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BRANCHING PROCESSES IN LÉVY PROCESSES: THE EXPLORATION PROCESS

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The main idea of the present work is to associate with a general continuous branching process an exploration process that contains the desirable information about the genealogical structure. The exploration process appears as a simple local time functional of a Lévy process with no negative jumps, whose Laplace exponent coincides with the branching mechanism function. This new relation between spectrally positive Lévy processes and continuous branching processes provides a unified perspective on both theories. In particular, we derive the adequate formulation of the classical Ray–Knight theorem for such Lévy processes. As a consequence of this theorem, we show that the path continuity of the exploration process is equivalent to the almost sure extinction of the branching process.

1. Introduction. It has been known for several years that a class of nonlinear operators of the type $-\Delta u + u^\alpha$ can be interpreted and studied through the measure-valued branching processes called superprocesses [15]. In the special case $\alpha = 2$, corresponding to the quadratic branching mechanism, a natural construction of the associated superprocesses involves the path-valued process known as the Brownian snake [32, 33]. This construction yields detailed information about the genealogical structure of the superprocess and has interesting applications to the potential theory of the nonlinear operator. A question remaining open was to extend the Brownian snake construction to superprocesses associated with more general branching mechanisms. It is the purpose of the present work to develop the tools that are needed for this extension. This requires a deep understanding of the genealogical structure of continuous-state branching processes, and involves a new connection between branching processes and spectrally positive Lévy processes, which is of independent interest. The discrete form of this connection is a pathwise construction relating left-continuous random walks and Galton–Watson branching trees, which can also be interpreted in terms of a last-in first-out queue. This construction can be viewed as a variant of the classical relation between queues and branching processes, which was pointed out by Kendall [26] and has been used since by many authors.

A key role in this work is played by the notion of the exploration process. Consider first a discrete-time Galton–Watson tree, describing the genealogy of an ordinary Galton–Watson branching process with offspring distribution ν.

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The exploration process at time n, denoted by H_n, is the generation of the individual visited at time n, assuming that the individuals of the population are visited successively in the lexicographical order (for the usual coding of the tree), or equivalently according to the law of primogeniture. This exploration process is closely related to a left-continuous random walk (in the terminology of Spitzer [42], page 21) with jump distribution $\mu(k) = \nu(k+1)$, $k = -1, 0, 1, \dots$. Precisely, the value W_n of the random walk at time n is obtained by adding the number of younger brothers of the individual visited at time n and of all his ancestors. Conversely, the exploration process is recovered as a simple functional of the random walk:

$$H_n = \mathrm{Card}\Big\{j;\ 0 \le j < n,\ W_j = \inf_{j \le l \le n} W_l\Big\}. \tag{1.1}$$

We are here interested in continuous versions of this correspondence. It is well known [28] that continuous-state branching processes are the possible scaling limits of Galton–Watson branching processes. A fundamental idea of the present work is that one can associate with a general continuous-state branching process an exploration process that contains all the desirable information about the genealogical structure. In this setting, the role of the left-continuous random walk W is played by a spectrally positive Lévy process X.

To understand our construction, it is worth considering first the simple case when the Lévy process $X = (X_t,\ t \ge 0)$ is a sum of (positive) discrete jumps and a drift part $-\alpha t$, where $\alpha > 0$. We can interpret this Lévy process in terms of a last-in first-out (LIFO) queue with one server: the jumps of X correspond to arrival times of customers, the size of a jump is the service required by the corresponding customer and α represents the output rate of the server. According to the LIFO rule, each new customer is served in priority by the server, independently of the customers already in the line. This interpretation yields a branching structure on the set of jumps of the Lévy process X: the jump, or customer, arriving at time t is a descendant of the jump s if and only if the customer s is still in the line at the arrival of the customer t. Analytically, this is equivalent to the condition

$$X_{s-} < \inf_{s \le u \le t} X_u.$$

The generation N_t of the jump t is the number of customers in the queue at time t, or equivalently the number of instants $s < t$ such that the previous inequality holds [compare with formula (1.1)]. This branching structure yields a Galton–Watson process whose offspring distribution can be computed explicitly in terms of the Lévy measure of X (see the remark after Proposition 3.2). In addition, we also consider the process ρ_t which gives for every time t the state of the queue at this time, that is, the residual service times of all customers present in the queue, in their order of arrival. This process is a nice Markov process (in contrast to the number N_t of customers in the queue, which is not Markovian) and has interesting duality properties (Propositions 3.4 and 3.5).

Our main interest is in the case of a general spectrally positive Lévy process X. In the present paper, we assume for simplicity that X does not drift to $+\infty$ as $t \to \infty$ and has no Gaussian part. The distribution of X is then characterized by its "Laplace transform"

$$E(\exp -\lambda X_t) = \exp t\psi(\lambda), \qquad \lambda > 0,$$

where the Laplace exponent $\psi(\lambda)$ is of the form

$$\psi(\lambda) = \alpha\lambda + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r)\,\pi(dr),$$

where $\alpha \geq 0$ and the Lévy measure $\pi(dr)$ is a σ-finite measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} (r \wedge r^2)\,\pi(dr) < \infty.$$

The most interesting case is when X has infinite variation, which is equivalent to the condition $\int_{(0,1)} r\,\pi(dr) = \infty$. Then we can define for every t (not only for jump times) a height H_t that corresponds to the notion of generation in the discrete jump case. Precisely, consider the time-reversed process $\widehat{X}^{(t)}_s = X_t - X_{(t-s)-}$, for $0 \leq s \leq t$ (by convention $X_{0-} = 0$), and its associated supremum process $\widehat{S}^{(t)}_s = \sup_{[0,s]} \widehat{X}^{(t)}_r$. Then H_t is defined as the local time at level 0 at time t of the reflected Lévy process $\widehat{S}^{(t)} - \widehat{X}^{(t)}$. Informally, H_t accounts for the "number" of instants $s \leq t$ such that $\widehat{X}^{(t)}_s = \widehat{S}^{(t)}_s$ or equivalently $X_{(t-s)-} \leq \inf_{[t-s,t]} X_r$. This is of course analogous to the definition of N_t in the discrete jump case.

The height process $(H_t,\ t \geq 0)$ is the key to our constructions. It is interpreted as the exploration process associated with a continuous-state branching process Z whose branching mechanism function ψ is the Laplace exponent of X. Recall that the continuous-state branching process with branching mechanism ψ [in short, the CSBP(ψ)] is the strong Markov process Z in $\mathbb{R}_+$ whose transition kernels are characterized by their Laplace functional

$$E\big(\exp -\lambda Z_t \mid Z_0 = x\big) = \exp -x\,u_t(\lambda), \tag{1.2}$$

where $u_t(\lambda)$ is the unique nonnegative solution of the integral equation

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))\,ds = \lambda, \qquad \lambda \geq 0,\ t \geq 0. \tag{1.3}$$

The previous interpretation of the height process can be made rigorous in several ways. We show that H appears as the scaling limit of discrete exploration processes corresponding to Galton–Watson branching processes that converge to Z after rescaling. Alternatively, we obtain the following striking analogue of a classical Ray–Knight theorem about Brownian local times [27]. Let $x > 0$ and let τ_x be the hitting time of $-x$ by X. Then the random measure $\mathscr{Z}_x$ on

$\mathbb{R}_+$ defined by

$$\langle \mathscr{Y}_x, \varphi \rangle = \int_0^{\tau_x} dt\, \varphi(H_t)$$

has a density which is distributed as the process Z started at x (Theorem 4.2). Intuitively, the "time spent" at a given level by the exploration process corresponds to the size of the population of the tree at that level. The previous result can be stated for a more general class of Lévy processes [35] including the special case when X is linear Brownian motion. In that case, the result reduces to the Ray–Knight theorem, and Z is Feller's diffusion, which is the simplest continuous-state branching process [corresponding to $\psi(\lambda) = c\lambda^2$].

We use the previous result to investigate the continuity properties of the height process. We prove (Theorem 4.7) that H has a continuous version if and only if

$$\int^{\infty} \frac{d\lambda}{\psi(\lambda)} < \infty.$$

This condition is known to be equivalent to the almost sure extinction of Z. If it does not hold, the process H has a very wild behavior.

The height process H is not a Markov process. As in the discrete jump case, it can, however, be interpreted as the length of a generalized queue ρ_t, which is a nice Markov process. The existence and properties of this process ρ_t will be important in the construction of Markov snakes associated with general superprocesses. In fact, the results of the present work show that the snake construction of quadratic superprocesses [32] can be adapted formally to superprocesses with a general branching mechanism (see, e.g., [16] or [11] for the general theory of superprocesses). At the end of Section 3, we briefly give a discrete version of the method. Details of the construction will be presented in the forthcoming paper [35], where we also extend some results of the present work to more general spectrally positive Lévy processes.

The paper is organized as follows. In Section 2, we briefly present the exploration process associated with a Galton–Watson tree and explain its connection with a random walk. In Section 3, we deal with the case of (spectrally positive) Lévy processes with finite variation. Although more general, this case is essentially similar to the discrete jump case described previously. Section 4 contains the main results, concerning the infinite variation case. Finally, we give in Section 5 a limit theorem showing that the process H of the infinite variation case can be obtained as a scaling limit of discrete exploration processes.

Let us now comment on the relationship between the present work and the existing literature. Lamperti [29] showed that a general continuous-state branching process can be obtained from a spectrally positive Lévy process by a random time change. This observation was applied by Bingham [8] to investigate properties of continuous-state branching processes. Lamperti's result and the present work thus give two different transformations connecting a continuous-state branching process to the same Lévy process. Both these transformations preserve (the size of) the jumps, but of course not their location in time. The relation between Galton–Watson trees and left-continuous random walks which is described in Section 2 is a variant of Kendall's connection between queues and branching processes ([26], page 168; see also [3], page 64). This connection has been treated and exploited by numerous authors in different forms: see, in particular, [21] and [13], page 1020. A nice consequence of these relations between trees and random walks is the fact (already implicit in [26]) that the total progeny of a Galton–Watson process has the same distribution as a certain hitting time for an associated left-continuous random walk (cf. [14] for a different approach). However, we stress that, in contrast to Kendall [26] and most of the subsequent authors who were only interested in the size of the population at every time, we establish a correspondence between the *genealogical tree* of the population and the positive excursion of the random walk. In this connection, our use of the LIFO discipline in Section 3 is crucial. The first-in first-out (FIFO) discipline would give a genealogical structure different from the one considered here, and would not be suitable for the passage to the limit that we have in mind. We also refer to [7] and [9], for connections between spectrally positive Lévy processes and queueing systems, and to [43], for a recent study of trees associated with queues. The discrete result of Section 2 can also be seen as a generalization of the well-known relationship between simple random walk and the Galton–Watson branching tree with a geometric offspring distribution [23]. More generally, the discrete constructions connecting branching trees with random walks are often related to combinatorial results (see the recent review of Pitman [40] and the references therein). As a final remark, the connections between Lévy processes and branching processes that are studied in the present work are in the spirit of the numerous papers connecting linear Brownian motion to random trees (see in particular [38], [2] and [31]). In the terminology of Aldous [2], our exploration process H should be understood as the search-depth process describing the "continuum random tree" associated with a general continuous-state branching process. This extends the well-known case of Feller's diffusion, where the search-depth process is (reflecting) linear Brownian motion (see [2] and also [30]).

After the publication of the note [34] presenting our main results, we learnt of some recent preprints independent of the present work but closely related to Sections 2 and 3 below. Borovkov and Vatutin [10] give a slightly different version of the correspondence of Section 2. Bennies and Kersting [4] use the same correspondence to derive certain properties of Galton–Watson branching processes. Finally, there are some connections between Section 3 below and the recent work of Geiger [19, 20]. In particular, the pruned tree process of [20] is related to the process ρ of Section 3 in the discrete jump case.

2. Discrete Galton–Watson trees and random walks.

In this section, we introduce the exploration process of a discrete Galton–Watson tree and show that it can be written as a simple functional of an associated random walk. The proofs are elementary and will be merely sketched.

We first define the Galton–Watson tree with a given offspring distribution. This construction has been known for a long time (see in particular [39]). We follow the presentation of Neveu [37]. Let $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ be the set of positive integers, and let

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where by convention $(\mathbb{N}^*)^0 = \{\varnothing\}$. An element u of $(\mathbb{N}^*)^n$ is written $u = u_1 \cdots u_n$, and we set $|u| = n$. If $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ belong to U, we write $uv = u_1 \cdots u_m v_1 \cdots v_n$ for the concatenation of u and v. In particular $u\varnothing = \varnothing u = u$. We write $u < v$ for the lexicographical order on U: $\varnothing < 1 < 11 < 12 < 121$, for example.

A *tree* τ is a finite subset of U such that the following hold:

1. $\varnothing \in \tau$;
2. if $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$;
3. for every $u \in \tau$, there exists a number $k_u(\tau) \geq 0$ such that $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

We denote by $\mathbb{T}$ the set of all trees. If τ is a tree and $u \in \tau$, we define the shift of τ at u by $T_u\tau = \{v \in U,\ uv \in \tau\}$. Note that $T_u\tau \in \mathbb{T}$.

Let μ be a probability measure on $\mathbb{N}$. We assume that μ is critical or subcritical, meaning that

$$\sum_{k=1}^{\infty} k\,\mu(k) \leq 1$$

and $\mu(1) < 1$. The law of the Galton–Watson tree with offspring distribution μ is the unique probability measure $\mathbb{P}_\mu$ on $\mathbb{T}$ such that the following hold:

1. $\mathbb{P}_\mu(k_\varnothing = j) = \mu(j),\ j \in \mathbb{N}$;
2. for every $j \geq 1$ with $\mu(j) > 0$, the shifted trees $T_1\tau, \ldots, T_j\tau$ are independent under the conditional probability $\mathbb{P}_\mu(\cdot \mid k_\varnothing = j)$ and their conditional distribution is $\mathbb{P}_\mu$.

We next introduce the *exploration process*. Let $\tau \in \mathbb{T}$ and $\sigma = \sigma(\tau) = \mathrm{Card}(\tau)$. Denote by $u(0) = \varnothing < u(1) < \cdots < u(\sigma - 1)$ the elements of τ listed in lexicographical order. Let also Δ stand for a cemetery point. The exploration process $(H_n(\tau),\ n \geq 0)$ associated with τ is then defined by

$$H_n(\tau) = \begin{cases} |u(n)|, & \text{if } 0 \leq n \leq \sigma - 1, \\ \Delta, & \text{if } n \geq \sigma. \end{cases}$$

At an intuitive level, one visits successively all "individuals" of the tree in the lexicographical order, starting from $\varnothing$ at time 0, and H_n represents the generation of the individual visited at time n. It is easy to check that the function $n \to H_n(\tau)$ determines the tree τ.

In general, the process H_n is not Markovian under $\mathbb{P}_\mu$, and it is not clear how to describe its distribution. However, we can define a related Markov process as follows. If $u = u_1 \cdots u_p \in U$ and $k \le p$, we set $[u]_k = u_1 \cdots u_k$, so that $[u]_k$ is the ancestor of u in the kth generation of the tree. Then, for any $n \in \{0, 1, \dots, \sigma - 1\}$ and $j \in \{1, \dots, H_n\}$, we set

$$B_{n,j}(\tau) = \mathrm{Card}\big\{v \in \tau;\ |v| = j,\ [u(n)]_{j-1} = [v]_{j-1},\ u(n) < v\big\},$$

which represents the number of "younger" brothers of the ancestor of $u(n)$ in the jth generation.

PROPOSITION 2.1. *The process $(\rho_n,\ n \ge 0)$ defined by*

$$\rho_n = \begin{cases} (B_{n,1}, \dots, B_{n,H_n}), & \text{if } n < \sigma, \\ \Delta, & \text{if } n \ge \sigma, \end{cases}$$

is under $\mathbb{P}_\mu$ a Markov chain with values in $\bigcup_{n=0}^\infty \mathbb{N}^n \cup \{\Delta\}$. The transition kernel of this Markov chain is described as follows. If $b = (b_1, \dots, b_h) \in \bigcup_{n=0}^\infty \mathbb{N}^n$, then

$$Q(b, (b, k)) = \mu(k+1), \qquad k \in \mathbb{N},$$
$$Q(b, \tilde{b}) = \mu(0),$$

where $\tilde{b} = (b_1, \dots, b_{m-1}, b_m - 1)$ if $m = \sup\{j \in \{1, \dots, h\},\ b_j > 0\} > 0$, $\tilde{b} = \Delta$ if $b_1 = \cdots = b_h = 0$. Furthermore, $Q(\Delta, \Delta) = 1$.

The statement of the proposition is intuitively clear. Let us briefly sketch a heuristic argument. At time n, we are visiting the individual $u(n)$, and because of the lexicographical order of visits, we do not know yet whether this individual has children or not. For every $l \ge 1$, the individual $u(n)$ has l children with probability $\mu(l)$, and in that case $u(n+1) = u(n)1$ is the first child of $u(n)$ and $\rho_{n+1} = (\rho_n, l-1)$ by definition. On the other hand, with probability $\mu(0)$, $u(n)$ has no child. Then $u(n+1)$ will be the first younger (i.e., not yet visited) brother of the last ancestor of $u(n)$ [including $u(n)$ himself] that has at least one younger brother. In that case we get $\rho_{n+1} = \tilde{\rho}_n$.

COROLLARY 2.2. *Set*

$$W_n = \begin{cases} \displaystyle\sum_{j=1}^{H_n} B_{n,j}, & \text{if } n < \sigma, \\ \Delta, & \text{if } n \ge \sigma. \end{cases}$$

Then W is under $\mathbb{P}_\mu$ a random walk on the integers started at 0, with jump distribution $\nu(k) = \mu(k+1)$, $k = -1, 0, 1, 2, \dots,$ and killed at its first hitting time of -1. Furthermore, for $n < \sigma$,

$$H_n = \mathrm{Card}\Big\{j;\ 0 \le j < n,\ W_j = \inf_{j \le l \le n} W_l\Big\}. \tag{2.1}$$

The first part of the corollary is obvious from the form of the kernel Q in Proposition 2.1. As for (2.1), notice that the condition $W_j = \inf_{j \le l \le n} W_l$ holds if and only if $n < \inf\{k > j,\ W_k < W_j\} =: k_j$. However, it is clear from our construction that k_j is the time of the first visit of an individual that is not a descendant of $u(j)$. Hence, the right-hand side of (2.1) is exactly the number of ancestors of $u(n)$, that is, $|u(n)| = H_n$.

The point of the previous corollary is that the exploration process is determined as a simple functional of the left-continuous random walk W. This is of course similar to the classical correspondence between simple random walk and the geometric Galton–Watson tree [23]. One can also compare the results of this section with the well-known embedding of a Galton–Watson branching process in random walk (see, e.g., [36]), which is a discrete form of Lamperti's embedding. However, note that, in contrast to the previous construction, this embedding does not give access to the family structure of the branching process.

Much of what follows in Sections 3 and 4 is devoted to studying continuous analogues of the correspondence (2.1) and of the process $(\rho_n,\ n \in \mathbb{N})$. In this continuous setting, W is replaced by a spectrally positive Lévy process. The analogue of H studied in Section 4 can be interpreted as the exploration process associated with a continuous-state branching process. This interpretation is justified by the limit theorems proved in Section 5.

3. Lévy processes with finite variation.

3.1. *Assumptions and preliminaries.* In this section, we consider a Lévy process X on the real line with no negative jumps and paths of finite variation. We always assume that the process X starts at 0 under the probability measure $P = P_0$. The Lévy measure π of X is a measure supported on $(0, \infty)$ such that

$$\int_0^\infty (r \wedge 1)\, \pi(dr) < \infty.$$

The process X can be decomposed as the sum of a subordinator $Y_t = \sum_{0\le s\le t} \Delta X_s$ and a drift part $Z_t = \beta t,\ \beta \in \mathbb{R}$. Recall that

$$\sum_{s:\, \Delta X_s > 0} \delta_{(s,\, \Delta X_s)}$$

is a Poisson point measure on $\mathbb{R}_+^2$ with intensity $ds\, \pi(dr)$. In particular $E(\sum_{0\le s\le t} \Delta X_s) = t\int_0^\infty r\pi(dr)$ and, for $\lambda > 0$,

$$E(\exp(-\lambda X_t)) = \exp(t\psi(\lambda)),$$

where

$$\psi(\lambda) = -\beta\lambda + \int_0^\infty (e^{-\lambda r} - 1)\, \pi(dr).$$

We will only consider the case when X is recurrent or drifts to $-\infty$. This property holds if and only if $E(X_t) \le 0$, or equivalently

$$\int_0^\infty r\, \pi(dr) + \beta \le 0.$$

Note that in particular $\int_0^\infty r\,\pi(dr) < \infty$. We set $\alpha = -\beta \geq 0$. To avoid trivial cases we also assume that $\pi \neq 0$ and thus $\alpha > 0$.

We will need a few simple facts about the behavior of X. First, the point 0 is irregular for $(0,\infty)$ ([6], Corollary VII.5). A short argument can be given as follows. If $\gamma_{(t)} = \inf\{s > t,\ X_s > X_t\}$, $P(\gamma_{(t)} > t)$ is independent of t. Certainly $\gamma_{(t)} > t$ if the derivative of the map $s \to X_s$ at $s = t$ exists and is (strictly) negative. However, this derivative is dt-a.e. equal to $-\alpha$ by a standard derivation theorem. As a consequence, the supremum process of X only increases by discrete jumps.

Write $\gamma = \gamma_{(0)}$ for the hitting time of $(0,\infty)$. The joint distribution of $(X_\gamma, \Delta X_\gamma)$ is given by the following formula ([6], Theorem VII.17):

$$E(1_{\{\gamma<\infty\}} f(X_\gamma, \Delta X_\gamma)) = \alpha^{-1} \int \pi(dy) \int_0^y dx\, f(x, y). \tag{3.1}$$

Let us briefly give a proof of (3.1), as one of the intermediate steps will be needed later. First note that $\Delta X_\gamma > 0$, because otherwise the strong Markov property at γ gives a contradiction. By translation invariance, there exists a constant $c > 0$ such that, for every Borel subset B of $(-\infty, 0)$,

$$E\biggl(\int_0^\gamma 1_B(X_s)\, ds\biggr) = c\, m(B), \tag{3.2}$$

where m is Lebesgue measure. Then

$$\begin{aligned} E(1_{\{\gamma<\infty\}} f(X_\gamma, \Delta X_\gamma)) &= E\biggl(\sum_{s:\,\Delta X_s>0} 1_{\{s\leq\gamma\}} 1_{\{\Delta X_s > -X_{s-}\}} f(X_s, \Delta X_s)\biggr) \\ &= E\biggl(\int_0^\gamma ds \int_{(-X_{s-},\,\infty)} f(X_{s-} + r, r)\pi(dr)\biggr) \\ &= c\int_{-\infty}^0 dx \int_{(-x,\,\infty)} f(x+r, r)\pi(dr). \end{aligned}$$

Formula (3.1) will follow if we can check that $c = \alpha^{-1}$. By taking $f = 1$, we get $P(\gamma < \infty) = c\int r\,\pi(dr)$. In the recurrent case $[\alpha = \int r\,\pi(dr)]$, we have $P(\gamma < \infty) = 1$ and the desired result follows at once. In the transient case, we note that the potential kernel $B \to E(\int_0^\infty 1_B(X_s)\,ds)$ is on $(-\infty, 0]$ a multiple of Lebesgue measure m. The constant is easily computed from a renewal-type argument or some Fourier calculations:

$$E\biggl(\int_0^\infty 1_B(X_s)\,ds\biggr) = \biggl(\alpha - \int r\,\pi(dr)\biggr)^{-1} m(B).$$

On the other hand the Markov property at γ leads to

$$E\biggl(\int_0^\infty 1_B(X_s)\,ds\biggr) = (P(\gamma = \infty))^{-1} E\biggl(\int_0^\gamma 1_B(X_s)\,ds\biggr).$$

Combining these formulas with the identity $P(\gamma < \infty) = c\int r\,\pi(dr)$ yields the desired result $c = \alpha^{-1}$.

Denote by $I_t = \inf_{[0,t]} X_r$ the infimum process of X and by $S_t = \sup_{[0,t]} X_r$ its supremum process. Then $X-I$ and $S-X$ are strong Markov processes ([6], Chapter VI; this fact holds for a general Lévy process). From our assumptions and the previous remarks, it is immediate to check that 0 is a regular recurrent point for $X-I$. From the fact that 0 is irregular for $(0,\infty)$ (with respect to X) and a time-reversal argument we have also $P(X_t = I_t) = P(S_t = 0) > 0$. Hence a local time at 0 for $X-I$ is given by

$$\Lambda_t = \int_0^t 1_{\{X_s = I_s\}}\, ds.$$

We denote by N the excursion measure away from 0 for $X-I$ associated with this local time. The measure N can be computed explicitly:

$$\mathrm{N} = \int \pi(dr)\, P_r^0, \tag{3.3}$$

where P_r^0 stands for the law of X started at r and stopped at $T_0 = \inf\{s \ge 0,\ X_s = 0\}$. The first step to prove (3.3) is to observe that excursions of $X-I$ away from 0 must start with a jump. This follows via a time-reversal argument from the fact that the maximum process of X only increases by jumps: if an excursion of $X-I$ were leaving 0 continuously, then we could find a rational q such that the time-reversed process $\widehat{X}^{(q)}$ reaches continuously a (strictly) positive level, which is impossible. Once we know that excursions start with a jump, we apply the classical formulas of excursion theory, which give, for $\tau_{(1)} = \inf\{t;\ \Lambda_t = 1\}$,

$$\mathrm{N}(f(X_0)) = E\left(\int_0^{\tau_{(1)}} dt\, 1_{\{X_{t-} = I_t\}} f(\Delta X_t)\right) = \int f(r)\, \pi(dr),$$

and (3.3) follows easily.

3.2. *The queueing system representation.* We now explain how the Lévy process X can be interpreted as describing the evolution of a queue, which in turn will determine a branching structure. Consider first the simple special case where π is a finite measure, $Y_t = \sum_{0 \le s \le t} \Delta X_s$ is a compound Poisson process: The jumps of X occur on a discrete set of times, these jumps are distributed according to the law $\pi(dr)/\pi(\mathbb{R}_+)$ and the intervals between two successive jumps are exponentially distributed with parameter $\pi(\mathbb{R}_+)$. Therefore $(X_t,\ t \ge 0)$ belongs to the set $\mathscr{T}$ of all functions $(\omega(t),\ t \ge 0)$ of the type $\omega(t) = u(t) - \alpha t$, where u is a sum of discrete positive jumps . We can interpret the trajectory $(X(t),\ t \ge 0)$ as describing the evolution in time of a queue LIFO (last-in, first-out) with one server, whose service output rate is α. A jump of X at time t corresponds to the arrival of a new customer requiring a service equal to ΔX_t. The server will immediately start the service of this new customer and this service will be completed at time $t + \alpha^{-1}\Delta X_t$, unless it is interrupted by another new arrival, and so on. When the server has completed the service of a customer, he comes back to the service of the last arrived customer whose service is not completed (if there is any). The customer arrived at time t will still be in the system at time $t' > t$ if and only if

$$X_{t-} < \inf_{t \le r \le t'} X_r,$$

and the quantity $\inf_{t \le r \le t'} X_r - X_{t-}$ then represents the remaining part of his service at time t'. We denote by N_t the number of customers in the queue at time t:

$$N_t = \mathrm{Card}\left\{ s \in [0, t],\ X_{s-} < \inf_{s \le r \le t} X_r \right\}$$

and by $\mathscr{J}_t = \{s_t^1 \le \cdots \le s_t^{N_t}\}$ the set of arrival times of these customers. We also let $\rho_t(s_t^1), \dots, \rho_t(s_t^{N_t})$ be the corresponding services remaining at time t $[\rho_t(s) = \inf_{s \le r \le t} X_r - X_{s-}]$. Observe the easy identity

$$\sum_{s \in \mathscr{J}_t} \rho_t(s) = X_t - I_t.$$

Hence, $X_t - I_t$ exactly corresponds to the load of the server at time t. The quantity $-I_t/\alpha$ represents the total amount of time before t during which the server was idle.

In the general case when π may be an infinite measure, $(X_t,\ t \ge 0)$ belongs to the set $\bar{\mathscr{T}}$ of trajectories of the type $\omega(t) = u(t) - \alpha t$, where $u(t)$ is a sum of positive jumps. The previous interpretation still make sense. We again denote by $\mathscr{J}_t$ the set $\{s \le t,\ \rho_t(s) = \inf_{s \le r \le t} X_r - X_{s-} > 0\}$ and by N_t the cardinal of $\mathscr{J}_t$, which may now be infinite.

A truncation argument will be useful to study the case $\pi(\mathbb{R}_+) = \infty$. For every $\varepsilon > 0$, we set

$$X_t^\varepsilon = \sum_{0 \le s \le t} \Delta X_s \, 1_{\{\Delta X_s > \varepsilon\}} - \alpha t.$$

Then X_t^ε corresponds to the evolution of a LIFO queue where the services required by the customers are larger than ε. Decreasing ε means adding new customers with smaller services. Then $(X_t^\varepsilon,\ t \ge 0)$ belongs to $\mathscr{T}$ and converges to $(X_t,\ t \ge 0)$ uniformly on compact sets. With an obvious notation, $\mathscr{J}_t^\varepsilon$ increases toward $\mathscr{J}_t$ and thus N_t^ε increases towards N_t as ε goes to 0. Also, for every $s \in \mathscr{J}_t$, $\rho_t^\varepsilon(s)$ increases to $\rho_t(s)$, and

$$X_t - I_t = \lim_{\varepsilon \to 0} \uparrow (X_t^\varepsilon - I_t^\varepsilon)$$

again represents the load of the server at time t. We can thus interpret the pair $(\mathscr{J}_t, \rho_t)$ as describing the evolution of a generalized queue where the total number of customers in the system can be infinite.

REMARK. As was pointed to us by J. Pitman, the previous construction has some relation with the work [1] of Adhikari on skip free processes. In a somewhat different context, the sets M_n introduced in [1] correspond to the level sets of our process N_t.

3.3. *The branching structure.* The previous considerations applied to any element of $\mathscr{T}$ or $\bar{\mathscr{T}}$. We will now use the probabilistic structure of X to get more information.

LEMMA 3.1. *For every $t > 0$, $P(N_t < \infty) = 1$. Moreover, P-a.s., for every $t > 0$, the elements of $\mathscr{J}_t$ can be ordered in a strictly increasing sequence $s_t^1 < s_t^2 < \cdots$, which is finite if t is a time of jump for X.*

PROOF. Recall from Section 1 the notation $\widehat{X}_s^{(t)}$, $\widehat{S}_s^{(t)}$ for the time-reversed process at t and its supremum process. Notice that, for $0 < u \le t$, $t - u \in \mathscr{J}_t$ if and only if $\widehat{S}_u^{(t)} > \widehat{S}_{u-}^{(t)}$. Obviously, $(\widehat{S}_s^{(t)},\ 0 \le s \le t)$ and $(S_s,\ 0 \le s \le t)$ have the same distribution.

We saw that the set $\{s > 0,\ S_s > S_{s-}\}$ is discrete and thus $\{s \in (0, t],\ S_s > S_{s-}\}$ is a.s. finite, for every t. By the previous observation, this gives $N_t < \infty$ a.s. In particular, N_r is a.s. finite for every positive rational r.

Then notice that, for $s, s' \in \mathscr{J}_t$, $s < s'$ if and only if $X_{s-} < X_{s'-}$. The second assertion will thus follow if we can check that P-a.s. for every $t > 0$, $\varepsilon > 0$, $\{s \in \mathscr{J}_t,\ X_{s-} < X_t - \varepsilon\}$ is finite. To this end, note that by the right continuity of paths the latter set is contained in $\mathscr{J}_r$ for some rational $r > t$. If t is a time of jump, then it is clear that $\mathscr{J}_t = \{s \in \mathscr{J}_t,\ X_{s-} < X_t - \varepsilon\}$ for some $\varepsilon > 0$. This completes the proof. $\square$

For every $p \ge 0$, we define a continuous process Z^p by

$$Z_t^p = \int_0^t 1_{\{N_s = p\}}\, ds.$$

By Lemma 3.1, $\sum_{p=0}^{\infty} Z_t^p = t$. Let $Z^{\varepsilon, p}$ be associated with the truncated Lévy process Z^ε introduced previously. By dominated convergence, $Z_t^{\varepsilon, p} \to Z_t^p$ as ε decreases to zero. In particular, $Z_t^0 = \lim Z_t^{\varepsilon, 0} = \lim -I_t^\varepsilon/\alpha = -I_t/\alpha$. We also observe that $X_t = I_t$ if and only if $N_t = 0$. The only if part is trivial. For the reverse implication, observe that $I_t < X_t$ implies that $I_t^\varepsilon < X_t^\varepsilon$ for $\varepsilon > 0$ small, which in turn gives $N_t \ge N_t^\varepsilon \ge 1$.

For every $x > 0$, set $\tau_x = \inf\{t,\ X_t = -x\}$. Note that $\tau_x = \inf\{t,\ X_{t-} = -x\}$, a.s., and so it is clear that $\tau_x^\varepsilon \uparrow \tau_x$ as $\varepsilon \downarrow 0$, a.s. By the previous observations, $Z_{\tau_x}^0 = x/\alpha$. The next proposition gives for a fixed x the law of the process $(Z_{\tau_x}^p,\ p \ge 0)$.

PROPOSITION 3.2. *$(Z_{\tau_x}^p,\ p \ge 0)$ is a Markov chain in $\mathbb{R}_+$ whose transition kernel $\mathsf{P}(u, dv)$ is determined by its Laplace transform:*

$$\int \exp(-\lambda v)\mathsf{P}(u, dv) = \exp -u \int \left(1 - \exp\left(-\frac{\lambda r}{\alpha}\right)\right)\pi(dr).$$

REMARK. The kernel P verifies the branching property: $\mathsf{P}(u, \cdot) * \mathsf{P}(u', \cdot) = \mathsf{P}(u + u', \cdot)$.

PROOF OF PROPOSITION 3.2. By passing to the limit $\varepsilon \to 0$, we may assume in proving Proposition 3.2 that $\pi(\mathbb{R}_+) < \infty$. Then let $T_1 < T_2 < \cdots$ be the sequence of stopping times such that

$$\{T_1, T_2, \ldots\} = \{t > 0,\ X_{t-} = I_t,\ \Delta X_t > 0\}$$

and for every n let $U_n = \inf\{t > T_n,\ X_t = X_{T_n-}\}$. The strong Markov property of X shows that the variables $T_1, T_2 - U_1, \ldots, T_{n+1} - U_n, \ldots$ are independent and exponentially distributed with parameter $\pi(\mathbb{R}_+)$. Hence, the random variable

$$j_x := \sup\{n,\ T_1 + (T_2 - U_1) + \cdots + (T_n - U_{n-1}) \le x/\alpha\}$$

is Poisson with parameter $x\pi(\mathbb{R}_+)/\alpha$. Clearly $T_n \le \tau_x$ if and only if $n \le j_x$.

Then observe that the processes $X^i = (X_{T_i+t} - X_{T_i},\ 0 \le t \le U_i - T_i)$ are independent and distributed as $(X_t,\ t \le \tau_\xi)$, where ξ is a random variable independent of X with distribution $\pi/\pi(\mathbb{R}_+)$. These processes are also independent of the quantities $T_{n+1} - U_n,\ n \in \mathbb{N}$, hence of j_x. From our construction, if $t \in [T_i, U_i)$, $N_t = 1 + N^i_{t-T_i}$ (with an obvious notation) and $N_t = 0$ if $t \notin \bigcup_i [T_i, U_i)$. It follows that

$$(Z^p_{\tau_x},\ p = 1, 2, \ldots) \overset{(\mathrm{d})}{=} \left(\sum_{i=1}^{j_x} Z^{i,\,p-1}, p = 1, 2, \ldots\right),$$

where the processes $Z^i = (Z^{i,k},\ k = 0, 1, \ldots)$ are independent (and independent of j_x) and distributed as Z_{τ_ξ}. Using the obvious additivity property of the laws of Z_{τ_x}, we get that

$$(Z^p_{\tau_x},\ p = 1, 2, \ldots) \overset{(\mathrm{d})}{=} (Z^{p-1}_{\tau_U}, p = 1, 2, \ldots),$$

where U is independent of X and distributed as the sum of j_x independent copies of ξ:

$$E[\exp(-\lambda U)] = \exp\left(-\frac{x}{\alpha}\int (1 - \exp(-\lambda r))\pi(dr)\right).$$

Proposition 3.2 now follows easily. □

REMARKS. (a) Suppose that $\pi(\mathbb{R}_+) < \infty$ and denote by J^p_t the number of jumps of the process N from p to $p+1$ before time t (in particular $J^0_{\tau_x} = j_x$). In other words, J^p_t represents the number of arrivals, before time t, at times when the number of customers in the system is p. Then $(J^p_{\tau_x},\ p \ge 0)$ is a Galton–Watson branching process with offspring distribution

$$\nu(k) = \int \frac{(r\pi(\mathbb{R}_+)/\alpha)^k}{k!} \exp\left(-\frac{r\pi(\mathbb{R}_+)}{\alpha}\right) \frac{\pi(dr)}{\pi(\mathbb{R}_+)}.$$

(Compare with [3], page 64.) This follows from the arguments used in the previous proof: Conditionally on $J^0_{\tau_x} = k$, the process $(J^p_{\tau_x},\ p = 1, 2, \ldots)$ is distributed as the sum of k independent copies of $(J^{p-1}_{\tau_\xi},\ p = 1, 2, \ldots)$, and the distribution of $J^0_{\tau_\xi}$ is ν. Note that the mean of ν is $\sum_{k=0}^{\infty} k\nu(k) = \alpha^{-1}\int r\pi(dr)$, and so the Galton–Watson process $J^p_{\tau_x}$ is critical or subcritical according as X is recurrent or drifts to $-\infty$.

(b) When $\pi(\mathbb{R}_+) = \infty$, one easily checks that

$$P(Z^p_{\tau_x} > 0) = \lim_{\lambda \downarrow 0} E(1 - \exp -\lambda Z^p_{\tau_x}) = 1$$

for every p. This property, which obviously does not hold when $\pi(\mathbb{R}_+) < \infty$, is related to the fact that $\{t,\ N_t = \infty\}$ is not empty. Indeed, we can easily construct a (random) sequence (t_n) such that $\Delta X_{t_n} > 0$ and $t_n < t_{n+1} < \inf\{s > t_n,\ X_s = X_{t_n-}\}$. Clearly $N_{t_n} \geq n$ and $t = \lim \uparrow t_n$ satisfies $N_t = \infty$. This argument shows in fact that $\{t,\ N_t = \infty\}$ is dense in $\mathbb{R}_+$.

3.4. *The Markov process ρ.* We will now investigate the process $\rho_t = (\rho_t(s),\ s \in \mathscr{J}_t)$ which gives for every $t \geq 0$ the state of the queue at time t. By Lemma 3.1, the set $\mathscr{J}_t$ can be written as an increasing sequence $s_t^1 < s_t^2 < \cdots$ which may be finite or infinite. As usual, we denote by l^1 the Banach space of all sequences $a = (a^k,\ k = 1, 2, \ldots)$ of real numbers such that $\sum |a^k| < \infty$. Since $\sum_{s \in \mathscr{J}_t} \rho_t(s) = X_t - I_t < \infty$ for every $t \geq 0$, a.s., we can view $(\rho_t)_{t \geq 0}$ as a random process with values in l^1, taking $\rho_t^k = \rho_t(s_t^k)$ if $k \leq N_t$ and $\rho_t^k = 0$ otherwise. In particular, $\rho_t = 0$ if and only if $N_t = 0$. Lemma 3.1 shows that, for every fixed t, ρ_t a.s. belongs to the subset l^1_0 of finitely supported sequences.

We denote by $(\mathscr{F}_t)$ the canonical filtration of X augmented as usual by the negligible sets of $\mathscr{F}_\infty$.

PROPOSITION 3.3. *The process $(\rho_t,\ t \geq 0)$ is a cadlag strong Markov process with respect to the filtration $(\mathscr{F}_t)$, and 0 is a recurrent point for this process. An invariant measure for $(\rho_t,\ t \geq 0)$ is*

$$\begin{aligned}\nu(\Phi) &= \Phi(0) + \mathrm{N}\left(\int_0^{T_0} \Phi(\rho_s)\, ds\right)\\ &= \Phi(0) + \sum_{k=1}^{\infty} R^k \int \Phi(x_1, \ldots, x_k, 0, 0, \ldots)\, \overline{\pi}^{\otimes k}(dx_1 \cdots dx_k),\end{aligned}$$

where $\overline{\pi}(dr) = (\int r\,\pi(dr))^{-1} \pi([r, \infty))\, dr$ and $R = \alpha^{-1} \int r\,\pi(dr)$.

PROOF. We first introduce some notation. Denote by l^1_+ the subset of all elements of l^1 with nonnegative components. For $a \in l^1_+$, the "length" of a is $\|a\| = \sum_{j=1}^{\infty} a_j$. If $a \in l^1_+$ and $h > 0$, we let $k_h a$ be that member of l^1_0 whose length is $(\|a\| - h)^+$ and which agrees with a up to the last nonzero component of $k_h a$. More precisely, $k_h a = 0$ if $\|a\| \leq h$. If $\|a\| > h$, we may find a unique integer m such that

$$\sum_{i \geq m} a^i > h \geq \sum_{i > m} a^i$$

and we take $(k_h a)^i = 0$ if $i > m$, $(k_h a)^i = a^i$ if $i < m$ and, finally, $(k_h a)^m = \sum_{i\geq m} a^i - h$, in such a way that $\|k_h a\| = \|a\| - h$. Notice that $k_h a \in l_0^1$.

Next, let $a \in l_+^1 \cap l_0^1$ and $b \in l_+^1$. We let $[a, b]$ be the element of l_+^1 obtained by concatenating a and b: If $m = \sup\{i,\ a^i > 0\}$, $[a, b]^i = a^i$ if $i \leq m$ and $[a, b]^i = b^{i-m}$ if $i > m$.

For every $s, t \geq 0$, set $X_s^{(t)} = X_{t+s} - X_t$, $I_s^{(t)} = \inf_{[0,s]} X_r^{(t)}$. Then, a.s. for every $s > 0$, $t \geq 0$, we have $I_s^{(t)} < 0$ (if this were not the case, a time-reversal argument would give a contradiction with the fact that the maximum of X only increases by jumps). Let $(\rho_s^{(t)},\ s \geq 0)$ be the analogue of $(\rho_s,\ s \geq 0)$ for the shifted function $(X_s^{(t)},\ s \geq 0)$. Then a.s. for every $t \geq 0$, $s \geq 0$, $\rho_s^{(t)}$ can be seen as an element of l_+^1. We claim that, a.s. for every $t \geq 0$, $s > 0$,

$$\rho_{t+s} = \big[k_{-I_s^{(t)}}\rho_t, \rho_s^{(t)}\big]. \tag{3.4}$$

This identity is an elementary consequence of our definitions.

The right-continuity of the process ρ follows at once, observing that, a.s. for every $t \geq 0$,

$$\lim_{s\downarrow 0} I_s^{(t)} = 0, \qquad \lim_{s\downarrow 0} \|\rho_s^{(t)}\| = 0.$$

The existence of left limits follows similarly. In fact, ρ and X have the same discontinuity times and $\rho_t = [\rho_{t-}, \Delta X_t]$, where we abuse notation by writing ΔX_t for the sequence whose only nonzero element is the first one, which is equal to ΔX_t.

The identity (3.4) also shows how to define the process ρ started at an arbitrary element $a \in l_+^1$:

$$\rho_t^a = [k_{-I_t} a, \rho_t].$$

The strong Markov property for ρ then follows from (3.4) and the strong Markov property for X.

From the fact that 0 is regular and recurrent for $X - I$, it is immediate that 0 is a regular recurrent point for ρ, and it is also clear that a choice of the associated excursion measure is the law of $(\rho_t,\ t \geq 0)$ under N (to define properly ρ_t under N, it is necessary to take account of the initial "jump at time 0"). It is then classical (and easy to prove) that an invariant measure for ρ is

$$\begin{aligned}\nu(B) &= 1_B(0) + \mathrm{N}\left(\int_0^{T_0} 1_B(\rho_t)\,dt\right)\\ &= 1_B(0) + \int \pi(dr)\, E_r^0\left(\int_0^{T_0} 1_B(\rho_t)\,dt\right),\end{aligned}$$

by (3.3). By Lemma 3.1, ν is supported on l_0^1. Fix $k \geq 1$ and let B be a Borel subset of $(0, \infty)^k$, which we identify with the set of all sequences $(a_1, \dots, a_k, 0, 0, \dots)$ such that $(a_1, \dots, a_k) \in B$. To compute $\nu(B)$, denote by $\tilde{B}$ the set of all finite cadlag paths ω: $[0, t] \to \mathbb{R}$ whose maximum has exactly k successive jumps $\alpha_1, \dots, \alpha_k$ such that $(\alpha_k, \dots, \alpha_1) \in B$. Let $\gamma_1, \dots, \gamma_n, \dots$ be the successive times of increase of S. Using the convention $X_{0-}=0$ to define $\widehat{X}_t^{(t)}=X_t$, we have then

$$\begin{aligned}
\nu(B) &= \int \pi(dr)\,E_r^0\Big(\int_0^{T_0} 1_B(\rho_t)\,dt\Big)\\
&= \int \pi(dr)\,E_r^0\Big(\int_0^{T_0} 1_{\tilde B}(\widehat{X}^{(t)})dt\Big)\\
&= \int \pi(dr)\int_0^\infty dt\,E_0\Big(1_{\tilde B}([X_{[0,t)},X_t+r])\,1_{\{S_t-X_t<r\}}\Big)\\
&= \int \pi(dr)E_0\Big(1_{\{\gamma_{k-1}<\infty\}}\int_{\gamma_{k-1}}^{\gamma_k} dt\,1_B(X_t+r-S_t,X_{\gamma_{k-1}}-X_{\gamma_{k-2}},\dots,X_{\gamma_1})\\
&\qquad\qquad\times 1_{\{S_t-X_t<r\}}\Big)
\end{aligned}$$

where we used the notation $[X_{[0,t)},X_t+r]$ to denote the cadlag path on $[0,t]$ that coincides with X_s for $s<t$ and equals X_t+r for $s=t$. Conditionally on the event $\{\gamma_{k-1}<\infty\}$, the variables $X_{\gamma_1},\dots,X_{\gamma_{k-1}}-X_{\gamma_{k-2}}$ are independent and identically distributed. Furthermore, by (3.1), their common conditional distribution is $\overline{\pi}$, and $P(\gamma_{k-1}<\infty)=P(\gamma_1<\infty)^{k-1}=R^{k-1}$. Using also the strong Markov property at γ_{k-1}, we get

$$\begin{aligned}
\nu(B) &= R^{k-1}\int \overline{\pi}^{\otimes(k-1)}(dx_2\cdots dx_k)\\
&\qquad\times E\Big(\int_0^\gamma dt\int \pi(dr)\,1_{\{X_t>-r\}}1_B(X_t+r,x_2,\dots,x_k)\Big)\\
&= \alpha^{-1}R^{k-1}\int \overline{\pi}^{\otimes(k-1)}(dx_2\cdots dx_k)\int_{-\infty}^0 dy\int_{-y}^\infty \pi(dr)\,1_B(r+y,x_2,\dots,x_k)\\
&= R^k\int \overline{\pi}^{\otimes k}(dx_1\cdots dx_k)\,1_B(x_1,\dots,x_k),
\end{aligned}$$

using (3.2) (with $c=\alpha^{-1}$) for the second equality. This completes the proof of Proposition 3.3. □

3.5. *Duality properties.* We now proceed to investigate the dual process of ρ with respect to the reference measure ν. This leads to a process η which has also a simple interpretation in terms of the queueing system. In the same way as ρ represents the remaining service times due to the customers in the system at time t, η_t gives the service times already accomplished for these customers. More precisely, $\eta_t=(\eta_t^1,\eta_t^2,\dots)$, where $\eta_t^k=0$ if $k>N_t$ and, if $k\le N_t$,

$$\eta_t^k=\Delta X_{s_t^k}-\rho_t^k=X_{s_t^k}-\inf_{s_t^k\le r\le t}X_r,$$

in the notation of Lemma 3.1.

Because $\sum_{0\le s\le t}\Delta X_s<\infty$, it is clear that the process η lives in l_+^1. Obviously, $P(\eta_t\in l_0^1)=1$ for every $t\ge 0$.

PROPOSITION 3.4. *The pair (ρ, η) is a cadlag strong Markov process with values in $(l^1_+)^2$. An invariant measure for (ρ, η) is given by*

$$\Theta(\Phi) = \Phi(0) + \sum_{k=1}^{\infty} R^k \int \theta^{\otimes k}(dx_1\, dy_1 \ldots dx_k\, dy_k) \times \Phi((x_1, \ldots, x_k, 0, \ldots), (y_1, \ldots, y_k, 0, \ldots)),$$

where the measure θ is defined on $(0, \infty)^2$ by

$$\int \theta(dxdy)\, f(x, y) = \left(\int \pi(dz)\, z\right)^{-1} \int \pi(dz) \int_0^z dx\, f(x, z - x).$$

The dual process of (ρ, η) with respect to Θ is (η, ρ).

REMARK. The process η is not a strong Markov process: When π is finite, the strong Markov property fails at the first jump of X.

PROOF OF PROPOSITION 3.4. We first extend the identity (3.4). For $h > 0$ and $(a, b) \in (l^1_+)^2$ we define $\tilde{k}_h(a, b) = (a', b')$ as follows. If $\|a\| \le h$, $\tilde{k}_h(a, b) = 0$. If $\|a\| > h$, we take $a' = k_h a$, we then let $m \ge 1$ be as in the definition of $k_h a$ and we set

$$b'^i = \begin{cases} 0, & \text{if } i > m, \\ b^i, & \text{if } i < m, \\ b^m + \left(\sum_{i \ge m} a^i - h\right), & \text{if } i = m. \end{cases}$$

Then (3.4) is easily strengthened to

$$(\rho_{t+s}, \eta_{t+s}) = \left[\tilde{k}_{-I_s^{(t)}}(\rho_t, \eta_t), (\rho_s^{(t)}, \eta_s^{(t)})\right],$$

for every $t \ge 0$, $s > 0$, a.s. (the concatenation in the right-hand side acts separately on the components ρ and η). The strong Markov property of the pair (ρ, η) follows in a straightforward way, as well as the caldlag property of the paths.

The calculation of the invariant measure is similar to the proof of Proposition 3.3, observing now that the excursion measure of (ρ, η) away from 0 is the law of (ρ, η) under N. We use (3.1) again at the end of the proof.

The last assertion of Proposition 3.4 follows from the next proposition by standard arguments.

PROPOSITION 3.5. *The processes $(\rho_{(T_0-t)-}, \eta_{(T_0-t)-})_{0 \le t \le T_0}$ and $(\eta_t, \rho_t)_{0 \le t \le T_0}$ have the same distribution under* N.

PROOF. We first observe that we may restrict our attention to the case when π is finite. In fact, if $\rho^\varepsilon, \eta^\varepsilon$ denote the analogues of ρ, η for X^ε, we can argue as follows. Proposition 3.5 is equivalent to the assertion that

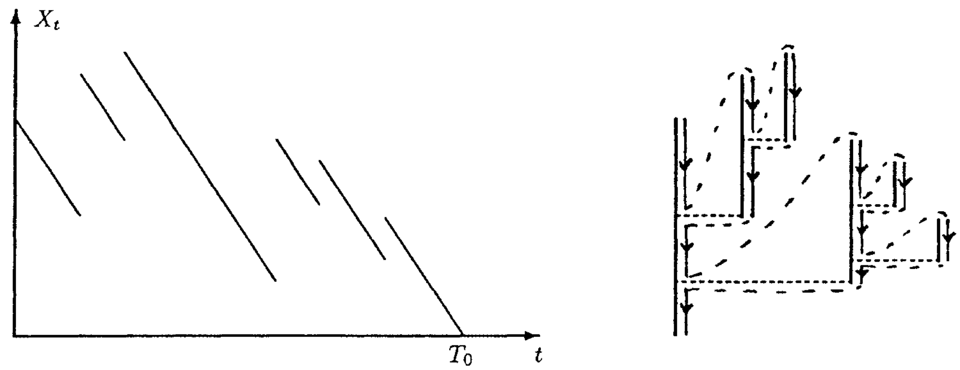

FIG. 1.

$(\rho_{(\tau_x-s)-}, \eta_{(\tau_x-s)-})_{0\le s\le\tau_x}$ and $(\eta_s, \rho_s)_{0\le s\le\tau_x}$ have the same distribution under P_0. If we know Proposition 3.5 when π is finite, we get that $(\rho^{\varepsilon}_{(\tau^{\varepsilon}_x-s)-}, \eta^{\varepsilon}_{(\tau^{\varepsilon}_x-s)-})_{0\le s\le\tau^{\varepsilon}_x}$ and $(\eta^{\varepsilon}_s, \rho^{\varepsilon}_s)_{0\le s\le\tau^{\varepsilon}_x}$ have the same distribution. We can then let ε go to 0, observing that $\tau^{\varepsilon}_x \to \tau_x$, and $\rho^{\varepsilon}_t \to \rho_t$, $\eta^{\varepsilon}_t \to \eta_t$ for every $t \ge 0$, P_0-a.s.

We thus assume that $\pi(\mathbb{R}_+) < \infty$. We then use a marked tree representation of the excursions of $X - I$, which is most easily described via the queueing system representation. Each customer or jump of X (including the initial one at time 0) is represented by one vertex of the tree. The initial customer corresponds to the root of the tree. Then the children of any customer are those customers who interrupt his service. In addition, each vertex is marked by two positive quantities. The first one is the service required by the given customer (the size of the jump). The second mark (defined except for the root) is the service already accomplished for his "father" before the given customer interrupts it. We think of the first mark as a vertical line segment (whose length is the size of the jump), which is divided into subintervals corresponding to periods of uninterrupted service. Each subdivision point therefore corresponds to exactly one child of the given customer (the second mark of this child is the distance between the subdivision point and the top of the segment) and the number of children is the number of subdivision points. See Figure 1 for the tree associated with a given path of $(X_t,\ 0 \le t \le T_0)$ under N.

To specify the distribution of the tree under the excursion measure N, we observe that the lengths of the different segments are independent and identically distributed according to $\pi/\pi(\mathbb{R}_+)$, and that for a given segment the subdivision points are distributed according to a Poisson point measure with intensity $\pi(\mathbb{R}_+)\alpha^{-1}\,dx$ (independently of the size of the segment and independently for the different segments). This description follows from a minor modification of the proof of Proposition 3.2 (conversely, this result also follows from the tree representation).

We can recover the processes ρ and η by considering the motion of a particle that runs along the segments at constant speed α, starting from the top of the root segment (cf. the right part of Figure 1). When the particle reaches a subdivision point of one segment, it jumps to the top of the corresponding "child" segment. On the other hand, when the particle has finished visiting a segment, it goes back to the corresponding subdivision point of the "father" segment. The particle stops when it reaches the bottom of the root segment. The process ρ_t is then obtained by considering for each ancestor (segment) of the segment visited at time t the length of the part that has not yet been visited. For η_t, one considers instead the length of the part that has already been visited at time t.

Suppose that we reverse time in the motion of the particle, which will now start from the bottom of the root segment. From the previous interpretation of ρ_t, η_t, the effect of this operation is to replace the pair $(\eta_t, \rho_t)_{0\le t\le T_0}$ by $(\rho_{(T_0-t)-}, \eta_{(T_0-t)-})_{0\le t\le T_0}$. However, the probabilistic structure of the tree shows that this time-reversal operation does not affect the law of the process $(\eta_t, \rho_t)_{0\le t\le T_0}$. The desired result follows. □

3.6. *Combining branching and spatial motion.* We will now explain a snakelike construction of certain branching Markov chains, which can be seen as a toy model for the construction of superprocesses in [35]. This should also be compared to the Brownian snake of [32] and [33]. We consider an auxiliary measurable space $(E, \mathscr{E})$ on which a transition kernel Q is given. Denote by $\mathbb{Q}_y$ the law of the Q-Markov chain started at $y \in E$. The canonical process on $E^{\mathbb{N}}$ is denoted by $(Y_n,\ n \in \mathbb{N})$.

Fix a point $y_0 \in E$. We can then define an extension $(\rho_t, q_t)_{t\ge 0}$ of the Markov process $(\rho_t)_{t\ge 0}$ as follows. For each $t \ge 0$, q_t is a (finite or infinite) sequence $(q_t^0, q_t^1, \dots)$ in E, and $q_t^0 = y_0$. If $N(q_t)$ denotes the cardinality of the sequence q_t, we have $N(q_t) = N_t+1$ for every $t \ge 0$ a.s. Furthermore, the process $(q_t)_{t\ge 0}$ is (time-inhomogeneous) Markov under the conditional distribution knowing $(\rho_t)_{t\ge 0}$, and its transition kernels are described as follows. Let $0 \le s < t$ and let $m_{s,t} = \inf_{[s,t]} N_r$. Then the conditional distribution of q_t, given q_s and the process $(\rho_r)_{r\ge 0}$, is as follows:

1. the distribution of $(Y_0, Y_1, Y_2, \dots, Y_{N_t})$ under $\mathbb{Q}_{y_0}$ if $m_{s,t} = 0$;
2. the distribution of $(q_s^0, q_s^1, \dots, q_s^{m_{s,t}}, Y_1, \dots, Y_{N_t - m_{s,t}})$ under $\mathbb{Q}_{q_s^{m_{s,t}}}$ if $m_{s,t} > 0$.

It is easy to construct a good version of the process $(q_t)_{t\ge 0}$ in such a way that $q_t^i = q_s^i$ for every $i \le m_{s,t}$ and every $s < t$, a.s. This version is unique up to indistinguishability.

For every integer $k \ge 0$, let $\widetilde{Z}_k^x$ be the random element of $M_f(E)$ defined by

$$\widetilde{Z}_k^x = \int_0^{\tau_x} 1_{\{N_t = k\}}\, \delta_{q_t^k}\, dt.$$

PROPOSITION 3.6. *The process $(\widetilde{Z}_k^x,\ k \in \mathbb{N})$ is a Markov chain in $M_f(E)$ started at $\alpha^{-1} x\, \delta_{y_0}$ and whose transition kernel $\widetilde{\mathrm{P}}(\zeta, d\xi)$ is characterized by its*

Laplace functional

$$\int \exp(-\langle \xi, \Lambda\rangle)\, \widetilde{\mathrm{P}}(\zeta, d\xi) = \exp\Big(-\int \zeta(dy) \int Q(y, dz) \int \pi(dr)\, (1 - \exp(-r\Lambda(z)/\alpha))\Big),$$

for any nonnegative measurable function Λ *on* E.

Proposition 3.6 can be proved in a way similar to Proposition 3.2 when $\pi(\mathbb{R}_+) < \infty$. When $\pi(\mathbb{R}_+) = \infty$ it is preferable not to use the approximations X^ε as in the proof of Proposition 3.2, but to rely instead on the excursion theory for $X - I$. We leave details to the reader.

4. The infinite variation case.

4.1. *Assumptions and preliminaries.* Let X be a general Lévy process on the real line with no negative jumps and no Gaussian part. According to [6], Chapter VII, we have then $E[\exp -\lambda X_t] < \infty$ for every $\lambda \geq 0$, $t \geq 0$, and this Laplace transform can be written in the form

$$E[\exp -\lambda X_t] = \exp t\psi(\lambda),$$

with

$$\psi(\lambda) = a\lambda + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r\, 1_{\{r \leq 1\}})\pi(dr),$$

where $a \in \mathbb{R}$ and π, the Lévy measure of X, is such that

$$\int_0^\infty (1 \wedge r^2)\, \pi(dr) < \infty.$$

As in the previous section, we assume that X does not drift to $+\infty$. This implies that X has first moments and thus the Lévy measure satisfies the stronger condition

$$\int_0^\infty (r \wedge r^2)\, \pi(dr) < \infty.$$

We can then rewrite ψ in the form

$$\psi(\lambda) = \alpha\lambda + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r)\pi(dr),$$

and by Corollary VII.2 of [6], X does not drift to $+\infty$ if and only if $\alpha \geq 0$, which we also assume from now on. Note that the function ψ is Lipschitz on compact subsets of $[0, \infty)$.

In contrast with Section 3, we now consider the case when the paths of X have infinite variation, which is equivalent to

$$\int_0^1 r\, \pi(dr) = +\infty.$$

The theory developed in Section 3 can be extended to this different (and more difficult) context.

We start by recalling some important facts about the Lévy process X. In contrast to the finite variation case of Section 3, 0 is now regular for $(0,\infty)$ ([6], Corollary VII.5). As a consequence, 0 is regular for itself, with respect to the strong Markov process $S-X$. We can thus consider the local time at 0 of $S-X$, denoted by $(L_t,\ t\geq 0)$, which is uniquely defined up to a multiplicative constant. In the case $\alpha>0$, where X drifts to $-\infty$, L_∞ is finite a.s. and has an exponential distribution.

Denote by $(s_i,t_i),\ i\in I$, the excursion intervals of $S-X$ away from 0. The point measure

$$\sum_{i\in I,\, t_i<\infty}\delta_{(L_{t_i},\,\Delta S_{t_i},\,\Delta X_{t_i})}(dl\,dx\,dy)$$

is distributed as $1_{\{l<\zeta\}}\mathscr{N}(dl\,dx\,dy)$, where $\mathscr{N}$ is a Poisson measure on $\mathbb{R}_+\times\mathbb{R}_+^2$ with intensity $dl\,n(dx\,dy)$, and ζ is an independent exponential time ($\zeta=\infty$ in the recurrent case $\alpha=0$). We may and will choose the normalizing factor in the definition of L so that

$$n(dx\,dy)=1_{[0,\,y]}(x)\,dx\,\pi(dy).$$

This result is indeed the analogue of (3.1) and can be proved by similar arguments. We refer the reader to [5], Corollary 1 (see also [41]).

Set $\beta_\varepsilon=n(\mathbb{R}_+\times[\varepsilon,\infty))=\int_{[\varepsilon,\infty)}y\pi(dy)$. Note that $\beta_\varepsilon\uparrow\infty$ as $\varepsilon\downarrow 0$. By standard arguments for Poisson measures, we have, a.s. for every $u\geq 0$,

$$\begin{aligned}L_\infty\wedge u&=\lim_{\varepsilon\downarrow 0}\frac{1}{\beta_\varepsilon}\mathrm{Card}\,\{i\in I,\ L_{t_i}\leq u,\ \Delta X_{t_i}\geq\varepsilon\}\\&=\lim_{\varepsilon\downarrow 0}\frac{1}{\beta_\varepsilon}\mathrm{Card}\,\{s\in[0,\infty),\ L_s\leq u,\ X_s>S_{s-},\ \Delta X_s\geq\varepsilon\}.\end{aligned}$$

It follows that, a.s. for every $t\geq 0$,

$$L_t=\lim_{\varepsilon\downarrow 0}\frac{1}{\beta_\varepsilon}\mathrm{Card}\,\{s\in[0,t],\ X_s>S_{s-},\ \Delta X_s\geq\varepsilon\}. \tag{4.1}$$

4.2. *The branching structure of discrete jumps.* Let $\mathscr{J}=\{s\geq 0,\ \Delta X_s>0\}$. As in Section 3, we can interpret each $s\in\mathscr{J}$ as the arrival time of a customer claiming a service ΔX_s. Note, however, that the total quantity of services claimed during any nontrivial finite interval will be infinite a.s., so that in some sense the output rate of the server must also be infinite.

For every $s\in\mathscr{J}$, set

$$R(s)=\inf\{t>s,\ X_t\leq X_{s-}\},$$

which in the previous interpretation corresponds to the time when the customer who arrived at s exits the system.

Let $\varepsilon>0$ and define $\mathscr{J}^\varepsilon=\{s\in\mathscr{J},\ \Delta X_s\geq\varepsilon\}$. For every $t\geq 0$, set

$$N_\varepsilon(t)=\mathrm{Card}\,\{s\in\mathscr{J}^\varepsilon,\ s\leq t<R(s)\}$$

and

$$Z_n^\varepsilon(t) = \int_0^t 1_{\{N_\varepsilon(s)=n\}}\, ds, \qquad n \in \mathbb{N}.$$

For $x > 0$, set $\tau_x = \inf\{t \geq 0,\ X_t = -x\}$. The next result is analogous to Proposition 3.2.

PROPOSITION 4.1. *Let*

$$\psi_\varepsilon(\lambda) = \alpha\lambda + \int_{[0,\,\varepsilon)} (e^{-\lambda r} - 1 + \lambda r)\pi(dr) + \lambda \int_{[\varepsilon,\,\infty)} r\pi(dr),$$

and let K_ε be the inverse function of ψ_ε [$K_\varepsilon(\psi_\varepsilon(\lambda)) = \lambda$ for $\lambda \in \mathbb{R}_+$]. The process $(Z_n^\varepsilon(\tau_x),\ n \geq 0)$ is a Markov chain in $\mathbb{R}_+$ whose initial distribution ν_ε and transition kernel P^ε are characterized as follows:

$$\int \nu_\varepsilon(du) \exp(-\lambda u) = \exp(-xK_\varepsilon(\lambda)),$$

$$\int \mathrm{P}^\varepsilon(u, dv) \exp(-\lambda v) = \exp\left(-u \int_{[\varepsilon,\,\infty)} (1 - \exp(-rK_\varepsilon(\lambda)))\pi(dr)\right).$$

PROOF. We first observe that, from the strong Markov property at τ_x, it is immediate to check that $(Z_n^\varepsilon(\tau_{x+y}),\ n \geq 0)$ has the same distribution as $(Z_n^\varepsilon(\tau_x) + \tilde{Z}_n^\varepsilon(\tilde{\tau}_y),\ n \geq 0)$, where $(\tilde{Z}_n^\varepsilon(\tilde{\tau}_y),\ n \geq 0)$ is an independent copy of $(Z_n^\varepsilon(\tau_y),\ n \geq 0)$.

Then consider the sequence of stopping times defined inductively by

$$T_1 = \inf\{t \geq 0,\ t \in \mathscr{J}^\varepsilon\},$$
$$T_{n+1} = \inf\{t \geq R(T_n),\ t \in \mathscr{J}^\varepsilon\}.$$

Write $R(T_0) = 0$ by convention. The processes

$$Y_t^i = X_{R(T_{i-1})+t} - X_{R(T_{i-1})}, \qquad 0 \leq t < T_i - R(T_{i-1}),$$

defined for every $i \geq 1$, are independent and identically distributed. Furthermore, the classical construction of Lévy processes shows that their common distribution is the law of the Lévy process with Laplace exponent ψ_ε (denoted by Y), killed at an independent exponential time with parameter $\pi([\varepsilon, \infty))$. Informally, Y is obtained from X by removing the jumps of size at least ε. Next observe that $N_\varepsilon(t) = 0$ if and only if $t \in [R(T_{i-1}),\ T_i)$ for some $i \geq 1$. If

$$\gamma^\varepsilon(t) = \inf\{s \geq 0,\ Z_0^\varepsilon(s) > t\},$$

it follows that the process $X_t^{(\varepsilon)} := X_{\gamma^\varepsilon(t)}$ has the same distribution as Y. Note that

$$Z_0^\varepsilon(\tau_x) = \inf\{t \geq 0,\ X_t^{(\varepsilon)} = -x\} =: \tau_x^{(\varepsilon)},$$

and by a classical result for spectrally positive Lévy processes ([6], Theorem VII.1) we have

$$E(\exp -\lambda Z_0^\varepsilon(\tau_x)) = \exp -xK_\varepsilon(\lambda), \tag{4.2}$$

which gives the desired formula for the Laplace functional of ν_ε.

By the strong Markov property again, we also know that the processes

$$X^i_t = X_{T_i+t} - X_{T_i}, \qquad 0 \le t < R(T_i) - T_i,$$

defined for $i \ge 1$, are independent (and independent of $X^{(\varepsilon)}$) and distributed as the process $(X_t,\ 0 \le t < \tau_\xi)$, where ξ is an independent variable with law $\pi([\varepsilon,\infty))^{-1}1_{\{x\ge\varepsilon\}}\pi(dx)$. Also notice that, for $t \in [T_i, R(T_i))$, we have

$$N_\varepsilon(t) = 1 + N^i_\varepsilon(t - T_i), \tag{4.3}$$

with an obvious notation.

Set

$$J^\varepsilon(\tau_x) = \mathrm{Card}\,\{i \ge 1,\ T_i < \tau_x\}.$$

By previous considerations, $J^\varepsilon(\tau_x)$ is the number of jumps over $[0,\tau_x^{(\varepsilon)})$ of a Poisson process with parameter $\pi([\varepsilon,\infty))$, independent of $X^{(\varepsilon)}$. Hence, conditionally on $X^{(\varepsilon)}$, $J^\varepsilon(\tau_x)$ has a Poisson distribution with parameter $\pi([\varepsilon,\infty))\tau_x^{(\varepsilon)}$. Also notice that the pair $(\tau_x^{(\varepsilon)}, J^\varepsilon(\tau_x))$ is independent of the processes X^i, $i \ge 1$.

It follows from the previous observations, and in particular from (4.3), that, for every $p \ge 1$,

$$Z^\varepsilon_p(\tau_x) = \sum_{i=1}^{J^\varepsilon(\tau_x)} Z^{\varepsilon,i}_{p-1},$$

where the processes $Z^{\varepsilon,i}$ are independent [and independent of the pair $(\tau_x^{(\varepsilon)}, J^\varepsilon(\tau_x))$] and distributed as $Z^\varepsilon(\tau_\xi)$. Recall that $Z^\varepsilon_0(\tau_x) = \tau_x^{(\varepsilon)}$. From the additivity property mentioned at the beginning of the proof, we get

$$\big(Z^\varepsilon_0(\tau_x), (Z^\varepsilon_p(\tau_x),\ p \ge 1)\big) \overset{(\mathrm{d})}{=} (Z^\varepsilon_0(\tau_x), (\tilde Z^\varepsilon_{p-1}(\tilde\tau_U),\ p \ge 1)), \tag{4.4}$$

where the notation $\tilde Z, \tilde\tau$ refers to an independent Lévy process $\tilde X$ distributed as X and the variable U is independent of $\tilde X$ and conditionally on X is distributed as the sum of $J^\varepsilon(\tau_x)$ independent copies of ξ. Hence, using (4.2),

$$\begin{aligned} E[\exp -\lambda Z^\varepsilon_1(\tau_x) \mid Z^\varepsilon_0(\tau_x)] &= E[\exp -\lambda \tilde Z^\varepsilon_0(\tilde\tau_U) \mid Z^\varepsilon_0(\tau_x)] \\ &= E[\exp -U K_\varepsilon(\lambda) \mid Z^\varepsilon_0(\tau_x)] \\ &= \exp\left(-Z^\varepsilon_0(\tau_x) \int_{[\varepsilon,\infty)} (1 - \exp(-r K_\varepsilon(\lambda)))\pi(dr)\right). \end{aligned}$$

The proof of Proposition 4.1 is then easily completed by an induction argument using the identity (4.4). □

4.3. *The height process.* We will now introduce the analogue of the process N_t of Section 3. We apply the considerations of the end of Section 4.1 to the time-reversed process $\widehat{X}^{(t)}$ and its supremum $\widehat{S}^{(t)}$. We denote by H_t the local time at 0, at time t, of the process $\widehat{S}^{(t)} - \widehat{X}^{(t)}$. The process $(H_t,\ t \geq 0)$ is called the *height process* associated with X. Note that, by (4.1) for L_t, we have for every $t \geq 0$, a.s.,

$$
\begin{aligned}
H_t &= \lim_{\varepsilon \downarrow 0} \frac{1}{\beta_\varepsilon} \mathrm{Card}\left\{s \in (0,t],\ X_{s-} < \inf_{s \leq r \leq t} X_r,\ \Delta X_s \geq \varepsilon\right\} \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\beta_\varepsilon} N_\varepsilon(t).
\end{aligned} \tag{4.5}
$$

This approximation allows us to take a measurable version of the process $(H_t, t \geq 0)$ in order to define the random measure $\mathscr{Q}_x$ (independent of the chosen version) that appears in the following theorem.

THEOREM 4.2. *Let $x > 0$ and let $\mathscr{Q}_x$ be the random measure on $\mathbb{R}_+$ defined by*

$$
\langle \mathscr{Q}_x, \varphi \rangle = \int_0^{\tau_x} \varphi(H_s)\, ds.
$$

The measure $\mathscr{Q}_x$ has a.s. a cadlag density $(Z_x(u),\ u \geq 0)$ with respect to Lebesgue measure on $\mathbb{R}_+$, and the process $(Z_x(u),\ u \geq 0)$ is a continuous-state branching process with branching mechanism ψ started at x.

PROOF. Let $(Z_u,\ u \geq 0)$ be a CSBP(ψ) started at x. We can easily extend (1.2) and (1.3) to get a formula for the finite-dimensional marginals of the process Z: for $0 \leq t_1 < \cdots < t_n$ and $\lambda_1, \ldots, \lambda_n \geq 0$,

$$
E\left(\exp - \sum_{j=1}^{n} \lambda_j Z_{t_j}\right) = \exp - x\, v(0),
$$

where $(v(t),\ t \geq 0)$ is the unique nonnegative solution of the integral equation

$$
v(t) + \int_t^\infty \psi(v(s))\, ds = \sum_{j=1}^{n} \lambda_j 1_{[0,\, t_j]}(t).
$$

(Note that the uniqueness of the solution is a straightforward consequence of Gronwall's lemma.) When $n = 1$, this is merely a rewriting of (1.2)–(1.3). In the general case, we argue by induction on n, using the Markov property of Z at time t_1. Then let φ be a nonnegative continuous function with compact support on $\mathbb{R}_+$. By approximating φ with suitable step functions, we deduce from the above that

$$
E\left(\exp - \int_0^\infty Z_s \varphi(s)\, ds\right) = \exp - x w(0),
$$

where $(w(t),\ t \geq 0)$ is the unique nonnegative solution of the integral equation

$$w(t) + \int_t^\infty \psi(w(s))\, ds = \int_t^\infty \varphi(s)\, ds. \tag{4.6}$$

We will verify that

$$E(\exp - \langle \mathscr{Z}_x, \varphi \rangle) = \exp - x w(0),$$

where w solves (4.6). It follows that the random measure $\mathscr{Z}_x$ has the same distribution as the random measure $\varphi \to \int_0^\infty Z_s \varphi(s)\, ds$. By standard arguments, this implies that $\mathscr{Z}_x$ has a cadlag density which is a CSBP(ψ) started at x.

In order to compute the Laplace functional of $\mathscr{Z}_x$, we observe that, from the approximation (4.5) for H_t,

$$\int_0^{\tau_x} \varphi(H_s)\, ds = \lim_{\varepsilon \downarrow 0} \int_0^{\tau_x} \varphi(\beta_\varepsilon^{-1} N_\varepsilon(s))\, ds = \lim_{\varepsilon \downarrow 0} \sum_{k=0}^\infty \varphi(k \beta_\varepsilon^{-1})\, Z_k^\varepsilon(\tau_x).$$

Therefore,

$$E\left[\exp - \int_0^{\tau_x} \varphi(H_s)\, ds\right] = \lim_{\varepsilon \downarrow 0} E\left[\exp - \sum_{k=0}^\infty \varphi(k\beta_\varepsilon^{-1})\, Z_k^\varepsilon(\tau_x)\right].$$

Proposition 4.1 allows us to compute the right-hand side of the previous formula. Set

$$w_0^\varepsilon = -\frac{1}{x} \log E\left[\exp - \sum_{k=0}^\infty \varphi(k\beta_\varepsilon^{-1})\, Z_k^\varepsilon(\tau_x)\right]$$

and, for $\lambda \geq 0$,

$$U_\varepsilon(\lambda) = \int_{[\varepsilon, \infty)} (1 - \exp(-r K_\varepsilon(\lambda)))\pi(dr).$$

By Proposition 4.1 and finitely many successive applications of the Markov property (recall that φ has compact support), we get

$$w_0^\varepsilon = K_\varepsilon\left(\varphi(0) + U_\varepsilon\left(\varphi\left(\frac{1}{\beta_\varepsilon}\right) + U_\varepsilon\left(\varphi\left(\frac{2}{\beta_\varepsilon}\right) + \cdots\right)\right)\right).$$

Then, for every integer $k \geq 0$, set

$$\tilde{w}_k^\varepsilon = \frac{1}{\beta_\varepsilon}\left(\varphi\left(\frac{k}{\beta_\varepsilon}\right) + U_\varepsilon\left(\varphi\left(\frac{k+1}{\beta_\varepsilon}\right) + U_\varepsilon\left(\varphi\left(\frac{k+2}{\beta_\varepsilon}\right) + \cdots\right)\right)\right)$$

and note that $\tilde{w}_k^\varepsilon = 0$ for all k sufficiently large. It is easy to verify that, for every integer $l \geq 0$,

$$\tilde{w}_l^\varepsilon = \frac{1}{\beta_\varepsilon} \sum_{k=l}^\infty \left(\varphi\left(\frac{k}{\beta_\varepsilon}\right) + \left(U_\varepsilon(\beta_\varepsilon \tilde{w}_{k+1}^\varepsilon) - \beta_\varepsilon \tilde{w}_{k+1}^\varepsilon\right)\right) = \frac{1}{\beta_\varepsilon} \sum_{k=l}^\infty \left(\varphi\left(\frac{k}{\beta_\varepsilon}\right) + \theta_\varepsilon(\tilde{w}_{k+1}^\varepsilon)\right),$$

where, for $\lambda \geq 0$,

$$\theta_\varepsilon(\lambda) = \int_{[\varepsilon, \infty)} (1 - \exp(-r K_\varepsilon(\beta_\varepsilon \lambda)) - r\lambda)\, \pi(dr).$$

For $t \in \mathbb{R}_+$, set $\tilde{w}^\varepsilon(t) = \tilde{w}^\varepsilon_{[\beta_\varepsilon t]}$, where $[s]$ denotes the integral part of s. If t is of the form $t = \beta_\varepsilon^{-1} l$, we have

$$\tilde{w}^\varepsilon(t) = \int_t^\infty \varphi\left(\frac{[s\beta_\varepsilon]}{\beta_\varepsilon}\right) ds + \int_{t+1/\beta_\varepsilon}^\infty \theta_\varepsilon(\tilde{w}^\varepsilon(s))\, ds. \tag{4.7}$$

Now notice that from our definitions

$$\psi_\varepsilon(K_\varepsilon(\beta_\varepsilon\lambda)) - \theta_\varepsilon(\lambda) = \psi(K_\varepsilon(\beta_\varepsilon\lambda)) + \beta_\varepsilon\lambda,$$

and $\psi_\varepsilon(K_\varepsilon(\beta_\varepsilon\lambda)) = \beta_\varepsilon\lambda$ by the definition of the function K_ε. Therefore,

$$\theta_\varepsilon(\lambda) = -\psi(K_\varepsilon(\beta_\varepsilon\lambda)),$$

and in particular θ_ε takes only nonpositive values. Then observe that

$$\lim_{\varepsilon\downarrow 0} \frac{\psi_\varepsilon(\lambda)}{\beta_\varepsilon\lambda} = 1,$$

uniformly on compact subsets of $[0,\infty)$. It easily follows that $K_\varepsilon(\beta_\varepsilon\lambda)$ converges to λ, hence $\theta_\varepsilon(\lambda)$ converges to $-\psi(\lambda)$ uniformly on compact sets. Note that the property $\theta_\varepsilon \le 0$ combined with (4.7) gives a uniform upper bound on the functions $\tilde{w}^\varepsilon$. From (4.7), Gronwall's lemma and the Lipschitz property of ψ, it is then a simple matter to verify that $\tilde{w}^\varepsilon$ converges uniformly to the unique nonnegative solution w of (4.6). In particular, $\tilde{w}^\varepsilon_0$ converges to $w(0)$ and $w^\varepsilon_0 = K_\varepsilon(\beta_\varepsilon\tilde{w}^\varepsilon_0)$ also converges to $w(0)$. This implies that $E[\exp -\langle \mathscr{Z}_x, \varphi\rangle] = E[\exp -\int_0^{\tau_x} \varphi(H_s)\, ds] = \exp -x\, w(0)$ as desired. □

4.4. *The Markov process ρ.* We will now introduce the analogue of the process ρ_t of Section 3. To this end, it will be convenient to derive some additional information about the process $(H_t,\ t \ge 0)$. For $\varepsilon > 0$ and $0 \le s \le t$, we set

$$I^s_t = \inf_{s \le r \le t} X_r,$$

$$N_\varepsilon(s,t) = \mathrm{Card}\,\{u \in (0,s],\ X_{u-} < I^u_t,\ \Delta X_u \ge \varepsilon\}.$$

In particular, $N_\varepsilon(t,t) = N_\varepsilon(t)$.

PROPOSITION 4.3. *Almost surely for every $t > 0$ and every $s \in [0,t)$, the limit*

$$H^s_t = \lim_{\varepsilon\to 0} \beta_\varepsilon^{-1} N_\varepsilon(s,t)$$

exists and defines a continuous monotone increasing function of $s \in [0,t)$. The formulas

$$H'_t = \lim_{s\uparrow t,\, s<t} \uparrow H^s_t \qquad (t > 0), \qquad H'_0 = 0,$$

give a lower semicontinuous version of the process $(H_t,\ t \ge 0)$ with values in $[0,\infty]$.

PROOF. (a) We first fix $t > 0$. Let $L^{(t)} = (L_s^{(t)},\ 0 \le s \le t)$ be the local time at 0 of $\widehat{S}^{(t)} - \widehat{X}^{(t)}$, so that $H_t = L_t^{(t)}$. Then, from the approximation (4.1) for local time, we have, a.s. for every $s \in [0, t]$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \beta_\varepsilon^{-1} N_\varepsilon(s,t) &= \lim_{\varepsilon\downarrow 0} \beta_\varepsilon^{-1}\text{Card}\left\{u \in [t-s,t),\ \widehat{X}_u^{(t)} > \widehat{S}_{u-}^{(t)},\ \Delta\widehat{X}_u^{(t)} > \varepsilon\right\}\\ &= L_t^{(t)} - L_{t-s}^{(t)}.\end{aligned}$$

This shows that the first part of the proposition holds for a fixed $t > 0$, even with $s \in [0, t]$.

(b) By step (a), we may find a negligible set $\mathscr{N}$ such that the first part of the proposition holds for every rational $t > 0$ outside $\mathscr{N}$. We then argue outside $\mathscr{N}$. Let $t > 0$. First assume that $I_t^s < X_t$ for every $s < t$. Then, for any $\delta \in (0, t)$, we may find a rational $q > t$ such that $I_t^s = I_q^s$ for every $s \in [0, t-\delta)$. Clearly, we have also $N_\varepsilon(s,t) = N_\varepsilon(s,q)$ for every $s \in [0, t-\delta)$, and we see that the first part of the proposition holds for t because it does for q and δ is arbitrary.

If $I_t^r = X_t$ for some $r < t$, we argue differently. Set $\gamma_t = \sup\{s < t,\ X_s < X_t\}$ ($\sup\varnothing = 0$) and pick any rational $q \in (\gamma_t, t)$. Clearly, $N_\varepsilon(s,t) = N_\varepsilon(s \wedge \gamma_t, t) = N_\varepsilon(s \wedge \gamma_t, q)$ for every $s \in [0, t)$, and the desired result follows.

(c) Step (a) shows that, for every fixed $t > 0$, $H'_t = \lim \uparrow H_t^s = H_t^t = H_t$ a.s. Hence the process $(H'_t,\ t \ge 0)$ is a version of H.

(d) Let $t > 0$. For $s' < s < t$, it is obvious that $H_t^{s'} \le H_s^{s'} \le H'_s$. Hence,

$$\liminf_{s\uparrow t} H'_s \ge H_t^{s'}$$

and, by letting $s' \uparrow t$,

$$\liminf_{s\uparrow t} H'_s \ge H'_t.$$

(e) Let $t \ge 0$. As in step (b) consider $\gamma_t = \sup\{s < t,\ X_s < X_t\}$. From our definitions it is immediate that $H'_t = H'_{\gamma_t}$. On the other hand, if $s < \gamma_t$, $I_t^s < X_t$ and thus $I_t^s = I_r^s$ for every $r > t$ sufficiently close to t. Hence, for any $\delta > 0$, for $r > t$ sufficiently close to t, we have $N_\varepsilon(s,r) = N_\varepsilon(s,t) = N_\varepsilon(s,\gamma_t)$ for every $s \in [0, \gamma_t - \delta]$. It follows that

$$\liminf_{r\downarrow t} H'_r \ge H_{\gamma_t}^{\gamma_t-\delta}$$

and, since δ was arbitrary,

$$\liminf_{r\downarrow t} H'_r \ge H'_{\gamma_t} = H'_t. \qquad \square$$

From now on, we deal only with the lower semicontinuous version of H constructed in Proposition 4.3 but write H instead of H'. For every $t \ge 0$, we let ρ_t be the random measure on $\mathbb{R}_+$ defined by

$$\langle \rho_t, \varphi\rangle = \int_{[0,t]} d_s I_t^s\, \varphi(H_t^s),$$

where the notation $d_sI_t^s$ refers to the finite measure on $[0,t]$ associated with the cadlag increasing function $s \to I_t^s$, and by convention $H_t^t = H_t$. Obviously the total mass of ρ_t is $\langle \rho_t, 1\rangle = X_t - I_t^0$. Furthermore, a.s. for every $t > 0$ such that $\rho_t \neq 0$,

$$\operatorname{supp} \rho_t = [0, H_t],$$

where $\operatorname{supp}\rho_t$ denotes the topological support of ρ_t. The inclusion $\operatorname{supp}\rho_t \subset [0, H_t]$ is trivial. The reverse inclusion is easy for a fixed time $t > 0$, because the local time at 0 of $S - X$ only increases when S increases. The fact that the result holds simultaneously for all $t > 0$ follows from arguments similar to the proof of the previous proposition. We let $M_f(\mathbb{R}_+)$ be the set of all finite measures on $\mathbb{R}_+$, equipped with the weak topology.

PROPOSITION 4.4. *The process $(\rho_t,\ t \geq 0)$ is a cadlag strong Markov process in $M_f(\mathbb{R}_+)$.*

PROOF. We first explain how to define the process ρ started at an arbitrary $\mu \in M_f(\mathbb{R}_+)$. To this end, we introduce some notation analogous to Section 3. Let $\mu \in M_f(\mathbb{R}_+)$ and $a \geq 0$. If $a \leq \langle \mu, 1\rangle$, we let $k_a\mu$ be the unique finite measure on $\mathbb{R}_+$ such that, for every $r \geq 0$,

$$k_a\mu([0,r]) = \mu([0,r]) \wedge (\langle \mu, 1\rangle - a).$$

In particular, $\langle k_a\mu, 1\rangle = \langle \mu, 1\rangle - a$. If $a \geq \langle \mu, 1\rangle$, we take $k_a\mu = 0$.

If $\mu \in M_f(\mathbb{R}_+)$ has compact support and $\nu \in M_f(\mathbb{R}_+)$, we define the concatenation $[\mu, \nu] \in M_f(\mathbb{R}_+)$ by the formula

$$\int [\mu,\nu](dr)\,\varphi(r) = \int \mu(dr)\,\varphi(r) + \int \nu(dr)\,\varphi(m+r),$$

where $m = \sup(\operatorname{supp}\mu)$.

With this notation at hand, the law of the process ρ started at $\mu \in M_f(\mathbb{R}_+)$ is defined as the distribution of the process $\rho_t^\mu = [k_{-I_t}\mu, \rho_t]$ (this makes sense because, a.s. for every $t > 0$, $k_{-I_t}\mu$ has compact support).

We then verify that the process ρ has the stated properties. For simplicity we deal only with the case when the initial value is 0, that is, with the process $(\rho_t,\ t \geq 0)$ defined as previously (the results are then immediately extended using the formula for ρ_t^μ). Let $t \geq 0$ and let φ be a bounded continuous function on $\mathbb{R}_+$. Then

$$\langle \rho_t, \varphi\rangle = \int_{[0,t]} d_sI_t^s\,\varphi(H_t^s) = \int_{[0,t]} d_sI_t^s\,1_{\{X_{s-} < X_t\}}\,\varphi(H_t^s).$$

On the one hand, the measures $d_sI_{t'}^s$ converge in the variation norm to $d_sI_t^s$ as $t' \downarrow t$. On the other hand, from our definition of H_t^s, it is immediately checked that, for any $s \leq t$ such that $X_{s-} < X_t$, $H_{t'}^s = H_t^s$ for all $t' > t$ sufficiently close to t. It follows that

$$\lim_{t' \downarrow t} \langle \rho_{t'}, \varphi\rangle = \langle \rho_t, \varphi\rangle.$$

As for left limits, note that the measures $d_s I^s_{t'}$ converge to $1_{[0,t)}(s)\, d_s I^s_t$ as $t' \uparrow t$, $t' < t$, again in the variation norm. By an argument exactly similar to the above, it follows that

$$\lim_{t'\uparrow t, t'<t} \langle \rho_{t'}, \varphi\rangle = \int_{[0,t)} d_s I^s_t\, \varphi(H^s_t).$$

We see in particular that ρ and X have the same discontinuity times and that

$$\rho_t = \rho_{t-} + \Delta X_t\, \delta_{H_t}.$$

We now turn to the strong Markov property. Let T be a stopping time of the filtration $(\mathscr{F}_t)_{t\geq 0}$. We will express ρ_{T+t} in terms of ρ_T and the shifted process $X^{(T)}_r = X_{T+r} - X_T$ in a way similar to Section 3. We let $I^{(T)}_t = \inf_{[0,t]} X^{(T)}_t$ be the minimum process of $X^{(T)}$. We then claim that, for any (finite) stopping time T, we have, a.s. for every $t > 0$,

$$\rho_{T+t} = \left[k_{-I^{(T)}_t} \rho_T, \rho^{(T)}_t\right], \tag{4.8}$$

where $\rho^{(T)}_t$ obviously denotes the analogue of ρ_t when X is replaced by $X^{(T)}$. When we have proved (4.8), the strong Markov property of the process ρ follows by standard arguments, using the previous representation of the process started at μ.

For the proof of (4.8), let $t > 0$ and consider the case when $-I^{(T)}_t < \langle \rho_T, 1\rangle = X_T - I_T$. The other case is similar and easier. For simplicity, write $\eta = -I^{(T)}_t$. Set

$$r = \sup\{u \leq T,\ X_{u-} \leq X_T - \eta\}.$$

From our definitions, it is easy to verify that

$$I^s_{T+t} = \begin{cases} I^s_T, & \text{if } s < r, \\ X_T - \eta, & \text{if } r \leq s \leq T, \end{cases}$$

and

$$H^s_{T+t} = \begin{cases} H^s_T, & \text{if } s < r, \\ H^r_T, & \text{if } r \leq s \leq T. \end{cases}$$

It follows that

$$\begin{aligned} \int_{[0,T]} d_s I^s_{T+t} \varphi(H^s_{T+t}) &= \int_{[0,r)} d_s I^s_T \varphi(H^s_T) + (X_T - \eta - X_{r-})\varphi(H^r_T) \\ &= \langle k_\eta \rho_T, \varphi\rangle. \end{aligned} \tag{4.9}$$

On the other hand, denote by $\tilde{I}^s_t$ and $\tilde{H}^s_t$ the analogues of I^s_t and H^s_t for the shifted process $X^{(T)}$, and set

$$v = \sup\{u \leq t,\ \tilde{I}^u_t = -\eta\}$$

(v is the time at which $X^{(T)}$ reaches its minimum on $[0,t]$). It is again easy to check that

$$I_{T+t}^{T+s} = \begin{cases} X_T - \eta, & \text{if } 0 \le s < v, \\ X_T + \tilde{I}_T^s, & \text{if } v \le s \le t, \end{cases}$$

and

$$H_{T+t}^{T+s} = \begin{cases} H_T^r, & \text{if } 0 \le s < v, \\ H_T^r + \tilde{H}_t^s, & \text{if } v \le s \le t. \end{cases}$$

It follows that

$$\int_{(T,\,T+t]} d_s I_{T+t}^s \varphi(H_{T+t}^s) = \int_{[v,\,t]} d_s \tilde{I}_t^s \varphi(H_T^r + \tilde{H}_t^s) \tag{4.10}$$
$$= \int \rho_t^{(T)}(dx)\, \varphi(H_T^r + x).$$

Note that $H_T^r = \sup(\operatorname{supp} k_\alpha \rho_T)$ by the property $\operatorname{supp} \rho_T = [0, H_T]$. Formula (4.8) then follows from (4.9) and (4.10). □

REMARK. The property (4.8) [or (3.4)] is reminiscent of the evolution mechanism of the Brownian snake in [32] and [33]: the value of the process at time $T+t$ is obtained by "erasing" the value at time T on a certain length and then "extending" it independently.

We will not push further the study of the process ρ, although all results obtained in Section 3 can be extended to the present setting. In particular, one can give a description of the invariant measure of ρ analogous to Proposition 3.3 and state duality properties similar to Proposition 3.4. These results should be derived in a more general setting in a forthcoming paper.

4.5. *Continuity of the height process.* We will now get a necessary and sufficient condition for the continuity of the process $(H_t,\ t \ge 0)$. We need two preliminary lemmas, which are of independent interest. Recall the notation $X^{(T)}$ from the previous proof.

LEMMA 4.5 (Subadditivity property). *Let T be a finite stopping time. Then a.s. for every $t \ge 0$,*

$$H_{T+t} \le H_T + H_t^{(T)},$$

where $H^{(T)}$ denotes the height process associated with $X^{(T)}$.

PROOF. This is a straightforward consequence of (4.8). □

LEMMA 4.6. *Almost surely for every $t < t'$, the process H takes all values between H_t and $H_{t'}$ on the time interval $[t, t']$.*

PROOF. Suppose first that $H_{t'} > H_t$ and set

$$u = \sup\{r \in [t, t'],\ I^r_{t'} = I^t_{t'}\}.$$

Then $X_{u-} = I^t_{t'}$, and it is clear from our definitions that $H_u \le H_t$. For every $s \in [u, t')$, set

$$u(s) = \sup\big\{r \in [s, t'],\ I^r_{t'} = I^s_{t'}\big\},$$

and note that $X_{u(s)-} = I^s_{t'}$. From the definitions again it is easy to check that $H^s_{t'} = H_{u(s)}$. However, by Proposition 4.3, $\{H^s_{t'},\ u \le s < t'\}$ contains the interval $[H^u_{t'}, H_{t'}) = [H_u, H_{t'})$. The desired result follows.

In the case $H_t > H_{t'}$, we argue differently. By the lower semicontinuity of H, it is enough to consider one fixed value of $t > 0$. Then, for every $a \in [I^t_{t'}, X_t]$, set

$$u(a) = \inf\{r \ge t,\ X_{r-} \le a\} \le t'.$$

From (4.8), we have $\rho_{u(a)} = k_{X_t - a}\rho_t$. Hence, $H_{u(a)} = \sup(\operatorname{supp}\rho_{u(a)})$ varies continuously with a. However, $H_{u(X_t)} = H_t$ and it is trivial that $H_{u(I^t_{t'})} \le H_{t'}$. This completes the proof. □

THEOREM 4.7. *The process* $(H_t,\ t \ge 0)$ *is continuous if and only if*

$$\int^{\infty} \frac{d\lambda}{\psi(\lambda)} < \infty.$$

If this condition does not hold, then the set of values taken by H *on any nontrivial open interval contains a half-line* $[a, \infty)$.

REMARK. Theorem 4.7 is stated for the lower semicontinuous version of H given by Proposition 4.3. In the case $\int^\infty \psi(\lambda)^{-1} d\lambda = \infty$, it is, however, immediate from Theorem 4.7 that any version of H must be unbounded on any nontrivial interval.

PROOF OF THEOREM 4.7. We first recall some well-known facts. Let $(Z_u, u \ge 0)$ be a CSBP(ψ) started at $x > 0$, and let $\zeta = \inf\{u \ge 0,\ Z_u = 0\}$. By the strong Markov property, $Z_u = 0$ for every $u \ge \zeta$. Note that the solution $u_t(\lambda)$ of (1.3) satisfies, for $\lambda > 0$,

$$\int_{u_t(\lambda)}^{\lambda} \frac{du}{\psi(u)} = t.$$

By letting $\lambda \to \infty$, it easily follows that $P(\zeta < \infty) = 1$ or 0 according as $\int^\infty \psi(\lambda)^{-1}\, d\lambda$ is finite or infinite. Moreover, in the case $\int^\infty \psi(\lambda)^{-1}\, d\lambda < \infty$, we have, for every $\delta > 0$,

$$P(\zeta > \delta) = P(Z_\delta > 0) = 1 - \exp -x\, u_\delta(\infty), \tag{4.11}$$

where $0 < u_\delta(\infty) < \infty$, $u_\delta(\infty) \downarrow 0$ as $\delta \uparrow \infty$.

We apply these results in combination with Theorem 4.2. First assume that

$$\int^{\infty} \psi(\lambda)^{-1}\, d\lambda = \infty.$$

By Theorem 4.2 and the previous remarks, we have, for every $x > 0$,

$$\sup_{0\le s\le \tau_x} H_s = \infty \quad \text{a.s.}$$

Since $\tau_x \downarrow 0$ as $x \downarrow 0$ we also have

$$\sup_{0\le s\le a} H_s = \infty$$

for every $a > 0$ a.s. Then, if (a, b) is a nontrivial open subinterval of $(0, \infty)$, we can apply the last observation to the shifted process $X^{(a)}$. Since it is trivial that $H_{a+t} \ge H_t^{(a)}$ for every $t \ge 0$ a.s., we get

$$\sup_{a\le s\le b} H_s \ge \sup_{0\le s\le b-a} H_s^{(a)} = \infty \quad \text{a.s.}$$

The last assertion of Theorem 4.7 is then a consequence of Lemma 4.6.

Suppose now that $\int^{\infty} \psi(\lambda)^{-1}\, d\lambda < \infty$. From Theorem 4.2, the facts recalled at the beginning of the proof and the lower semicontinuity of H, it follows that, for every $x > 0$,

$$\sup_{0\le s\le \tau_x} H_s < \infty \quad \text{a.s.}$$

and more precisely, by (4.11),

$$P\Big[\sup_{0\le s\le \tau_x} H_s > \delta\Big] = 1 - \exp -x\, u_\delta(\infty).$$

This last identity shows that

$$\lim_{x\downarrow 0} P\Big[\sup_{0\le s\le \tau_x} H_s > \delta\Big] = 0$$

and thus

$$\lim_{t\downarrow 0} H_t = 0 \quad \text{a.s.} \tag{4.12}$$

The continuity of H will follow from Lemma 4.6 if we can check that the number of upcrossings of H along any fixed interval $[a, a+h]$, $h > 0$, is finite over a finite time interval. To this end, set $\sigma_0 = 0$ and define by induction, for every $n \ge 1$,

$$\tau_n = \inf\{t \ge \sigma_{n-1},\ H_t \ge a + h\},$$
$$\sigma_n = \inf\{t \ge \tau_n,\ H_t \le a\}.$$

Note that $H_{\sigma_n} \le a$ by the lower semicontinuity of H. From Lemma 4.5, we have a.s. for every $t \ge 0$,

$$H_{\sigma_n+t} \le H_{\sigma_n} + H_t^{(\sigma_n)},$$

and it follows that $\tau_{n+1} - \sigma_n \ge \eta_n$, where $\eta_n = \inf\{t \ge 0,\ H_t^{(\sigma_n)} \ge h\}$. The strong Markov property implies that the variables η_n are independent and identically distributed. Furthermore $\eta_n > 0$ a.s. by (4.12). It follows that $\sum_{n=0}^{\infty} \eta_n = \infty$ a.s., and thus $\tau_n \uparrow \infty$ as $n \uparrow \infty$. This completes the proof. □

5. From discrete exploration processes to the height process.

5.1. *Preliminaries.* In this section, we consider a continuous-state branching process $Z = (Z_t,\ t \ge 0)$ with initial value $Z_0 = 1$ and with branching mechanism ψ of the same form as in Section 4:

$$\psi(\lambda) = \alpha\,\lambda + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r)\pi(dr),$$

where the measure π is such that $\int_0^\infty (r \wedge r^2)\pi(dr) < \infty$, $\int_0^1 r\pi(dr) = \infty$ and $\alpha \ge 0$. We also denote by X the Lévy process with Laplace exponent ψ as in Section 4.

Our goal is to show that, for a sequence of rescaled Galton–Watson branching processes that converge in distribution to Z, the corresponding exploration processes, as defined in Section 2, also converge in distribution, after a suitable rescaling, to the height process H of Section 4 associated with the Lévy process X.

We assume that we are given a sequence (μ_p) of probability measures on $\mathbb{N}$ that are critical or subcritical [$\sum k\mu_p(k) \le 1$] and two sequences β_p, λ_p of positive numbers such that $\beta_p \to \infty$, $\lambda_p/\beta_p \to \infty$, $\lambda_p/\beta_p^2 \to 0$ and the following conditions hold:

(a)
$$\lim_{p\to\infty} \frac{\lambda_p}{\beta_p}\left(\sum_{k=0}^{\infty} k\,\mu_p(k) - 1\right) = -\alpha;$$

(b) for every continuous function φ from $\mathbb{R}_+$ into $\mathbb{R}$ such that $|\varphi(r)| \le C(r \wedge r^2)$ for some constant C, we have

$$\lim_{p\to\infty} \lambda_p \sum_{k=0}^{\infty} \varphi\left(\frac{k}{\beta_p}\right)\mu_p(k) = \int_0^\infty \varphi(r)\,\pi(dr).$$

These conditions are stronger than what we really need (see [12] for similar statements under weaker hypotheses). It is, however, easy to construct examples where they hold. We may in fact start from any sequence β_p converging to ∞, then define λ_p by the relation

$$\sum_{k=1}^{\infty} \frac{k}{\beta_p}\,\pi\left(\left[\frac{k}{\beta_p}, \frac{k+1}{\beta_p}\right)\right) = \frac{\lambda_p}{\beta_p} - \alpha, \tag{5.1}$$

and μ_p by

$$\mu_p(k) = \frac{1}{\lambda_p}\,\pi\left(\left[\frac{k}{\beta_p}, \frac{k+1}{\beta_p}\right)\right) \quad \text{if } k \geq 1,$$

$$\mu_p(0) = 1 - \sum_{k=1}^{\infty} \mu_p(k).$$

The properties $\lambda_p/\beta_p \to \infty$, $\lambda_p/\beta_p^2 \to 0$ are easy from (5.1) and our assumptions on π. Condition (a) is trivial from (5.1). Furthermore, (b) follows from dominated convergence and the identity

$$\lambda_p \sum_{k=0}^{\infty} \varphi\left(\frac{k}{\beta_p}\right)\mu_p(k) = \sum_{k=0}^{\infty} \varphi\left(\frac{k}{\beta_p}\right)\pi\left(\left[\frac{k}{\beta_p}, \frac{k+1}{\beta_p}\right)\right).$$

For every p, let ν_p be the probability measure on $\{-1, 0, 1, 2, \ldots\}$ defined by $\nu_p(k) = \mu_p(k+1)$. Denote by $W^p = (W_k^p,\ k \geq 0)$ a random walk started at 0 with jump distribution ν_p, and by $Y^p = (Y_k^p,\ k \geq 0)$ a Galton–Watson branching process with offspring distribution μ_p started at $Y_0^p = [\beta_p]$ (recall that $[x]$ denotes the integral part of x).

PROPOSITION 5.1. *We have*

$$\lim_{p\to\infty}\left(\beta_p^{-1} W^p_{[\lambda_p t]},\ t \geq 0\right) = (X_t,\ t \geq 0)$$

and

$$\lim_{p\to\infty}\left(\beta_p^{-1} Y^p_{[(\lambda_p/\beta_p t)]},\ t \geq 0\right) = (Z_t,\ t \geq 0),$$

where both convergences hold in distribution in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$.

PROOF. The first convergence is a special case of well-known limit theorems for random walks or processes with independent increments (see, e.g., [25]). Let f be a truncation function, that is, a continuous function with compact support from $\mathbb{R}$ into $\mathbb{R}$ such that $f(x) = x$ for every x belonging to a neighborhood of 0. It easily follows from (a) and (b) that

$$\lim_{p\to\infty} \lambda_p \sum_{k=-1}^{\infty} f\left(\frac{k}{\beta_p}\right)\nu_p(k) = -\alpha + \int_0^{\infty} (f(r) - r)\,\pi(dr),$$

$$\lim_{p\to\infty} \lambda_p \sum_{k=-1}^{\infty} f\left(\frac{k}{\beta_p}\right)^2 \nu_p(k) = \int_0^{\infty} f(r)^2\,\pi(dr)$$

and, for every bounded continuous function φ that vanishes on a neighborhood of 0,

$$\lim_{p\to\infty} \lambda_p \sum_{k=-1}^{\infty} \varphi\left(\frac{k}{\beta_p}\right)\nu_p(k) = \int_0^{\infty} \varphi(r)\,\pi(dr).$$

The first part of the proposition then follows as a very special case of Theorem VII.3.4 in [25].

The second assertion of the proposition can be obtained from the first one and the results in Section 3 of [22] (in particular Theorems 3.1 and 3.4). Grimwall [22] takes $\lambda_p = p\beta_p$, but our statement can be deduced easily from this special case by renumbering the sequences λ_p, β_p. An alternative approach is to derive the second convergence from the first one by using the (easy) discrete form of Lamperti's embedding (see [24] and [17], page 390, for this method). □

5.2. *The basic limit theorem.* We now turn to the convergence of exploration processes. For every $p \geq 1$, we denote by $H^p = (H^p_n,\ n \geq 0)$ the exploration process of a sequence of independent Galton–Watson trees with offspring distribution μ_p. More precisely, this means in the notation of Section 2 that

$$H^p_k = H_{k-(\sigma(\tau_1)+\cdots+\sigma(\tau_{n-1}))}(\tau_n)$$
$$\text{if } \sigma(\tau_1)+\cdots+\sigma(\tau_{n-1}) \leq k < \sigma(\tau_1)+\cdots+\sigma(\tau_n),$$

where $(\tau_n,\ n \geq 1)$ is a sequence of independent trees distributed according to $\mathbb{P}_{\mu_p}$.

As in Section 4, we denote by $H = (H_t,\ t \geq 0)$ the height process associated with the Lévy process X.

PROPOSITION 5.2. *We have*

$$\lim_{p\to\infty}\left(\frac{\beta_p}{\lambda_p} H^p_{[\lambda_p t]},\ t \geq 0\right) = (H_t,\ t \geq 0),$$

in the sense of weak convergence of the finite-dimensional marginals.

PROOF. From an immediate extension of the results of Section 2, we may assume that

$$H^p_k = \mathrm{Card}\left\{ j \in \{0, 1, \ldots, k-1\},\ W^p_j = \inf_{j \leq l \leq k} W^p_l \right\},$$

where the random walks W^p are as previously. Then, for each fixed k, H^p_k has the same law as Λ^p_k, where

$$\Lambda^p_k = \mathrm{Card}\left\{ j \in \{1, \ldots, k\},\ W^p_j = \sup_{0 \leq l \leq j} W^p_l \right\}.$$

From Proposition 5.1 and Skorokhod's representation theorem, we may assume that

$$\lim_{p\to\infty}\left(\beta_p^{-1} W^p_{[\lambda_p t]},\ t \geq 0\right) = (X_t,\ t \geq 0) \tag{5.2}$$

a.s. in the sense of Skorokhod's topology on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. We will then prove that, for every fixed $t > 0$,

$$\lim_{p\to\infty} \frac{\beta_p}{\lambda_p} \Lambda^p_{[\lambda_p t]} = L_t \tag{5.3}$$

in probability, where L stands for the local time at 0 of $S - X$ as in Section 4.

To this end, we introduce the weak ladder times of the random walk W^p: $T^p_0 = 0$ and

$$T^p_1 = \inf\{n > 0,\ W^p_n \geq 0\},$$
$$T^p_{m+1} = \inf\{n > T^p_m,\ W^p_n \geq W^p_{T^p_m}\},$$

where $\inf \varnothing = \infty$. We then consider the corresponding ladder heights $W^p_{T^p_m}$. By the strong Markov property, conditionally on the event $\{T^p_m < \infty\}$, the random variable $1_{\{T^p_{m+1}<\infty\}}(W^p_{T^p_{m+1}} - W^p_{T^p_m})$ is independent of the past of W^p up to time T^p_m, and its conditional distribution is the law of $1_{\{T^p_1<\infty\}} W^p_{T^p_1}$. Using a discrete form of the arguments involved in the proof of (3.1), it is elementary to check that

$$P\big(T^p_1 < \infty,\ W^p_{T^p_1} = j\big) = \nu_p([j,\infty)), \qquad j \geq 0. \tag{5.4}$$

This identity is a special case of the Wiener–Hopf factorization for random walks (see [18], Exercise 3, Section XII.10, for a closely related result). In particular,

$$P(T^p_1 < \infty) = \sum_{j=0}^{\infty} \nu_p([j,\infty)) = \sum_{k=0}^{\infty} k\mu_p(k) \tag{5.5}$$

converges to 1 as $p \to \infty$ by assumption (a).

Let $\delta > 0$ and set

$$\gamma(\delta) = \int_\delta^\infty \pi([x,\infty))\,dx,$$
$$\gamma_p(\delta) = \frac{\sum_{j>\beta_p\delta} \nu_p([j,\infty))}{\sum_{j\geq 0} \nu_p([j,\infty))} = P\big(W^p_{T^p_1} > \beta_p\delta \mid T^p_1 < \infty\big),$$
$$L^\delta_t = \mathrm{Card}\{s \leq t,\ X_s > S_{s-} + \delta\},$$
$$l^{p,\delta}_k = \mathrm{Card}\{j < k,\ W^p_{j+1} > \overline{W}^p_j + \beta_p\delta\},$$

where $\overline{W}^p_j = \sup\{W^p_i,\ 0 \leq i \leq j\}$. Note that, with probability 1, $L^\delta_t = L^{\delta'}_t$ for all δ' in a neighborhood of δ. From the convergence (5.2), it is then easy to obtain

$$\lim_{p\to\infty} l^{p,\delta}_{[\lambda_p t]} = L^\delta_t \quad \text{a.s.} \tag{5.6}$$

On the other hand, by the same argument as for (4.1), using the form of the measure $n(dx\,dy)$ in Section 4.1, we know that

$$\lim_{\delta\to 0} \gamma(\delta)^{-1} L^\delta_t = L_t \quad \text{a.s.} \tag{5.7}$$

Finally, we can also relate $l^{p,\delta}_{[\lambda_p t]}$ to $\Lambda^p_{[\lambda_p t]}$. Trivially, $\Lambda^p_{T^p_k} = k$ on $\{T^p_k < \infty\}$ and, on the other hand, from previous remarks and formulas (5.4) and (5.5) we know that

$$\left(1_{\{T^p_k<\infty\}} l^{p,\delta}_{T^p_k}, k = 1, 2, \ldots\right) \overset{\text{(d)}}{=} \left(1_{\{k\le\zeta\}} \sum_{i=1}^{k} \varepsilon_i, k = 1, 2, \ldots\right), \tag{5.8}$$

where the variables ε_i are independent Bernoulli random variables with $P(\varepsilon_i = 1) = \gamma_p(\delta)$, and ζ is an integer-valued random variable, independent of the sequence (ε_i) and such that $P(\zeta \ge k) = P(T^p_1 < \infty)^k$ for every $k \ge 0$.

Notice that

$$\gamma_p(\delta) = \frac{\sum_{k=0}^{\infty} (k - 1 - [\beta_p \delta])^+ \mu_p(k)}{\sum_{k=0}^{\infty} k\, \mu_p(k)}.$$

By assumption (a), $\sum_{k=0}^{\infty} k\, \mu_p(k)$ converges to 1. Using assumption (b) with $\varphi(r) = (r - \delta)^+$, we easily get

$$\lim_{p\to\infty} \frac{\lambda_p}{\beta_p} \gamma_p(\delta) = \int_0^\infty (r-\delta)^+ \, \pi(dr) = \gamma(\delta). \tag{5.9}$$

For a fixed $A > 0$, set $A_p = [A\lambda_p/\beta_p] + 1$. Then

$$\sup_{j \le T^p_{A_p}} \left| \frac{\beta_p}{\lambda_p} \left(\Lambda^p_j - \gamma_p(\delta)^{-1} l^{p,\delta}_j\right)\right| = \sup_{0\le k\le A_p} 1_{\{T^p_k<\infty\}} \left| \frac{\beta_p}{\lambda_p}\left(k - \gamma_p(\delta)^{-1} l^{p,\delta}_{T^p_k}\right)\right|,$$

and, using (5.8),

$$\begin{aligned}
&E\left(\sup_{0\le k\le A_p} 1_{\{T^p_k<\infty\}} \left|\frac{\beta_p}{\lambda_p}\left(k - \gamma_p(\delta)^{-1} l^{p,\delta}_{T^p_k}\right)\right|^2\right)\\
&\quad\le E\left(\sup_{0\le k\le A_p} \left|\frac{\beta_p}{\lambda_p \gamma_p(\delta)}\left(\sum_{i=1}^{k} (\varepsilon_i - E(\varepsilon_i))\right)\right|^2\right)\\
&\quad\le 4\left(\frac{\beta_p}{\lambda_p}\right)^2 \gamma_p(\delta)^{-2} E\left(\left(\sum_{i=1}^{A_p} (\varepsilon_i - E(\varepsilon_i))\right)^2\right)\\
&\quad\le 8(A+1)\frac{\beta_p}{\lambda_p} \gamma_p(\delta)^{-1}.
\end{aligned}$$

From (5.9), we obtain

$$\limsup_{p\to\infty} E\left(\sup_{j\le T^p_{A_p}} \left|\frac{\beta_p}{\lambda_p}\left(\Lambda^p_j - \gamma_p(\delta)^{-1} l^{p,\delta}_j\right)\right|^2\right) \le \frac{8(A+1)}{\gamma(\delta)}. \tag{5.10}$$

To complete the argument, fix $\varepsilon > 0$. We first choose A large enough so that $P(L_t \ge A - 3\varepsilon) < \varepsilon$. By (5.7) and (5.10), we can then choose $\delta > 0$ and $p_0 = p_0(\delta)$ such that

$$P\left(\left|\gamma(\delta)^{-1} L^\delta_t - L_t\right| > \varepsilon\right) < \varepsilon \tag{5.11}$$

and

$$P\left(\sup_{j\le T^p_{A_p}}\left|\frac{\beta_p}{\lambda_p}\left(\Lambda^p_j-\gamma_p(\delta)^{-1}l^{p,\delta}_j\right)\right|>\varepsilon\right)<\varepsilon \quad \text{if } p\ge p_0. \tag{5.12}$$

By (5.6) and (5.9), we have also, for $p\ge p_1(\delta)$,

$$P\left(\left|\frac{\beta_p}{\lambda_p\gamma_p(\delta)}\,l^{p,\delta}_{[\lambda_p t]}-\gamma(\delta)^{-1}L^\delta_t\right|>\varepsilon\right)<\varepsilon. \tag{5.13}$$

Combining the previous estimates gives, for $p\ge p_0\vee p_1$,

$$P\left(\left|\frac{\beta_p}{\lambda_p}\,\Lambda^p_{[\lambda_p t]}-L_t\right|>3\varepsilon\right)\le 3\varepsilon+P\big([\lambda_p t]>T^p_{A_p}\big).$$

Moreover, by using (5.12) and then (5.11) and (5.13) again, we have for p sufficiently large

$$\begin{aligned}P\big(T^p_{A_p}<[\lambda_p t]\big)&\le P\Big(T^p_{A_p}<\infty,\ l^{p,\delta}_{[\lambda_p t]}\ge l^{p,\delta}_{T^p_{A_p}}\Big)\\&\le \varepsilon+P\left(\frac{\beta_p}{\lambda_p\gamma_p(\delta)}\,l^{p,\delta}_{[\lambda_p t]}\ge A-\varepsilon\right)\\&\le 3\varepsilon+P(L_t\ge A-3\varepsilon)\\&\le 4\varepsilon,\end{aligned}$$

by the choice of A. This completes the proof of (5.3).

We may then replace W^p by the time-reversed random walk $\widehat{W}^p_k=W^p_{[\lambda_p t]}-W^p_{[\lambda_p t]-k}$, for $0\le k\le[\lambda_p t]$. Under (5.2), we have also

$$\lim_{p\to\infty}\big(\beta_p^{-1}\widehat{W}^p_{[\lambda_p s]},\ 0\le s\le t\big)=(\widehat{X}^{(t)}_s,\ 0\le s\le t),$$

a.s. in $\mathbb{D}([0,t],\mathbb{R})$. Since $H^p_{[\lambda_p t]}=\widehat{\Lambda}^p_{[\lambda_p t]}$, with obvious notation, we get from (5.3) that

$$\lim_{p\to\infty}\frac{\beta_p}{\lambda_p}\,H^p_{[\lambda_p t]}=\widehat{L}^{(t)}_t=H_t,$$

in probability. This completes the proof of Proposition 5.2. □

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REFERENCES

[1] Adhikari, A. (1986). Skip free processes. Ph.D. dissertation, Univ. California, Berkeley.
[2] Aldous, D. J. (1993). The continuum random tree III. *Ann. Probab.* 21 248–289.
[3] Athreya, K. B. and Ney, P. (1973). *Branching Processes.* Springer, Berlin.
[4] Bennies, J. and Kersting, G. (1995). A random walk approach to Galton–Watson trees. Preprint.

[5] BERTOIN, J. (1992). An extension of Pitman's theorem for spectrally positive Lévy processes. *Ann. Probab.* 20 1464–1483.
[6] BERTOIN, J. (1996). *Lévy Processes.* Cambridge Univ. Press.
[7] BINGHAM, N. (1975). Fluctuation theory in continuous time. *Adv. in Appl. Probab.* 7 705–766.
[8] BINGHAM, N. (1976). Continuous branching processes and spectral positivity. *Stochastic Process. Appl.* 4 217–242.
[9] BOROVKOV, A. A. (1976). *Stochastic Processes in Queueing Theory.* Springer, Berlin.
[10] BOROVKOV, K. A. and VATUTIN, V. A. (1996). On distribution tails and expectations of maxima in critical branching processes. *J. Appl. Probab.* 33 614–622.
[11] DAWSON, D. A. and PERKINS, E. A. (1991). Historical processes. *Mem. Amer. Math. Soc.* 454 1–179.
[12] DUQUESNE, T. (1997). Théorèmes limites pour le processus d'exploration d'arbres de Galton–Watson. Unpublished manuscript.
[13] DURRETT, R. T. (1984). Oriented percolation in two dimensions. *Ann. Probab.* 12 999–1040.
[14] DWASS, M. (1969). The total progeny in a branching process. *J. Appl. Probab.* 6 682–686.
[15] DYNKIN, E. B. (1993). Superprocesses and partial differential equations. *Ann. Probab.* 21 1185–1262.
[16] DYNKIN, E. B. (1994). *An Introduction to Branching Measure-Valued Processes.* Amer. Math. Soc., Providence, RI.
[17] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence.* Wiley, New York.
[18] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* 2. Wiley, New York.
[19] GEIGER, J. (1995). Contour processes of random trees. In *Stochastic Partial Differential Equations* (A. Etheridge, ed.) 72–96. Cambridge Univ. Press.
[20] GEIGER, J. (1996). Size-biased and conditioned random splitting trees. *Stochastic Process. Appl.* 65 187–207.
[21] GREEN, P. J. (1996). The maximum and time to absorption of a left-continuous random walk. *J. Appl. Probab.* 13 444–454.
[22] GRIMVALL, A. (1974). On the convergence of a sequence of branching processes. *Ann. Probab.* 2 1027–1045.
[23] HARRIS, T. E. (1974). First passage and recurrence distributions. *Trans. Amer. Math. Soc.* 73 471–486.
[24] HELLAND, I. S. (1978). Continuity of a class of random time transformations. *Stochastic Process. Appl.* 7 79–99.
[25] JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes.* Springer, Berlin.
[26] KENDALL, D. G. (1951). Some problems in the theory of queues. *J. Roy. Statist. Soc. Ser. B* 13 151–185.
[27] KNIGHT, F. B. (1963). Random walks and a sojourn density of Brownian motion. *Trans. Amer. Math. Soc.* 109 56–86.
[28] LAMPERTI, J. (1967). The limit of a sequence of branching processes. *Z. Wahrsch. Verw. Gebiete* 7 271–288.
[29] LAMPERTI, J. (1967). Continuous-state branching processes. *Bull. Amer. Math. Soc.* 73 382–386.
[30] LE GALL, J. F. (1991). Brownian excursions, trees and measure-valued branching processes. *Ann. Probab.* 19 1399–1439.
[31] LE GALL, J. F. (1993). The uniform random tree in a Brownian excursion. *Probab. Theory Related Fields* 96 369–383.
[32] LE GALL, J. F. (1993). A class of path-valued Markov processes and its applications to superprocesses. *Probab. Theory Related Fields* 95 25–46.
[33] LE GALL, J. F. (1995). The Brownian snake and solutions of $\Delta u = u^2$ in a domain. *Probab. Theory Related Fields* 102 393–432.
[34] LE GALL, J. F. and LE JAN, Y. (1995). Arbres aléatoires et processus de Lévy. *C.R. Acad. Sci. Paris Sér. I Math.* 321 1241–1244.

[35] LE GALL, J. F. and LE JAN, Y. (1997). Branching processes in Lévy processes: Laplace functionals of snakes and superprocesses. Preprint.

[36] LINDVALL, T. (1976). On the maximum of a branching process. *Scand. J. Statist.* 3 209–214.

[37] NEVEU, J. (1986). Arbres et processus de Galton–Watson. *Ann. Inst. H. Poincaré Sér. B* 22 199–207.

[38] NEVEU, J. and PITMAN, J. (1989). The branching process in a Brownian excursion. *Séminaire de Probabilités XXIII. Lecture Notes in Math.* 1372 248–257. Springer, Berlin.

[39] OTTER, R. (1949). The multiplicative process. *Ann. Math. Statist.* 20 206–224.

[40] PITMAN, J. (1997). Enumerations of trees and forests related to branching processes and random walks. Preprint.

[41] ROGERS, L. C. G. (1984). A new identity for real Lévy processes. *Ann. Inst. H. Poincaré* (*Probab. Statist.*) 20 21–34.

[42] SPITZER, F. (1964). *Principles of Random Walk*. Van Nostrand, Princeton.

[43] TAKÁCS, L. (1993). Limit distributions for queues and random rooted trees. *J. Appl. Math. Stochastic Anal.* 6 189–216.

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