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Branching Processes in Markovian Environments

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Summary. A positive recurrent Markov transition matrix $\underset{\sim}{P}$ generates a Markovian sequence $\{V_n\}$ of states called *omens*. A branching process $\{Z_n\}$ develops in the usual way except that families born to the n-th generation are all (independently) governed by a pgf $\phi_{\zeta_n}(s)$, $0 \le s \le 1$, where $\{\zeta_n\}$ is the sequence of *environmental variables*. The distribution of ζ_n depends on the *omen* V_n but the environment variables are otherwise independent. A necessary and sufficient condition for *immortality* of the process, that is $\lim_{n\to\infty} P\{Z_n > 0\} > 0$, is given. This assumes a particularly simple form when $\underset{\sim}{P}$ is finite, involving $\underset{\sim}{P}$ only through its stationary distribution $\{\omega_j\}$, say. This simple form of the condition is also shown to be *sufficient* for immortality, even when $\underset{\sim}{P}$ is infinite. However, an example is given of two related processes, which demonstrates that when $\underset{\sim}{P}$ is infinite there cannot be a necessary and sufficient condition for immortality involving no more specific information about $\underset{\sim}{P}$ than is contained in the stationary distribution $\{\omega_j\}$.

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1. Introduction.

In this paper, we shall generalize the theory given in Smith and Wilkinson (1969) for the Branching Process with Random Environments (BPRE) to cover situations where a certain kind of Markov dependence holds between successive environmental variables. We shall call the new process a *Branching Process with Markovian Environments* (BPME).

We imagine first an irreducible Markov transition matrix $\underset{\sim}{P} = ||p_{ij}||$ $(i,j = 1,2,3,...)$ of a positive recurrent chain $\{V_n\}$, which we shall call the *guiding chain*; the states of this chain shall be called the *omens*. We shall assume V_0 to be given and non-random.

For each $i = 1,2,...$, let $\{y(i,j)\}$ $(j = 1,2,..)$ be a sequence of iid random variables taking values in some subset Θ of a finite dimensional Euclidean space†, and call the points θ in Θ the *environments*. Let us further assume that the random variables comprising the double sequence $\{y(i,j)\}$ $(i,j = 1,2,...)$ are mutually independent.

To simplify notation as much as possible we shall write $y(i)$ for a random variable with the same d.f. as any one of the $\{y(i,j)\}$ $(j = 1,2,...)$, whenever no confusion can thereby arise. Thus $y(i)$ represents a "typical" member of $\{y(i,j)\}$.

We now define a sequence of environmental variables $\{\zeta_n\}$ by the equation

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† Since the set of all pgf's can be put into (1,1) correspondence with a subset of the interval [0,1], the reader will soon see that Θ could be taken as uni-dimensional; we adopt, however, what seems a more convenient assumption.

(1.1) $$\zeta_n = y(V_n, n+1), \quad n = 0,1,2,\ldots .$$

A very special case of the above process which merits particular mention is the *pure* case. This arises when the omen V_n uniquely determines the environment. In other words, to each omen i $(i = 1,2,..)$ corresponds an environment θ_i in Θ and $P\{y(i,j) = \theta_i\} = 1$ for all $j = 1,2,\ldots$ If the BPME is not pure we shall, when the emphasis is desirable, call it *mixed*.

Corresponding to each environment $\theta \in \Theta$ is to be a pgf $\phi_\theta(s)$, $0 \le s \le 1$, of a non-negative integer-valued random variable. We can then define the expected value of this random variable by

$$\xi(\theta) = \lim_{s \uparrow 1} \frac{1 - \phi_\theta(s)}{1 - s}$$

and we shall assume that $\xi(\theta) < \infty$ for all $\theta \in \Theta$ †.

We shall assume that $\phi_\theta(s)$, for each fixed s, is a Borel measurable function of θ. Some consequences of this assumption are: (i) for fixed s, $\{\phi_{y(i,j)}(s)\}$ is a double sequence of mutually independent random variables; (ii) for fixed s and i, $\{\phi_{y(i,j)}(s)\}$ is a sequence of iid variables; (iii) $\{\xi(y(i,j))\}$, $i = 1,2,\ldots,$ $j = 1,2,\ldots$ are mutually independent random variables and any subsequence obtained by holding i fast will be a sequence of iid random variables.

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† This for convenience; by suitable incantations about events of probability zero it could be "generalized" to an a.s. condition.

We imagine the BPME to develop as follows. Initially there are Z_0 particles and the omen for the environment is V_0. Every particle in this zero-th generation then experiences an environment $\zeta_0 = y(V_0,1)$, and the sizes of their families, though independent, are to be governed by the same pgf $\phi_{\zeta_0}(s)$. In this way, Z_1 new particles, forming generation one, are created; the families of these Z_1 particles will then, in their turn, be produced under the influence of omen V_1. The process then develops in an obvious way.

If $P\{Z_n > 0\} \to 0$, we say the process $\{Z_n\}$ is *mortal*; otherwise it is *immortal*. We shall see that mortality cannot be affected by choice of the initial omen or by the initial size Z_0 (so long as it be finite).

2. Necessary and Sufficient conditions for Mortality of the BPME.

Let us select an arbitrary omen i and let us suppose $V_0 = i$. By the term *i-cycle* we mean any *finite* sequence of omens $\{v_1, v_2, \ldots, v_\ell\}$ for which $v_1 = i$ and $v_k \neq i$ for every k, $2 \leq k \leq \ell$. We call ℓ the *length* of the i-cycle.

Holding the omen i fixed, let $\{\pi_r^{(i)}\}$, $r = 1,2,\ldots,$ be any enumeration of the i-cycles. Suppose that, in fact,

$$\pi_r^{(i)} = \{i, v_2, v_3, \ldots, v_\ell\},$$

and define

$$p_r^{(i)} = p_{iv_2} p_{v_2 v_3} \cdots p_{v_\ell i}.$$

Since the guiding chain is positive recurrent, the omen i will, almost surely, recur infinitely often. It is clear that $p_r^{(i)}$ is the probability that, given that the guiding chain is initially in omen i, it will move through the omens of the i-cycle $\pi_r^{(i)}$ in returning to omen i. Evidently $\sum_{r=1}^{\infty} p_r^{(i)} = 1$.

We shall study the $\{Z_n\}$ process with the aid of these i-cycles. When the guiding chain enters omen i we shall say an epoch $\mathcal{Q}_i$ occurs. Let $\mathcal{Q}_i$ occur at generations $n_0 = 0, n_1, n_2, \ldots,$ etc., and call its initial occurrence the zero-th. Then between the k-th and (k+1)-th occurrence of $\mathcal{Q}_i$ will be some i-cycle, c_k say, the length of which is $\ell_k = n_{(k+1)} - n_k$. Let the environments encountered in this i-cycle be

$$(\zeta_{n_k}, \zeta_{n_k+1}, \ldots, \zeta_{n_k+\ell_k-1})$$

and denote this vector by $\underset{\sim}{\zeta}_k$. Then we define

$$\phi^{(i)}_{\underset{\sim}{\zeta}_k}(s) = \phi_{\zeta_{n_k}}(\phi_{\zeta_{n_k+1}}(\ldots(\phi_{\zeta_{n_k+\ell_k-1}}(s)\ldots))).$$

Routine, but tedious, measure-theoretic discussion will establish that $\{\phi^{(i)}_{\underset{\sim}{\zeta}_k}(s)\}$, $k = 0,1,2,\ldots,$ is a sequence of independent and identically distributed random variables for each fixed s, $0 \le s \le 1$.

If the guiding chain had only one state, the environmental variables $\{\underset{\sim}{\zeta}_n\}$ would merely be a sequence of iid random variables and the BPME would become the BPRE discussed in Smith and Wilkinson (1969). The latter paper coupled with Smith (1968) provide the following Theorem A. Let a BPRE be based on iid environmental variables $\{\zeta_n\}$, taking values in some abstract space Θ, with associated pgf's $\{\phi_{\zeta_n}(s)\}$ assumed to be random variables for each fixed s, $0 \le s \le 1$. Let $\phi_{\zeta_n}(s) = \sum_{r=0}^{\infty} p_r(n)s^n$ be the Taylor expansion of $\phi_{\zeta_n}(s)$ and set $\xi_n = \sum_{r=0}^{\infty} rp_r(\zeta_n)$. To avoid triviality, assume the following conditions

A(i) $P\{p_0(\zeta_n) < 1\} = 1$

A(ii) $P\{p_0(\zeta_n) + p_1(\zeta_n) < 1\} > 0.$

We can now state:

THEOREM A. *Suppose that $P\{\xi_n < \infty\} = 1$ and that $E|\log\xi_n| < \infty$. Then, subject to A(i), (ii) above, it is necessary and sufficient for immortality of the BPRE that both the following conditions hold:*

C(i) $\quad E\,\ell og\ \xi_n > 0$

C(ii) $\quad E\,\ell og\ [1 - \phi_{\zeta_n}(0)] > -\infty.$

Let us now return to our BPME $\{Z_n\}$. It should be clear that, if we observe this BPME only at successive epochs $\mathcal{Q}_i$, then we are effectively observing a BPRE; i.e., $\{Z_{n_k}\}$, $k = 0,1,2,\ldots,$ is a BPRE, and Theorem A may be applied. The sequence of environmental variables for this BPRE is $\{\underset{\sim}{\zeta}_k\}$ and the corresponding sequence of pgf's is $\{\Phi^{(i)}_{\underset{\sim}{\zeta}_k}(s)\}$, from which the other relevant quantities must be derived. For ease of writing, we shall let $\Phi^{(i)}(s)$ stand for $\Phi^{(i)}_{\underset{\sim}{\zeta}_0}(s)$ and suppose $\Phi^{(i)}(s) = \sum_{r=0}^{\infty} p_r s^r$. Necessarily, each p_r, $r = 0,1,2,\ldots,$ is a random variable. It also transprires, after a fairly obvious argument, that if $\xi = \sum_{r=1}^{\infty} rp_r$, then necessarily $P\{\xi < \infty\} = 1$.

We are thus in a position to apply Theorem A to $\{Z_{n_k}\}$ if we can find a suitable parallel to the triviality-avoiding conditions A(i), (ii). To this end, it is helpful at this point to introduce the stationary distribution $\{\omega_v\}$, say, $v = 1,2,\ldots,$ over the omens of the positive recurrent guiding chain. It is well known that this distribution has the property that $\omega_v > 0$ for all v. Suppose $A(V,y(V))$ is any event defined by a random omen V and the corresponding random variable $y(V)$. Then we shall write

$$\mathbf{P}\{A(V,y(V))\} \equiv \sum_{v=1}^{\infty} \omega_v P\{A(v,y(v))\}.$$

Thus, if V_0 were assigned the stationary distribution $\{\omega_v\}$ instead of being fixed at i, then $\mathbf{P}\{A(V,y(V))\}$ would be the stationary probability of the event $A(V_n,y(V_n))$. Similarly, if $f(V,y(V))$ is any Borel function of the omen V and $y(V)$, then we write

$$\mathcal{E}f(V,y(V)) \equiv \sum_{v=0}^{\infty} \omega_v Ef(v,y(v))$$

for the stationary expectation of $f(V_n,y(V_n))$. We can now introduce the following conditions on the BPME:

B(i) $\mathcal{P}\{p_0(y(V)) < 1\} = 1$

B(ii) $\mathcal{P}\{p_0(y(V)) + p_1(y(V)) < 1\} > 0.$

Because $\omega_v > 0$ for all v, it will be seen that the above conditions prevent the same trivial situations for the BPME as did A(i), (ii) for the BPRE. In particular, if the BPME satisfies B(i), (ii), it is easy to see that the BPRE $\{Z_{n_k}\}$ satisfies A(i), (ii). We now prove:

THEOREM 2.1. <u>Suppose that B(i), (ii) hold and that</u> $\mathcal{E}|\log\xi(y(V))| < \infty$. <u>Then it is necessary and sufficient for immortality of the BPME that the following conditions both hold</u>

D(i) $\mathcal{E} \log \xi(y(V)) > 0$

D(ii) For some initial omen i,

$E \log [1 - \Phi^{(i)}(0)] > -\infty,$

<u>where</u> $\Phi^{(i)}(s)$ <u>is defined with reference to i-cycles as explained above</u>.

PROOF. It is apparent that the BPME $\{Z_n\}$ is immortal if and only if the BPRE $\{Z_{n_k}\}$ is immortal. Thus we can deduce from Theorem A necessary and sufficient conditions for immortality of the BPME. Condition C(ii) translates directly into D(ii). To see that the other conditions translate as claimed, we prove:

LEMMA 2.1. If we define the random variable

$$\Xi = \lim_{s \uparrow 1} \frac{1-\phi^{(i)}(s)}{1-s} .$$

and if we assume that $\mathcal{E}|\ell og\,\xi(y(V))| < \infty$ then: (a) $E|\ell og\,\Xi| < \infty$ and: (b) $E\,\ell og\,\Xi = 1/\omega_i\, \mathcal{E}\,\ell og\,\xi(y(V))$.

PROOF OF LEMMA. Let us write

$$p_{ij}(n) = P\{V_n=j|V_0=i\}.$$

Then, since the guiding chain is positive recurrent (though not necessarily aperiodic) there exists

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} p_{ij}(r) = \omega_j,$$

where the $\{\omega_j\}$ constitute the stationary distribution over the omens. Of course, when the chain *is* aperiodic we can replace these Cesàro limits by ordinary ones. It is known that $\omega_j = 1/L_j$, where L_j is the mean recurrence time of the epoch $\mathcal{Q}_j$.

Let $n_j^{(i)}(r)$ be the number of occurrences of omen j in the i-cycle $\pi_r^{(i)}$. Then it can be shown that

$$\text{(2.1)} \qquad \frac{\omega_j}{\omega_i} = \sum_{r=1}^{\infty} p_r^{(i)} n_j^{(i)}(r) ,$$

a relation we shall need presently.

Suppose the initial i-cycle is $c_0 \equiv (i,v_1,v_2,\ldots,v_{\ell-1})$, say, and suppose that $\underset{\sim}{\zeta}_0 = (\zeta_0,\zeta_1,\ldots,\zeta_{\ell-1})$. Then, conditional upon c_0 the

ζ_j's, $0 \le j \le \ell-1$, are independent random variables and ζ_j has a distribution depending only on the omen v_j (with $v_0 = i$). Let $\xi_j = \lim_{s\to 1}[1-\phi_{\zeta_j}(s)]/[1-s]$. Then, given $\underset{\sim}{\zeta}_0$, by a familiar property of iterated pgf's we have that $\Xi = \xi_1\xi_2\cdots\xi_{\ell-1}$ and that

$$\ell og\ \Xi = \sum_{j=1}^{\ell-1} \ell og\ \xi_j ,$$

$$|\ell og \Xi| \le \sum_{j=1}^{\ell-1} |\ell og \xi_j| .$$

Thus, if $c_0 = \pi_r^{(i)}$ we see that

$$\text{(2.2)} \qquad E\{\ell og \Xi | c_0 = \pi_r^{(i)}\} = \sum_{j=1}^{\infty} n_j^{(i)}(r) E \ell og \xi(y(j))$$

$$\text{(2.3)} \qquad E\{|\ell og \Xi| \,|\, c_0 = \pi_r^{(i)}\} \le \sum_{j=1}^{\infty} n_j^{(i)}(r) E|\ell og \xi(y(j))| .$$

From (2.3) we infer that

$$E|\ell og \Xi| \le \sum_{j=1}^{\infty} \sum_{r=1}^{\infty} p_r^{(i)} n_j^{(i)}(r)\ E|\ell og \xi(y(j))|$$

$$= \frac{1}{\omega_i} \mathcal{E}|\ell og \xi(y(V))| .$$

Since $\omega_i > 0$, it is clear that (a) is proved. To prove (b), we merely repeat on (2.2) the argument we have used on (2.3) and appeal to the absolute convergence we have now proved to justify the reversal of the double summation.

We now make some observations on Theorem 2.1.

Observation 1. The theorem makes no reference to Z_0, the initial number of particles, which is merely supposed to be a (finite) strictly positive integer. This is reasonable because, from the theory of the BPRE, we know that the BPRE is immortal if and only if it is so for the special case $Z_0 = 1$. Thus, since the BPME is immortal if and only if the BPRE $\{Z_{n_k}\}$ is immortal, then the BPME is immortal if and only if it is so when $Z_0 = 1$.

Observation 2. Suppose v is any omen distinct from the initial omen i. Since the guiding chain is irreducible positive recurrent, the omen v will almost surely arise after a finite number of generations. Thus, bearing Observation 1 in mind, if the BPME is mortal when it starts in v (with any initial number of particles) it must be mortal when its initial omen is i. We, therefore, have the following:

COROLLARY 2.1.1. If D(ii) holds for one initial omen i it holds for any initial omen.

Observation 3. The assumption $\mathcal{E}|\ell og\,\xi(y(V))| < \infty$ is needed to justify a number of arguments in the theory of the BPRE. Suppose however that

$$\mathcal{E}\,\ell og^{+}\xi(y(V)) = +\infty$$

and

$$\mathcal{E}|\ell og^{-}\xi(y(V))| < \infty$$

while D(ii) is satisfied. Select a large integer Δ and construct a new "truncated" BPME from the given one as follows. If

$$\phi_\zeta(s) = \sum_{n=0}^{\infty} p_n(\zeta)s^n$$

is a pgf of the given BPME then replace it by the pgf

$$\psi_\zeta(s) = \sum_{n=0}^{\Delta-1} p_n(\zeta)s^n + \left(\sum_{n=\Delta}^{\infty} p_n(\zeta)\right)s^\Delta .$$

In other words, whenever a family size exceeds Δ then imagine it reduced *by fiat* to Δ. If $\xi'(y(V))$ denotes a typical mean family size in the "truncated" process then we can choose Δ finite, but so large that

$$\mathcal{E}|\ell og\xi'(y(V))| < \infty$$

and

$$\mathcal{E}\ \ell og\ \xi'(y(V)) > 0 .$$

Clearly condition D(ii) is unaffected, so the "truncated" process will be immortal. Plainly, whatever the initial omen and Z_0, the probability of ultimate extinction of the "truncated" process will not be less than that of the original process. Thus the original process is immortal, and we have:

COROLLARY 2.1.2. Condition D(i) can be replaced by

$$\text{D(iii)} \quad \mathcal{E}|\ell og^-\xi(V)| < \mathcal{E}\ \ell og^+\xi(V) \leq +\infty.$$

Theorem 2.1 and the two Corollaries 2.1.1 and 2.1.2 are the most we have been able to achieve in the way of necessary and sufficient conditions for immortality of the BPME. The obvious drawback to any application of these results is the checking of D(ii); presumably such a check would be difficult. Until the counterexample described in the next section was obtained, we devoted some considerable effort to a search for a substitute for D(ii), one which would be easier to verify. We are now inclined to believe that such a more amenable condition does not exist. However, if we ask for reasonably verifiable *sufficient* conditions for immortality, the picture is brighter and we have the following.

THEOREM 2.2. Suppose the BPME satisfies B(i) and (ii) and that $\mathcal{E}|\log\xi(y(V))| < \infty$. Then

a) If $\mathcal{E}\log\xi(y(V)) \le 0$ the process is *mortal*, i.e., $P\{Z_n > 0\} \to 0$ as $n \to \infty$;

b) If the following two conditions hold:

$$\mathcal{E}\log\xi(y(V)) > 0, \tag{2.4}$$

$$\mathcal{E}\log[1-\phi_{y(V)}(0)] > -\infty, \tag{2.5}$$

then $P\{Z_n > 0\}$ tends to some strictly positive limit as $n \to \infty$, which will depend on Z_0 and V_0, i.e., the BPME is immortal.

Note. As explained in Observation 3, we can again deal with the situation where $\mathcal{E}\log^+\xi(y(V)) = +\infty$, provided $\mathcal{E}|\log^-\xi(y(V))| < \infty$.

Proof. Part (a) of the theorem follows at once from Theorem 2.1. To prove part (b) we have to show that (2.5) implies D(ii). To this end, for each $\theta \in \Theta$ define

$$\psi_\theta(s) = \phi_\theta(0) + [1-\phi_\theta(0)]s .$$

Evidently,

(2.6) $$\phi_\theta(s) \le \psi_\theta(s), \quad 0 \le s \le 1, \quad \theta \in \Theta.$$

Furthermore,

$$\psi_{\theta_1}(\psi_{\theta_2}(0)) = \phi_{\theta_1}(0) + [1-\phi_{\theta_1}(0)]\phi_{\theta_2}(0)$$

$$= 1 - [1-\phi_{\theta_1}(0)][1-\phi_{\theta_2}(0)]$$

and, by an induction argument,

(2.7) $$\psi_{\theta_1}(\psi_{\theta_2}(\ldots(\psi_{\theta_k}(0))\ldots) = 1 - \prod_{j=1}^{k} [1-\phi_{\theta_j}(0)] ,$$

for $k = 1,2,\ldots$

Construct a pgf $\Psi^{(i)}_{\underset{\sim}{\zeta}_0}(s)$ from the pgf's $\{\psi_\theta(s)\}$ in the same way as $\Phi^{(i)}_{\underset{\sim}{\zeta}_0}(s)$ is constructed from the pgf's $\{\phi_\theta(s)\}$ and, for ease, let $\Psi(s)$ stand for $\Psi^{(i)}_{\underset{\sim}{\zeta}_0}(s)$ as $\Phi^{(i)}(s)$ stands for $\Phi_{\underset{\sim}{\zeta}_0}(s)$. From (2.6) it follows that

$$\Phi^{(i)}(0) \le \Psi(0),$$

so that

$$|\log(1-\Phi^{(i)}(0))| \le |\log(1-\Psi(0))|.$$

Hence, if c_0 again denote the initial i-cycle,

$$E|\log(1-\Phi^{(i)}(0))|$$

$$\le \sum_{r=1}^{\infty} p_r^{(i)} \; E\{|\log(1-\Psi(0))| \,|\, c_0 = \pi_r^{(i)}\}$$

$$= \sum_{r=1}^{\infty} p_r^{(i)} \, E\{ | \sum_{j=0}^{\ell_r^{(i)}-1} \ell og[1-\phi_{\zeta_j}(0)] | c_0 = \pi_r^{(i)} \},$$

by (2.7), when we write $\ell_r^{(i)}$ for the length of the i-cycle $\pi_r^{(i)}$. But the last expression must equal

$$\sum_{r=1}^{\infty} p_r^{(i)} \sum_{j=1}^{\infty} n_j^{(i)}(r) \, E|\ell og[1-\phi_{y(j)}(0)]| = \frac{1}{\omega_i} \mathbf{\mathcal{E}} |\ell og[1-\phi_{y(v)}(0)]|,$$

by (2.1). Thus (2.5) implies D(ii) and the theorem is proved.

Observation 4. It should be remarked that D(ii) does not imply (2.5) and, in general, *(2.5) is not necessary*. The counterexample of §3 underscores this remark. However, when the guiding chain has a finite number of omens we can show (2.5) *is* necessary, and we obtain the following satisfactory corollary.

COROLLARY 2.2.1. If the BPME satisfies B(i), and (ii) and if there are only finitely-many omens then (2.4) and (2.5) are necessary and sufficient for immortality of the BPME.

PROOF. We have merely to show that, when there are finitely many omens, D(ii) implies (2.5). But it is easy to see that

$$\Phi^{(i)}(0) \ge \phi_{y(i,1)}(0)$$

and hence that

$$E \, \ell og[1-\phi_{y(i)}(0)] \ge E \, \ell og[1-\Phi^{(i)}(0)]$$

$$> -\infty.$$

Corollary 2.1.1 shows that if D(ii) holds for one omen i it holds for all values of i. Thus

$$E\ \log[1-\phi_{y(i)}(0)] > -\infty$$

for all i, and, since there are finitely many such, (2.5) must hold. Thus the Corollary is proved.

Let A_n be the event $\{Z_n > 0\}$. We close this section with the proof of a theorem which shows that when the BPME is immortal A_n and V_n are asymptotically independent.

THEOREM 2.3. _If the guiding chain is aperiodic and if_ $P\{Z_n > 0\} \to c > 0$ _as_ $n \to \infty$, _then, for every_ v,

$$P\{Z_n > 0,\ V_n = v\} \to c\omega_v,$$

where $\omega_v = \lim_{n\to\infty} P\{V_n = v\}$.

PROOF. Let the improper random variable N be the least integer k such that $Z_k = 0$. Then, if Δ is fixed and large, and $n > \Delta$,

$$\text{(2.8)}\qquad P\{Z_n=0,\ V_n=v\} = \sum_{j=1}^{n} P\{N=j,\ V_n=v\}$$

$$\geq \sum_{j=1}^{\Delta} P\{N_j,\ V_n=v\}$$

$$\geq \sum_{j=1}^{\Delta} \sum_{r=1}^{\Delta'} P\{N=j,\ V_j=r,\ V_n=v\},$$

where Δ' is a second large number. But

$$P\{N=j,\ V_j=r,\ V_n=v\} = P\{N=j,\ V_j=r\}P\{V_n=v \mid V_j=r\}$$

and, by the familiar properties of an ergodic chain, this means that, as $n \to \infty$,

$$P\{N=j,\ V_j=r,\ V_n=v\} \ \to\ \omega_v P\{N=j,\ V_j=r\}.$$

It therefore follows from (2.8) that

$$\liminf_{n \to \infty} P\{Z_n=0,\ V_n=v\} \ \geq\ \omega_v \sum_{j=1}^{\Delta} \sum_{r=1}^{\Delta'} P\{N=j,\ V_j=r\}\ .$$

If we let $\Delta' \to \infty$ on the right, and then $\Delta \to \infty$, we obtain

$$\begin{aligned}\liminf_{n \to \infty} P\{Z_n=0,\ V_n=v\} &\geq \omega_v\ P\{N < \infty\} \\ &= \omega_v(1 - c).\end{aligned}$$

Hence

$$\limsup_{n \to \infty} P\{Z_n=0,\ V_n=v\} \ \leq\ \omega_v c. \tag{2.9}$$

However, since $P\{Z_n > 0\} \downarrow$, from the equation

$$P\{Z_n>0\} \ =\ \sum_{k=1}^{\infty} P\{Z_n>0,\ V_n=k\}$$

we may infer that for any large fixed Δ,

$$c \ \leq\ \sum_{k=1}^{\Delta} P\{Z_n>0,\ V_n=k\} + P\{V_n>\Delta\}.$$

Thus, from (2.9), which is true for all v, for any fixed $\ell \leq \Delta$:

$$c \ \leq\ \sum_{\substack{k=1 \\ k\neq\ell}}^{\Delta} c\omega_k + \liminf_{n \to \infty} P\{Z_n>0,\ V_n=\ell\} + \sum_{k=\Delta+1}^{\infty} \omega_k,$$

and this implies, since $\sum_{k=1}^{\infty} \omega_k = 1$, that

$$\liminf_{n \to \infty} P\{Z_n>0,\ V_n=\ell\} \;\geq\; c\omega_\ell + (1-c) \sum_{k=\Delta+1}^{\infty} \omega_k \,.$$

Now let $\Delta \to \infty$ and we infer that, for any omen ℓ,

$$\liminf_{n \to \infty} P\{Z_n>0,\ V_n=\ell\} \;\geq\; c\omega_\ell . \tag{2.10}$$

The theorem follows from (2.9) and (2.10).

A similar theorem will hold in the periodic case; its proof will differ little from the above.

3. Some important counterexamples.

The conditions for immortality of the BPME in Theorem 2.1 do not involve the transition probabilities (p_{ij}), but only the numbers $\{\omega_j\}$, which, when the guiding chain is aperiodic, constitute the unique stationary distribution of the guiding chain. Bearing this observation in mind, and comparing Theorem 2.1 with Theorem A, one is tempted to conjecture that there are necessary and sufficient conditions for immortality of the BPME which involve the transition matrix $||p_{ij}||$ only through the numbers $\{\omega_j\}$. In this section, we shall show that such a conjecture is false. We shall construct two BPME's which are identical in all respects except for their guiding chains. In each case, the guiding chain is aperiodic and, what is important, *they have identical stationary distributions* $\{\omega_j\}$. However, one of these BPME's is mortal, the other immortal.

Therefore, we are forced to conclude that any necessary and sufficient condition for immortality of the BPME must involve more specific information about the matrix $||p_{ij}||$ than is contained in the stationary distribution $\{\omega_j\}$ alone.

We find the existence of these two BPME's somewhat surprising. The long-term averages of occurrence of the various pgf's of family size are the same in each process, but in some strange way it is the pattern of these occurrences that is crucial.

For convenient reference, we shall refer to these two processes as the *ladder process* and the *mock process*, the choice of names having been suggested by their particular constructions.

Ladder Process. Let $\{V_m\}$ be a Markov chain on the state space consisting of the positive integers and with a transition probability matrix of the form

$$\underset{\sim}{P} = \begin{bmatrix} 1-r_1 & r_1 & 0 & 0 & \cdots \\ 1-r_2 & 0 & r_2 & 0 & \cdots \\ 1-r_3 & 0 & 0 & r_3 & \cdots \\ & & \cdots & & \end{bmatrix}$$

The probabilities $\{r_i\}$ are defined by the equations

$$(3.1) \qquad \begin{aligned} 1 - r_1 &= c \\ r_1(1-r_2) &= c/2^3 \\ r_1 r_2 (1-r_3) &= c/3^3 \\ &\cdots \\ r_1 r_2 \cdots r_{n-1}(1-r_n) &= c/n^3 , \\ &\cdots \end{aligned}$$

where $c = 1/(\sum_{n=1}^{\infty} 1/n^3)$. Adding the first n equations in (3.1), we obtain

$$1 - r_1 r_2 \cdots r_n = c \sum_{i=1}^{n} 1/i^3 ,$$

so that

$$(3.2) \qquad r_1 r_2 \cdots r_n = c \sum_{i=n+1}^{\infty} 1/i^3 .$$

Since the right-hand side of (3.2) is strictly positive for all n, it follows that $r_i > 0$ for all i. From the equations (3.1), it follows that $1-r_i > 0$ for all i. Hence $0 < r_i < 1$ for all i, so that $\underset{\sim}{P}$ is indeed a transition probability matrix.

We now want to define a pure BPME with $\{V_m\}$ as the guiding chain. We shall assume that $V_0 = 1$ and, for notational convenience, shall drop the superscript denoting the initial omen. If π_n represents the 1-cycle $\{1,2,\ldots,n-1\}$, then

(3.3) $$p_n = r_1 r_2 \ldots r_{n-1}(1-r_n) = c/n^3, \quad n = 1,2,\ldots \quad .$$

The pgf $\phi_n(s)$ determined by omen n is defined by

$$\phi_n(s) = \begin{cases} s & \text{if} \quad n = 1 \\ (1-e^{-n}) + e^{-n}s^{\lambda_n}, & \text{if} \quad n = 2,3\ldots \end{cases} ,$$

where, with $\{x\}$ denoting the greatest integer in $x+1$, $\lambda_n = \{2e^{n+1}\ell og2\}$. Thus $\phi_1'(1) = 1$ and $\phi_n'(1) = e^{-n}\{2e^{n+1}\ell og2\}$, $n \geq 2$, so that $\phi_n'(1) > 1$ for $n \geq 2$ and

$$\phi_n'(1) \sim 2e\ell og2 \quad \text{as} \quad n \to \infty.$$

Furthermore, if $\Phi_n(s)$ is defined by

$$\Phi_n(s) = \phi_1(\phi_2(\ldots\phi_n(s)\ldots)),$$

we have

$$\Phi_n'(1) = \prod_{j=2}^{n} e^{-j}\{2e^{j+1}\ell og2\} ,$$

from which it follows that

$$\ell og\Phi_n'(1) \sim (n-1)\ell og(2e\ell og2), \quad \text{as} \quad n \to \infty.$$

Hence, using (3.3), we see that

$$p_n \log \phi_n'(1) \sim c(n-1)\log(2e\log 2)/n^3, \quad \text{as} \quad n \to \infty.$$

The series $\sum_{n=1}^{\infty} p_n \log \phi_n'(1)$ therefore converges; its sum is clearly positive.

If for each positive integer n, we define

$$y_n = (1-\tfrac{1}{2}e^{-(n+1)})^{\lambda_n},$$

then

$$\begin{aligned} \log y_n &= \lambda_n \log(1-\tfrac{1}{2}e^{-(n+1)}) \\ &< -\tfrac{1}{2}\lambda_n e^{-(n+1)} \\ &= -\tfrac{1}{2}\{2e^{n+1}\log 2\}e^{-(n+1)} \\ &< -\log 2. \end{aligned}$$

Hence for all n, $y_n < \tfrac{1}{2}$. Using this inequality, we see that for $n \geq 2$,

$$\begin{aligned} \phi_n(1-\tfrac{1}{2}e^{-(n+1)}) &= (1-e^{-n}) + e^{-n}(1-\tfrac{1}{2}e^{-(n+1)})^{\lambda_n} \\ &< 1 - \tfrac{1}{2}e^{-n}. \end{aligned}$$

Since $\phi_n(0) < 1-\tfrac{1}{2}e^{-n}$, we obtain

$$\begin{aligned} &\phi_2(\phi_3(\ldots\phi_n(0)\ldots)) \\ &\quad < \phi_2(\phi_3(\ldots\phi_{n-1}(1-\tfrac{1}{2}e^{-n})\ldots)) \\ &\quad < \phi_2(\phi_3(\ldots\phi_{n-2}(1-\tfrac{1}{2}e^{-(n-1)})\ldots)) \end{aligned}$$

$$< \quad \ldots$$

$$< \quad \phi_2(1-\tfrac{1}{2}e^{-3})$$

$$< \quad 1 - \tfrac{1}{2}e^{-2}.$$

Hence

$$\begin{aligned} \Phi_n(0) &= \phi_1(\phi_2(\ldots\phi_n(0)\ldots)) \\ &= \phi_2(\phi_3(\ldots\phi_n(0)\ldots)) \\ &< 1 - \tfrac{1}{2}e^{-2} \ , \end{aligned}$$

so that

$$\ell og(1-\Phi_n(0)) > \ell og\tfrac{1}{2}e^{-2} \ ,$$

and, consequently,

$$\sum_{n=1}^{\infty} p_n \ell og(1-\Phi_n(0)) > \ell og\tfrac{1}{2}e^{-2} > -\infty.$$

It therefore follows from Theorem A that the ladder process is immortal.

If $\{\omega_n\}$ is the stationary distribution of the guiding chain $\{V_m\}$, then

$$\begin{aligned} \omega_n &= \sum_{i=n}^{\infty} p_i / \sum_{i=1}^{\infty} ip_i \\ &= \frac{6}{\pi^2} \sum_{i=n}^{\infty} 1/i^3 \ , \end{aligned}$$

since $p_i = c/i^3$ and thus $\sum_{i=1}^{\infty} ip_i = \pi^2 c/6$. We note here, for later reference, that $\omega_1 > \frac{1}{2}$. Furthermore,

$$\omega_n \sim \frac{6}{\pi^2} \int_n^\infty x^{-3} dx = \frac{3}{\pi^2 n^2} ,$$

so that

$$\omega_n \log(1-\phi_n(0)) \sim -\frac{3}{\pi^2 n} ,$$

and therefore

$$\sum_{n=1}^\infty \omega_n \log(1-\phi_n(0)) = -\infty .$$

<u>Mock Process</u>. For this process, the guiding chain $\{\bar{V}_m\}$ is defined as a Markov chain on the state space consisting of the positive integers and with a transition probability matrix of the form

$$\bar{\underset{\sim}{P}} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n & \cdots \\ 1 & 0 & \cdots & 0 & \cdots \\ & & \cdots & & \\ 1 & 0 & \cdots & 0 & \cdots \\ & & \cdots & & \end{bmatrix}$$

We again assume that $V_0 = 1$; the 1-cycles in this case may be represented by $\pi_1 \equiv \{1\}$ and $\pi_n \equiv \{1,n\}$, $n = 2,3,\ldots$. The probability $\bar{p}_n$ associated with the n-th 1-cycle is then given simply by

$$\bar{p}_n = a_n, \quad n = 1,2,\ldots .$$

If $n_j(r)$ is the number of occurrences of omen j in π_r, then

In particular, then, we can choose the sequence $\{a_j\}$ so that the guiding chain of the mock process has the same stationary distribution $\{\omega_j\}$ as has the guiding chain of the ladder process. With the sequence $\{a_j\}$ thereby obtained, we define a BPME with $\{\overline{V}_m\}$ as guiding chain by using the same sequence $\{\phi_n(s)\}$ of pgf's as were defined in the ladder process.

For the mock process, we have

$$\overline{\Phi}_n(s) = \phi_1(\phi_n(s)) = \phi_n(s),$$

since $\phi_1(s) = s$. Thus $\overline{\Phi}_1'(1) = 1$, and for $n \geq 2$,

$$\overline{\Phi}_n'(1) = e^{-n}\{2e^{n+1}\ell og 2\} \sim 2e\ell og 2, \quad \text{as} \quad n \to \infty.$$

Since $\omega_n \sim 3/\pi^2 n^2$, it follows that

$$\overline{p}_n = a_n \sim 3/\pi^2 \omega_1 n^2 , \tag{3.5}$$

and thus $\sum_{n=1}^{\infty} \overline{p}_n \ell og \overline{\Phi}_n'(1)$ is positive and finite. Furthermore, since

$$\ell og\ (1-\overline{\Phi}_n(0)) = \ell og\ (1-\phi_n(0)) = -\ n$$

for $n \geq 2$, it follows from (3.5) that

$$\sum_{n=1}^{\infty} \overline{p}_n\ \ell og\ (1-\overline{\Phi}_n(0)) = -\ \infty.$$

By Theorem A, the particular mock process we have considered is mortal.

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13. ABSTRACT

A positive recurrent Markov transition matrix $\underset{\sim}{P}$ generates a Markovian sequence $\{V_n\}$ of states called *omens*. A branching process $\{Z_n\}$ develops in the usual way except that families born to the n-th generation are all (independently) governed by a pgf $\phi_{\zeta_n}(s)$, $0 \le s \le 1$, where $\{\zeta_n\}$ is the sequence of *environmental variables*. The distribution of ζ_n depends on the *omen* V_n but the environment variables are otherwise independent. A necessary and sufficient condition for *immortality* of the process, that is $\lim_{n\to\infty} P\{Z_n>0\} >0$, is given. This assumes a particularly simple form when $\underset{\sim}{P}$ is finite, involving $\underset{\sim}{P}$ only through its stationary distribution $\{\omega_j\}$, say. This simple form of the condition is also shown to be *sufficient* for immortality, even when $\underset{\sim}{P}$ is infinite. However, an example is given of two related processes, which demonstrates that when $\underset{\sim}{P}$ is infinite there cannot be a necessary and sufficient condition for immortality involving no more specific information about $\underset{\sim}{P}$ than is contained in the stationary distribution $\{\omega_j\}$.

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