BRANCHED COVERINGS WITHOUT REGULAR POINTS OVER BRANCH POINT IMAGES

BY

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1. Introduction

The purpose of this paper is to describe the branch sets B_f [1, p. 528] of those light open maps $f : S^n \to S^n$ (where S^n denotes the n-sphere) for which $f^{-1}f(B_f) = B_f$ and $\dim f(B_f) \leq n - 2$. It will be proved that, in the cases $n = 2$ and $n = 3$, numerous different maps are possible whereas certain restrictions occur on the nature of B_f in higher dimensions. The hypothesis that $f^{-1}f(B_f) = B_f$ is a natural one. It holds for example if f is the orbit map of a finite group acting on the n-sphere. Furthermore, while the examples in [2] show the complications possible in the general case, in the regular Montgomery-Samelson case ($f^{-1}fB_f = B_f$ and f is a homeomorphism there—abbreviated M-S) it is possible to find some structure [4]. (The reader should be warned that the hypothesis of regularity is invalidly omitted in [4].) The maps considered in this paper are an intermediate class between the M-S and the general light open maps.

Throughout, $f : M^n \to N^n$ will be a light open map of n-manifolds for which $\dim f(B_f) \leq n - 2$ and hence [1, corollary 2.3, p. 531] $\dim B_f \leq n - 2$. In dimension 2, even without further hypotheses, the Stoilow-Whyburn theory guarantees that B_f and $f(B_f)$ are finite sets.

2. The case of the two-sphere

Throughout this section, we consider maps $f : S^2 \to S^2$.

THEOREM 1. *If $f^{-1}f(B_f) = B_f \neq \emptyset$, then either $f(B_f) = S^0 = B_f$ or else $f(B_f)$ is a set consisting of three points. In the latter case the degree of f cannot be less than 4; for $d = 4$ both B_f and the local behavior of f at B_f is uniquely determined; for $d = 5$ there is no such map; and for $d > 5$ there are various possibilities.*

Proof. Let $f(B_f) = \{q_1, \cdots, q_k\}$; let $f^{-1}(q_j) = \{p_{1j}, \cdots, p_{m_j,j}\}$ and let the *exceptionality* [2, p. 608] of f at p_{ij} be $e_{ij} > 0$. In this manner every element of $f^{-1}(q_j)$ becomes a branch point. Since the degree d is obtainable by computing for any y in the range of f the sum of the local degrees at the points of $f^{-1}(y)$, it follows that

$$d = \textstyle\sum_i (e_{ij} + 1) = \sum_i e_{ij} + m_j . \qquad (1)$$

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Since the local degree at p_{ij} is at least 2, $m_j \leq \frac{1}{2} d$. Hence

$$\frac{1}{2} d \leq \sum_i e_{ij} \text{ and } \tfrac{1}{2}kd \leq \sum_i \sum_j e_{ij}. \tag{2}$$

From the Hurwitz-Riemann formula [3, p. 275], which is the 2-dimensional case of Tucker's formula [7], it follows that

$$2 + \sum_j \sum_i e_{ij} = 2d. \tag{3}$$

Hence from (2) and (3) it follows that

$$k \leq 4 - 4/d. \tag{4}$$

Thus, for maps with the prescribed properties, k is either 1, 2, or 3. If one solves the last inequality for d instead of k, one obtains

$$d \geq 4/(4 - k) \tag{5}$$

from which it follows that $d \geq 2$ if $k = 1$ or 2 and $d \geq 4$ when $k = 3$.

From the second part of (2) and from (1) it follows that

$$\tfrac{1}{2}kd \leq \sum_{ij} e_{ij} = \sum_j (d - m_j) = kd - \sum_j m_j \tag{6}$$

and hence that $kd \geq 2\sum_{j=1} m_j$. Since $m_j \geq 1$, it follows that $d \geq 2$.

From (1) and (3) it follows that

$$2d - 2 = kd - \sum_j m_j. \tag{7}$$

When $k = 2$, $m_1 + m_2 = 2$ and B_f consists precisely of two points. Thus the case $k = 2$ is the case in which the restriction $f \mid {}^{-1}f(B_f)$ is a homeomorphism. For all degrees $d \geq 2$, the complex function $f(z) = z^d$ yields such a map, and topologically these are the only such maps.

When $k = 3$ and the functions under discussion exist, equation (6) yields $\sum m_j = d + 2$ and the number of branch points is seen to depend upon the degree. For large degree there are a great many different functions of this type with various collections of exceptionalities for the branch points.

For even degree the functions of degree $2n$ defined by

$$g(z) = (z^{2n} + 6z^n + 1)/4z^n, \quad n = 2, 3, \cdots$$

provide examples. A computation will show that

$$g^{-1}g(B_g) = B_g = \{z \mid z^{2n} = 1\} \text{ together with } 0 \text{ and } \infty,$$
$$g(B_g) = \{1, 2, \infty\},$$

where

$$g^{-1}(\infty) = \{0, \infty\}, \quad g^{-1}(1) = n\text{th roots of } -1, \quad g^{-1}(2) = n\text{th roots of } +1$$

and the exceptionalities are as follows:

$$e(0) = e(\infty) = n - 1, \qquad e(+1) = e(-1) = 1.$$

It will now be proved that the values $d = 5$ and $k = 3$ cannot occur together. If they did, the values of e_{ij} would be at most 4. If $e_{ij} = 4$, then $m_j = 1$. The case $e_{ij} = 3$ cannot occur, for it would mean that the local degree of f at p_{ij} would be 4 and that the other point in $f^{-1}(q_j)$ would be outside B_f. If $e_{ij} = 2$, there is just one other element in $f^{-1}(q_j)$ and it has exceptionality 1. Hence, for each j, $\sum_i e_{ij}$ is either 3 or 4 and $\sum_{ij} e_{ij} \geq 9$. In equation (3) this would mean that $2 + 9 \leq 10$ which is false.

If $d = 2n + 1$ and $n > 2$, there are examples. In the case $d = 7$ and $k = 3$, there is topologically precisely one such map. For higher degrees there are many. This question is dealt with for both even and odd degree in the thesis of Carl Shepardson [5], to which we refer the reader for these examples.

In the case $k = 2$, the sets $f^{-1}(q_j)$, $j = 1, 2$, are homeomorphic. When $k = 3$, one obtains the following:

Remark. If $k = 3$, and the sets $f^{-1}(q_j)$, $j = 1, 2, 3$, are homeomorphic, then $d \equiv 4 \bmod 3$.

Proof. Let $m_j = m$, $j = 1, 2, 3$. Then from 1 and 3, an elimination of $\sum_{ij} e_{ij}$ yields $d = 3m - 2$. If, in addition, one requires that the exceptionalities be the same, say e at all branch points, then from (1), $d = m(e + 1)$. This cannot occur, therefore, at prime degrees.

3. Higher dimensions

We consider maps $f: S^3 \to S^3$. Let p and q be positive integers and let S^1 and D be the unit circle and unit disk in the complex plane respectively. Let

$$g_{pq}: S^1 \times D \to S^1 \times D$$

be defined by $g_{pq}(z, w) = (z^p, w^q)$. Appropriate identification of the boundaries of two such solid tori, one the domain for g_{pq} and the other for g_{qp} produces a map $f: S^3 \to S^3$ satisfying the hypotheses of this paper. The set B_f is the disjoint union of two copies of S^1 and they are linked; $f(B_f)$ has the same structure. Certain aspects of this situation are valid in higher dimensions, to which we now turn.

The rest of this section will be devoted to the case $\dim M = \dim N = n > 2$. The singular homology (and cohomology) theory with integer coefficients will be employed. Let M and N be compact orientable manifolds without boundary whose homology vanishes in dimensions 1 and 2. Let B_i and $f(B_i)$ be orientable $(n\text{-}2)$-manifolds such that $B_i = f^{-1}f(B_i)$, B_i and $f(B_i)$ are isolated tamely embedded components of B_f and $f(B_f)$ respectively and let d_i be the local degree on B_i.

LEMMA 1. *Let x be a point of B_i and let U be a Euclidean neighborhood of x in M such that $U \cap B_i$ is a Euclidean neighborhood of x in B_i and $U \cap B_f = U \cap B_i$. Let $V = B_i \cap U$. Then diagram* A *is a commutative diagram of groups and homomorphisms in which φ is the Lefschetz duality isomorphism, δ*

is the coboundary homomorphism and i denotes inclusion. Furthermore, the vertical arrows represent isomorphisms.

$$\begin{array}{ccc} H_1(U - V) & \xrightarrow{i_i^*} & H_1(M - B_i) \\ \varphi\downarrow & & \varphi\downarrow \\ H^{n-1}[M, M - (U - V)] & \xrightarrow{i_2^*} & H^{n-1}(M, B_i) \\ \delta\uparrow & & \delta\uparrow \\ H^{n-2}[(M - U)\cup V] & \xrightarrow{i_3^*} & H^{n-2}(B_i) \end{array}$$

DIAGRAM A

Proof. We know that B_i and $(U - V)$ are tautly embedded in M [6, Theorem 10, p. 290] and hence φ is an isomorphism [6, Theorem 19, p. 297] in both cases. In the exact cohomology sequences for (M, B_i) and $(M, U - V)$, the groups $H^{n-1}(M)$ and $H^{n-2}(M)$ are zero by the Poincaré duality theorem [6, Theorem 18, p. 297] and the fact that the homology of M vanishes in dimensions 1 and 2. Therefore δ is an isomorphism in both cases. Diagram A is commutative, the bottom square by the naturality of the exact sequence for a pair and the top square by the naturality of $\bar{\gamma}_U$ and the inclusions appearing in the proof of [6, Theorem 19, p. 297]. The naturality of $\bar{\gamma}_U$ with respect to inclusions is established at [6, p. 292].

We remark that $U - V$ is homotopically equivalent to S^1 which implies that

$$Z = H_1(U - V) = H^{n-2}[(M - U) \cap V].$$

Since B_i is an orientable $(n - 2)$-manifold, $H^{n-2}(B_i) = Z$.

LEMMA 2. *In Diagram* A, *the horizontal arrows represent isomorphisms.*

Proof. It suffices to prove that i_3^* is an isomorphism.

Notice that $\bar{V} \cap (M - U) = S^{n-3} = (B_i - U) \cap \bar{V}$. Since $B_i - U$ is a manifold with boundary, $H^{n-2}(B_i - U) = 0$. The following commutative diagram with exact rows is a consequence of the inclusion of

$$(B_i, B_i - U, \bar{V}) \text{ in } (M - U\cup V, M - U, \bar{V})$$

and of the Mayer-Vietoris theorem [6, p. 239].

$$\begin{array}{ccccc} Z & & Z & & 0 \\ \wr\Vert & & \wr\Vert & & \wr\Vert \\ H^{n-3}((B_i - U) \cap \bar{V}) & \xrightarrow{\Delta^*} & H^{n-2}(B_i) & \rightarrow & H^{n-2}(B_i - U) \oplus H^{n-2}(\bar{V}) \\ \uparrow i^* & & \uparrow i_3^* & & \\ H^{n-3}((M - U) \cap \bar{V}) & \xrightarrow{\Delta^*} & H^{n-2}((M - U)\cup V) \cong Z & & \end{array}$$

Here the maps i^* and i_3^* are induced by the inclusion. The homomorphism i^* is an isomorphism since it is induced by the identity map. The homomorphism Δ^* is an epimorphism, and since its domain and range are copies of Z, it is an isomorphism. Hence $i_3 \circ \Delta^*$ is an isomorphism and thus i_3^* is an isomorphism.

LEMMA 3. *There is a 1-cycle z whose carrier is a simple closed curve linking V in U; and on the homology class $\{z\} \in H_1(U - V)$ the map f_* is a multiplication by d_i, where d_i is the local degree of f at B_i.*

Proof. We know [1, Theorem 4, p. 533] that there is a euclidean neighborhood U such that f is topologically equivalent to the natural winding map around the tamely embedded $(n - 2)$-cell V. Let this be the one employed in Diagram A. Thus there is a 1-cycle z whose carrier $|z|$ is a simple closed curve linking V in U, and this carrier can be chosen so that it has as an image a simple closed curve on which it winds d_i times. If $\bar{z}$ is the cycle carried by $f(|z|)$ and if $\{z\}$ is the homology class at z, then $f_*(\{z\}) = d_i\{\bar{z}\}$. Since $U - V$ is contractible to $|z|$, the homology class $\{z\}$ is a generator of the group $H_1(U - V)$ and the action of f_* on $H_1(U - V)$ is merely a multiplication by d_i.

LEMMA 4. *The homomorphism $f_*: H_1(M - B_i) \to H_1[N - f(B_i)]$ is a multiplication by d_i.*

Proof. Consider the following commutative diagram in which the vertical arrows are seen by the argument on Diagram A to be isomorphisms.

$$\begin{array}{ccc} H_1(U - V) & \xrightarrow{f_*} & H_1(f(U - V)) \\ \downarrow i_* & & \downarrow i_* \\ H_1(M - B_i) & \xrightarrow{f_*} & H_1(N - f(B_i)) \end{array}$$

It is immediate that $f_*: H_1(M - B_i) \to H_1(N - f(B_i))$ is a multiplication by d_i.

THEOREM 2. *Let $f: M \to N$ be a light open map of compact, oriented n-manifolds with vanishing homology in dimensions 1 and 2. Suppose $\dim B_f = n - 2$, $n > 2$, and B_f contains as an isolated component a tamely embedded orientable $(n - 2)$-manifold B_i whose image $f(B_i)$ is also an isolated tamely embedded orientable $(n - 2)$-manifold such that $f^{-1}f(B_i) = B_i$. Let B_j be an arc-connected component of B_f for which $f^{-1}f(B_j) = B_j$ and $f(B_j) \cap f(B_i) = \emptyset$. Then $f(B_j)$ carries a 1-cycle which represents a nonzero class in $H_1[N - f(B_i)]$.*

Proof. Suppose that no 1-cycle in $f(B_j)$ belongs to a nonzero class in $H_1[N - f(B_i)]$. Let α be a generator of $H_1[N - f(B_i)]$ chosen as follows: Let β be an arc from a point y_1 of $f(B_j)$ to a point y_2 of the cycle $|\bar{z}|$ of the proof of Lemma 3 such that β is disjoint from $f(B_f)$ except at y_1. Let α be the path that proceeds along β from y_1 to y_2 then around $|\bar{z}|$ and finally back to y_1 along the reverse of β; i.e. $\alpha = \beta\bar{z}\beta^{-1}$. Let $x_1 \in f^{-1}(y_1) \cap B_j$. Let $\tilde{\alpha}$ be a lift through f of α starting at x_1 and proceeding around a part of $|z|$ and returning from a point x_2' of $f^{-1}(y_2) \cap |z|$ to a point x_1' of $f^{-1}(y_1) \cap B_j$.

Let γ be an arc in B_j joining x_1' to x_1. The paths $f(\gamma)$ and $\tilde{\alpha}\gamma$ are closed. That the cycle carried by $\tilde{\alpha}\gamma$ is non-trivial in $M - B_i$ can be seen as follows. Let $\{\tilde{\alpha}\gamma\}$ be the homology class of $\tilde{\alpha}\gamma$. Then

$$f_*(\{\tilde{\alpha}\gamma\}) = \{\alpha f(\gamma)\} = \{\alpha\} + \{f(\gamma)\}.$$

Since $f(\gamma) \subset f(B_j)$ and no cycle of $f(B_j)$ links $f(B_i)$, it follows that

$$\{f(\gamma)\} = 0 \in H_1(N - f(B_i)).$$

Thus

$$f_*(\{\tilde{\alpha}\gamma\}) = \{\alpha\} = \{\bar{z}\} \neq 0.$$

On the other hand, by Lemma 4, f_* is a multiplication by d_i on $H_1(M - B_i)$. Hence $\{\bar{z}\}$ is a d_i multiple of some element of $H_1(N - f(B_i))$ which in turn is a multiple of $\{\bar{z}\}$. This is impossible, and thus there is a 1-cycle in $f(B_j)$ that links $f(B_i)$.

Theorem 2 can be extended and applied in various directions. Here is a sample.

THEOREM 3 *Under the hypotheses of Theorem 2, if $f \mid B_j$ is a covering map, then B_j carries a cycle which represents a generator in $H_1(M - B_i)$.*

Proof. If g is the degree of the covering map $f \mid B_j$ and $\bar{z}$ is the cycle guaranteed to exist by Theorem 2, there is a cycle z carried by B_j such that $f(z) = g\bar{z}$. Consider

$$f_*\colon H_1(M - B_i) \to H_1[N - f(B_i)].$$

Then $f_*\{z\} = g\{\bar{z}\} \neq 0$ since $\{\bar{z}\} \neq 0$ and $H_1[N - f(B_i)] = Z$. Now z is some multiple of a generator of $H_1(M - B_i)$, so the generator is also carried by B_j.

It is known that the homology of B_f cannot be more complicated than that of M for certain regular M-S coverings and certain coefficient domains [4]. Theorem 3 allows a strong statement about B_f for certain branched coverings:

COROLLARY. *Let $f\colon M \to N$ be a branched covering, $n > 3$. Suppose for each component B of B_f, $f^{-1}f(B) = B$. Then B_f does not contain two disjoint copies of S^{n-2}.*

Proof. One copy of S^{n-2} cannot link the other in an n-manifold, $n > 3$, contrary to Theorem 3.

Notice that if we drop the requirement that B_i and $f(B_i)$ be orientable and replace integral coefficients by coefficients in Z_2, Lemmas 1-4 remain valid. A minor modification of the proof of Theorem 2 then yields the following theorem.

THEOREM 2′. *Omit the hypothesis of orientability in Theorem 2. Suppose that the local degree on B_i is even. Then $f(B_j)$ carries a representative of a non-zero class in $H_1[N - f(B_i); Z_2]$.*

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