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In this paper, we consider the Time-Dependent Vehicle Routing Problem with Time Windows (TDVRPTW). Travel times are time-dependent, meaning that depending on the departure time from a customer a different travel time is incurred. Because of time-dependency, vehicles' dispatch times from the depot are crucial as road congestion might be avoided. Due to its complexity, all existing solutions to the TDVRPTW are based on (meta-) heuristics and no exact methods are known for this problem. In this paper, we propose the first exact method to solve the TDVRPTW. The MIP formulation is decomposed into a master problem that is solved by means of column generation, and a pricing problem. To insure integrality, the resulting algorithm is embedded in a Branch and Cut framework. We aim to determine the set of routes with the least total travel time. Furthermore, for each vehicle, the best dispatch time from the depot is calculated.

Key words: vehicle routing problem; column generation; time-dependent travel times; branch and cut
History:

1. Introduction

The vehicle routing problem with time windows (VRPTW) concerns the determination of a set of routes starting and ending at a depot, in which the demand of a set of geographically scattered customers is fulfilled. Each route is traversed by a vehicle with a fixed and finite capacity, and each customer must be visited exactly once. The total demand delivered in each route should not exceed the vehicle's capacity. At customers time windows are imposed, meaning that service at a customer is only allowed to start within its time window. The solution to the VRPTW consists of the set of routes with the least traveled distance.

Due to its practical relevance, the VRPTW has been extensively studied in the literature (Toth and Vigo 2002). Consequently, many (meta-) heuristics and exact methods have been successfully developed to solve it. However, most of the existing models are time-independent, meaning that a vehicle is assumed to travel with constant speed throughout its operating period. Because of road congestion, vehicles hardly travel with constant speed. Obviously, solutions derived from time-independent models to the VRPTW could be infeasible when implemented in real-life. In fact, in real-life road congestion results in tremendous delays. Consequently, it is unlikely that a vehicle respects customers' time windows. Therefore, it is highly important to consider time-dependent travel times when dealing with the VRPTW.

In this paper, we consider the time-dependent vehicle routing problem with time windows (TDVRPTW). We take road congestion into account by assuming time-dependent travel times:

depending on the departure time at a customer a different travel time is incurred. We divide the planning horizon into time zones (e.g. morning, afternoon, etc.) where a different speed is associated with each of these zones. The resulting stepwise speed function is translated into travel time functions that satisfy the First-In First-Out (FIFO) principle (see also Ichoua et al. (2003)). Because of the time-dependency, the vehicles' dispatch times from the depot are crucial. In fact, a later dispatch time from the depot might result in a reduced travel time as congestion might be avoided. In this paper, we aim to determine the set of routes with the least total travel time. Furthermore, for each vehicle, the best dispatch time from the depot is calculated.

Despite numerous publications dealing with the vehicle routing problem, very few addressed the inherent time-dependent nature of this problem. Additionally, to our knowledge, all existing algorithms are based on (meta-) heuristics, and no exact approach has been provided for the TDVRPTW. In this paper, we solve the TDVRPTW exactly. We use the flow arc formulation of the VRPTW which is decomposed into a master problem (set partitioning problem) and a pricing problem. While the master problem remains unchanged, compared to that of the VRPTW (as time-dependency is implicitly included in the set of feasible solutions) the pricing problem is translated into a time-dependent elementary shortest path problem with resource constraints (TDESPPRC), where time windows and capacity are the constrained resources. The relaxation of the master problem is solved by means of column generation. To guarantee integrality, the resulting column generation algorithm is embedded in a branch-and-bound framework. Furthermore, in each node, we use cutting planes in the pricing problem to obtain better lower bounds and hence reduce the size of branching trees. This results in a branch-and-cut-and-price (BCP) algorithm. Time-dependency in travel times increases the complexity of the pricing problem. In fact, the set of feasible solutions increases as the cost of a generated column (*i.e.* route) does not depend only on the visited customers, but also on the vehicles' dispatch time from the depot. The pricing problem in case of the VRPTW is usually solved by means of a labeling algorithm (Desrochers 1986). However, the labeling algorithm designed for the VRPTW is incapable to deal with time-dependency in travel times and needs to be adapted. In this paper, we develop a time-dependent labeling (TDL) algorithm such that in each label the arrival time function (*i.e.* function of the departure time from the depot) of the corresponding partial path is stored. the TDL generates columns that have negative reduced cost together with their best dispatch time from the depot. To accelerate the BCP algorithm, two heuristics based on the TDL algorithm are designed to quickly find columns with negative reduced cost. Moreover, new dominance criteria are introduced to discard labels that do not lead to routes in the final optimal solution. Furthermore, we relax the pricing problem by allowing non-elementary paths. The resulting pricing problem is a time-dependent shortest path problem with resource constraints (TDSPPRC). Although the TDSPPRC results in worse lower bounds, it is easier to solve and integrality is still guaranteed by branch-and-bound. Moreover, TDSPPRC should work well for instances with tight time windows. The pricing problem is explained in more details in section 5. Over the last decades, BCP proved to be the most successful exact method for the VRPTW. Hence, our choice for a BCP framework to solve the TDVRPTW is well motivated.

The main contributions of this paper are summarized as follows. First, we present an exact method for the TDVRPTW. We propose a branch-and-cut-and price algorithm to determine the set of routes with the least total travel time. Contrary to the VRPTW, the pricing problem is translated into a TDESPPRC and solved by a time-dependent labeling algorithm. Second, we capture road congestion by incorporating time-dependent travel times. Because of time dependency, vehicles' dispatch times from the depot are crucial. In this paper, dispatch times from the depot are also optimized. In the literature as well as in practice, dispatch time optimization is approached as a post-processing step, *i.e.* given the routes, the optimal dispatch times are determined (Kok et al. 2007). In this paper, the scheduling (dispatch time optimization) and routing are simultaneously performed. Third, ...

The paper is organized as follows...

2. Literature Review

An abundant number of publications is devoted to the vehicle routing problem (see Laporte (1992), Toth and Vigo (2002), and Laporte (2007) for good reviews). Specifically, the VRPTW has been extensively studied. For good reviews on the VRPTW, the reader is referred to Bräysy and Gendreau (2005a), and Bräysy and Gendreau (2005b). The majority of these publications assume a time-independent environment where vehicles travel with a constant speed throughout their operating period. Perceiving that vehicles operate in a stochastic and dynamic environment, more researchers moved their effort towards the optimization of the time-dependent vehicle routing problems. Nevertheless, literature on this subject remains scarce.

In the context of dynamic vehicle routing, we mention the work of Bertsimas and Simchi-Levi (1996), Bertsimas and Ryzin (1991) and Bertsimas and Ryzin (1993a) where a probabilistic analysis of the vehicle routing problem with stochastic demand and service time is provided. Malandraki and Dial (1996), Hill and Benton (1992) and Ichoua et al. (2003) tackle the vehicle routing problem where vehicles' travel time depends on the time of the day, and Malandraki and Daskin (1992) considers a time-dependent traveling salesman problem. Time-dependent travel times has been modeled by dividing the planning horizon into a number of zones, where a different speed is associated with each of these time zones (see Ichoua et al. (2003) and Jabali et al. (2009)). In Van Woensel et al. (2008), traffic congestion is captured using a queuing approach. Malandraki and Dial (1996) and Malandraki and Daskin (1992) models travel time using stepwise function, such that different time zones are assigned different travel times. Fleischmann et al. (2004) emphasized that modeling travel times as such leads to the undesired effect of passing. That is, a later start time might lead to an earlier arrival time. As in Ichoua et al. (2003), we consider travel time functions that adhere to the FIFO principle. Such travel time functions does not allow passing.

While several successful (meta-) heuristics and exact algorithms have been developed to solve the VRPTW, algorithms designed to deal with the TDVRPTW are somewhat limited to (meta-) heuristics. In fact, most of the existing algorithms are based on tabu search (Ichoua et al. (2003), Van Woensel et al. (2008), Jabali et al. (2009) and Maden et al. (2010)). In Malandraki and Dial (1996) mixed integer linear formulations the time-dependent vehicle routing problem are presented and several heuristics based on nearest neighbor and cutting planes are provided. Donati et al. (2008) proposes an algorithm based on a multi ant colony system and Haghani and Jung (2005) presents a genetic algorithm. In Hashimoto et al. (2008) a local search algorithm for the TDVRPTW is developed and a dynamic programming is embedded in the local search to determine the optimal starting for each route. Androutsopoulos and Zografos (2009) considers a multi-criteria routing problem, they propose an approach based on the decomposition of the problem into a sequence of elementary itinerary subproblems that are solved by means of dynamic programming. Malandraki and Daskin (1992) presents a restricted dynamic programming for the time-dependent traveling salesman problem. In each iteration of the dynamic programming, only a subset with a predefined size and consisting of the best solutions is kept and used to compute solutions in the next iteration. Tang (2008) emphasizes the difficulty of implementing route improvement procedures in case of time-dependent travel times and proposes efficient ways to deal with that issue. In this paper, we attempt to solve the TDVRPTW to optimality using column generation. To the best of our knowledge, this is the first time an exact method for the TDVRPTW is presented.

Column generation has been successfully implemented for the VRPTW. For a overview of column generation algorithms, the reader is referred to Lübbecke and Desrosiers (2005). in the context of the VRPTW, Kohl et al. (1999) designed an efficient column generation algorithm where they applied subtour elimination constraints and 2-path cuts. This has been improved by Cook and Rich (1999) by applying k-path cuts. Jespen et al. (2008) proposes a column generation algorithm by applying subset-row inequalities to the master problem (set partitioning). Although, adding subset-row inequalities to the master problem increases the complexity of the pricing problem, Jespen et al. (2008) shows that better lower bounds can be obtained from the linear relaxation of the master problem. To accelerate the pricing problem solution, Desaulniers et al. (2008) proposes a tabu search heuristic for the ESPPRC. Furthermore, elmentarity is relaxed for a subset of nodes and generalized k-inequalities are introduced. Recently, Baldacci et al. (2010) introduce a new route relaxation, called ng-route, used to solve the pricing problem. Their framework proves to be very effective in solving difficult instances of the VRPTW with wide time windows. Fleischmann et al. (2004) argued that existing algorithms for the VRPTW fail to solve the TDVRPTW. One major drawback of the existing algorithms is the incapability to deal with the dynamic nature of travel times. Therefore, existing algorithms for the VRPTW can not be applied to the TDVRPTW without a radical modification of their structure. In this paper, a branch-and-cut-and-price framework is modified such that time-dependent travel times can be incorporated.

3. Problem Description

We consider a graph $G(V,A)$ on which the problem is defined. $V = \{0,1,...,n,n+1\}$ is the set of all nodes such that $V_c = V/\{0,n+1\}$ represents the set of customers that need to be served. Moreover, 0 is the start deport and $n+1$ is the end depot. $A = \{(i,j) : i \neq j \text{ and } i,j \in V\}$ is the set of all arcs between the nodes. Let K be the set of homogeneous vehicles such that each vehicle has a finite capacity Q and q_i demand of customer $i \in V_c$. We assume $q_0 = q_{n+1} = 0$ and $|K|$ is unbounded. Let a_i and b_i be respectively the opening and closing time of node's i time window. At node i, a service time s_i is needed. We denote t_i departure time from node $i \in V$ and $\tau_{ij}(t_i)$ travel time from node i to node j which depend on the departure time at node i. Table 1 summarizes the notation used in this paper.

Table 1 Notation used in this paper.

Variable	Description
V	: Set of nodes
V_c	: Set of customers
K	: Set of vehicles
Q	: Capacity of a vehicle
t_i	: Departure time at node i
$t_i^l(L)$	: Latest possible departure time at a node i visited on the partial path represented by L
q_i	: Demand at node$i$
s_i	: Service time at node i
x_{ijk}	: Binary variable. Is one if and only if arc (i,j) is traversed by vehicle k
$\gamma^+(S)$	: Arcs originating from the set $S \subseteq V$. We write $\gamma^+(i)$ instead of $\gamma^+(\{i\})$
$\gamma^-(S)$	: Arcs ending in the set $S \subseteq V$. We write $\gamma^-(i)$ instead of $\gamma^-(\{i\})$
$\tau_{ij}(t_i)$	: Travel time from node i to node j when departure time at i is t_i
$\delta_{v(L)}(t_j)$	: Piecewise linear function measuring the arrival at the current node $v(L)$ of the partial path represented by L when departure at the start node j is t_j
Ω	: Set of all feasible routes
s_p	: Start time of route $p \in \Omega$
e_p	: End time of route $p \in \Omega$
c_p	: cost of route $p \in \Omega$. It is defined as as $e_p - s_p$
a_{ip}	: Is one if node i is visited by path p and zero otherwise
π_i	: Dual variable associated with row i of the master problem
$\bar{c}_p$	: Reduced cost of route $p \in \Omega$
$[a_i,b_i]$	: Time window at node i
$\lvert X\rvert$	: Size of the set X

3.1. Travel Time and Arrival Time Functions

We divide the planning horizon into time zones where a different speed is associated with each of these zones. The resulting stepwise speed function is translated into travel time functions that satisfy the First-In First-Out (FIFO) principle. Usually traffic networks have a morning and an afternoon congestion period. Therefore, we consider speed profiles that have two periods with relatively low speeds. In the rest of the planning horizon, speeds are relatively high. This complies with data collected for a Belgian highway (Van Woensel and Vandaele (2006)). Figure 1 depicts the speed profile for each start time for an arbitrary link. Moreover, it shows how the speed profile is translated into a travel time function. We call the points *a, b, c, d* and *e* where speeds change *speed breakpoints.* Speed breakpoints are also breakpoints in the travel time function. The other *travel time breakpoints* are determined as the start time to arrive exactly at a speed breakpoint (*e.g, a'* is the start time to exactly arrive at time *a*) using the procedure as described in Ichoua et al. (2003). While the slopes in the travel time function mean that the traveled distance is traversed using

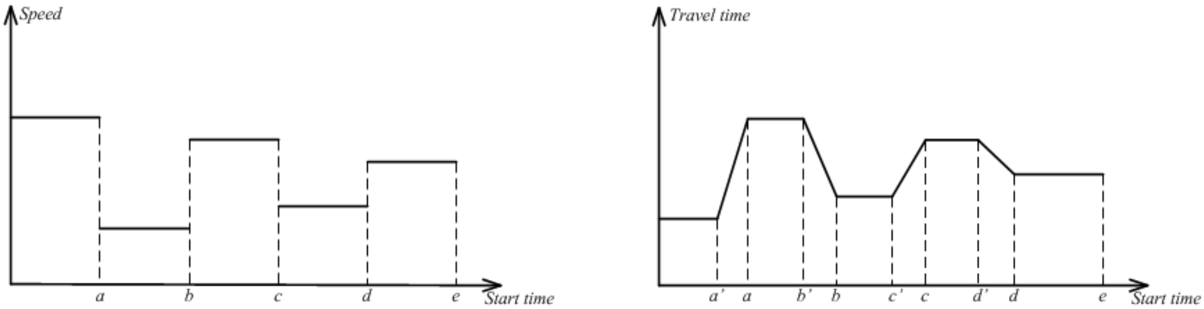

Figure 1 Speed and travel time functions.

several speeds, the horizontal segments mean that it is traversed using only one speed. Clearly, for large distances we might have travel time functions without any horizontal segments. Travel time functions are stepwise linear functions in which every two consecutive travel time breakpoints define a zone. Given any start time within a zone, travel time can easily be computed using the breakpoints defining that zone. Therefore, travel time functions can be completely represented by their breakpoints.

Given a partial path P_i starting at the depot 0 and ending at some node i, the arrival time at i depends on the dispatch time t_0 at the depot. Due to the FIFO property of the travel time functions, a later dispatch at the depot will result in a later arrival at node i. Therefore, if route P_i is unfeasible for some dispatch time t_0 at the depot (*i.e.* time windows are violated), P_i will be unfeasible for any dispatch time at the depot that is later than t_0. Moreover, If we define $\delta_i(t_0)$ as the arrival time function at node i given a dispatch time t_0 at the depot, $\delta_i(t_0)$ will be non-decreasing in t_0. We call the parent node j of node i, the node that is visited directly before node i on route P_i. $\delta_j(t_0)$ is the arrival time at j given a dispatch time t_0 at the depot, and $\tau_{ji}(\delta_j(t_0))$ is the incurred travel time from j to i. Consequently, for every $i \in V$, $\delta_i(t_0)$ is recursively calculated as follows:

$$\delta_0(t_0) = t_0 \quad \text{and} \quad \delta_i(t_0) = \delta_j(t_0) + \tau_{ji}(\delta_j(t_0)) \tag{1}$$

Where $\delta_0(t_0)$ is a sort of dummy function representing the arrival time at the depot given a dispatch time t_0 at the same depot. Formula (1) shows that an arrival time function is the sum of two linear stepwise functions (travel time function and arrival time function of the parent node), hence it is also a linear stepwise function. Figure 2 depicts the recursive calculation of the arrival time functions using equation (1). Again, we can completely represent an arrival time function using the *arrival time function breakpoints* resulting from either breakpoints of travel time functions, breakpoints of the arrival time function of the parent node, or from time windows. The cost of a path is equal to its duration $\delta_i(t_0) - t_0$. Clearly, the departure time t_0^* from the depot that results in the shortest path duration belongs to a breakpoint. That is:

$$t_0^* = \min_{t_0 \in B_i} \{\delta_i(t_0) - t_0\} \tag{2}$$

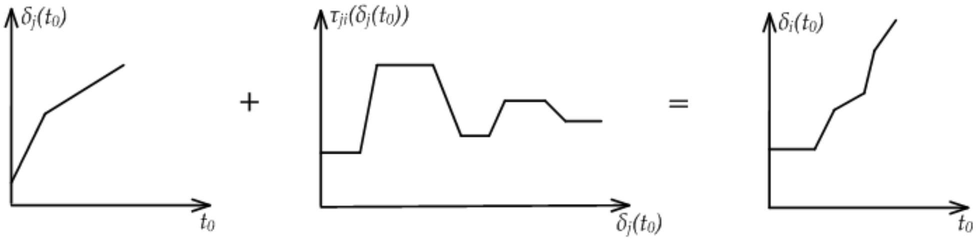

Figure 2 **Arrival time functions.**

B_i is the set of breakpoints of the arrival time function $\delta_i(t_0)$.

3.2. The Mathematical Formulation

If ω_{ik} is the departure time of vehicle k at customer i and x_{ijk} is a binary variable that takes the value 1 if and only if arc (i,j) is traversed by vehicle k, the objective function for the TDVRPTW is as follows:

$$\sum_{k\in K}\sum_{(i,j)\in A} \tau_{ij}(\omega_{ik})x_{ijk} \tag{3}$$

For every arc (i,j), we denote Z_{ij} as the set of zones of the corresponding travel function $\tau_{ij}(t_i)$. A zone $Z_m \in Z_{ij}$, is defined by two consecutive travel time breakpoints, $Z_m = [r_m, r_{m+1}[$. A slope θ_m and an intersection η_m with the y-axis can be calculated using r_m, r_{m+1}, $\tau_{ij}(r_m)$ and $\tau_{ij}(r_{m+1})$. Therefore, for some $Z_m \in Z_{ij}$, the travel time $\tau_{ij}(\omega_{ik})$ from i to j given departure time ω_{ik} at i is:

$$\tau_{ij}(\omega_{ik}) = \theta_m\omega_{ik} + \eta_m \tag{4}$$

The objective function can be re-written as follows:

$$\sum_{k\in K}\sum_{(i,j)\in A}\sum_{m=1}^{|Z_{ij}|}(\theta_m\omega_{ik} + \eta_m)x_{ijk}^m \tag{5}$$

Where, x_{ijk}^m is a binary variable that takes the value 1 if and only arc (i,j) is traversed by vehicle k and departure time from customer i is within zone Z_m. Obviously, the non-linear term $\omega_{ik}x_{ijk}^m$ will appear in the objective function. However, if we define the variable:

$$\omega_{ik}^m = \begin{cases} \omega_{ik} & \text{if} \quad x_{ijk}^m = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$\omega_{ik}x_{ijk}^m$ can be replaced by ω_{ik}^m. Furthermore, we denote Z_{ij}^+ and Z_{ij}^- respectively as the set of zones with positive slope and the set of zones with absolutely negative slope. The MIP formulation for TDVRPTW can be written as follows:

$$\min z = \sum_{k\in K}\sum_{(i,j)\in A}\sum_{m=1}^{|Z_{ij}|}(\theta_m\omega_{ik}^m + \eta_m x_{ijk}^m) \tag{7}$$

subject to:

$$\sum_{k \in K} x^k(\gamma^+(i)) = 1 \qquad \forall i \in V/\{n+1\} \tag{8}$$
$$x^k(\gamma^+(0)) = 1 \qquad \forall k \in K \tag{9}$$
$$x^k(\gamma^+(j)) = x^k(\gamma^-(j)) \qquad \forall k \in K, \forall j \in V/\{n+1\} \tag{10}$$
$$x^k(\gamma^-(n+1)) = 1 \qquad \forall k \in K \tag{11}$$
$$(1+\theta_m)\omega_{ik}^m - s_i + \eta_m \le \omega_{jk}^m - s_j + (1 - x_{ijk}^m)M \qquad \forall k \in K, \forall (i,j) \in A, \forall m \in |Z_{ij}| \tag{12}$$
$$\omega_{ik}^m \ge \omega_{ik} - (1 - x_{ijk}^m)M \qquad \forall k \in K, \forall (i,j) \in A, \forall m \in |Z_{ij}^+| \tag{13}$$
$$\omega_{ik}^m \le min(\omega_{ik}, M x_{ijk}^m) \qquad \forall k \in K, \forall (i,j) \in A, \forall m \in |Z_{ij}^-| \tag{14}$$
$$a_i + s_i \le \omega_{ik}^m \le b_i + s_i \qquad \forall k \in K, \forall i \in V \tag{15}$$
$$\sum_{i \in N} q_i x^k(\gamma^+(i)) \le Q \qquad \forall k \in K \tag{16}$$
$$x_{ijk}^m \in \{0,1\} \qquad \forall k \in K, \forall (i,j) \in A, \forall m \in |Z_{ij}| \tag{17}$$
$$r_m \le \omega_{ik}^m < r_{m+1} \qquad \forall k \in K, \forall i \in V, \forall m \in |Z_{ij}| \tag{18}$$

When departure time is within a zone with positive slope, w_{ik}^m will appear with a positive sign in the objective function (7), and the optimization will attempt to set it as low as possible to reduce travel time. This is taken care of by means of constraint (13). However, when departure time is within a zone with a negative slope, w_{ik}^m will appear with negative sign in the objective function, and the optimization will attempt to set it as large as possible through constraint (14).
Obviously, the number of decision variable has increased. However, we don't have to decide on all of them. In fact, due to the FIFO assumption, waiting at customers will not result in better solutions. Therefore, we only have to decide on departure time at the depot. Departure times at customers take place immediately after finishing service which is computable given the sequence of visited customers.

4. Column Generation

To derive the set partitioning formulation for the TDVRPTW, we define Ω as the set of feasible paths satisfying constraints (9)-(18) (the index k is dropped since we are considering a homogeneous fleet). A feasible path is defined by the sequence of customers visited along it and the dispatch time at the depot. To each path $p \in \Omega$, we associate the cost c_p which is simply its duration. Hence:

$$c_p = e_p - s_p \tag{19}$$

Where e_p and s_p are respectively the end time and the start time of path p. Furthermore, if y_p is a binary variable that takes the value 1 if and only if the path p is included in the solution, the TDVRPTW is formulated as the following set partitioning problem:

$$\min z_M = \sum_{p \in \Omega} c_p y_p \tag{20}$$

subject to:

$$\sum_{p \in \Omega} a_{ip} y_p = 1 \qquad \forall i \in V \tag{21}$$
$$y_p \in \{0,1\} \qquad \forall p \in \Omega. \tag{22}$$

The objective function (20) minimize the duration of the chosen routes. Constraint (21) guarantees that each node is visited only once. Solving the LP-relaxation , resulting from relaxing the integrality constraints of the variables y_p, of the master problem provides a lower bound on its optimal value. The set of feasible paths Ω is usually very large making it hard to solve the LP-relaxation of the master problem. Therefore, we have recourse to column generation. In column generation, a restricted master problem is solved by considering only a subset $\Omega' \subseteq \Omega$ of feasible paths. Additional paths with negative reduced cost are generated after solving a pricing subproblem. The pricing problem for the TDVRPTW is (the index k is dropped):

$$\min z_P = \sum_{(i,j)\in A} \overline{\tau}_{ij}(\omega_i) x_{ij} \tag{23}$$

subject to constraints (9)-(18). Furthermore, $\overline{\tau}_{ij}(\omega_i) = \tau_{ij}(\omega_i) - \pi_i$ is the arc reduced cost, where π_i is the dual variable associated with servicing node i. In the master problem, π_i results from the constraint corresponding to node i in the set of constraints (21). The objective function of the pricing problem can be expressed as:

$$z_P = e_p - s_p - \sum_{i\in V_c} a_{ip}\pi_i \tag{24}$$

or in the variables x_{ij} as:

$$z_P = e_p - s_p - \sum_{i\in V_c} \left(\pi_i \sum_{j\in\gamma^+(i)} x_{ij} \right) \tag{25}$$

The problem with the objective function (24) and constraints (9)-(18) is called the time-dependent elementary shortest path problem with resource constraints (TDESPPRC). In this paper the only resources we consider are time windows. Capacity is relaxed in the pricing problem and handled using valid inequalities. Therefore, a feasible solution to the pricing problem must only respect time windows.In the next section the pricing problem is addressed in more details and it is shown how it is solved by means of a time-dependent labeling algorithm.

4.1. Capacity Cuts

5. The Pricing Problem

Solving the pricing problem involves finding columns (*i.e.* routes) with negative reduced cost that improve the objective function of master problem. In case of the TDVRPTW, this corresponds to solving the TDESPPRC and finding paths with negative cost. The TDESPPRC is a generalization of the ESPPRC in which costs are time-dependent. In this paper, we solve the pricing problem by means of a time-dependent labeling (TDL) algorithm which is a modification of the labeling algorithm applied to the ESPPRC. To speed up the TDL algorithm , a bi-directional search is performed in which labels are extended both forward from the depot (*i.e.* node 0) to its successors, and backward from the depot (*i.e.* node n+1) to its predecessors. While forward labels are extended to some fixed time t_m (*e.g.* the middle of the planning horizon) but not further, backward labels are extended to, but are allowed to directly cross, t_m. Forward and backward labels are finally merged to construct complete tours. The running time of a labeling algorithm depends on the length of partial paths associated with its labels. A bi-directional search avoids generating long paths and therefore limits running times.

5.1. The Forward TDL Algorithm

In the forward TDL algorithm, labels are extended from the depot (*i.e.* node 0) to its successors. The extension to a node is allowed if it is feasible and if the earliest arrival time (including waiting and service time) at that node is no further than t_m. We associate the following components to a Label L_f:

The set of feasible extensions $E(L_f)$ of L_f is the set of partial paths that when departing at node $v(L_f)$ at time $\delta_{v(L_f)}(0)$, they reach the depot (*i.e.* node n+1) without violating time windows.

$v(L_f)$	the current node visited on the partial path represented by L_f
$c(L_f)$	the sum of the dual variables associated with nodes visited along the partial path represented by L_f
$\delta_{v(L_f)}(t_0)$	arrival time at $v(L_f)$ through the partial path represented by L_f when the departure time at the depot is t_0. It includes both waiting time and service time at $v(L_f)$
$S(L_f)$	set of nodes visited along the partial path represented by L_f

If $L \in E(L_f)$, we denote $L_f \oplus L$ as the label resulting from extending L_f by L. If label L'_f is the parent label of label L_f, the arrival time function associated with label L_f is extended as follows:

$$\delta_{v(L_f)}(t_0) = \delta_{v(L'_f)}(t_0) + \tau_{v(L'_f)v(L_f)}(\delta_{v(L'_f)}(t_0)) \tag{26}$$

Furthermore, we have:

$$S(L_f) = S(L'_f) \bigcup \{v(L_f)\} \quad \text{and} \quad c(L_f) = c(L'_f) - \pi_{v(L_f)} \tag{27}$$

Where $\pi_{v(L_f)}$ is the dual variable corresponding to visiting node $v(L_f)$. Given the FIFO assumption, the earliest arrival time at $v(L_f)$ corresponds to the earliest possible dispatch time at the depot, $t_0 = 0$:

$$\delta_{v(L_f)}(0) = \delta_{v(L'_f)}(0) + \tau_{v(L'_f)v(L_f)}(\delta_{v(L'_f)}(0)) \tag{28}$$

The extension of label L'_f to label L_f is feasible if:

$$\delta_{v(L_f)}(0) \leq min(t_m, b_{v(L_f)} + s_{v(L_f)}) \tag{29}$$

In case of the ESPPRC, only the arrival time corresponding to a departure time $t_0 = 0$ from the depot is stored. Obviously, in case of the TDESPPRC, computing and storing arrival time functions is more complicated. The TDL algorithm is a complete enumeration in which, for every label, all possible extensions are derived and stored. It ends when all labels are processed. However, the number of labels that can be processed might be very large. Consequently, the labeling algorithm might be computationally very expensive. To reduce the number of labels, dominance criteria are introduced. In case of the forward TDL algorithm, dominance is defined as follows:

Definition 1. Label L^2_f is dominated by label L^1_f if:

1. $E(L^2_f) \subseteq E(L^1_f)$
2. $\overline{c}(L^1_f \oplus L) \leq \overline{c}(L^2_f \oplus L), \forall L \in E(L^2_f)$

Definition 1 states that any feasible extension of label L^2_f is also feasible for label L^1_f. Furthermore, extending L^1_f should always result in a better route. However, it is not straightforward to verify the conditions of Definition 1 as it requires the computation and the evaluation of all feasible extensions of both labels L^1_f and L^2_f. Therefore, sufficient dominance criteria that that are computationally less expensive are desirable. In Proposition 1, the sufficient conditions (3.), (4.) and (5.) are introduced. Condition (3.) is needed because of the elementarity of paths. Condition (4.), in addition to the FIFO assumption, guarantees that time windows of nodes visited along any feasible extension of L^2_f are respected when reached through L^1_f. Conditions (5.) ensures that no cheaper route can be obtained by extending L^2_f regardless of departure time at the depot. If we denote $t^l_0(L_f)$ as the latest feasible start time at the depot of the partial path represented by label L_f, Proposition 1 is formally stated as follows:

Proposition 1. *Label L_f^2 is dominated by label L_f^1 if:*

1. $v(L_f^1) = v(L_f^2)$
2. $c(L_f^1) \leq c(L_f^2)$
3. $S(L_f^1) \subseteq S(L_f^2)$
4. $\delta_{v(L_f^1)}(t_0) \leq \delta_{v(L_f^2)}(t_0), \quad \forall t_0 \in [0, t_0^l(L_f^2)]$
5. $t_0^l(L_f^2) \leq t_0^l(L_f^1)$

Proof of Proposition1: First we prove that $E(L_f^2) \subseteq E(L_f^1)$.
Let $L \in E(L_f^2)$, then $S(L) \bigcap S(L_f^2) = \emptyset$. As $S(L_f^1) \subseteq S(L_f^2)$, we should also have $S(L) \bigcap S(L_f^1) = \emptyset$. Now we will show that customers' time windows along the partial path represented by L are respected when reached trough L_f^1.
Let i be a node visited on the parrtial path represented by L, and $L_i \subseteq L$ be the partial path with i as the current node and the same start node as L. Furthermore, let $t_0 \leq t_0^l(L_f^2)$ be some start time at the depot.

$$\begin{aligned}
\delta_{v(L_f^1 \oplus L_i)}(t_0) &= \delta_{v(L_f^1)}(t_0) + \delta_{v(L_i)}(\delta_{v(L_f^1)}(t_0)) \\
&\leq \delta_{v(L_f^2)}(t_0) + \delta_{v(L_i)}(\delta_{v(L_f^2)}(t_0)) \\
&= \delta_{v(L_f^2 \oplus L_i)}(t_0) \\
&\leq b_i
\end{aligned}$$

Now we will show that $\overline{c}(L_f^1 \oplus L) \leq \overline{c}(L_f^2 \oplus L)$

$$\begin{aligned}
\delta_{v(L_f^1 \oplus L)}(t_0) &= \delta_{v(L_f^1)}(t_0) + \delta_{v(L)}(\delta_{v(L_f^1)}(t_0)) \\
&\leq \delta_{v(L_f^2)}(t_0) + \delta_{v(L)}(\delta_{v(L_f^2)}(t_0)) \\
&= \delta_{v(L_f^2 \oplus L)}(t_0)
\end{aligned}$$

Furthermore, we know that: $c(L_f^1) \leq c(L_f^2)$. Hence,

$$\begin{aligned}
c(L_f^1 \oplus L) &= c(L_f^1) + c(L) \\
&\leq c(L_f^2) + c(L) \\
&= c(L_f^2 \oplus L)
\end{aligned}$$

We conclude that for all $t_0 \leq t_0^l(L_f^2)$:

$$\delta_{v(L_f^1 \oplus L)}(t_0) - t_0 + c(L_f^1 \oplus L) \leq \delta_{v(L_f^2 \oplus L)}(t_0) - t_0 + c(L_f^2 \oplus L)$$

Hence, and since $t_0^l(L_f^2) \leq t_0^l(L_f^1)$,

$$\underset{t_0 \leq t_0^l(L_f^1)}{min} \left\{ \delta_{v(L_f^1 \oplus L)}(t_0) - t_0 \right\} + c(L_f^1 \oplus L) \leq \underset{t_0 \leq t_0^l(L_f^2)}{min} \left\{ \delta_{v(L_f^2 \oplus L)}(t_0) - t_0 \right\} + c(L_f^2 \oplus L)$$

Dominance as introduced in Proposition 1 is weak and will probably not sufficiently reduce the number of labels processed by the TDL algorithm. In fact, $S(L_f^1) \subseteq S(L_f^2)$ implies $c(L_f^1) \geq c(L_f^2)$ which contradicts the second condition. Hence, conditions (2.) and (3.) are only both true in case of equality. Furthermore, very cheap labels representing partial paths with a very long duration, that does not lead to a route in the optimal solution will probably not be dominated. In Figure 3, the numbers associated with the arcs represent travel times and the numbers associated with the nodes represents dual variables. Because of Condition (2.), the label representing partial path P_2 will not be dominated by the one representing partial path P_1. However, a path's reduced cost is equal to its duration reduced by the sum of the dual variables corresponding to the nodes visited along that path. Therefore, extending P_1 clearly results in a better final route. Another pitfall of Proposition 1 is that cheap labels are not able to dominate more expensive labels with, for some departure time at the depot, a shorter duration. In Figure 4, because of Condition (4.), the label representing partial path P_2, with cost -100, will not be dominated by the one representing partial path P_1 with cost -3000. The range of dispatch times at the depot, in which partial path P_2 has a shorter duration, has a width of 500 time units. Clearly, for any starting time at the depot in this range, it is possible to find an earlier (but no more than 500 time units earlier) starting time at the depot that results in the same arrival time at the end node for both P_1 and P_2. Leaving the depot earlier might increase P_1's duration. However, given P_1's new start time, its duration will be no more than 500 time units longer than P_2's duration. Therefore, the extension of P_1 will result in a better final route.

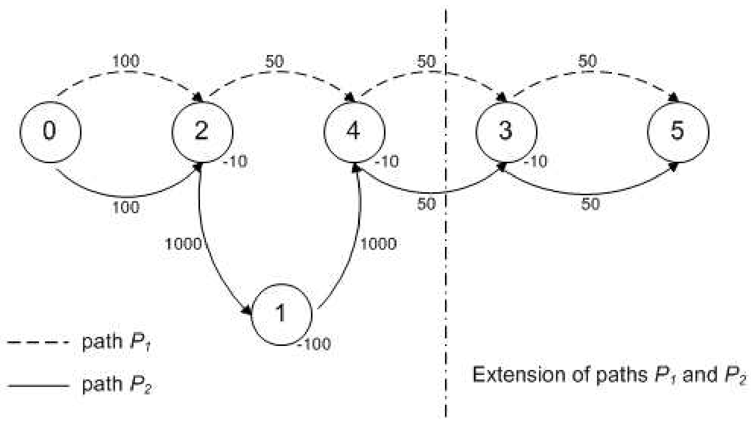

Figure 3

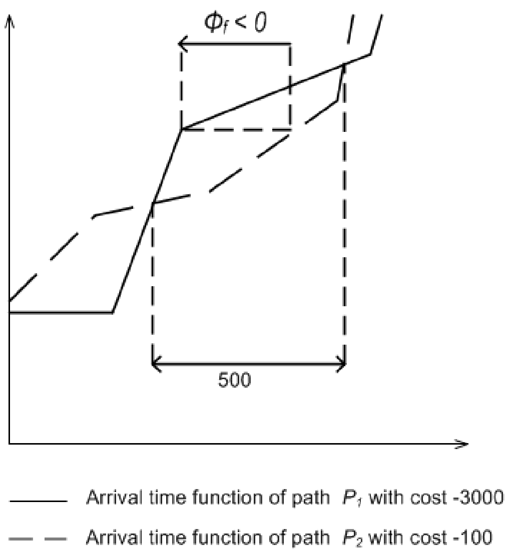

Figure 4

In Proposition 2, we improve dominance in two directions. First, for every label L_f, we extend $S(L_f)$ to the set $\widetilde{S}(L_f)$ by adding nodes that are unreachable from $v(L_f)$. The triangle inequality is not satisfied for time varying travel times as traveling directly to a node is not necessarily the shortest path. Consequently, a node that can not be directly reached from the end node might be indirectly reached via a diverted route. However, if we calculate the earliest arrival time to all nodes as in formula (28) and take the minimum, all nodes with a close time smaller than that minimum will not be reachable from $v(L_f)$. This can be done quickly, although we might fail to find all unreachable nodes. Second, we relax Condition (2.) by adding the quantity ϕ_f to the cost $c(L_f^2)$ of label L_f^2. ϕ_f is a real number related to how much the start time of the partial path represented by label L_f^1, can be postponed (in case ϕ_f is positive) or expedited (in case ϕ_f is negative) and still arrive at the end node at the same time as when reaching the end node through the partial path represented by label L_f^2. ϕ_f is illustrated in Figure 4. For every label L_f, let $\delta^{-1}_{v(L_f)}(t_a) = max\{t \leq t_0^l(L_f) : \delta_{v(L_f)}(t) = t_a\}$. The function $\delta^{-1}_{v(L)}(t_a)$ is defined on the domain $A_{\delta^{-1}_{v(L_f)}} = \{t_a \in \mathbb{R} : \exists t \leq t_0^l(L_f) : \delta_{v(L)}(t) = t_a\}$. Proposition 2 is stated as follows:

PROPOSITION 2. *Label L_f^2 is dominated by label L_f^1 if:*

1. $v(L_f^1) = v(L_f^2)$
2. $c(L_f^1) \leq c(L_f^2) + \phi_f$
3. $S(L_f^1) \subseteq \widetilde{S}(L_f^2)$
4. $\delta_{v(L_f^1)}(0) \leq \delta_{v(L_f^2)}(0)$

$$\phi_f = min\left\{t_0^l(L_f^1) - t_0^l(L_f^2), \min_{t\in A}\left\{\delta^{-1}_{v(L_f^1)}(t) - \delta^{-1}_{v(L_f^2)}(t)\right\}\right\} \quad and \quad A = A_{\delta^{-1}_{v(L_f^1)}} \bigcap A_{\delta^{-1}_{v(L_f^2)}}$$

Proof of Proposition 2: We will prove Proposition 2 for the case $\phi_f \geq 0$.
Similarly to Proposition 1, and by using the fact that $\delta_{v(L_f^1)}(0) \leq \delta_{v(L_f^2)}(0)$ and $S(L_f^1) \subseteq \widetilde{S}(L_f^2)$, we can prove that any feasible extension to L_f^2 is also feasible for L_f^1.
Let $L \in E(L_f^2)$, and $t_0 \leq t_0^l(L_f^2)$ be some start time at the depot.
Now, let t^* be such that:

$$t^* = \begin{cases} \delta^{-1}_{v(L_f^1)}(t_0) - t_0 & \text{if} \quad \delta_{v(L_f^2)}(t_0) \in A_{\delta^{-1}_{v(L_f^1)}} \\ t_0^l(L_f^1) - t_0^l(L_f^2) & \text{otherwise} \end{cases}$$

t^* is illustrated in Figure 5, and can also be written as:

$$t^* = \begin{cases} \delta^{-1}_{v(L_f^1)}(t_0) - \delta^{-1}_{v(L_f^1)}(\delta_{v(L_f^1)}(t_0)) & \text{if} \quad \delta_{v(L_f^2)}(t_0) \in A_{\delta^{-1}_{v(L_f^1)}} \\ t_0^l(L_f^1) - t_0^l(L_f^2) & \text{otherwise} \end{cases}$$

Postponing the start time of L_f^1 at the depot by t^* (*i.e.* the start time at the depot is $t_0 + t^*$ instead of t_0) results in a arrival time at the current node that is smaller than arrival time at the same current node reached through L_f^2, and when the start time at the depot is t_0 . Furthermore, $t_0 + t^* \leq t_0^l(L_f^1)$. Therefore:

$$\delta_{v(L_f^2)}(t_0) \geq \delta_{v(L_f^1)}(t_0 + t^*)$$

Consequently:

$$\begin{aligned} \delta_{v(L_f^2 \oplus L)}(t_0) &= \delta_{v(L_f^2)}(t_0) + \delta_{v(L)}(\delta_{v(L_f^2)}(t_0)) \\ &\geq \delta_{v(L_f^1)}(t_0 + t^*) + \delta_{v(L)}(\delta_{v(L_f^1)}(t_0 + t^*)) \\ &= \delta_{v(L_f^1 \oplus L)}(t_0 + t^*) \end{aligned}$$

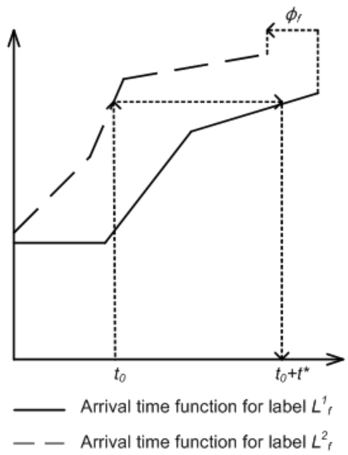

Figure 5

Now we will show that $\overline{c}(L_f^1 \oplus L) \leq \overline{c}(L_f^2 \oplus L)$
Obviously $\phi_f \leq t^*$, Hence,

$$\begin{aligned}\delta_{v(L_f^1 \oplus L)}(t_0+t^*) - (t_0+t^*) &\leq \delta_{v(L_f^2 \oplus L)}(t_0) - t_0 - t^* \\ &\leq \delta_{v(L_f^2 \oplus L)}(t_0) - t_0 - \phi_f\end{aligned}$$

Furthermore, we know that: $\phi_f \geq c(L_f^1) - c(L_f^2)$.
Hence,

$$\delta_{v(L_f^1 \oplus L)}(t_0+t^*) - (t_0+t^*) + c(L_f^1 \oplus L) \leq \delta_{v(L_f^2 \oplus L)}(t_0) - t_0 + c(L_f^2 \oplus L)$$

We conclude that for all $t_0 \leq t_0^l(L_f^2)$, there exists $\widetilde{t}_0 = t_0 + t^* \leq t_0^l(L_f^1)$ such that:

$$\delta_{v(L_f^1 \oplus L)}(\widetilde{t}_0) - (\widetilde{t}_0) + c(L_f^1 \oplus L) \leq \delta_{v(L_f^2 \oplus L)}(t_0) - t_0 + c(L_f^2 \oplus L)$$

Hence,

$$\min_{t_0 \leq t_0^l(L_f^1)} \left\{\delta_{v(L_f^1 \oplus L)}(t_0) - t_0\right\} + c(L_f^1 \oplus L) \leq \min_{t_0 \leq t_0^l(L_f^2)} \left\{\delta_{v(L_f^2 \oplus L)}(t_0) - t_0\right\} + c(L_f^2 \oplus L)$$

5.2. The Backward TDL Algorithm

In the backward TDL algorithm, labels are extended from the depot (*i.e.* node $n+1$) to its predecessors. The extension of a label is allowed if it is feasible and if the latest possible departure time at the end node is no further than t_m. To a Label L_b, we associate the following components:

$v(L_b)$	the first node visited on the partial path represented by L_b
$c(L_b)$	the sum of the dual variables associated with nodes visited along the partial path represented by L_b
$\delta_{n+1}(t_{v(L_b)})$	arrival time at the depot through the partial path represented by L_b and when leaving node $v(L_b)$ at time $t_{v(L_b)}$
$S(L_b)$	set of nodes visited along the partial path represented by L_b

The set of feasible extensions $E(L_b)$ of L_b is the set of partial paths departing at the depot (*i.e.* node 0) at some time $t_0 \geq 0$ and reaching node $v(L_b)$ at some time $t_{v(L_b)} > t_0$ ($t_{v(L_b)}$ includes waiting and service at $v(L_b)$) without violating time windows. Going back to the depot through the partial path represented by label L_b should be feasible given that the departure time at $v(L_b)$ is $t_{v(L_b)}$. If label L'_b is the parent label of label L_b, the arrival time function corresponding to label L_b is computed as follows:

$$\delta_{n+1}(t_{v(L_b)}) = \delta_{n+1}(t_{v(L'_b)} = t_{v(L_b)} + \tau_{v(L_b)v(L'_b)}(t_{v(L_b)})) \tag{30}$$

Furthermore, we have:

$$S(L_b) = S(L'_b) \bigcup \{v(L_b)\} \quad \text{and} \quad c(L_b) = c(L'_b) - \pi_{v(L_b)} \tag{31}$$

The latest departure time $t^l_{v(L_b)}$ at node $v(L_b)$, such that the arrival at node $v(L'_b)$ is exactly its latest possible departure time, can be calculated using the procedure as described in Ichoua et al. (2003).

The extension of L'_b with node $v(L_b)$ is feasible if:

$$t^l_{v(L_b)} \leq a_{v(L_b)} + s_{v(L_b)} \quad \text{and} \quad t^l_{v(L'_b)} \geq t_m \tag{32}$$

Again, as illustrated in Figure 6, arrival time functions are non-decreasing linear stepwise functions. Moreover, arrival time functions are completely defined by their breakpoints. Arrival time function breakpoints result from travel time functions breakpoints, breakpoints calculated as departure time at the start node to hit a breakpoint on the arrival time function of the destination node, or from time windows. Furthermore, dominance can be defined in the same way as in the case of the forward TDL algorithm. To avoid redundancy, we only present the improved dominance criteria as it is slightly different.

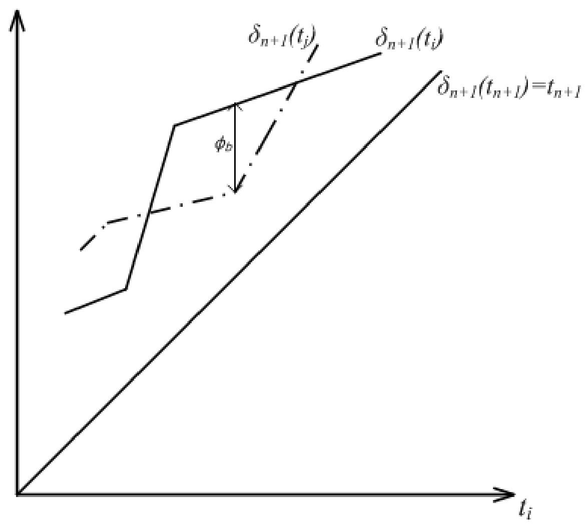

Figure 6 **The arrival time function.**

In Proposition 3, $\widetilde{S}(L_b)$ denotes the set of visited nodes along the partial path represented by label L_b extended by nodes that are unreachable from $v(L_b)$. In fact, the latest departure from all nodes, such that arrival time at $v(L_b)$ is its latest possible start time, is calculated using the procedure as described in Ichoua et al. (2003), and the maximum is taken. All nodes with an opening time (service time included) larger than that maximum will not be reachable from $v(L_b)$. Furthermore, we relax Condition (2.) by adding the quantity ϕ_b to the cost $c(L^2_b)$. ϕ_b is a real number related to, given a departure time at node $v(L^1_b)$, how early (in case ϕ_b is negative) or late (in case ϕ_b is positive) arrival at the depot takes place when traversing the partial path represented by label L_b^1 instead of the partial path represented by label L_b^2. Note that ϕ_b is conceptually different from ϕ_f as it is related to arrival time at the end node (*i.e.* the depot) instead of departure time at the depot. In the forward search, we can not relate ϕ_f to the arrival time at the end node as this might be different from the depot. Therefore, any gains in terms of arrival time does not guarantee a gain in the final complete tour. In fact, gains can easily be lost by possible waiting time due to time windows. If denote $D_{\delta_{L_b}}$ as the definition domain of the arrival time function $\delta_{v(L_f)}(t_0)$, we state Proposition 3 as follows:

PROPOSITION 3. *Label L_b^2 is dominated by label L_b^1 if:*

1. $v(L_b^1) = v(L_b^2)$
2. $c(L_b^1) \leq v(L_b^2) + \phi_b$
3. $S(L_b^1) \subseteq \widetilde{S}(L_b^2)$
4. $\delta_{n+1}^{-1}(t^l_{v(L_b^1)}) \leq \delta_{n+1}^{-1}(t^l_{v(L_b^2)})$

$$\phi_b = min\left\{\delta_{n+1}(t^l_{v(L_b^1)}) - \delta_{n+1}(t^l_{v(L_b^2)}), \min_{t\in D}\left\{\delta_{n+1}(t_{v(L_b^1)} = t) - \delta_{n+1}(t_{v(L_b^2)} = t)\right\}\right\} \quad and \quad D = D_{\delta_{L_b^1}} \bigcap D_{\delta_{L_b^2}}$$

Proof: see appendix

5.3. Merging Forward and backward Labels

After all forward and backward labels are processed, they are joined to construct feasible tours with negative reduced cost. A forward label L_f and a backward label L_b are joined if $v(L_f) = v(L_b)$, $S(L_f) \bigcap S(L_b)/\{i\} = \emptyset$, and there exists at least one possible dispatch time t_0 at the depot for which $\delta_{n+1}(t_{v(L_b)} = \delta_{v(L_f)}(t_0))$ is defined.
The attributes of label L resulting from merging a forward L_f and a backward label L_b are calculated as follows:

- $v(L) = n+1$
- $c(L) = c(L_f) + c(L_b)$
- $S(L) = S(L_f) \bigcup S(L_b)$
- $B_L = B_{L_f} \bigcup B_{L_b^{-1}}$

B_L is the set of breakpoints defining the arrival time function $\delta_{v(L)}(t_0)$ associated with label L. It is the union of the set B_{L_f} corresponding the breakpoints of the arrival time function $\delta_{v(L_t)}(t_0)$ associated with label L_f, and $B_{L_b^{-1}} = \{\delta^{-1}_{v(L_f)}(t_{v(L_b)}) : t_{v(L_b)} \in B_{L_b}\}$ where B_{L_b} is the set of breakpoints defining the arrival time function $\delta_{n+1}(t_{v(L_b)})$ associated with label L_b.

PROPOSITION 4. *For every route R in the optimal solution, there exist a forward path P_f and backward path P_b such that the route R is obtained by merging P_f and P_b.*

5.4. The Pricing Problem Heuristics

Branch-and-price algorithms can be accelerated using heuristics to solve the pricing problem. In fact, the heuristic will search for paths with negative reduced cost and add them to the master problem. When the heuristics fails to find any more paths with negative reduced cost, the exact algorithm is called. Ideally, for every node in the branching tree, the exact algorithm is called only once to check that no more paths with negative reduced cost exist. In our BCP framework, we use two heuristics. First, a greedy heuristic that extend each label to the node with the smallest travel time. Second, a truncated labeling heuristic in which only a limited number of labels is stored. Moreover, for the truncated heuristic, relaxed dominance criteria are used. In fact, we relax the condition on the sets of visited customers. Furthermore, we dominate label L_2 by label L_1 if:

$$\min_{t_0 \in B_{L_1}} \left\{ \delta_{v(L_1)}(t_0) - t_0 \right\} \le \min_{t_0 \in B_{L_2}} \left\{ \delta_{v(L_2)}(t_0) - t_0 \right\} \tag{33}$$

and

$$\min_{t_0 \in B_{L_1}} \left\{ \delta_{v(L_1)}(t_0) - t_0 \right\} + c(L_1) \le \min_{t_0 \in B_{L_2}} \left\{ \delta_{v(L_2)}(t_0) - t_0 \right\} + c(L_2) \tag{34}$$

Where B_{L_i} is the set of breakpoints defining $\delta_{v(L_i)}(t_0)$, $i = 1, 2$. $min_{t_0 \in B_{L_i}} \left\{ \delta_{v(L_i)}(t_0) - t_0 \right\}$ is the minimum duration of the partial path represented by label L_i, $i = 1, 2$. The number of stored labels can be increased each time the heuristic fails to find paths with negative reduced cost (*e.g.* we start with 250, then we increase the number of labels to 500 labels and finally to 1000 labels).

6. Computational Results

The BCP algorithm is implemented on a (**mention properties of the machine**). The open source framework COIN is used to solve the linear programming relaxation of the master problem. For our numerical study, we use the well known Solomon's data sets (Solomon (1987)) that follow a naming convention of $DTm.n$. D is the geographic distribution of the customers which can be R (Random), C (Clustered) or RC (Randomly Clustered). T is the instance type which can be either 1 (instances with tight time windows) or 2 (instances with wide time windows). m denotes the number of the instance and n the number of customers that need to be served. Road congestion is taken into account by assuming that vehicles travel through the network using different speed profiles. We consider speed profiles with two congested periods. Speeds in the rest of the planning horizon (*i.e.* the depot's time window) are relatively high. We consider speed profiles that comply with data from real life. Furthermore, we assume three types of links: fast, normal and slow. Slow links might represent links within the city center, fast links might represent highways and normal links might represent the transition from highways to city centers. Moreover, without loss of generality, we assume that breakpoints are the same for all speed profiles as congestion tends to happen around the same time regardless of the link's type (*e.g.* rush hours).The choice for a link type is done randomly and remains the same for all instances. The following speed profiles are considered:

Table 2 **Speed Profiles.**

	Zone1	Zone2	Zone3	Zone4	Zone5
Fast	1.5	1	1.67	1.17	1.33
Normal	1.17	0.67	1.33	0.83	1
Slow	1	0.33	0.67	0.5	0.83

Speed breakpoints are such that: $a = 0.2b_{n+1}$, $b = 0.3b_{n+1}$, $c = 0.7b_{n+1}$, $d = 0.8b_{n+1}$ and $e = b_{n+1}$. a, b, c, d and e are depicted in Figure 1, and b_{n+1} is the upper bound of the depot's time window. Travel time breakpoints are calculated using the procedure as described in Ichoua et al. (2003). Figures 7 and 8 illustrate respectively two travel time functions for a link from an R instance and a link from an RC instance.

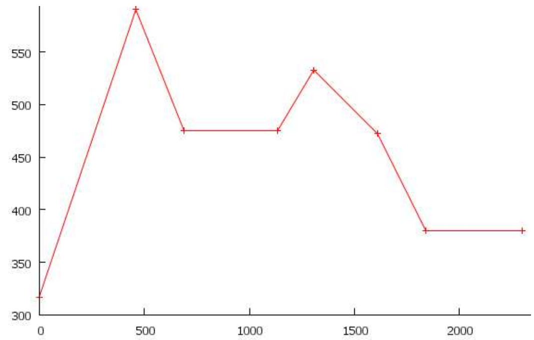

Figure 7 **Travel time function for an R instance.**

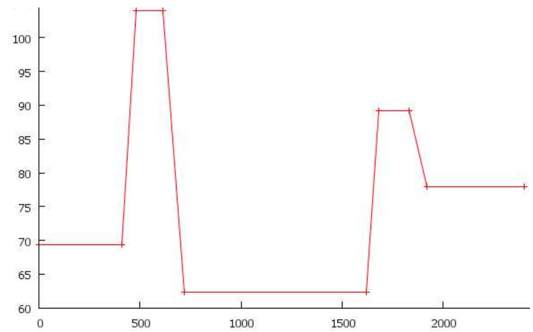

Figure 8 **Travel time function for an RC instance.**

6.1. TDESPPRC vs. TDSPPRC

6.2. Bi-directional TDL vs. Monodirectional TDL

7. Conclusions and Future Research

Acknowledgments
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Appendix.

In progress

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