \oplus A^+ \hookrightarrow \ker d_X$  induces an isomorphism  $R \oplus A^+ \cong H(\mathcal{X})$ . This implies the desired result.  $\square$

As a trivial example, let  $\mathcal{A} = (A, 0)$  be the dga given by  $A$  with zero differential, and take the  $A_\infty$ -morphism  $\gamma : A \rightarrow \mathcal{A}$  given by  $\gamma_1 = \text{id} : A^+ \rightarrow A$ ,  $\gamma_q = 0$  for all  $q \geq 2$ . Then Lemma 4.8 shows that the corresponding map  $\Gamma : \mathcal{X} \rightarrow \mathcal{A}$  is a quasi-isomorphism of dgas.

The next lemma is an instance of “homological perturbation theory” (see, e.g., [18]). Let  $A$  be an augmented graded algebra, let  $\mathcal{B}$  be a dga, and let  $\phi : A \rightarrow H(\mathcal{B})$  be a homomorphism of graded algebras. This makes the cohomology  $H(\mathcal{B})$  into a graded  $A$ -bimodule.

LEMMA 4.9

*Assume that  $HH^q(A, H(\mathcal{B})[2 - q]) = 0$  for all  $q > 2$ . Then there is an  $A_\infty$ -morphism  $\gamma : A \rightarrow \mathcal{B}$  such that the induced map  $(\gamma_1)_* : A^+ \rightarrow H(\mathcal{B})$  is equal to  $\phi|_{A^+}$ .*

*Proof*

Choose a map of graded  $R$ -bimodules  $\gamma_1 : A^+ \rightarrow \ker d_B \subset B$  which induces  $\phi|_{A^+}$ .

Since  $\gamma_1$  is multiplicative on cohomology, we can find a map  $\gamma_2$  such that  $(E_2)$  is satisfied. From here onwards the construction is inductive. Suppose that  $\gamma_1, \dots, \gamma_{q-1}$ , for some  $q \geq 3$ , are maps such that  $(E_1), \dots, (E_{q-1})$  hold. Denote the right-hand side of equation  $(E_q)$  for these maps by  $\psi : (A^+)^{\otimes R^q} \rightarrow B[2-q]$ . One can compute directly that

$$d_B \psi(a_1, \dots, a_q) = 0 \quad (4.3)$$

for all  $a_1, \dots, a_q \in A^+$  and that

$$\begin{aligned} & \gamma_1(a_1) \psi(a_2, \dots, a_{q+1}) + \sum_{i=1}^q (-1)^{\epsilon_i} \psi(a_1, \dots, a_i a_{i+1}, \dots, a_{q+1}) \\ & \quad - (-1)^{\epsilon_q} \psi(a_1, \dots, a_q) \gamma_1(a_{q+1}) \\ & = d_B \left( \sum_{i=1}^q (-1)^{\epsilon_i} \gamma_i(a_1, \dots, a_i) \gamma_{q+1-i}(a_{i+1}, \dots, a_{q+1}) \right). \end{aligned} \quad (4.4)$$

By (4.3),  $\psi$  induces a map  $\bar{\psi} : (A^+)^{\otimes R^q} \rightarrow H(\mathcal{B})[2-q]$ , which is just an element of the Hochschild chain group  $C^q(A, H(\mathcal{B})[2-q])$ . Equation (4.4) says that  $\bar{\psi}$  is a Hochschild cocycle. By assumption, there is an  $\bar{\eta} \in C^{q-1}(A, H(\mathcal{B})[2-q])$  such that  $\partial^{q-1} \bar{\eta} = \bar{\psi}$ . Choose any map of graded  $R$ -bimodules  $\eta : (A^+)^{\otimes R^{q-1}} \rightarrow (\ker d_B)[1-q]$  which induces  $\bar{\eta}$ , and set  $\gamma_{q-1}^{\text{new}} = \gamma_{q-1} - \eta$ . The equations  $(E_1), \dots, (E_{q-1})$  continue to hold if one replaces  $\gamma_{q-1}$  by  $\gamma_{q-1}^{\text{new}}$ . Moreover, if  $\psi^{\text{new}}$  denotes the right-hand side of  $(E_q)$  after this replacement, one computes that

$$\begin{aligned} (\psi - \psi^{\text{new}})(a_1, \dots, a_q) &= (-1)^{\deg(a_1)} \gamma_1(a_1) \eta(a_2, \dots, a_{q-1}) \\ & \quad + \sum_{i=1}^{q-1} (-1)^{\epsilon_i} \eta(a_1, \dots, a_i a_{i+1}, \dots, a_q) \\ & \quad - (-1)^{\epsilon_q} \eta(a_1, \dots, a_{q-1}) \gamma_1(a_q). \end{aligned} \quad (4.5)$$

This means that  $\bar{\psi}^{\text{new}} = \bar{\psi} - \partial^{q-1} \bar{\eta} = 0$ . Clearly, the vanishing of  $\bar{\psi}^{\text{new}}$  ensures that one can extend the sequence  $\gamma_1, \dots, \gamma_{q-2}, \gamma_{q-1}^{\text{new}}$  by a map  $\gamma_q$  such that  $(E_q)$  holds. This completes the induction step.

Note that in the  $q$ th step only the  $(q-1)$ st of the given maps  $\gamma_i$  is changed. Therefore the sequence that we construct does indeed converge to an  $A_\infty$ -morphism  $\gamma$ .  $\square$

#### *Proof of Theorem 4.7*

Let  $\mathcal{B}$  be a dga whose cohomology algebra is isomorphic to  $A$ . Choose an isomorphism  $\phi : A \rightarrow H(\mathcal{B})$ . By Lemma 4.9, there is an  $A_\infty$ -morphism  $\gamma : A \rightarrow \mathcal{B}$  such that  $\gamma_1$  induces  $\phi \mid A^+$ . This obviously means that  $(\iota_B \oplus \gamma_1)_* : R \oplus A^+ \rightarrow H(\mathcal{B})$  is an isomorphism. Hence, by Lemma 4.8, the induced map  $\Gamma : \mathcal{X} \rightarrow \mathcal{B}$  is a

quasi-isomorphism of dgas. We have already seen that there is a quasi-isomorphism  $\mathcal{X} \rightarrow \mathcal{A} = (A, 0)$ . This shows that  $\mathcal{B}$  is quasi-isomorphic to  $\mathcal{A}$  and hence is formal.  $\square$

#### 4.3. The graded algebras $A_{m,n}$

We assume from now on that  $m \geq 2$ ; this assumption is retained throughout this section and the following one. In addition, choose an  $n \geq 1$ . Let  $\Gamma$  be a quiver (an oriented graph) with vertices numbered  $1, \dots, m$ , and with a “degree” (an integer label) attached to each edge. One can associate to it a graded algebra  $k[\Gamma]$ , the path algebra, as follows. As a  $k$ -vector space,  $k[\Gamma]$  is freely generated by the set of all paths (not necessarily closed, of arbitrary length greater than or equal to zero) in  $\Gamma$ . The degree of a path is the sum of all “degrees” of the edges along which it runs. The product of two paths is their composition if the endpoint of the first one coincides with the starting point of the second one, and zero otherwise. The map  $\iota_{k[\Gamma]} : R \rightarrow (k[\Gamma])^0$  maps  $e_i$  to the path of length zero at the  $i$ th vertex.

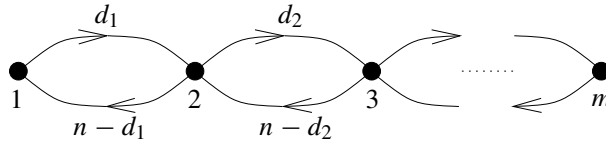


Figure 4

The example we are interested in is the quiver  $\Gamma_{m,n}$  shown in Figure 4. Paths of length  $l \geq 0$  in this quiver correspond to  $(l + 1)$ -tuples  $(i_0 | \dots | i_l)$  with  $i_v \in \{1, \dots, m\}$  and  $|i_{v+1} - i_v| = 1$ . The product of two paths in  $k[\Gamma_{m,n}]$  is given by  $(i_0 | \dots | i_l)(i'_0 | \dots | i'_l) = (i_0 | \dots | i_l | i'_1 | \dots | i'_l)$  if  $i_l = i'_0$ , or zero otherwise. The grading is  $\deg(i) = 0$ ,  $\deg(i | i + 1) = d_i$ ,  $\deg(i + 1 | i) = n - d_i$ , where we set

$$d_i = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even,} \\ \frac{1}{2}(n + (-1)^i) & \text{if } n \text{ is odd.} \end{cases} \quad (4.6)$$

We introduce a 2-sided homogeneous ideal  $J_{m,n} \subset k[\Gamma_{m,n}]$  as follows. If  $m \geq 3$ , then  $J_{m,n}$  is generated by  $(i | i - 1 | i) - (i | i + 1 | i)$ ,  $(i - 1 | i | i + 1)$ , and  $(i + 1 | i | i - 1)$  for all  $i = 2, \dots, m - 1$ ; in the remaining case  $m = 2$ ,  $J_{m,n}$  is generated by  $(1 | 2 | 1 | 2)$  and  $(2 | 1 | 2 | 1)$ . Now define  $A_{m,n} = k[\Gamma_{m,n}] / J_{m,n}$ . This is again a graded algebra. It is finite-dimensional over  $k$ ; an explicit basis is given by the  $(4m - 2)$  elements

$$\left\{ \begin{array}{l} (1), \dots, (m), \\ (1|2), \dots, (m-1|m), \\ (2|1), \dots, (m|m-1), \\ (1|2|1), (2|3|2) = (2|1|2), \dots, (m-1|m|m-1) \\ \qquad \qquad \qquad = (m-1|m-2|m-1), (m|m-1|m). \end{array} \right. \quad (4.7)$$

Here we have used the same notation for elements of  $k[\Gamma_{m,n}]$  and their images in  $A_{m,n}$ . We will continue to do so in the future; in particular,  $(i|i \pm 1|i)$  is used to denote the image of both  $(i|i+1|i)$  and  $(i|i-1|i)$  in  $A_{m,n}$ .

We now explain why these algebras are relevant to our problem. Let  $\mathfrak{K}$  be a category as in Definition 2.2, and let  $E_1, \dots, E_m \in \mathfrak{K}$  be an  $(A_m)$ -configuration of  $n$ -spherical objects.

LEMMA 4.10

*Suppose that for each  $i = 1, \dots, m-1$  the 1-dimensional space  $\text{Hom}^*(E_{i+1}, E_i)$  is concentrated in degree  $d_i$ . Then the cohomology algebra of the dga  $\text{end}(E)$  is isomorphic to  $A_{m,n}$ .*

We should say that the assumption on  $\text{Hom}^*(E_{i+1}, E_i)$  is not really restrictive since, given an arbitrary  $(A_m)$ -configuration, it can always be achieved by shifting each  $E_i$  suitably.

*Proof*

Since each  $E_i$  is  $n$ -spherical, the pairings

$$\begin{aligned} \text{Hom}^*(E_{i+1}, E_i) \otimes \text{Hom}^*(E_i, E_{i+1}) &\longrightarrow \text{Hom}^n(E_i, E_i) \cong k, \\ \text{Hom}^*(E_i, E_{i+1}) \otimes \text{Hom}^*(E_{i+1}, E_i) &\longrightarrow \text{Hom}^n(E_{i+1}, E_{i+1}) \cong k \end{aligned} \quad (4.8)$$

are nondegenerate for  $i = 1, \dots, m-1$ . Hence  $\text{Hom}^*(E_i, E_{i+1}) \cong k$  is concentrated in degree  $n - d_i$ . Choose nonzero elements  $\alpha_i \in \text{Hom}^*(E_{i+1}, E_i)$  and  $\beta_i \in \text{Hom}^*(E_i, E_{i+1})$ . Then, again because of the nondegeneracy of (4.8), one has

$$\alpha_i \beta_i = c_i (\beta_{i-1} \alpha_{i-1}) \quad (4.9)$$

in  $\text{Hom}^*(E, E)$  for some nonzero constants  $c_2, \dots, c_{m-1} \in k$ . Without changing notation, we multiply each  $\beta_i$  with  $c_2 c_3 \dots c_i$ ; then the same equations (4.9) hold with all  $c_i$  equal to 1. Since  $\text{Hom}^*(E_i, E_j) = 0$  for all  $|i - j| \geq 2$ , we also have  $\beta_i \beta_{i-1} = 0$ ,  $\alpha_{i-1} \alpha_i = 0$  for all  $i = 2, \dots, m-1$ . If  $m \geq 3$ , then this shows that there is a homomorphism of graded algebras  $A_{m,n} \rightarrow \text{Hom}^*(E, E)$  which maps  $(i)$  to  $\text{id}_{E_i}$ ,  $(i|i+1)$  to  $\alpha_i$ , and  $(i+1|i)$  to  $\beta_i$ . One sees easily that this is an isomorphism. In the remaining case  $m = 2$ , one has to consider

$$\beta_1 \alpha_1 \beta_1 \in \text{Hom}^{2n-d_1}(E_1, E_2), \quad \alpha_1 \beta_1 \alpha_1 \in \text{Hom}^{n+d_1}(E_2, E_1). \quad (4.10)$$

By assumption  $\text{Hom}^*(E_1, E_2)$  is concentrated in degree  $n - d_1 < 2n - d_1$ , and  $\text{Hom}^*(E_2, E_1)$  is concentrated in degree  $d_1 < n + d_1$ . Hence both elements in (4.10) are zero, which allows one to define  $A_{m,n} \rightarrow \text{Hom}^*(E, E)$  as before. The proof that this is an isomorphism is again straightforward.  $\square$

An inspection of the preceding proof shows that the result remains true for any other choice of numbers  $d_i$  in the definition of  $A_{m,n}$ . Our particular choice (4.6) makes the algebra as “highly connected” as possible;  $A_{m,n}/R \cdot 1$  is concentrated in degrees greater than or equal to  $[n/2]$ . This is useful in the Hochschild cohomology computations of Section 4.5.

Let  $\mathcal{A}_{m,n}$  be the dga given by  $A_{m,n}$  with zero differential. We now consider the properties of the functors  $t_i$  on the category  $D(\mathcal{A}_{m,n})$ .

LEMMA 4.11

*The functors  $t_i : D(\mathcal{A}_{m,n}) \rightarrow D(\mathcal{A}_{m,n})$ ,  $1 \leq i \leq m$ , are exact equivalences.*

*Proof*

This is closely related to the parallel statements in [28] and in Section 2.2. The strategy, as in Proposition 2.10, is to introduce a left adjoint  $t'_i$  of  $t_i$ , and then to prove that the canonical natural transformations  $\text{Id} \rightarrow t_i t'_i, t'_i t_i \rightarrow \text{Id}$  are isomorphisms.

Set  $\mathcal{A} = \mathcal{A}_{m,n}$  and  $\mathcal{Q}_i = \mathcal{P}_i[n] \in \text{Dgm}(\mathcal{A})$ . Define functors  $t'_i$  ( $1 \leq i \leq m$ ) from  $\text{Dgm}(\mathcal{A})$  to itself by

$$t'_i(\mathcal{M}) = \{ \mathcal{M} \xrightarrow{\eta_i} \mathcal{M} e_i \otimes_k \mathcal{Q}_i \},$$

where  $\mathcal{M}$  is placed in degree zero, and  $\eta_i(x) = x(i|i \pm 1|i) \otimes (i) + x(i+1|i) \otimes (i|i+1) + x(i-1|i) \otimes (i|i-1) + x(i) \otimes (i|i \pm 1|i)$ . (In this formula, the second term should be omitted for  $i = m$  and the third term for  $i = 1$ ; the same convention is used again later on.) To understand why  $\eta_i$  is a module homomorphism, it is sufficient to notice that the element

$$(i|i \pm 1|i) \otimes (i) + (i+1|i) \otimes (i|i+1) + (i-1|i) \otimes (i|i-1) + (i) \otimes (i|i \pm 1|i) \in Ae_i \otimes e_i A \quad (4.11)$$

is central, in the sense that left and right multiplication (with respect to the obvious  $A$ -bimodule structure of  $Ae_i \otimes e_i A$ ) with any  $a \in A$  have the same effect on it. The same argument as for  $t_i$  shows that  $t'_i$  descends to exact functors on  $K(\mathcal{A})$  and  $D(\mathcal{A})$ . For any  $\mathcal{M} \in \text{Dgm}(\mathcal{A})$ , consider the complex of dgms

$$\mathcal{C}_{-1} = \mathcal{M} e_i \otimes \mathcal{P}_i \xrightarrow{\delta_{-1}} \mathcal{C}_0 = \mathcal{M} \oplus (\mathcal{M} e_i \otimes e_i A e_i \otimes \mathcal{Q}_i) \xrightarrow{\delta_0} \mathcal{C}_1 = \mathcal{M} e_i \otimes \mathcal{Q}_i,$$

where  $\delta_{-1}(x \otimes a) = (xa, x \otimes a(i|i \pm 1|i) \otimes (i) + x \otimes a(i+1|i) \otimes (i|i+1) + x \otimes a(i-1|i) \otimes (i|i-1) + x \otimes a(i) \otimes (i|i \pm 1|i)) = (xa, x \otimes (i) \otimes (i|i \pm 1|i)a + x \otimes (i|i \pm 1|i) \otimes a)$  and  $\delta_0(x, y \otimes a \otimes b) = (\eta_i(x) - ya \otimes b)$ . The reason why the second expression for  $\delta_{-1}$  is equal to the first one is again that the element (4.11) is central. A straightforward computation (including some tedious sign checking) shows that the dgm  $\mathcal{C}$  obtained by collapsing this complex is equal to  $t'_i t_i(\mathcal{M})$ .

The algebra  $e_i A e_i = k(i) \oplus k(i|i \pm 1|i)$  is simply a 2-dimensional graded  $k$ -vector space, nontrivial in degrees zero and  $n$ . Take the homomorphism of dgms

$$\begin{aligned} \mathcal{C}_0 &= \mathcal{M} \oplus (\mathcal{M} e_i \otimes e_i A e_i \otimes \mathcal{P}_i) \longrightarrow \mathcal{M}, \\ (x, y_1 \otimes (i) \otimes b_1 + y_2 \otimes (i|i \pm 1|i) \otimes b_2) &\longmapsto x - y_2 b_2. \end{aligned} \quad (4.12)$$

Extending this by zero to  $\mathcal{C}_{-1}$ ,  $\mathcal{C}_1$  yields a dgm homomorphism  $\psi_{\mathcal{M}}: \mathcal{C} = t'_i t_i(\mathcal{M}) \rightarrow \mathcal{M}$  because (4.12) vanishes on the image of  $\delta_{-1}$ . This homomorphism is surjective for any  $\mathcal{M}$ , and a computation similar to that in Proposition 2.10 shows that the kernel is always an acyclic dgm. Since  $\psi_{\mathcal{M}}$  is natural in  $\mathcal{M}$ , we have indeed provided an isomorphism  $t'_i t_i \cong \text{Id}_{D(\mathcal{A})}$ . The proof that  $t_i t'_i \cong \text{Id}_{D(\mathcal{A})}$  is parallel.  $\square$

LEMMA 4.12

*The functors  $t_i$  on  $D(\mathcal{A}_{m,n})$  satisfy the braid relations (up to graded natural isomorphism)*

$$\begin{aligned} t_i t_{i+1} t_i &\cong t_{i+1} t_i t_{i+1} \quad \text{for } i = 1, \dots, m-1, \\ t_i t_j &\cong t_j t_i \quad \text{for } |i-j| \geq 2. \end{aligned}$$

*Proof*

The second relation is easy (it follows immediately from the fact that  $e_i A_{m,n} e_j = 0$  for  $|i-j| \geq 2$ ), and we therefore concentrate on the first one. Moreover, we only explain the salient points of the argument. (A different version of it is described in [28] with full details.) Note that the approach taken in Proposition 2.13 cannot be adapted directly to the present case since we have not developed a general theory of twist functors on derived categories of dgms.

Set  $\mathcal{A} = \mathcal{A}_{m,n}$  and  $\mathcal{R}_i = \mathcal{P}_i[-n]$ . For any  $\mathcal{M} \in \text{Dgm}(\mathcal{A})$ , consider the complex of dgms

$$\mathcal{C}_{-3} \xrightarrow{\delta_{-3}} \mathcal{C}_{-2} \xrightarrow{\delta_{-2}} \mathcal{C}_{-1} \xrightarrow{\delta_{-1}} \mathcal{C}_0, \quad (4.13)$$

where

$$\begin{aligned} \mathcal{C}_{-3} &= \mathcal{M} e_i \otimes \mathcal{R}_i, \\ \mathcal{C}_{-2} &= (\mathcal{M} e_i \otimes e_i A e_i \otimes \mathcal{P}_i) \oplus (\mathcal{M} e_i \otimes e_i A e_{i+1} \otimes \mathcal{P}_{i+1}) \\ &\quad \oplus (\mathcal{M} e_{i+1} \otimes e_{i+1} A e_i \otimes \mathcal{P}_i), \end{aligned}$$

$$\begin{aligned}\mathcal{C}_{-1} &= (\mathcal{M}e_i \otimes \mathcal{P}_i) \oplus (\mathcal{M}e_{i+1} \otimes \mathcal{P}_{i+1}) \oplus (\mathcal{M}e_i \otimes \mathcal{P}_i), \\ \mathcal{C}_0 &= \mathcal{M},\end{aligned}$$

and

$$\begin{aligned}\delta_{-3} : (x \otimes a) &\mapsto \begin{pmatrix} -x \otimes (i|i+1|i) \otimes a \\ x \otimes (i|i+1) \otimes (i+1|i)a \\ x(i|i+1) \otimes (i+1|i) \otimes a \end{pmatrix}, \\ \delta_{-2} : \begin{pmatrix} x_1 \otimes a_1 \otimes b_1 \\ x_2 \otimes (i|i+1) \otimes b_2 \\ x_3 \otimes (i+1|i) \otimes b_3 \end{pmatrix} &\mapsto \begin{pmatrix} x_1 \otimes a_1 b_1 + x_2 \otimes (i|i+1)b_2 \\ -x_2(i|i+1) \otimes b_2 + x_3 \otimes (i+1|i)b_3 \\ -x_1 a \otimes b_1 - x_3(i+1|i) \otimes b_3 \end{pmatrix}, \\ \delta_{-1} : \begin{pmatrix} x_1 \otimes a_1 \\ x_2 \otimes a_2 \\ x_3 \otimes x_3 \end{pmatrix} &\mapsto x_1 a_1 + x_2 a_2 + x_3 a_3.\end{aligned}$$

As in the proof of Lemma 4.11, one can contract this complex to a single dgm, which is in fact canonically isomorphic to  $t_i t_{i+1} t_i(\mathcal{M})$ . Now, one can map the whole complex (4.13) surjectively to an acyclic complex (concentrated in degrees  $-3$  and  $-2$ )

$$\mathcal{M}e_i \otimes \mathcal{R}_i \xrightarrow{\text{id}} \mathcal{M}e_i \otimes \mathcal{R}_i.$$

This is done by taking the identity map on  $\mathcal{C}_{-3}$  together with the homomorphism  $\mathcal{C}_{-2} \supset \mathcal{M}e_i \otimes e_i A e_i \otimes \mathcal{P}_i \rightarrow \mathcal{M}e_i \otimes \mathcal{R}_i$ ,  $m_1 \otimes (i) \otimes b_1 + m_2 \otimes (i|i+1|i) \otimes b_2 \mapsto -m_2 \otimes b_2$  and extending this by zero to the other summands of  $\mathcal{C}_{-2}$  and to  $\mathcal{C}_{-1}$ ,  $\mathcal{C}_0$ . The kernel of the dgm homomorphism defined in this way is a certain subcomplex of (4.13). When writing this down explicitly (which we do not do here), one notices that it contains an acyclic subcomplex isomorphic to

$$\mathcal{M}e_i \otimes \mathcal{P}_i \xrightarrow{\text{id}} \mathcal{M}e_i \otimes \mathcal{P}_i,$$

located in degrees  $-2$  and  $-1$ . If one divides out this acyclic subcomplex, what remains is the complex

$$\begin{aligned}& (\mathcal{M}e_i \otimes e_i A e_{i+1} \otimes \mathcal{P}_{i+1}) \oplus (\mathcal{M}e_{i+1} \otimes e_{i+1} A e_i \otimes \mathcal{P}_i) \\ & \xrightarrow{\delta'_{-1}} (\mathcal{M}e_i \otimes \mathcal{P}_i) \oplus (\mathcal{M}e_{i+1} \otimes \mathcal{P}_{i+1}) \xrightarrow{\delta'_0} \mathcal{M},\end{aligned}\tag{4.14}$$

with  $\delta'_{-1}(x_1 \otimes (i|i+1) \otimes b_1, x_2 \otimes (i+1|i) \otimes b_2) = (x_1 \otimes (i|i+1)b_1 - x_2(i+1|i) \otimes b_2, -x_1(i|i+1) \otimes b_1 + x_2 \otimes (i+1|i)b_2)$ ,  $\delta'_0(x_1 \otimes a_1, x_2 \otimes a_2) = x_1 a_1 + x_2 a_2$ . The remarkable fact about (4.14) is that it is symmetric with respect to exchanging  $i$  and  $i+1$ . Indeed, one can arrive at the same complex by starting with  $t_{i+1} t_i t_{i+1}(\mathcal{M})$  and removing acyclic parts. This shows that  $t_{i+1} t_i t_{i+1}(\mathcal{M})$  and  $t_i t_{i+1} t_i(\mathcal{M})$  are quasi-

isomorphic for all  $\mathcal{M}$ . We leave it to the reader to verify that the argument provides a chain of exact functors and graded natural isomorphisms between them, with  $t_i t_{i+1} t_i$  and  $t_{i+1} t_i t_{i+1}$  at the two ends of the chain.  $\square$

#### 4.4. Geometric intersection numbers

Consider the weak braid group action  $\rho_{m,n} : B_{m+1} \rightarrow \text{Auteq}(D(\mathcal{A}_{m,n}))$  generated by  $t_1, \dots, t_m$ . The aim of this section is prove a strong form of faithfulness for it.

##### THEOREM 4.13

Let  $R_{m,n}^g$  be a functor representing  $\rho_{m,n}(g)$  for some  $g \in B_{m+1}$ . If  $R_{m,n}^g(\mathcal{P}_j) \cong \mathcal{P}_j$  for all  $j$ , then  $g$  must be the identity element.

We begin by looking at the center of  $B_{m+1}$ . It is infinite cyclic and generated by an element that, in terms of the standard generators  $g_1, \dots, g_m$ , can be written as  $(g_1 g_2 \cdots g_m)^{m+1}$ .

##### LEMMA 4.14

For any  $1 \leq j \leq m$ ,  $(t_1 t_2 \cdots t_m)^{m+1}(\mathcal{P}_j)$  is isomorphic to  $\mathcal{P}_j[2m - (m+1)n]$  in  $D(\mathcal{A}_{m,n})$ .

*Proof*

For each  $1 \leq j \leq m$  there is a short exact sequence of dgms

$$0 \longrightarrow \mathcal{P}_j[-n] \xrightarrow{\alpha} e_j A_{m,n} e_j \otimes_k \mathcal{P}_j \xrightarrow{\text{multiplication}} \mathcal{P}_j \longrightarrow 0,$$

where  $\alpha(x) = (j|j \pm 1|j) \otimes x - (j) \otimes (j|j \pm 1|j)x$ . This implies that the cone of the multiplication map, which is  $t_j(\mathcal{P}_j)$ , is isomorphic to  $\mathcal{P}_j[1-n]$  in  $D(\mathcal{A}_{m,n})$ . Note also that  $t_i(\mathcal{P}_j) \cong \mathcal{P}_j$  whenever  $|i-j| \geq 2$ .

Consider the  $m+1$  differential graded modules

$$\mathcal{M}_0 = \left\{ \mathcal{P}_1[n-1] \longrightarrow \mathcal{P}_2[2n-1-d_1] \longrightarrow \cdots \longrightarrow \mathcal{P}_m[mn-1-d_1-\cdots-d_{m-1}] \right\},$$

$$\mathcal{M}_1 = \mathcal{P}_1,$$

$$\mathcal{M}_2 = \mathcal{P}_2[1-d_1],$$

$$\mathcal{M}_3 = \mathcal{P}_3[2-d_1-d_2],$$

$$\dots$$

$$\mathcal{M}_m = \mathcal{P}_m[m-1-d_1-\cdots-d_{m-1}].$$

The definition of  $\mathcal{M}_0$  is given by collapsing the complex of dgms in which  $\mathcal{P}_1[n-1]$  is placed in degree zero, and where the maps are given by left multiplication with  $(i+1|i)$ . We prove that

$$\begin{cases} (t_1 t_2 \cdots t_m)(\mathcal{M}_0) \cong \mathcal{M}_1, \\ (t_1 t_2 \cdots t_m)(\mathcal{M}_i) \cong \mathcal{M}_{i+1} \quad \text{for } 1 \leq i < m, \\ (t_1 t_2 \cdots t_m)(\mathcal{M}_m) \cong \mathcal{M}_0[2m - (m+1)n], \end{cases} \quad (4.15)$$

which clearly implies the desired result. By the definitions of  $t_i$  and  $t'_i$ , the second of which is given in the proof of Lemma 4.11, one has

$$t_{i+1}(\mathcal{P}_i) = \{\mathcal{P}_{i+1}[-d_i] \longrightarrow \mathcal{P}_i\} \cong t'_i(\mathcal{P}_{i+1})[1 - d_i],$$

where  $\mathcal{P}_i$  is placed in degree zero and the arrow is left multiplication with  $(i|i+1)$ . This shows that  $t_i t_{i+1}(\mathcal{P}_i) \cong \mathcal{P}_{i+1}[1 - d_i]$ , and since  $t_i(\mathcal{P}_j) \cong \mathcal{P}_j$  whenever  $|i - j| \geq 2$ , it proves the second equation in (4.15). To verify the other two equations one computes

$$\begin{aligned} & (t_1 t_2 \cdots t_m)(\mathcal{P}_m[n-1]) \\ & \cong (t_1 t_2 \cdots t_{m-1})(\mathcal{P}_m) \\ & \cong (t_1 t_2 \cdots t_{m-2})(\{\mathcal{P}_{m-1}[-n + d_{m-1}] \longrightarrow \mathcal{P}_m\}) \\ & \cong (t_1 t_2 \cdots t_{m-3}) \\ & \quad \cdot (\{\mathcal{P}_{m-2}[-2n + d_{m-2} + d_{m-1}] \longrightarrow \mathcal{P}_{m-1}[-n + d_{m-1}] \longrightarrow \mathcal{P}_m\}) \\ & \cong \cdots \cong \mathcal{M}_0[m(1-n) + d_1 + \cdots + d_{m-1}] \end{aligned}$$

and

$$\begin{aligned} & (t'_m \cdots t'_2 t'_1)(\mathcal{P}_1) \\ & \cong (t'_m \cdots t'_2)(\mathcal{P}_1[n-1]) \\ & \cong (t'_m \cdots t'_3)(\{\mathcal{P}_1[n-1] \longrightarrow \mathcal{P}_2[2n-1-d_1]\}) \\ & \cong (t'_m \cdots t'_4) \\ & \quad \cdot (\{\mathcal{P}_1[n-1] \longrightarrow \mathcal{P}_2[2n-1-d_1] \longrightarrow \mathcal{P}_3[3n-1-d_1-d_2]\}) \\ & \cong \cdots \cong \mathcal{M}_0. \end{aligned} \quad \square$$

It seems likely that  $(t_1 t_2 \cdots t_m)^{m+1}$  is in fact isomorphic to the translation functor  $[2m - (m+1)n]$ , but we have not checked this.

Before proceeding further, we need to recall some basic notions from the topology of curves on surfaces. Let  $D$  be a closed disc, and let  $\Delta \subset D \setminus \partial D$  be a set of  $m+1$  marked points.  $\text{Diff}(D, \partial D; \Delta)$  denotes the group of diffeomorphisms  $f : D \rightarrow D$  which satisfy  $f|_{\partial D} = \text{id}$  and  $f(\Delta) = \Delta$ . We write  $f_0 \simeq f_1$  for isotopy within this group. By a curve in  $(D, \Delta)$ , we mean a subset  $c \subset D \setminus \partial D$  which can be represented as the image of a smooth embedding  $\gamma : [0; 1] \rightarrow D$  such that  $\gamma^{-1}(\Delta) = \{0; 1\}$ . In other words,  $c$  is an unoriented embedded path in  $D \setminus \partial D$  whose endpoints lie

in  $\Delta$ , and which does not meet  $\Delta$  anywhere else. There is an obvious notion of isotopy for curves, denoted again by  $c_0 \simeq c_1$ . For any two curves  $c_0, c_1$  there is a geometric intersection number  $I(c_0, c_1) \geq 0$ , which is defined by  $I(c_0, c_1) = |(c'_0 \cap c_1) \setminus \Delta| + (1/2)|(c'_0 \cap c_1) \cap \Delta|$  for some  $c'_0 \simeq c_0$  which has minimal intersection with  $c_1$ . (This means, roughly speaking, that  $c'_0$  is obtained from  $c_0$  by removing all unnecessary intersection points with  $c_1$ .) We refer to [28, Section 2a] for the proof that this is well defined. Once one has shown this, the following properties are fairly obvious:

- (I1)  $I(c_0, c_1)$  depends only on the isotopy classes of  $c_0$  and  $c_1$ ;
- (I2)  $I(c_0, c_1) = I(f(c_0), f(c_1))$  for all  $f \in \text{Diff}(D, \partial D; \Delta)$ ;
- (I3)  $I(c_0, c_1) = I(c_1, c_0)$ .

Note that in general  $I(c_0, c_1)$  is only a half-integer because of the weight  $1/2$  which the common endpoints of  $c_0$  and  $c_1$  contribute. The next lemma, whose proof we omit, is a modified version of [13, Proposition III.16].

LEMMA 4.15

Let  $c_0, c_1$  be two curves in  $(D, \Delta)$  such that  $I(d, c_0) = I(d, c_1)$  for all  $d$ . Then  $c_0 \simeq c_1$ .

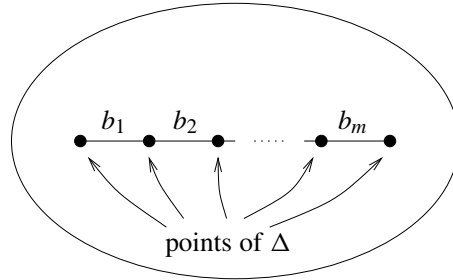


Figure 5

From now on, fix a collection of curves  $b_1, \dots, b_m$  as in Figure 5, as well as an orientation of  $D$ . Then one can identify  $\pi_0(\text{Diff}(D, \partial D; \Delta))$  with the braid group by mapping the standard generators  $g_1, \dots, g_m \in B_{m+1}$  to positive half-twists along  $b_1, \dots, b_m$ .

LEMMA 4.16

Let  $f \in \text{Diff}(D, \partial D; \Delta)$  be a diffeomorphism that satisfies  $f(b_j) \simeq b_j$  for all  $1 \leq j \leq m$ . The corresponding element  $g \in B_{m+1}$  must be of the form  $g = (g_1 g_2 \cdots g_m)^{v(m+1)}$  for some  $v \in \mathbb{Z}$ .

*Proof*

Since  $f(b_j) \simeq b_j$ ,  $f$  commutes up to isotopy with the half-twist along  $b_j$  and hence with any element of  $\text{Diff}(D, \partial D; \Delta)$ . This implies that  $g$  is central.  $\square$

The next lemma, which is far more substantial than the previous ones, establishes a relationship between the topology of curves in  $(D, \Delta)$  and the algebraically defined braid group action  $\rho_{m,n}$ .

LEMMA 4.17

For  $g \in B_{m+1}$ , let  $f \in \text{Diff}(D, \partial D; \Delta)$  be a diffeomorphism in the isotopy class corresponding to  $g$ , and let  $R_{m,n}^g$  be a functor that represents  $\rho_{m,n}(g)$ . Then

$$\sum_{r \in \mathbb{Z}} \dim_k \text{Hom}_{D(\mathcal{A}_{m,n})}(\mathcal{P}_i, R_{m,n}^g(\mathcal{P}_j)[r]) = 2I(b_i, f(b_j))$$

for all  $1 \leq i, j \leq m$ .

A statement of the same kind, concerning a category and braid group action slightly different from ours, has been proved in [28, Theorem 1.1]. In principle, the proof given there can be adapted to our situation, but verifying all the details is a rather tedious business. For this reason we take a slightly different approach, which is to derive the result as stated here from its counterpart in [28]. To do this, we first need to recall the situation considered in that paper. In order to avoid confusion, objects that belong to the setup of [28] are denoted by overlined symbols.

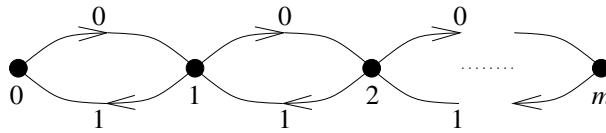


Figure 6

Consider the quiver  $\overline{\Gamma}_m$  in Figure 6 with vertices numbered  $0, \dots, m$  and whose edges are labeled with “degrees” zero or 1. Paths of length  $l$  in  $\overline{\Gamma}_m$  are described by  $(l+1)$ -tuples of numbers  $i_0, \dots, i_l \in \{0, \dots, m\}$ ; we use the notation  $(\overline{i_0} | \dots | \overline{i_l})$  for them. The path algebra  $k[\overline{\Gamma}_m]$  is a graded algebra, whose ground ring is  $\overline{R} = k^{m+1}$ . Let  $\overline{J}_m$  be the homogeneous 2-sided ideal in it, generated by the elements  $(\overline{i} - 1 | \overline{i} + 1)$ ,  $(\overline{i} + 1 | \overline{i} - 1)$ ,  $(\overline{i} | \overline{i} + 1 | \overline{i}) - (\overline{i} | \overline{i} - 1 | \overline{i})$  ( $1 \leq i \leq m-1$ ), and  $(\overline{0} | \overline{1} | \overline{0})$ . The quotient  $\overline{A}_m = k[\overline{\Gamma}_m] / \overline{J}_m$  is a finite-dimensional graded algebra; a concrete basis is given by the  $4m+1$  elements

$$\left\{ \begin{array}{l} (\bar{0}), \dots, (\bar{m}), (\bar{0|1}), \dots, (\overline{m-1|m}) \text{ of degree zero, and} \\ (\bar{1|0}), \dots, (\overline{m|m-1}), (\bar{1|2|1}) = (\bar{1|0|1}), \dots, (\overline{m-1|m-2|m-1}) \\ \quad = (\overline{m-1|m|m-1}), (\overline{m|m-1|m}) \text{ of degree 1.} \end{array} \right. \quad (4.16)$$

$\bar{A}_m$  is evidently a close cousin of our algebras  $A_{m,n}$ . We now make the relationship precise on the level of categories. Let  $\bar{A}_m\text{-mod}$  be the abelian category of finitely generated graded right modules over  $\bar{A}_m$ , and let  $D^b(\bar{A}_m\text{-mod})$  be its bounded derived category. (In contrast to the situation in Section 4.1, this is the derived category in the ordinary sense, not in the differential graded one.) There is an automorphism  $\{1\}$  that shifts the grading of a module up by 1. This descends to an automorphism of  $D^b(\bar{A}_m\text{-mod})$ , which is not the same as the translation functor. In particular, for any  $X, Y \in D^b(\bar{A}_m\text{-mod})$  there is a bigraded vector space

$$\bigoplus_{r_1, r_2} \text{Hom}_{D^b(\bar{A}_m\text{-mod})}(X, Y\{r_1\}[r_2]).$$

We denote by  $\bar{P}_i \in \bar{A}_m\text{-mod}$  the projective modules  $(\bar{i})\bar{A}_m$  for  $0 \leq i \leq m$ . Let  $\mathfrak{P} \subset \bar{A}_m\text{-mod}$  be the full subcategory whose objects are direct sums of  $\bar{P}_i\{r\}$  for  $i = 1, \dots, m$  and  $r \in \mathbb{Z}$ ; the important thing is that  $\bar{P}_0$  is not allowed. We write  $K^b(\mathfrak{P})$  for the full subcategory of  $K^b(\bar{A}_m\text{-mod})$  whose objects are finite complexes in  $\mathfrak{P}$ . This is an abuse of notation since  $\mathfrak{P}$  is not an abelian category; however,  $K^b(\mathfrak{P})$  is still a triangulated category because it contains the cone of any homomorphism.

LEMMA 4.18

There is an exact functor  $\Pi : K^b(\mathfrak{P}) \rightarrow D(\mathcal{A}_{m,n})$  with the following properties:

- (1)  $\Pi(\bar{P}_i)$  is isomorphic to  $\mathcal{P}_i$  up to some shift;
- (2) there is a canonical isomorphism of functors  $\Pi \circ \{1\} \cong [-n] \circ \Pi$ ;
- (3) the natural map, which exists in view of property 2,

$$\bigoplus_{r_2 = nr_1} \text{Hom}_{K^b(\mathfrak{P})}(X, Y\{r_1\}[r_2]) \longrightarrow \text{Hom}_{D(\mathcal{A}_{m,n})}(\Pi(X), \Pi(Y)),$$

is an isomorphism for all  $X, Y \in K^b(\mathfrak{P})$ .

*Proof*

As a first step, consider the functor  $\Pi' : \mathfrak{P} \rightarrow \text{Dgm}(\mathcal{A}_{m,n})$  defined as follows. The object  $\bar{P}_i\{r\}$  goes to the dgm  $\mathcal{P}_i[\sigma_i - nr]$ , where  $\sigma_i = -d_1 - d_2 - \dots - d_{i-1}$ , and this is extended to direct sums in the obvious way. Let  $\bar{A}_m^d$  be the space of elements of degree  $d$  in  $\bar{A}_m$ . Homomorphisms of graded modules  $\bar{P}_i\{r\} \rightarrow \bar{P}_j\{s\}$  correspond in a natural way to elements of  $(\bar{j})\bar{A}_m^{r-s}(\bar{i})$ . On the other hand, dgm homomorphisms between  $\mathcal{P}_i[\sigma_i - nr]$  and  $\mathcal{P}_j[\sigma_j - ns]$  correspond to elements of degree  $\sigma_j - \sigma_i - n(s - r)$

in  $(j)A_{m,n}(i)$ . There is an obvious isomorphism, for any  $1 \leq i, j \leq m$  and  $d \in \mathbb{Z}$ ,

$$(\bar{j})\bar{A}_m^d(\bar{i}) \cong (j)A_{m,n}^{\sigma_j - \sigma_i + nd}(i) \quad (4.17)$$

which sends any basis element in (4.16) of the form  $(\bar{i}_0 | \cdots | \bar{i}_v)$  to the corresponding element  $(i_0 | \cdots | i_v) \in A_{m,n}$ ; one needs to check, case by case, that the degrees turn out right. We use (4.17) to define  $\Pi'$  on morphisms; this is obviously compatible with composition, so that the outcome is indeed a functor. Note that  $\Pi' \circ \{1\} \cong [-n] \circ \Pi'$ .

Now take a finite chain complex in  $\mathfrak{P}$ . Applying  $\Pi'$  to each object in the complex yields a chain complex in  $\text{Dgm}(\mathcal{A}_{m,n})$ , which one can then collapse into a single dgm. This procedure yields a functor  $K^b(\mathfrak{P}) \rightarrow K(\mathcal{A}_{m,n})$ , which is exact since it carries cones to cones. We define  $\Pi$  to be the composition of this with the quotient functor  $K(\mathcal{A}_{m,n}) \rightarrow D(\mathcal{A}_{m,n})$ . Properties (1) and (2) are now obvious from the definition of  $\Pi'$ . The remaining property (3) can be reduced, by repeated use of the five lemma, to the case when  $X = \bar{P}_i\{r\}$ ,  $Y = \bar{P}_i\{s\}$ ; then it comes down to the fact that (4.17) is an isomorphism.  $\square$

Define exact functors  $\bar{t}_1, \dots, \bar{t}_m$  from  $D^b(\bar{A}_m\text{-mod})$  to itself by

$$\bar{t}_i(X) = \{X(\bar{i}) \otimes_k \bar{P}_i \longrightarrow X\}. \quad (4.18)$$

Here  $X(\bar{i})$  is considered as a complex of graded  $k$ -vector spaces; tensoring with  $\bar{P}_i$  over  $k$  makes this into a complex of graded  $\bar{A}_m$ -modules, and the arrow is the multiplication map. We now state the results of [28].

LEMMA 4.19

*The functors  $\bar{t}_1, \dots, \bar{t}_m$  are exact equivalences and generate a weak braid group action  $\bar{\rho}_m : B_{m+1} \rightarrow \text{Auteq}(D^b(\bar{A}_m\text{-mod}))$ .*

LEMMA 4.20

*For  $g \in B_{m+1}$ , let  $f \in \text{Diff}(D, \partial D; \Delta)$  be a diffeomorphism in the isotopy class corresponding to  $g$ , and let  $\bar{R}_m^g$  be a functor that represents  $\bar{\rho}_m(g)$ . Then*

$$\sum_{r_1, r_2} \dim_k \text{Hom}_{D^b(\bar{A}_m\text{-mod})}(\bar{P}_i, \bar{R}_m^g(\bar{P}_j)\{r_1\}[r_2]) = 2I(b_i, f(b_j))$$

*for all  $1 \leq i, j \leq m$ .*

Lemma 4.19 essentially summarizes the contents of [28, Section 3], and Lemma 4.20 is [28, Theorem 1.1]. The notation here is slightly different. (Our  $\bar{A}_m$ ,  $\bar{P}_i$ , and  $\bar{t}_i$  are the  $A_m$ ,  $P_i$ , and  $\mathcal{R}_i$  of that paper.) We have also modified the definitions very slightly; namely, we use right modules instead of left modules as in [28], and the coefficients

are  $k$  instead of  $\mathbb{Z}$ . These changes do not affect the results at all. (A very conscientious reader might want to check that inversion of paths defines an isomorphism between  $\overline{A}_m$  and its opposite, and that a result similar to Lemma 4.18 can be proved for an algebra  $\overline{A}_m$  defined over  $\mathbb{Z}$ .)

*Proof of Lemma 4.17*

Since the modules  $\overline{P}_i$  are projective, the obvious exact functor  $K^b(\mathfrak{P}) \rightarrow D^b(\overline{A}_m\text{-mod})$  is full and faithful. To save notation, we consider  $K^b(\mathfrak{P})$  simply as a subcategory of  $D^b(\overline{A}_m\text{-mod})$ . An inspection of (4.18) shows that the  $\bar{t}_i$  preserve this subcategory, and the same is true of their inverses, defined in [28]. In other words, the weak braid group action  $\bar{\rho}_m$  restricts to one on  $K^b(\mathfrak{P})$ . It follows from the definition of  $\Pi$  that  $\Pi \circ \bar{t}_i|_{K^b(\mathfrak{P})} \cong t_i \circ \Pi$ . Hence, if  $\bar{R}_m^g$  and  $R_{m,n}^g$  are functors representing  $\bar{\rho}_m(g)$ , respectively,  $\rho_{m,n}(g)$ , the diagram

$$\begin{array}{ccc} K^b(\mathfrak{P}) & \xrightarrow{\bar{R}_m^g} & K^b(\mathfrak{P}) \\ \Pi \downarrow & & \downarrow \Pi \\ D(\mathcal{A}_{m,n}) & \xrightarrow{R_{m,n}^g} & D(\mathcal{A}_{m,n}) \end{array}$$

commutes up to isomorphism. Using this, Lemma 4.18(3), and Lemma 4.20, one sees that

$$\begin{aligned} & \sum_r \dim_k \operatorname{Hom}_{D(\mathcal{A}_{m,n})}(\mathcal{P}_i, R_{m,n}^g(\mathcal{P}_j)[r]) \\ &= \sum_r \dim_k \operatorname{Hom}_{D(\mathcal{A}_{m,n})}(\Pi(\overline{P}_i), \Pi \bar{R}_m^g(\overline{P}_j)[r]) \\ &= \sum_{r_1, r_2} \dim_k \operatorname{Hom}_{D^b(\overline{A}_m\text{-mod})}(\overline{P}_i, \bar{R}_m^g(\overline{P}_j)\{r_1\}[r_2]) \\ &= 2I(b_i, f(b_j)). \end{aligned} \quad \square$$

*Proof of Theorem 4.13*

For  $g \in B_{m+1}$ , choose  $f$  and  $R_{m,n}^g$  as in Lemma 4.17. Take also another element  $g' \in B_{m+1}$  and, correspondingly,  $f'$  and  $R_{m,n}^{g'}$ . Applying Lemma 4.17 to  $(g')^{-1}g$  shows that

$$\begin{aligned} I(f'(b_i), f(b_j)) &= I(b_i, (f')^{-1}f(b_j)) \\ &= \frac{1}{2} \sum_r \dim_k \operatorname{Hom}(\mathcal{P}_i, (R_{m,n}^{g'})^{-1} R_{m,n}^g(\mathcal{P}_j)) \end{aligned}$$

and, assuming that  $R_{m,n}^g(\mathcal{P}_j) \cong \mathcal{P}_j$  for all  $j$ ,

$$\begin{aligned}
&= \frac{1}{2} \sum_r \dim_k \operatorname{Hom}(\mathcal{P}_i, (R_{m,n}^{g'})^{-1}(\mathcal{P}_j)) \\
&= I(b_i, (f')^{-1}(b_j)) = I(f'(b_i), b_j).
\end{aligned}$$

Since  $i$  and  $f'$  can be chosen arbitrarily, it follows from Lemma 4.15 that  $f(b_j) \simeq b_j$  for all  $j$ . Hence, by Lemma 4.16,  $g = (g_1 g_2 \cdots g_m)^{v(m+1)}$  for some  $v \in \mathbb{Z}$ . But then  $R_{m,n}^g(\mathcal{P}_j) \cong \mathcal{P}_j[v(2m - (m+1)n)]$  by Lemma 4.14. In view of the assumption that  $R_{m,n}^g(\mathcal{P}_j) \cong \mathcal{P}_j$ , this implies that  $v = 0$  and hence that  $g = 1$ .  $\square$

#### 4.5. Conclusion

The graded algebras  $A_{m,n}$  are always augmented. For  $n \geq 2$  they are even connected, so that there is only one choice of augmentation map. This makes it possible to apply Theorem 4.7.

##### LEMMA 4.21

$A_{m,n}$  is intrinsically formal for all  $m, n \geq 2$ .

The proof is by a straight computation of Hochschild cohomology. (It would be nice to have a more conceptual explanation of the result.) Its difficulty depends strongly on the parameter  $n$ . The easy case is when  $n > 2$ , since then already the relevant Hochschild cochain groups are zero; this is no longer true for  $n = 2$ . At first sight the computation may appear to rely on our specific choice (4.6) of degrees  $d_i$ , but in fact this only serves to simplify the bookkeeping; the Hochschild cohomology remains the same for any other choice. Throughout, we write  $\Gamma, A$  instead of  $\Gamma_{m,n}, A_{m,n}$ .

##### *Proof for $n > 2$*

Note that the “degree” label on any edge of  $\Gamma$  is greater than or equal to  $\lfloor n/2 \rfloor$ . Moreover, the labels on any two consecutive edges add up to  $n$ . These two facts imply that the degree of any nonzero path  $(i_0 | \cdots | i_l)$  of length  $l$  in  $k[\Gamma]$  is greater than or equal to  $\lfloor (nl)/2 \rfloor$ . Now, any element of  $(A^+)^{\otimes_{R^q}}$  can be written as a sum of expressions of the form

$$c = (i_{1,0} | \cdots | i_{1,l_1}) \otimes (i_{2,0} | \cdots | i_{2,l_2}) \otimes \cdots \otimes (i_{q,0} | \cdots | i_{q,l_q}),$$

with all  $l_q > 0$ . Because the tensor product is over  $R$ , such a  $c$  can be nonzero only if the paths  $(i_{v,0} | \cdots | i_{v,l_v})$  match up, in the sense that  $i_{v,l_v} = i_{v+1,0}$ . Then, using the observation made above, one finds that

$$\deg(c) = \deg(i_{1,0} | \cdots | i_{1,l_1} | i_{2,1} | \cdots | i_{2,l_2} | i_{3,1} | \cdots | i_{q,l_q}) \geq \left\lceil n \frac{(l_1 + \cdots + l_q)}{2} \right\rceil.$$

Hence  $(A^+)^{\otimes_{R^q}}$  is concentrated in degrees greater than or equal to  $\lfloor (nq)/2 \rfloor$ . On the

other hand,  $A[2 - q]$  is concentrated in degrees less than or equal to  $n + q - 2$ , which implies that

$$C^q(A, A[2 - q]) = \text{Hom}_{R-R}((A^+)^{\otimes_{R^q}}, A[2 - q]) = 0 \quad \text{if } n \geq 4 \text{ or } q \geq 4.$$

We now focus on the remaining case  $(n, q) = (3, 3)$ . Then  $(A^+)^{\otimes_{R^3}}$  is concentrated in degrees greater than or equal to 4, while  $A[-1]$  is concentrated in degrees less than or equal to 4. The degree 4 part of  $(A^+)^{\otimes_{R^3}}$  is spanned by elements  $c = (i_0|i_1) \otimes (i_1|i_2) \otimes (i_2|i_3)$ , which obviously satisfy  $i_3 \neq i_0$ . It follows that as an  $R$ -bimodule, the degree 4 part satisfies  $e_i((A^+)^{\otimes_{R^3}})^4 e_i = 0$ . On the other hand, the degree 4 part of  $A[-1]$  is spanned by the elements  $(i|i \pm 1|i)$ , so it satisfies  $e_i A[-1]^4 e_j = 0$  for all  $i \neq j$ . This implies that there can be no nonzero  $R$ -bimodule maps between  $(A^+)^{\otimes_{R^3}}$  and  $A[-1]$  and hence that  $C^3(A, A[-1])$  is after all trivial.  $\square$

*Proof for  $n = 2$*

Consider the relevant piece of the Hochschild complex,

$$C^{q-1}(A, A[2 - q]) \xrightarrow{\partial^{q-1}} C^q(A, A[2 - q]) \xrightarrow{\partial^q} C^{q+1}(A, A[2 - q]);$$

$C^{q+1}(A, A[2 - q])$  is zero for degree reasons. In fact, since all edges in  $\Gamma$  have “degree” labels 1, paths are now graded by their length, so that  $(A^+)^{\otimes_{R^{q+1}}}$  is concentrated in degrees greater than or equal to  $q + 1$ , while  $A[2 - q]$  is concentrated in degrees less than or equal to  $q$ . In contrast,  $C^q(A, A[2 - q])$  is nonzero for all even  $q$ . To give a more precise description of this group, we use the basis of  $A$  from (4.7) and the basis of  $(A^+)^{\otimes_{R^q}}$  derived from that. Let  $(i_0|\cdots|i_q)$ ,  $i_q = i_0$ , be a closed path of length  $q$  in  $\Gamma$ . Define  $\phi_{i_0, \dots, i_m} \in C^q(A, A[2 - q])$  by setting

$$\phi_{i_0, \dots, i_q}(c) = \begin{cases} (i_0|i_0 \pm 1|i_0) & \text{if } c = (i_0|i_1) \otimes \cdots \otimes (i_{q-1}|i_q), \\ 0 & \text{on all other basis elements } c. \end{cases}$$

We claim that the elements defined in this way, with  $(i_0|\cdots|i_q)$  ranging over all closed paths, form a basis of  $C^q(A, A[2 - q])$ . To prove this, note that there is only one degree, which is  $q$ , where both  $(A^+)^{\otimes_{R^q}}$  and  $A[2 - q]$  are nonzero. The degree  $q$  part of  $(A^+)^{\otimes_{R^q}}$  is spanned by expressions  $c = (i_0|i_1) \otimes \cdots \otimes (i_{q-1}|i_q)$ , with  $i_q$  not necessarily equal to  $i_0$ . The degree  $q$  part of  $A[2 - q]$  is spanned by elements  $(i|i \pm 1|i)$ . Hence, an argument using the  $R$ -bimodule structure shows that if  $i_q \neq i_0$ , then  $\phi(c) = 0$  for all  $\phi \in C^q(A, A[2 - q])$ . This essentially implies what we have claimed.

We now turn to  $C^{q-1}(A, A[2 - q])$ ; for this group we do not need a complete description, but only some sample elements. Given a closed path  $(i_0|\cdots|i_q)$  as before in  $\Gamma$ , we define  $\phi' \in C^{q-1}(A, A[2 - q])$  by setting  $\phi'(c) = (i_0|i_{q-1})$  if  $c = (i_0|i_1) \otimes \cdots \otimes (i_{q-2}|i_{q-1})$ , and zero on all other basis elements  $c$ . A simple computa-

tion shows that  $\delta^{q-1}(\phi') = -\phi_{i_0, \dots, i_q} - \phi_{i_{q-1}, i_0, i_1, \dots, i_{q-1}}$ . Also, for any closed path  $(i_0 | \dots | i_q)$  with  $i_2 = i_0$  and  $i_1 = i_0 + 1$ , define  $\phi'' \in C^{q-1}(A, A[2-q])$  by setting  $\phi''(c) = (i_0 | i_0 \pm 1 | i_0)$  for  $c = (i_0 | i_1 | i_2) \otimes (i_2 | i_3) \otimes \dots \otimes (i_{q-1} | i_q)$ , and again zero for all other basis elements  $c$ . Then  $\delta^{q-1}(\phi'')$  is equal to  $-\phi_{i_0, i_1, \dots, i_q} - \phi_{i_0, i_1-2, i_2, \dots, i_q}$  for  $i_0 > 1$ , and to  $-\phi_{i_0, i_1, \dots, i_q}$  for  $i_0 = 1$ .

To summarize, we have now established that the following relations hold in  $HH^q(A, A[2-q])$ :

- (1)  $[\phi_{i_0, \dots, i_q}] = -[\phi_{i_{q-1}, i_q, i_1, \dots, i_{q-1}}]$  for all closed paths  $(i_0 | \dots | i_q)$  in the quiver  $\Gamma$ ;
- (2)  $[\phi_{i_0, \dots, i_q}] = -[\phi_{i_0, i_1-2, i_2, \dots, i_q}]$  whenever  $i_0 = i_2 \geq 2$  and  $i_1 = i_0 + 1$ ;
- (3)  $[\phi_{i_0, \dots, i_q}] = 0$  whenever  $i_0 = i_2 = 1$  and  $i_1 = 2$ .

Take an arbitrary element  $\phi_{i_0, \dots, i_q}$ . By applying (1) repeatedly, one can find another element  $\phi_{i'_0, \dots, i'_q}$  that represents the same Hochschild cohomology class, up to a sign, and such that  $i'_1$  is maximal among all  $i'_v$ . This implies that  $i'_0 = i'_2 = i'_1 - 1$ . If  $i'_1 = 2$ , then we can apply (3) to show that our Hochschild cohomology class is zero. Otherwise, pass to  $\phi_{i'_0, i'_1-2, \dots, i'_q}$ , which represents the same Hochschild cohomology class up to sign due to (2), and repeat the argument. The iteration terminates after finitely many moves because the sum of the  $i_v$  decreases by 2 in each step. Hence  $HH^q(A, A[2-q])$  is zero for all  $q \geq 1$ .  $\square$

### Proof of Theorem 2.18

We first need to dispose of the trivial case  $m = 1$ . In that case, choose a resolution  $F_1 \in \mathfrak{K}$  of  $E_1$ . Pick a nonzero morphism  $\phi : F_1 \rightarrow F_1[n]$ . This, together with  $\text{id}_{F_1}$ , determines an isomorphism of graded vector spaces  $\text{Hom}^*(F_1, F_1) \cong k \oplus k[-n]$  and hence an isomorphism in  $\mathfrak{K}$  between  $F_1 \oplus F_1[-n]$  and  $\text{Hom}^*(F_1, F_1) \otimes F_1$ . Consider the commutative diagram

$$\begin{array}{ccccc}
 T_{F_1}(F_1)[-1] & \longrightarrow & \text{Hom}^*(F_1, F_1) \otimes F_1 & \xrightarrow{\text{ev}} & F_1 \\
 & & \uparrow \cong & & \uparrow \text{id} \\
 F_1[-n] & \xrightarrow{(-\phi, \text{id})} & F_1 \oplus F_1[-n] & \xrightarrow{(\text{id}, \phi)} & F_1
 \end{array}$$

The upper row is a piece of the exact triangle which comes from the definition of  $T_{F_1}$  as a cone, and the lower row is obviously also a piece of an exact triangle. By the axioms of a triangulated category, the diagram can be filled in with an isomorphism between  $F_1[-n]$  and  $T_{F_1}(F_1)[-1]$ . Transporting the result to  $D^b(\mathfrak{S}')$  yields  $T_{E_1}(E_1) \cong E_1[1-n]$ . Since  $n \geq 2$  by assumption, it follows that  $T_{E_1}^r(E_1) \not\cong E_1$  unless  $r = 0$ .

From now on, suppose that  $m \geq 2$ . After shifting each  $E_i$  by some amount, we may assume that  $\text{Hom}^*(E_{i+1}, E_i)$  is concentrated in degree  $d_i$  for  $i = 1, \dots, m-1$ . (Shifting does not affect the statement because  $T_{E_i[j]}$  is isomorphic to  $T_{E_i}$  for any

$j \in \mathbb{Z}$ .) Choose resolutions  $E'_1, \dots, E'_m \in \mathfrak{K}$  for  $E_1, \dots, E_m$ . Lemma 4.10 shows that the endomorphism dga  $\text{end}(E')$  has  $H(\text{end}(E')) \cong A_{m,n}$ . By Lemma 4.21,  $\text{end}(E')$  must be quasi-isomorphic to  $\mathcal{A}_{m,n}$ . Define an exact functor  $\Psi$  to be the composition

$$D^b(\mathfrak{S}') \xleftarrow{\cong} \mathfrak{K} \xrightarrow{\Psi_{E'}} D(\text{end}(E')) \xrightarrow{\cong} D(\mathcal{A}_{m,n}).$$

The first arrow is the standard equivalence, and the last one is the equivalence induced by some sequence of dgas and quasi-isomorphisms. By construction,  $\Psi(E_i) \cong \mathcal{P}_i$  for  $i = 1, \dots, m$ . In the diagram

$$\begin{array}{ccccccc} D^b(\mathfrak{S}') & \xleftarrow{\cong} & \mathfrak{K} & \xrightarrow{\Psi_{E'}} & D(\text{end}(E')) & \xrightarrow{\cong} & D(\mathcal{A}_{m,n}) \\ T_{E_i} \downarrow & & T_{E'_i} \downarrow & & t_i \downarrow & & t_i \downarrow \\ D^b(\mathfrak{S}') & \xleftarrow{\cong} & \mathfrak{K} & \xrightarrow{\Psi_{E'}} & D(\text{end}(E')) & \xrightarrow{\cong} & D(\mathcal{A}_{m,n}) \end{array}$$

the first square commutes because that is the definition of  $T_{E_i}$ , the second square by Lemma 4.3, and the third one by Lemma 4.2. Now let  $g$  be an element of  $B_{m+1}$ ,  $R^g : D^b(\mathfrak{S}') \rightarrow D^b(\mathfrak{S}')$ , a functor that represents  $\rho(g)$ , and  $R_{m,n}^g : D(\mathcal{A}_{m,n}) \rightarrow D(\mathcal{A}_{m,n})$ , a functor that represents  $\rho_{m,n}(g)$ . By applying the previous diagram several times, one sees that

$$R_{m,n}^g \circ \Psi \cong \Psi \circ R^g.$$

Assume that  $R_g(E_i) \cong E_i$  for all  $i$ ; then also  $R_{m,n}^g(\mathcal{P}_i) = R_{m,n}^g \Psi(E_i) \cong \Psi R^g(E_i) \cong \Psi(E_i) \cong \mathcal{P}_i$ . By Theorem 4.13, it follows that  $g$  must be the identity.  $\square$

We have not tried to compute the Hochschild cohomology of  $A_{m,n}$  for  $n = 1$ . However, an indirect argument using the nonfaithful  $B_4$ -action of Section 3.4 shows that  $A_{3,1}$  cannot be intrinsically formal. More explicitly, if one takes the sheaves  $\mathcal{O}_x, \mathcal{O}, \mathcal{O}_y$  used in that example and if one chooses injective resolutions by quasi-coherent sheaves for them, then the resulting dga  $\text{end}(E')$  is not formal. One can give a more direct proof of the same fact by using essentially the same Massey product computation as Polishchuk in [44, p. 3].

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*Addendum.* The results here were first announced at the Harvard Winter School on Mirror Symmetry in January of 1999 (see [53]). In the meantime, a preprint by R. Horja [22] has appeared which is inspired by similar mirror symmetry considerations. While there is little actual overlap ([22] does not operate in the derived category), Horja uses monodromy calculations to predict corresponding conjectural mirror Fourier-Mukai transforms that ought to be connected to our work, linking it to the toric construction of mirror manifolds.

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