Branch Flow Model: Relaxations and Convexification—Part II

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***Abstract*—We propose a branch flow model for the analysis and optimization of mesh as well as radial networks. The model leads to a new approach to solving optimal power flow (OPF) that consists of two relaxation steps. The first step eliminates the voltage and current angles and the second step approximates the resulting problem by a conic program that can be solved efficiently. For radial networks, we prove that both relaxation steps are always exact, provided there are no upper bounds on loads. For mesh networks, the conic relaxation is always exact but the angle relaxation may not be exact, and we provide a simple way to determine if a relaxed solution is globally optimal. We propose convexification of mesh networks using phase shifters so that OPF for the convexified network can always be solved efficiently for an optimal solution. We prove that convexification requires phase shifters only outside a spanning tree of the network and their placement depends only on network topology, not on power flows, generation, loads, or operating constraints. Part I introduces our branch flow model, explains the two relaxation steps, and proves the conditions for exact relaxation. Part II describes convexification of mesh networks, and presents simulation results.**

***Index Terms*— Convex relaxation, load flow control, optimal power flow, phase control, power system management.**

I. Introduction

In Part I of this two-part paper [2], we introduce a branch flow model that focuses on branch variables instead of nodal variables. We formulate optimal power flow (OPF) within the branch flow model and propose two relaxation steps. The first step eliminates phase angles of voltages and currents. We call the resulting problem OPF-ar which is still nonconvex. The second step relaxes the feasible set of OPF-ar to a second-order cone. We call the resulting problem OPF-cr which is convex, indeed a second-order cone program (SOCP) when the objective function is linear. We prove that the conic relaxation OPF-cr is always exact *even* for mesh networks, provided there are no upper bounds on real and reactive loads, i.e., any optimal solution of OPF-cr is also optimal for OPF-ar. Given an optimal solution of OPF-ar, whether we can derive an optimal solution to the original OPF depends on whether we can recover the voltage and current angles correctly from the given OPF-ar solution. We characterize the exact condition (the angle recovery condition) under which this is possible, and present two angle recovery algorithms. It turns out that the angle recovery condition has a simple interpretation: any solution of OPF-ar implies a phase angle difference across a line, and the angle recovery condition says that the implied phase angle differences sum to zero (mod 2π) around each cycle. For a radial network, this condition holds trivially and hence solving the conic relaxation OPF-cr always produces an optimal solution for the original OPF. For a mesh network, the angle recovery condition may not hold, and our characterization can be used to check if a relaxed solution yields an optimal solution for OPF.

In this paper, we prove that, by placing phase shifters on some of the branches, *any* relaxed solution of OPF-ar can be mapped to an optimal solution of OPF for the convexified network, with an optimal cost that is no higher than that of the original network. Phase shifters thus convert an NP-hard problem into a simpler problem. Our result implies that when the angle recovery condition holds for a relaxed branch flow solution, not only is the solution optimal for the OPF without phase shifters, but the addition of phase shifters cannot further reduce the cost. On the other hand, when the angle recovery condition is violated, then the convexified network may have a strictly lower optimal cost. Moreover, this benefit can be attained by placing phase shifters only outside an *arbitrary* spanning tree of the network graph.

There are in general many ways to choose phase shifter angles to convexity a network, depending on the number and location of the phase shifters. While placing phase shifters on each link outside a spanning tree requires the minimum number of phase shifters to guarantee exact relaxation, this strategy might require relatively large angles at some of these phase shifters. On the other extreme, one can choose to minimize (the Euclidean norm of) the phase shifter angles by deploying phase shifters on every link in the network. We prove that this minimization problem is NP-hard. Simulations suggest, however, that a simple heuristic works quite well in practice.

These results lead to an algorithm for solving OPF when there are phase shifters in mesh networks, as summarized in Fig. 1.

Since power networks in practice are very sparse, the number of lines not in a spanning tree can be relatively small compared to the number of buses squared, as demonstrated in simulations in Section V using the IEEE test systems with 14, 30, 57, 118, and 300 buses, as well as a 39-bus model of a New England power system and two models of a Polish power system with more than 2000 buses. Moreover, the placement of these phase shifters depends only on network topology, but not on power

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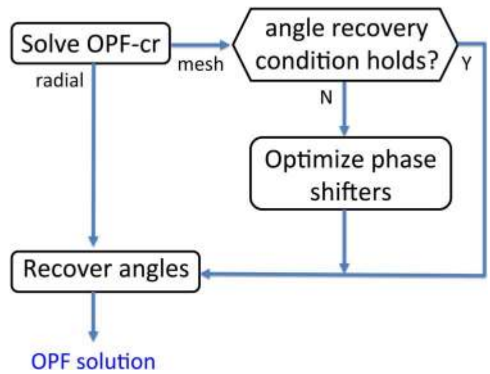

Fig. 1. Proposed algorithm for solving OPF with phase shifters in mesh networks. The details are explained in this two-part paper.

flows, generations, loads, or operating constraints. Therefore only one-time deployment cost is required to achieve subsequent simplicity in network operation. Even when phase shifters are not installed in the network, the optimal solution of a convex relaxation is useful in providing a lower bound on the true optimal objective value. This lower bound serves as a benchmark for other heuristic solutions of OPF.

The paper is organized as follows. In Section II, we extend the branch flow model of [2] to include phase shifters. In Section III, we describe methods to compute phase shifter angles to map any relaxed solution to an branch flow solution. In Section IV, we explain how to use phase shifters to simplify OPF. In Section V, we present our simulation results.

II. Branch Flow Model With Phase Shifters

We adopt the same notations and assumptions A1–A4 of [2].

A. Review: Model Without Phase Shifters

For ease of reference, we reproduce the *branch flow model* of [2] here:

$$I_{ij} = y_{ij}(V_i - V_j) \tag{1}$$
$$S_{ij} = V_i I_{ij}^* \tag{2}$$
$$s_j = \sum_{k:j\to k} S_{jk} - \sum_{i:i\to j}\left(S_{ij} - z_{ij}|I_{ij}|^2\right) + y_j^*|V_j|^2. \tag{3}$$

Recall the set $\mathbb{X}(s)$ of branch flow solutions given s defined in [2]:

$$\mathbb{X}(s) := \{x := (S, I, V, s_0) | x \text{ solves } (1)-(3) \text{ given } s\} \tag{4}$$

and the set $\mathbb{X}$ of all branch flow solutions:

$$\mathbb{X} := \bigcup_{s\in\mathbb{C}^n} \mathbb{X}(s). \tag{5}$$

To simplify notation, we often use $\mathbb{X}$ to denote the set defined either in (4) or in (5), depending on the context. In this section we study power flow solutions and hence we fix an s. All quantities, such as $x, \hat{y}, \mathbb{X}, \hat{\mathbb{Y}}, \overline{X}, \overline{X}_T$, are with respect to the given s, even though that is not explicit in the notation. In the next section, s is also an optimization variable and the sets $\mathbb{X}, \hat{\mathbb{Y}}, \overline{X}, \overline{X}_T$ are for any s.

Given a relaxed solution $\hat{y}$, define $\beta := \beta(\hat{y})$ by

$$\beta_{ij} := \angle\left(v_i - z_{ij}^* S_{ij}\right), \qquad (i,j) \in E. \tag{6}$$

It is proved in [2, Theorem 2] that a given $\hat{y}$ can be mapped to a branch flow solution in $\mathbb{X}$ if and only if there exists a (θ, k) that solves

$$B\theta = \beta + 2\pi k \tag{7}$$

for some integer vector $k \in \mathbb{N}^n$. Moreover if (7) has a solution, then it has a countably infinite set of solutions (θ, k), but they are relatively unique, i.e., given k, the solution θ is unique, and given θ, the solution k is unique. Hence (7) has a unique solution (θ_*, k_*) with $\theta_* \in (-\pi, \pi]^n$ if and only if

$$B_\perp B_T^{-1}\beta_T = \beta_\perp \pmod{2\pi} \tag{8}$$

which is equivalent to the requirement that the (implied) voltage angle differences sum to zero around any cycle c:

$$\sum_{(i,j)\in c} \tilde{\beta}_{ij} = 0 \pmod{2\pi}$$

where $\tilde{\beta}_{ij} = \beta_{ij}$ if $(i,j) \in E$ and $\tilde{\beta}_{ij} = -\beta_{ji}$ if $(j,i) \in E$.

B. Model With Phase Shifters

Phase shifters can be traditional transformers or Flexible AC Transmission Systems (FACTS) devices. They can increase transmission capacity and improve stability and power quality [3], [4]. In this paper, we consider an idealized phase shifter that only shifts the phase angles of the sending-end voltage and current across a line, and has no impedance nor limits on the shifted angles. Specifically, consider an idealized phase shifter parametrized by ϕ_{ij} across line (i,j), as shown in Fig. 2. As before, let V_i denote the sending-end voltage. Define I_{ij} to be the *sending-end* current leaving node i towards node j. Let k be the point between the phase shifter ϕ_{ij} and line impedance z_{ij}. Let V_k and I_k be the voltage at k and current from k to j, respectively. Then the effect of the idealized phase shifter is summarized by the following modeling assumption:

$$V_k = V_i e^{\mathbf{i}\phi_{ij}} \quad \text{and} \quad I_k = I_{ij} e^{\mathbf{i}\phi_{ij}}.$$

The power transferred from nodes i to j is still (defined to be) $S_{ij} := V_i I_{ij}^*$ which, as expected, is equal to the power $V_k I_k^*$ from nodes k to j since the phase shifter is assumed to be lossless. Applying Ohm's law across z_{ij}, we define the *branch flow model with phase shifters* as the following set of equations:

$$I_{ij} = y_{ij}\left(V_i - V_j e^{-\mathbf{i}\phi_{ij}}\right) \tag{9}$$
$$S_{ij} = V_i I_{ij}^* \tag{10}$$
$$s_j = \sum_{k:j\to k} S_{jk} - \sum_{i:i\to j}\left(S_{ij} - z_{ij}|I_{ij}|^2\right) + y_j^*|V_j|^2. \tag{11}$$

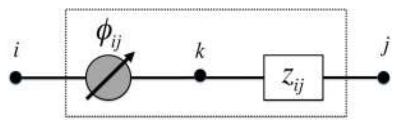

Fig. 2. Model of a phase shifter in line (i, j).

Without phase shifters ($\phi_{ij} = 0$), (9)–(11) reduce to the branch flow model (1)–(3).

The inclusion of phase shifters modifies the network and enlargers the solution set of the (new) branch flow equations. Formally, let

$$\overline{\mathbb{X}} := \{x | x \text{ solves } (9) - (11) \text{ for some } \phi\}. \tag{12}$$

Unless otherwise specified, all angles should be interpreted as being modulo 2π and in $(-\pi, \pi]$. Hence we are primarily interested in $\phi \in (-\pi, \pi]^m$. For any spanning tree T of G, let "$\phi \in T^\perp$" stand for "$\phi_{ij} = 0$ for all $(i, j) \in T$", i.e., ϕ involves only phase shifters in branches not in the spanning tree T. Define

$$\overline{\mathbb{X}}_T := \left\{x | x \text{ solves } (9) - (11) \text{ for some } \phi \in T^\perp\right\}. \tag{13}$$

Since (9)–(11) reduce to the branch flow model when $\phi = 0$, $\mathbb{X} \subseteq \overline{\mathbb{X}}_T \subseteq \overline{\mathbb{X}}$.

III. Phase Angle Setting

Given a relaxed solution $\hat{y}$, there are in general many ways to choose angles ϕ on the phase shifters to recover a feasible branch flow solution $x \in \overline{\mathbb{X}}$ from $\hat{y}$. They depend on the number and location of the phase shifters.

A. Computing ϕ

For a network with phase shifters, we have from (9) and (10)

$$S_{ij} = V_i \frac{V_i^* - V_j^* e^{\mathbf{i}\phi_{ij}}}{z_{ij}^*}$$

leading to $V_i V_j^* e^{\mathbf{i}\phi_{ij}} = v_i - z_{ij}^* S_{ij}$. Hence $\theta_i - \theta_j = \beta_{ij} - \phi_{ij} + 2\pi k_{ij}$ for some integer k_{ij}. This changes the angle recovery condition in [2, Theorem 2] from whether there exists (θ, k) that solves (7) to whether there exists (θ, ϕ, k) that solves

$$B\theta = \beta - \phi + 2\pi k \tag{14}$$

for some integer vector $k \in (-2\pi, 2\pi]^m$. The case without phase shifters corresponds to setting $\phi = 0$.

We now describe two ways to compute ϕ: the first minimizes the required number of phase shifters, and the second minimizes the size of phase angles.

1) Minimize Number of Phase Shifters: Our first key result implies that, given a relaxed solution $\hat{y} := (S, \ell, v, s_0) \in \hat{\mathbb{Y}}$, we can always recover a branch flow solution $x := (S, I, V, s_0) \in \overline{\mathbb{X}}$ of the convexified network. Moreover it suffices to use phase shifters in branches only outside a spanning tree. This method requires the smallest number $(m - n)$ of phase shifters.

Given any d-dimensional vector α, let $\mathcal{P}(\alpha)$ denote its projection onto $(-\pi, \pi]^d$ by taking modulo 2π componentwise.

Theorem 1: Let T be *any* spanning tree of G. Consider a relaxed solution $\hat{y} \in \hat{\mathbb{Y}}$ and the corresponding β defined by (6) in terms of $\hat{y}$.

1) There exists a unique $(\theta_*, \phi_*) \in (-\pi, \pi]^{n+m}$ with $\phi_* \in T^\perp$ such that $h_{\theta_*}(\hat{y}) \in \overline{\mathbb{X}}_T$, i.e., $h_{\theta_*}(\hat{y})$ is a branch flow solution of the convexified network. Specifically

$$\begin{aligned} \theta_* &= \mathcal{P}\left(B_T^{-1}\beta_T\right) \\ \phi_* &= \mathcal{P}\left(\begin{bmatrix} 0 \\ \beta_\perp - B_\perp B_T^{-1}\beta_T \end{bmatrix}\right). \end{aligned}$$

2) $\mathbb{Y} = \overline{\mathbb{X}} = \overline{\mathbb{X}}_T$ and hence $\hat{\mathbb{Y}} = \hat{h}(\overline{X}) = \hat{h}(\overline{\mathbb{X}}_T)$.

Proof: For the first assertion, write $\phi = [\phi_T^t \ \phi_\perp^t]^t$ and set $\phi_T = 0$. Then (14) becomes

$$\begin{bmatrix} B_T \\ B_\perp \end{bmatrix}\theta = \begin{bmatrix} \beta_T \\ \beta_\perp \end{bmatrix} - \begin{bmatrix} 0 \\ \phi_\perp \end{bmatrix} + 2\pi \begin{bmatrix} k_T \\ k_\perp \end{bmatrix}. \tag{15}$$

We now argue that there always exists a unique (θ_*, ϕ_*), with $\theta_* \in (-\pi, \pi]^n$, $\phi_* \in (-\pi, \pi]^m$ and $\phi_* \in T^\perp$, that solves (15) for some $k \in \mathbb{N}^m$.

The same argument as in the proof of [2, Theorem 2] shows that a vector (θ_*, ϕ_*, k_*) with $\theta_* \in (-\pi, \pi]^n$ and $\phi_* \in T^\perp$ is a solution of (15) if and only if

$$B_\perp B_T^{-1}\beta_T = \beta_\perp - [\phi_*]_\perp + 2\pi[\hat{k}_*]_\perp$$

where $[\hat{k}_*]_\perp := [k_*]_\perp - B_\perp B_T^{-1}[k_*]_T$ is an integer vector. Clearly this can always be satisfied by choosing

$$[\phi_*]_\perp - 2\pi[\hat{k}_*]_\perp = \beta_\perp - B_\perp B_T^{-1}\beta_T. \tag{16}$$

Note that given θ_*, $[k_*]_T$ is uniquely determined since $[\phi_*]_T = 0$, but $([\phi_*]_\perp, [k_*]_\perp)$ can be freely chosen to satisfy (16). Hence we can choose the unique $[k_*]_\perp$ such that $[\phi_*]_\perp \in (-\pi, \pi]^{m-n}$.

Hence we have shown that there always exists a unique (θ_*, ϕ_*), with $\theta_* \in (-\pi, \pi]^n$, $\phi_* \in (-\pi, \pi]^m$ and $\phi_* \in T^\perp$, that solves (15) for some $k_* \in \mathbb{N}^m$. Moreover this unique vector (θ_*, ϕ_*) is given by the formulae in the theorem.

The second assertion follows from assertion 1. ■

2) Minimize Phase Angles: The choice of (θ_*, ϕ_*) in Theorem 1 has the advantage that it requires the minimum number of phase shifters (only on links outside an arbitrary spanning tree T). It might however require relatively large angles $[\phi_*]_e$ at some links e outside T. On the other extreme, suppose we have phase shifters on every link. Then one can choose (θ_*, ϕ_*) such that the phase shifter angles are minimized.

Specifically we are interested in a solution (θ, ϕ, k) of (14) that minimizes $\|\mathcal{P}(\phi)\|^2$ where $\|\cdot\|$ denotes the Euclidean norm of ϕ after taking mod 2π componentwise. Hence we are interested in solving the following problem: given B, β

$$\min_{\theta,\phi,k,l} \quad \|\phi - 2\pi l\|^2 \tag{17}$$

$$\text{subject to} \quad B\theta = \beta - \phi + 2\pi k \tag{18}$$

where $k, l \in \mathbb{N}^m$ are integer vectors.

Theorem 2: The problem (17), (18) of minimum phase angles is NP-hard.

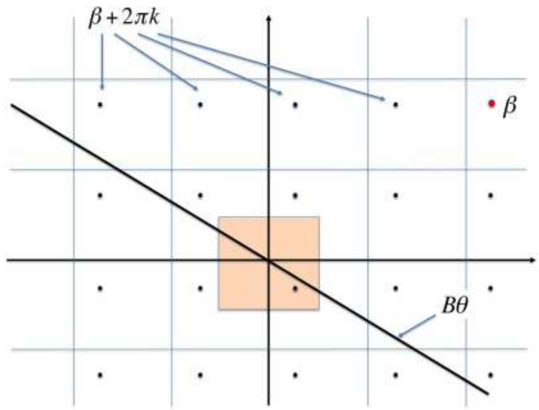

Fig. 3. Each lattice point corresponds to $2\pi k$ for a $k \in \mathbb{N}^m$. The constrained optimization (19) is to find a lattice point that is closest to the range space $\{B\theta | \theta \in \mathbb{R}^n\}$ of B. The shaded region around the origin is $(-\pi,\pi]^m$ and contains a point $\beta' := \beta + 2\pi k$ for exactly one $k \in \mathbb{N}^m$. Our approximate solution corresponds to solving (20) for this fixed k.

Proof: Clearly the problem (17), (18) is equivalent to the following unconstrained minimization [eliminate ϕ from (17), (18)]:

$$\min_{k\in\mathbb{N}^m} \min_{\theta\in\mathbb{R}^n} \|(\beta + 2\pi k) - B\theta\|^2 . \tag{19}$$

It thus solves for a lattice point $\beta + 2\pi k$ that is closest to the range space $\{B\theta | \theta \in \mathbb{R}^n\}$ of B, as illustrated in Fig. 3.

Fix any $k \in \mathbb{N}^m$. Consider $\beta' := \beta + 2\pi k$ and the inner minimization in (19):

$$\min_{\theta\in\mathbb{R}^m} \quad \|\beta' - B\theta\|^2. \tag{20}$$

This is the standard linear least-squares estimation where β' represents an observed vector that is to be estimated by an vector in the range space of B in order to minimize the normed error squared. The optimal solution is

$$\theta_* := (B^tB)^{-1} B^t \beta' \tag{21}$$

$$\beta' - B\theta_* = \left(I - B(B^tB)^{-1}B^t\right)\beta'. \tag{22}$$

Substituting (22) and (20) into (19), (19) becomes

$$\min_{k\in\mathbb{N}^m} \quad \|\gamma + 2\pi A k\|^2 \tag{23}$$

where $\gamma := A\beta \in \mathbb{R}^m$ and $A := I - B(B^tB)^{-1}B^t$ is the orthogonal complement of the range space of B. But (23) is the *closest lattice vector problem* and is known to be NP-hard [5].[1] ■

Remark 1: Since the objective function is strictly convex, the phase angles $\phi_* = (\beta + 2\pi k_*) - B\theta_*$ at optimality will lie in $(-\pi,\pi]^m$. Moreover, if an optimal solution exists, then there is always an optimal solution with θ_* in $(-\pi,\pi]^n$: if (θ,k) is optimal for (19) with $\theta \notin (-\pi,\pi]^n$, then by writing $k =: B\alpha + k'$ for integer vectors $\alpha \in \mathbb{N}^n$, $k' \in \mathbb{N}^m$, the objective function in (19) becomes

$$(\beta + 2\pi k') - B(\theta - 2\pi\alpha)$$

[1]We thank Babak Hassibi for pointing out (23) is the closest lattice vector problem studied in the literature.

i.e., we can always choose (k_*, α_*) so that $\theta_* := \theta - 2\pi\alpha_*$ lies in $(-\pi,\pi]^n$ and $k = B\alpha_* + k_*$. Therefore, given an optimal solution (θ,k) with $\theta \notin (-\pi,\pi]^n$, we can find another point (θ_*, k_*) with $\theta_* = \mathcal{P}(\theta) \in (-\pi,\pi]^n$ that is also optimal.

Many algorithms have been proposed to solve the closest lattice vector problem. See [6] for state-of-the-art algorithms. Given β, there is a unique k such that $\beta' := \beta + 2\pi k$ is in $(-\pi,\pi]^m$, as illustrated in the shaded area of Fig. 3. A simple heuristic that provides an upper bound on (19) is to solve (20) for this fixed k. From (21)–(22), the heuristic solution is

$$\theta_* := \mathcal{P}\left((B^tB)^{-1}B^t\beta'\right)$$

$$\phi_* := \left(I - B(B^tB)^{-1}B^t\right)\beta'.$$

This approximate solution is illustrated in Section V and seems to be effective in reducing the phase shifter angles ($k = 0$ in all our test cases).

B. Arbitrary Network of Phase Shifters

More generally, consider a network with phase shifters on an arbitrary subset of links. Given a relaxed solution $\hat{y}$, under what condition does there exists a θ such that the inverse projection $h_\theta(\hat{y})$ is a branch flow solution in $\mathbb{X}$? If there is a spanning tree T such that all links outside T have phase shifters, then Theorem 1 says that such a θ always exists, with an appropriate choice of phase shifter angles on non-tree links. Suppose no such spanning tree exists, i.e., given any spanning tree T, there is a set $E_{\perp'} \subseteq E \setminus E_T$ of links that contain no phase shifters. Let $B_{\perp'}$ and $\beta_{\perp'}$ denote the submatrix of B and subvector of β, respective, corresponding to these links. Then a necessary and sufficient condition for angle recovery is: there exists a spanning tree T such that the associated $B_{\perp'}$ and $\beta_{\perp'}$ satisfy

$$B_{\perp'} B_T^{-1} \beta_T = \beta_{\perp'} \pmod{2\pi}. \tag{24}$$

This condition reduces to (8) if there are no phase shifters in the network ($E_{\perp'} = E \setminus E_T$) and is always satisfied if every link outside any spanning tree has a phase shifter ($E_{\perp'} = \emptyset$). It requires that the angle differences implied by $\hat{y}$ sum to zero (mod 2π) around any loop that contains no phase shifter (c.f. [2, Theorem 2(1) and Remark 4]). After such a T is identified, the above two methods can be used to compute the required phase shifts.

C. Other Properties

We close this section by discussing two properties of ϕ. First, the voltage angles are $\theta = \mathcal{P}(B_T^{-1}(\beta_T - \phi_T))$ and the angle recovery condition (8) becomes

$$B_\perp B_T^{-1}(\beta_T - \phi_T) = \beta_\perp - \phi_\perp \pmod{2\pi} \tag{25}$$

which can always be satisfied by appropriate (nonunique) choices of ϕ. A similar argument to the proof of Theorem 2(2) leads to the following interpretation of (25). For any link $(i,j) \in E$, (14) says that the phase angle difference from node i to node j is β_{ij} and consists of the voltage angle difference $\theta_i - \theta_j = \beta_{ij} - \phi_{ij}$ and the phase shifter angle ϕ_{ij}. Fix any link $(i,j) \in E \setminus E_T$ not in tree T. The left-hand side $[B_\perp B_T^{-1}(\beta_T - \phi_T)]_{ij}$ of (25) represents the sum of the voltage angle differences from node i to node j along the unique path in T, *not* including the phase shifter angles along the path. This must be equal to the voltage angle difference $[\beta_\perp - \phi_\perp]_{ij}$ across (the non-tree) link (i,j), *not* including the phase shifter angle across (i,j). Therefore (25) has the same interpretation as before that the voltage angle differences sum to zero (mod 2π) around any cycle, though, with phase shifters, the voltage angle differences are now $\beta_{ij} - \phi_{ij}$ instead of β_{ij}. This in particular leads to a relationship between any two solutions (θ_*, ϕ_*) and $(\hat\theta, \hat\phi)$ of (14).

In particular, let (θ_*, ϕ_*) be the solution in Theorem 1 where $\phi_* \in T^\perp$, and $(\hat\theta, \hat\phi)$ any other solution. Then applying (25) to both ϕ_* and $\hat\phi$ leads to a relation between them on every basis cycle. Specifically, let $i \to j$ be a link not in the spanning tree T, let $T(0 \rightsquigarrow k)$ be the unique path in T from node 0 to any node k. Then for each link $i \to j$ in E that is not in T, we have (equalities to be interpreted as mod 2π)

$$\begin{aligned}[\phi_*]_{ij} &= \hat\phi_{ij} - \sum_{(k,l)\in T(0\rightsquigarrow j)} \hat\phi_{kl} + \sum_{(k,l)\in T(0\rightsquigarrow i)} \hat\phi_{kl} \\ &= \beta_{ij} - \sum_{(k,l)\in T(0\rightsquigarrow j)} \beta_{kl} + \sum_{(k,l)\in T(0\rightsquigarrow i)} \beta_{kl}.\end{aligned}$$

Second, Theorem 1 implies that given any relaxed solution $\hat y$, there exists a $\phi \in T^\perp$ such that its inverse projection $x := h_\theta(\hat y)$ is a branch flow solution, i.e., (x,ϕ) satisfies (9)–(11). We now give an alternative direct construction of such a solution (x,ϕ) from any given branch flow solution $\tilde x$ and phase shifter setting $\tilde\phi$ that may have nonzero angles on some links in T. It exhibits how the effect of phase shifters in a tree is equivalent to changes in voltage angles.

Fix any spanning tree T. Given any $(\tilde x, \tilde\phi)$, partition $\tilde\phi^t = [\tilde\phi_T \ \tilde\phi_\perp]$ with respect to T. Define $\alpha \in (-\pi, \pi]^n$ by $B_T\alpha = \tilde\phi_T$ or $\alpha := B_T^{-1}\tilde\phi_T$. Then define the mapping $(x,\phi) = g(\tilde x, \tilde\phi)$ by

$$V_i := \tilde V_i e^{\mathbf{i}\alpha_i}, \quad I_{ij} := \tilde I_{ij} e^{\mathbf{i}\alpha_i}, \quad S_{ij} := \tilde S_{ij} \tag{26}$$

and

$$\phi_{ij} := \begin{cases} 0 & \text{if } (i,j) \in E_T \\ \tilde\phi_{ij} - (\alpha_i - \alpha_j) & \text{if } (i,j) \in E \setminus E_T \end{cases} \tag{27}$$

i.e., ϕ is nonzero only on non-tree links. It can be verified that $\alpha_i - \alpha_j = \sum_{e\in T(i\rightsquigarrow j)} \tilde\phi_e$ where $T(i \rightsquigarrow j)$ is the unique path in tree T from node i to node j. Note that $|V_i| = |\tilde V_i|$, $|I_{ij}| = |\tilde I_{ij}|$ and $S = \tilde S$. Hence if $h(\tilde x)$ is a relaxed branch flow solution, so is $h(x)$. Moreover, the effect of phase shifters in T is equivalent to adding α_i to the phases of V_i and I_{ij}.

Theorem 3: Fix any tree T. If $(\tilde x, \tilde\phi)$ is a solution of (9)–(11), so is $(x,\phi) = g(\tilde x, \tilde\phi)$ defined in (26) and (27).

Proof[2]*:* Since $|V_i| = |\tilde V_i|$, $|I_{ij}| = |\tilde I_{ij}|$ and $S = \tilde S$, (x,ϕ) satisfies (10) and (11). For any link $(i,j) \in E_T$ in tree T, (26) and (27) imply

$$\begin{aligned}V_i - V_j e^{-\mathbf{i}\phi_{ij}} &= \left(\tilde V_i - \tilde V_j e^{-\mathbf{i}(\alpha_i - \alpha_j)}\right) e^{\mathbf{i}\alpha_i} \\ &= \left(\tilde V_i - \tilde V_j e^{-\mathbf{i}\tilde\phi_{ij}}\right) e^{\mathbf{i}\alpha_i}\end{aligned}$$

where the second equality follows from $B_T\alpha = \tilde\phi_T$. For any link $(i,j) \in E \setminus E_T$ not in T, (26) and (27) imply

$$V_i - V_j e^{-\mathbf{i}\phi_{ij}} = \left(\tilde V_i - \tilde V_j e^{-\mathbf{i}\tilde\phi_{ij}}\right) e^{\mathbf{i}\alpha_i}.$$

But $(\tilde V_i - \tilde V_j e^{-\mathbf{i}\tilde\phi_{ij}}) = \tilde I_{ij}$ since $(\tilde x, \tilde\phi)$ satisfies (9). Therefore $V_i - V_j e^{-\mathbf{i}\phi_{ij}} = I_{ij}$, i.e., (x,ϕ) satisfies (9) on every link. ■

[2]A less direct proof is to observe that (25) and $\alpha = B_T^{-1}\tilde\phi_T$ imply

$$B_\perp B_T^{-1}\beta_T = \beta_\perp - (\tilde\phi_\perp - B_\perp\alpha) = \beta_\perp - \phi_\perp \pmod{2\pi}$$

which means (x,ϕ) satisfies (14).

IV. OPF in Convexified Network

Theorem 1 suggests using phase shifters to convexify a mesh network so that *any* solution of OPF-ar can be mapped into an optimal solution of OPF of the convexified network. Convexification thus modifies an NP-hard problem into a simple problem without loss in optimality; moreover this requires an one-time deployment cost for subsequent operational simplicity, as we now show.

We will compare four OPF problems: the original OPF defined in [2]:

OPF:

$$\begin{aligned}&\min_{x,s} \quad f\left(\hat h(x), s\right) \\ &\text{subject to} \quad x \in \mathbb{X}, \quad (S, v, s_0, s) \in \mathbb{S}\end{aligned}$$

the relaxed OPF-ar defined in [2]:

OPF-ar:

$$\begin{aligned}&\min_{x,s} \quad f\left(\hat h(x), s\right) \\ &\text{subject to} \quad x \in \mathbb{Y}, \quad (S, v, s_0, s) \in \mathbb{S}\end{aligned}$$

the following problem where there is a phase shifter on every line ($\phi \in (-\pi, \pi]^m$):

OPF-ps:

$$\begin{aligned}&\min_{x,s,\phi} \quad f\left(\hat h(x), s\right) \\ &\text{subject to} \quad x \in \overline{\mathbb{X}}, (S, v, s_0, s) \in \mathbb{S}\end{aligned}$$

and the problem where, given any spanning tree T, there are phase shifters only outside T:

OPF-ps(T):

$$\begin{aligned}&\min_{x,s,\phi} \quad f\left(\hat h(x), s\right) \\ &\text{subject to} \quad x \in \overline{\mathbb{X}}_T, (S, v, s_0, s) \in \mathbb{S}, \phi \in T^\perp\end{aligned}.$$

Let the optimal values of OPF, OPF-ar, OPF-ps, and OPF-ps(T) be f_*, f_{ar}, f_{ps}, and f_T, respectively.

Theorem 1 implies that $\mathbb{X} \subseteq \mathbb{Y} = \overline{\mathbb{X}} = \overline{\mathbb{X}}_T$ for any spanning tree T. Hence we have

TABLE I
LOSS MINIMIZATION. MIN LOSS WITHOUT PHASE SHIFTERS (PS) WAS COMPUTED USING SDP RELAXATION OF OPF; MIN LOSS WITH PHASE SHIFTERS WAS COMPUTED USING SOCP RELAXATIONS OPF-CR OF OPF-AR. THE "(%)" INDICATES THE NUMBER OF PS AS A PERCENTAGE OF #LINKS

Test cases	# links (m)	No PS: Min loss (OPF, MW)	With phase shifters (PS): Min loss (OPF-cr, MW)	# required PS ($m-n$)	# active PS $\lvert\phi_i\rvert > 0.1^\circ$	Min #PS ($^\circ$) $[\phi_{\min}, \phi_{\max}]$	Min $\lVert\phi\rVert^2$ ($^\circ$) $[\phi_{\min}, \phi_{\max}]$
IEEE 14-Bus	20	0.546	0.545	7 (35%)	2 (10%)	$[-2.09, 0.58]$	[-0.63, 0.12]
IEEE 30-Bus	41	1.372	1.239	12 (29%)	3 (7%)	$[-0.20, 4.47]$	[-0.95, 0.65]
IEEE 57-Bus	80	11.302	10.910	24 (30%)	19 (24%)	$[-3.47, 3.15]$	[-0.99, 0.99]
IEEE 118-Bus	186	9.232	8.728	69 (37%)	36 (19%)	$[-1.95, 2.03]$	[-0.81, 0.31]
IEEE 300-Bus	411	211.871	197.387	112 (27%)	101 (25%)	$[-13.3, 9.40]$	[-3.96, 2.85]
New England 39-Bus	46	29.915	28.901	8 (17%)	7 (15%)	$[-0.26, 1.83]$	[-0.33, 0.33]
Polish (case2383wp)	2,896	433.019	385.894	514 (18%)	373 (13%)	$[-19.9, 16.8]$	[-3.07, 3.23]
Polish (case2737sop)	3,506	130.145	109.905	770 (22%)	395 (11%)	$[-10.9, 11.9]$	[-1.23, 2.36]

TABLE II
LOADABILITY MAXIMIZATION. MAX LOADABILITY WITHOUT PHASE SHIFTERS (PS) WAS COMPUTED USING SDP RELAXATION OF OPF; MAX LOADABILITY WITH PHASE SHIFTERS WAS COMPUTED USING SOCP RELAXATIONS OPF-CR OF OPF-AR. THE "(%)" INDICATES THE NUMBER OF PS AS A PERCENTAGE OF #LINKS

Test cases	No PS: Max load. (OPF)	With phase shifters (PS): Max load. (OPF-cr)	# required PS ($m-n$)	# active PS $\lvert\phi_i\rvert > 0.1^\circ$	Min #PS ($^\circ$) $[\phi_{\min}, \phi_{\max}]$	Min $\lVert\phi\rVert^2$ ($^\circ$) $[\phi_{\min}, \phi_{\max}]$	Simu. time (seconds)
IEEE 14-Bus	195.0%	195.2%	7 (35%)	6 (30%)	$[-0.51, 1.35]$	[-0.28, 0.23]	1.92
IEEE 30-Bus	156.7%	158.7%	12 (29%)	9 (22%)	$[-0.42, 12.4]$	[-2.68, 1.86]	3.86
IEEE 57-Bus	108.2%	118.3%	24 (30%)	24 (30%)	$[-13.1, 23.2]$	[-4.12, 4.12]	7.13
IEEE 118-Bus	203.7%	204.9%	69 (37%)	64 (34%)	$[-8.47, 17.6]$	[-4.61, 5.36]	15.96
IEEE 300-Bus	106.8%	112.8%	112 (27%)	103 (25%)	$[-15.0, 16.5]$	[-4.28, 6.31]	34.6
New England 39-Bus	109.1%	117.0%	8 (17%)	5 (11%)	$[-1.02, 1.28]$	[-0.28, 0.18]	2.82
Polish (case2383wp)	101.4%	106.6%	514 (18%)	435 (15%)	$[-19.6, 19.4]$	[-4.06, 4.32]	434.5
Polish (case2737sop)	127.6%	132.5%	770 (22%)	420 (12%)	$[-13.9, 17.1]$	[-2.07, 3.62]	483.7

Corollary 4: For any spanning tree T, $f_* \geq f_{ar} = f_{ps} = f_T$, with equality if there is a solution $(\hat{y}_{ar}, s_{ar})$ of OPF-ar that satisfies (8).

Corollary 4 has several important implications:

1) Theorem 1 in [2] implies that we can solve OPF-ar efficiently through conic relaxation to obtain a relaxed solution $(\hat{y}_{ar}, s_{ar})$. An optimal solution of OPF may or may not be recoverable from it. If $\hat{y}_{ar}$ satisfies the angle recovery condition (8) with respect to s_{ar}, then Theorem 2 in [2] guarantees a unique $\theta_* \in (-\pi, \pi]^n$ such that the inverse projection $(\hat{h}_{\theta_*}(\hat{y}_{ar}), s_{ar})$ is indeed optimal for OPF.
2) In this case, Corollary 4 implies that adding any phase shifters to the network cannot further reduce the cost since $f_* = f_{ar} = f_{ps}$.
3) If (8) is not satisfied, then $\hat{y}_{ar} \notin \hat{h}(\mathbb{X})$ and there is no inverse projection that can recover an optimal solution of OPF from $(\hat{y}_{ar}, s_{ar})$. In this case, $f_* \geq f_{ar}$. Theorem 1 implies that if we allow phase shifters, we can always attain $f_{ar} = f_{ps}$ with the relaxed solution $(\hat{y}_{ar}, s_{ar})$, with potentially strict improvement over the network without phase shifters (when $f_* > f_{ar}$).
4) Moreover, Corollary 4 implies that such benefit can be achieved with phase shifters only in branches outside an arbitrary spanning tree T.

Remark 2: The choice of the spanning tree T does not affect the conclusion of the theorem. Different choices of T correspond to different choices of n linearly independent rows of B and the resulting decomposition of B and β into B_T and β_T. Therefore T determines the phase angles θ_* and ϕ_* according to the formulae in the theorem. Since the objective $f(\hat{h}(x), s)$ of OPF is independent of the phase angles θ_*, for the same relaxed solution $\hat{y}$, OPF-ps achieves the same objective value regardless of the choice of T.

V. SIMULATIONS

For radial networks, results in Part I (Theorem 4) guarantees that both the angle relaxation and the conic relaxation are exact. For mesh networks, the angle relaxation may be inexact and phase shifters may be needed to implement a solution of the conic relaxation. We now explore through numerical experiments the following questions:

- How many phase shifters are typically needed to convexify a mesh network?
- What are typical phase shifter angles to implement an optimal solution for the convexified network?

Test Cases: We explore these questions using the IEEE benchmark systems with 14, 30, 57, 118, and 300 buses, as well as a 39-bus model of a New England power system and two models of a Polish power system with 2383 and 2737 buses. The data for all the test cases were extracted from the library of built in models of the MATPOWER toolbox [7] in Matlab. The test cases involve constraints on bus voltages as well as limits on the real and reactive power generation at every generator bus. The New England and the Polish power systems also involve MVA limits on branch power flows. All these systems are mesh networks, but very sparse.

Objectives: We solve the test cases for two scenarios:

- *Loss minimization*. In this scenario, the objective is to minimize the total active power loss of the circuit given constant load values, which is equivalent to minimizing the total active power generation. The results are shown in Table I.

- *Loadability maximization*. In this scenario, the objective is to determine the maximum possible load increase in the system while satisfying the generation, voltage and line constraints. We have assumed all real and reactive loads grow uniformly, i.e., by a constant multiplicative factor called the *max loadability* in Table II.

Solution Methods: We use the "SEDUMI" solver in Matlab [8]. We first solved the SOCP relaxation OPF-cr for a solution $(\hat{y}, s)$ of OPF-ar. In all test cases, equality was attained at optimality for the second-order cone constraint, and hence OPF-cr was exact, as [2, Theorem 1] would suggest. Recall however that the load values were constants in all the test cases. Even though this violated our condition that there are no upper bounds on the loads OPF-cr turned out to be exact with respect to OPF-ar in all cases. This confirms that the no-upper-bound condition is sufficient but not necessary for the conic relaxation to be exact.

Using the solution $(\hat{y}, s)$ of OPF-ar, we checked if the angle recovery condition (8) was satisfied. In all test cases, the angle recovery condition failed and hence no $(h_\theta(\hat{y}), s)$ was feasible for OPF without phase shifters. We computed the phase shifter angles ϕ using both methods explained in Section III-A and the corresponding unique $(h_\theta(\hat{y}), s)$ that was an optimal solution of OPF for the convexified network. For the first method that minimizes the number of required phase shifters, we have used a minimum spanning tree of the network where the weights on the lines are their reactance values. For the second method, we solve an approximation to the angle minimization that optimizes over θ for the fixed k that shifts β to $(-\pi, \pi]^m$.

In Tables I and II, we report the number $m - n$ of phase shifters potentially required, the number of active phase shifters (i.e., those with a phase angles greater than 0.1°), and the range of the phase angles at optimality using both methods. In Table II, we also report the simulation time on an Intel 1.8 GHz Core i5 CPU.

We report the optimal objective values of OPF with and without phase shifters in Tables I and II. The optimal values of OPF without phase shifters were obtained by implementing the SDP formulation and relaxation proposed in [9] for solving OPF. In all test cases, the solution matrix was of rank one and hence the SDP relaxation was exact. Therefore the values reported here are indeed optimal for OPF.

The SDP relaxation requires the addition of small resistances $(10^{-6}\ \mathrm{pu})$ to every link that has a zero resistance in the original model, as suggested in [10]. This addition is, on the other hand, not required for the SOCP relaxation: OPF-cr is tight with respect to OPF-ar with or without this addition. For comparison, we report the results where the same resistances are added for both the SDP and SOCP relaxations.

Summary: From Tables I and II:

1) Across all test cases, the convexified networks have higher performance (lower minimum loss and higher maximum loadability) than the original networks. More important than the modest performance improvement is design for simplicity: it guarantees an efficient solution for OPF.
2) The networks are (mesh but) very sparse, with the ratios $m/(n+1)$ of the number of lines to the number of buses varying from 1.2 to 1.6 (Table I). The numbers $m - n$ of phase shifters potentially required on every link outside a spanning tree for convexification vary from 17% of the numbers m of links to 37%.
3) The numbers of active phase shifters in the test cases vary from 7% of the numbers m of links to 25% for loss minimization, and 11% to 34% for loadability maximization. The phase angles required at optimality is no more than 20° in magnitude with the minimum number of phase shifters. With the maximum number of phase shifters, the range of the phase angles is much smaller (less than 7°).
4) The simulation times range from a few seconds to minutes. This is much faster than SDP relaxation. Furthermore they appear linear in network size.

VI. Conclusion

We have presented a branch flow model and demonstrated how it can be used for the analysis and optimization of mesh as well as radial networks. Our results confirm that radial networks are computationally much simpler than mesh networks. For mesh networks, we have proposed a simple way to convexify them using phase shifters that will render them computationally as simple as radial networks for power flow solution and optimization. The addition of phase shifters thus convert a nonconvex problem into a different, simpler problem.

We have proposed a solution strategy for OPF that consists of two steps:

1) Compute a relaxed solution of OPF-ar by solving its conic relaxation OPF-cr.
2) Recover from a relaxed solution an optimal solution of the original OPF using an angle recovery algorithm.

We have proved that, for radial networks, both steps are always exact, provided there are no upper bounds on loads, so this strategy guarantees a globally optimal solution. For mesh networks the angle recovery condition may not hold but can be used to check if a given relaxed solution is globally optimal.

Since practical power networks are very sparse, the number of required phase shifters may be relatively small. Moreover, their placement depends only on network topology, but not on power flows, generations, loads, or operating constraints. Therefore an one-time deployment cost is required to achieve the subsequent simplicity in network and market operations.

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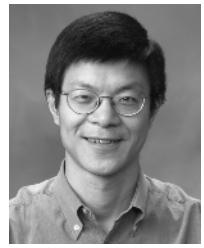

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