Brans-Dicke cosmology with a scalar field potential

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Abstract. – Three solutions of the Brans-Dicke theory with a self-interacting quartic potential and perfect fluid distribution are presented for a spatially flat FRW geometry. The physical behavior is consistent with the recent cosmological scenario favored by supernova type-Ia observations, indicating an accelerated expansion of the Universe.

Introduction. – Recent observations of type-Ia supernovae with redshift up to about $z \lesssim 1$ provided evidence that we may live in a low–mass-density Universe, with the contribution of the non-relativistic matter (baryonic plus dark) to the total energy density of the Universe of the order of $\Omega_m \sim 0.3$ [1]. The value of Ω_m is significantly less than unity [2], and, consequently, either the Universe is open or there is some additional energy density ρ sufficient to reach the value $\Omega_{\text{total}} = 1$, predicted by inflationary theory. Observations also show that the deceleration parameter of the Universe q is in the range $-1 \leq q < 0$, and the present-day Universe undergoes an accelerated expansionary evolution.

Several physical models have been proposed to give a consistent physical interpretation to these observational facts. Photons propagating in extra-galactic magnetic fields can oscillate into very light axions, so the supernovae appear dimer and more distant than they really are [3]. For the missing energy one candidate is the vacuum energy density, or the cosmological constant Λ [4]. A charged Universe, in which the long-range force carriers have a small mass, has been considered as a cause for the cosmic acceleration in [5]. Another possibility are cosmologies based on a mixture of cold dark matter and quintessence, a slowly varying, spatially inhomogeneous component [6]. An example of implementation of the idea of quintessence is the suggestion that it is the energy associated with a scalar field Q with self-interaction potential $V(Q)$. If the potential energy density is greater than the kinetic one, then the pressure associated to the Q-field is negative [7]. Quintessence models that accommodate the present-day acceleration tend to accelerate eternally, and, as a consequence, the resulting space-times exhibit event horizons [8]. A wide class of quintessence models, with eternal acceleration, associated with static metrics, have been presented in [9].

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Brans-Dicke (BD) theory may explain the present accelerated expansion of the Universe without resorting to a cosmological constant or quintessence matter [10]. The conditions under which the dynamics of a self-interacting BD field can account for the accelerated expansion have been considered in [11–14]. Accelerated expanding solutions can be obtained with a quadratic self-coupling of the BD field and a negative coupling constant ω [11]. A cosmic fluid obeying a perfect-fluid–type equation of state cannot support the acceleration [13]. The nature of the scalar field potential compatible with a power law expansion in a self-interacting BD cosmology with a perfect-fluid background has been analyzed in [14]. Models with non-minimal coupling of the scalar field have been considered in [15]. Complicated cosmological scenarios, with a four-dimensional effective action connected with supergravity and string theory, have been obtained in [16].

It is the purpose of the present letter to consider some exact classes of solutions of the field equations in the framework of BD theory with a quartic potential, $V(\phi) \sim \phi^4$. By means of some appropriate transformations, the field equations can be reduced to a system of two independent Riccati's type differential equations. Three classes of exact solutions of the field equations are presented, and their physical properties are investigated in detail.

Field equations, geometry and consequences. –   The physical model we are considering is the Brans-Dicke action, along with a self-interacting potential $V(\phi)$, coupled to the matter field Lagrangian L_m via the action

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} \phi^{,\alpha} \phi_{,\alpha} - V(\phi) + L_m \right),$$ (1)

where ω is the BD coupling parameter. For $\omega = -1$ this action is identical to the low-energy string theory action. In the present letter we use units so that $8\pi G = c = 1$.

For a homogeneous flat space-time, with scale factor a, filled with a perfect fluid, with pressure p_m and energy density ρ_m, the Einstein-Brans-Dicke gravitational field equations are

$$3H^2 = \frac{\rho_m + \rho_\phi}{\phi}, \qquad 2\dot{H} + 3H^2 = -\frac{p_m + p_\phi}{\phi},$$ (2)

where ρ_ϕ and p_ϕ are the energy density and pressure associated to the BD scalar field, given by

$$\rho_\phi \phi^{-1} = \frac{\omega}{2}\psi^2 + \frac{V(\phi)}{2\phi} - 3H\psi, \qquad p_\phi \phi^{-1} = \left(1 + \frac{\omega}{2}\right)\psi^2 - \frac{V(\phi)}{2\phi} + \dot{\psi} + 2H\psi,$$ (3)

where we denoted $H = \dot{a}a^{-1}$ and $\psi = \dot{\phi}\phi^{-1}$. ψ gives the rate of change of the gravitational constant $G(t)$, $\psi = -\frac{\dot{G}(t)}{G(t)}$.

The wave equation for the BD field takes the form

$$\dot{\psi} + \psi^2 + 3H\psi = \frac{(\rho_m - 3p_m)\phi^{-1}}{(2\omega + 3)} - \frac{1}{(2\omega + 3)}\left(\frac{2V(\phi)}{\phi} - \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}\right).$$ (4)

The energy conservation of the matter implies $\dot{\rho}_m + 3(\rho_m + p_m)H = 0$.

We assume that the self-interaction potential $V(\phi)$ has the form $V(\phi) = V_0\phi^4$, $V_0 = $ const. This form corresponds to chaotic inflation model, where the constant V_0 is subject to the constraint $V_0 < 10^{-12}$, coming from the observational limits on the amplitude of fluctuations in the cosmic microwave background [17].

With this choice, and with the use of eqs. (2)-(4), we obtain the following equation describing the dynamics of the Universe:

$$\dot{\psi} - \frac{3}{\omega}\dot{H} = \frac{6}{\omega}H^2 - \frac{1}{2}\psi^2 - 3H\psi.$$ (5)

We introduce the following matrices:

$$L = \left(\dot{\psi} - \frac{3}{\omega}\dot{H} \right), \qquad M^T = \left(\psi \quad H \right), \qquad A = \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{3}{2} \\ -\dfrac{3}{2} & +\dfrac{6}{\omega} \end{pmatrix}. \tag{6}$$

Then eq. (5) can be written as a matrix equation in the form $L = M^T A M$.

We define two new variables $\hat{\psi}, \hat{H}$, which are obtained by means of a linear transformation, described by the matrix K, and applied to the matrix $\hat{M}$,

$$M = \begin{pmatrix} \psi \\ H \end{pmatrix} = K\hat{M} = \begin{pmatrix} f_+ K_+^{-1} & f_- K_-^{-1} \\ K_+^{-1} & K_-^{-1} \end{pmatrix} \begin{pmatrix} \hat{\psi} \\ \hat{H} \end{pmatrix}, \tag{7}$$

where f_+, f_-, K_+ and K_- are real numbers.

In the new variables $(\hat{\psi}, \hat{H})$ we have $L = \hat{M}^T K^T A K \hat{M}$. We choose the elements of the matrix K so that $K^T A K$ is a diagonal matrix, with the diagonal elements $\lambda_\mp$ given by the eigenvalues of A. The eigenvectors and the eigenvalues of the matrix A are given by $\left(f_+ K_+^{-1} \quad K_+^{-1} \right)^T, \left(f_- K_-^{-1} \quad K_-^{-1} \right)^T$, where $\lambda_\mp = \frac{12-\omega\mp s}{4\omega}$, $f_\mp = \frac{12+\omega\mp s}{6\omega}$, $K_\mp = \sqrt{f_\mp^2 + 1}$ and $s = \sqrt{144 + 24\omega + 37\omega^2}$. s satisfies the condition $s \geq 0, \forall \omega$.

Therefore the equations satisfied by the new unknown functions $\hat{\psi}, \hat{H}$ are

$$a_- \frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t} - \lambda_- \hat{\psi}^2 = a_+ \frac{\mathrm{d}\hat{H}}{\mathrm{d}t} + \lambda_+ \hat{H}^2 = F(t), \tag{8}$$

where we have introduced a solution generating function $F(t)$ and denoted $a_\mp = \varepsilon_\pm K_\pm^{-1}(f_\pm - 3\omega^{-1})$, with $\varepsilon_\pm = \pm 1$.

Classes of exact solutions of the field equations. – In order to obtain some classes of general solutions of eqs. (8), we assume first that $\hat{\psi}$ and $\hat{H}$ are given by $\hat{\psi} = c_- t^{-1}$, $\hat{H} = c_+ t^{-1}$, $c_\pm = \text{const.}$ These functional forms of the new variables satisfy eqs. (8) if the consistency condition $\alpha_- = \alpha_+$ relating the parameters $a_\mp$, $c_\mp$ and $\lambda_\mp$ holds, with $\alpha_\mp = -a_\mp c_\mp + \varepsilon_\mp \lambda_\mp c_\mp^2$. By choosing $F(t) = \alpha_- t^{-2} = \alpha_+ t^{-2}$, we separate eqs. (8) into a system of Riccati's differential equations given as

$$a_- \frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t} - \lambda_- \hat{\psi}^2 = \frac{\alpha_-}{t^2}, \qquad a_+ \frac{\mathrm{d}\hat{H}}{\mathrm{d}t} + \lambda_+ \hat{H}^2 = \frac{\alpha_+}{t^2}. \tag{9}$$

The mathematical form of $F(t)$ has been chosen in order to obtain an exact solution of eqs. (8). The system (9) has the particular solutions $\hat{\psi}_0 = c_- t^{-1}$ and $\hat{H}_0 = c_+ t^{-1}$. With the help of the standard transformations $\hat{\psi} = u^{-1} + \hat{\psi}_0$, $\hat{H} = v^{-1} + \hat{H}_0$ Riccati's differential equations are transformed into two linear Bernoulli's equations.

Therefore, a first class of general solutions of eqs. (9) is given by

$$\hat{\psi} = \frac{t^{2n_-}}{d_- - \frac{n_-}{c_-(1+2n_-)}t^{2n_-+1}} + c_- t^{-1}, \qquad \hat{H} = \frac{t^{-2n_+}}{d_+ + \frac{n_+}{c_+(1-2n_+)}t^{-2n_++1}} + c_+ t^{-1}, \tag{10}$$

where we have denoted $n_\mp = c_\mp \lambda_\mp a_\mp^{-1}$. $d_\mp$ are arbitrary constants of integration.

With the use of the linear transformation (7) the solution for (ψ, H) is obtained in the form

$$\psi = \frac{f_+ K_+^{-1} c_- + f_- K_-^{-1} c_+}{t} + f_+ K_+^{-1} \frac{t^{2n_-}}{d_- - \frac{n_-}{c_-(1+2n_-)} t^{2n_-+1}} +$$

$$+ f_- K_-^{-1} \frac{t^{-2n_+}}{d_+ + \frac{n_+}{c_+(1-2n_+)} t^{-2n_++1}}, \tag{11}$$

$$H = \frac{K_+^{-1} c_- + K_-^{-1} c_+}{t} + K_+^{-1} \frac{t^{2n_-}}{d_- - \frac{n_-}{c_-(1+2n_-)} t^{2n_-+1}} +$$

$$+ K_-^{-1} \frac{t^{-2n_+}}{d_+ + \frac{n_+}{c_+(1-2n_+)} t^{-2n_++1}}. \tag{12}$$

The BD scalar field and the scale factor of the Universe are

$$\phi = \phi_0 t^{f_+ K_+^{-1} c_- + f_- K_-^{-1} c_+} \left[d_- - \frac{n_-}{c_-(1+2n_-)} t^{2n_-+1} \right]^{-\frac{f_+ K_+^{-1} c_-}{n_-}} \times$$

$$\times \left[d_+ + \frac{n_+}{c_+(1-2n_+)} t^{-2n_++1} \right]^{\frac{f_- K_-^{-1} c_+}{n_+}}, \tag{13}$$

$$a = a_0 t^{K_+^{-1} c_- + K_-^{-1} c_+} \left[d_- - \frac{n_-}{c_-(1+2n_-)} t^{2n_-+1} \right]^{-\frac{K_+^{-1} c_-}{n_-}} \times$$

$$\times \left[d_+ + \frac{n_+}{c_+(1-2n_+)} t^{-2n_++1} \right]^{\frac{K_-^{-1} c_+}{n_+}}, \tag{14}$$

where $a_0 > 0$ and $\phi_0 > 0$ are constants of integration.

Another class of exact solutions of the gravitational field equations in the BD theory with matter fluid is obtained by assuming for the generating function the form $F(t) = \beta = \text{const}$. In this case eqs. (8) take the form

$$a_- \frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t} = \lambda_- \hat{\psi}^2 + \beta, \qquad a_+ \frac{\mathrm{d}\hat{H}}{\mathrm{d}t} = -\lambda_+ \hat{H}^2 + \beta. \tag{15}$$

Depending on the sign of the parameters λ_-, λ_+ and β, we obtain two distinct classes of solutions. In the first case we assume that $\lambda_- > 0$, $-\lambda_+ > 0$ and $\beta < 0$. With this choice the general solutions of eqs. (15) are given by

$$\hat{\psi} = -\sqrt{|\beta|\lambda_-^{-1}} \coth \left[N_-(t - t_0) \right], \qquad \hat{H} = -\sqrt{-|\beta|\lambda_+^{-1}} \coth \left[N_+(t - t_0) \right], \tag{16}$$

where $N_\pm = \sqrt{\epsilon_\mp \lambda_\pm |\beta|} a_\pm^{-1}$. This solution is mathematically consistent for values of $\hat{\psi}$ and $\hat{H}$ so that $\hat{\psi} > \sqrt{\frac{|\beta|}{\lambda_-}}, \hat{\psi} < -\sqrt{\frac{|\beta|}{\lambda_-}}$ and $\hat{H} > \sqrt{\frac{|\beta|}{-\lambda_+}}$.

With the use of the transformation (7) we obtain

$$\psi = -f_+ K_+^{-1} \sqrt{|\beta|\lambda_-^{-1}} \coth \left[N_-(t - t_0) \right] - f_- K_-^{-1} \sqrt{-|\beta|\lambda_+^{-1}} \coth \left[N_+(t - t_0) \right], \tag{17}$$
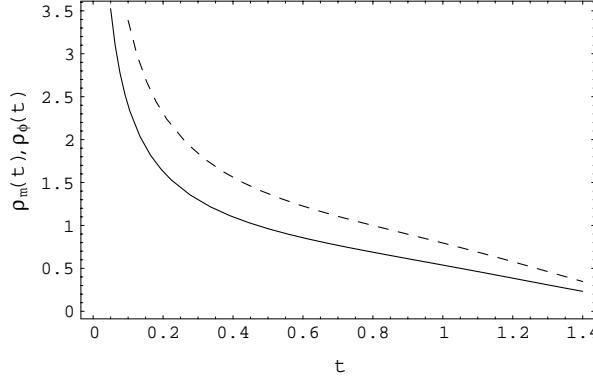
Fig. 1 – Time evolution of the energy density of the matter ρ_m (solid curve) and of the scalar field ρ_ϕ (dashed curve) (in units of $10^{-47}\,\mathrm{GeV}^4$) for the first class of solutions for $\omega = -2$, $d_- = 0.32$, $d_+ = 0.53$, $c_- = -1.95$, $V_0 = 0.0064$ and $c_+ = -0.54$. The time is expressed in units of $10\,\mathrm{Gy}$.

and

$$H = -K_+^{-1}\sqrt{|\beta|\lambda_-^{-1}}\coth\left[N_-(t-t_0)\right] - K_-^{-1}\sqrt{-|\beta|\lambda_+^{-1}}\coth\left[N_+(t-t_0)\right]. \qquad (18)$$

On integration, we obtain the BD scalar field and the scale factor:

$$\phi = \phi_0 \sinh^{-\frac{f_+ K_+^{-1}\sqrt{|\beta|\lambda_-^{-1}}}{N_-}}\left[N_-(t-t_0)\right]\sinh^{-\frac{f_- K_-^{-1}\sqrt{-|\beta|\lambda_+^{-1}}}{N_+}}\left[N_+(t-t_0)\right], \qquad (19)$$

$$a = a_0 \sinh^{-\frac{K_+^{-1}\sqrt{|\beta|\lambda_-^{-1}}}{N_-}}\left[N_-(t-t_0)\right]\sinh^{-\frac{K_-^{-1}\sqrt{-|\beta|\lambda_+^{-1}}}{N_+}}\left[N_+(t-t_0)\right]. \qquad (20)$$

A third class of solutions is obtained by assuming $\lambda_- > 0$, $-\lambda_+ > 0$ and $\beta > 0$. The solution can be formally obtained from the previous one by means of the substitution $\beta \to -\beta$, $\coth ix \to \frac{1}{i}\cot x$, etc. In all these cases the energy density and pressure of the baryonic matter and of the scalar field follow from the field equations (2), (3).

Discussions and final remarks. –   The first class of solutions depends on the set of five arbitrary constants c_-, $d_\pm$, V_0 and ω. To determine them from the actual observational
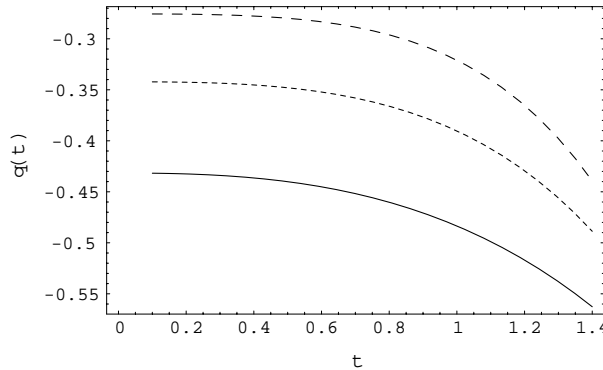

Fig. 2 – Dynamics of the deceleration parameter for the first class of solutions for $\omega = -1.8$ (solid curve), for $\omega = -2$ (dotted curve), and for $\omega = -2.2$ (dashed curve). We have used the values $d_- = 0.32$, $d_+ = 0.53$, $c_- = -1.95$, $V_0 = 0.0064$ and $c_+ = -0.54$. The time is expressed in units of $10\,\mathrm{Gy}$.
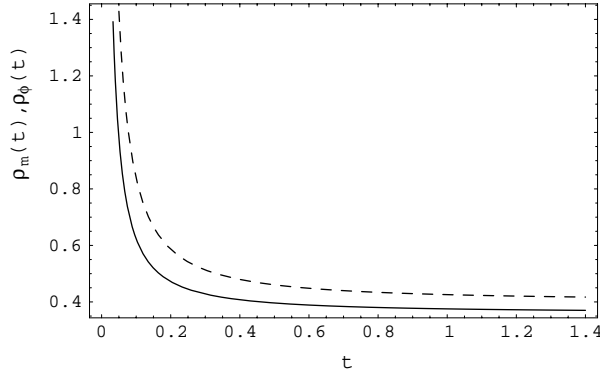
Fig. 3 – Time evolution of the energy density of the matter ρ_m (solid curve) and of the scalar field ρ_ϕ (dashed curve) (in units of $10^{-47}\,\mathrm{GeV}^4$) for the second class of solutions, for $\omega = -2$, $\beta = 0.001$ and $V_0 = 0.0073$. The time is expressed in units of $10\,\mathrm{Gy}$.

data we take the matter pressure to be zero, $p_m = 0$. For the age of the Universe and the Hubble constant we adopt the values $t = 14\,\mathrm{Gy}$ and $H_0 = 65\,\mathrm{km\ s^{-1}Mpc^{-1}}$ [1]. The density parameters of the matter and of the BD scalar fields are defined as $\Omega_m = \frac{\rho_m}{3H^2\phi} = 0.25$ and $\Omega_\phi = \frac{\rho_\phi}{3H^2\phi} = 0.75$, respectively [1]. For the deceleration parameter $q = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{H} - 1$, which is an indicator of the accelerating behavior, we assume an actual value of $q = -0.5$. Therefore, by taking ω as a free parameter, we have four observational constraints to be satisfied by the model. Thus the numerical values of the constants c_-, $d_\pm$ and V_0 can be obtained from fitting the model with the observations. Generally, a physical solution for the resulting non-linear system of algebraic equations can be obtained only for small negative values of ω.

The variation of the energy density of the matter and of the BD field is represented, for the first class of solutions, in fig. 1. Generally, the energy density of the scalar field dominates the matter energy density, thus providing the dominant contribution to the total energy density. The deceleration parameter is represented, for different values of $\omega < 0$, in fig. 2. q is negative, with values in the range $-1 < q < 0$, indicating an accelerating evolution of the Universe.

The second class of solutions depends (by choosing $t_0 = 0$) on three parameters, β, V_0 and ω. Their numerical values can be obtained so that the solution fits the actual observational
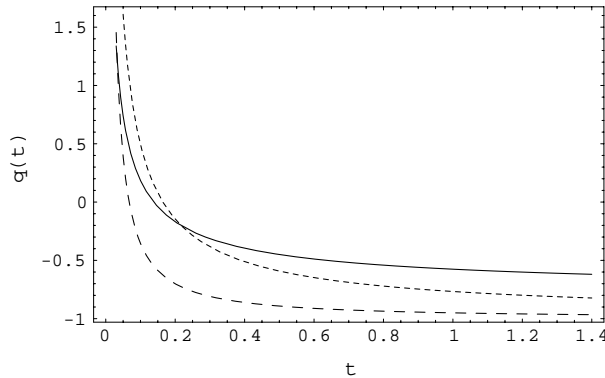
Fig. 4 – Dynamics of the deceleration parameter in the second class of solutions for $\omega = -1.8$ (solid curve), $\omega = -2$ (dotted curve) and $\omega = -2.2$ (dashed curve). In all cases we have used the values $\beta = 0.001$ and $V_0 = 0.0073$. The time is expressed in units of $10\,\mathrm{Gy}$.

values of H, Ω_m and Ω_ϕ. The resulting algebraic system of equations has also a physical solution only for small negative values of ω.

The time variations of the energy density of the matter and of the BD field, and of q for the second class of solutions are represented in figs. 3 and 4. The energy density of the scalar field dominates the matter energy density, and the evolution is accelerating.

Therefore, these two classes of solutions are compatible with the observed cosmological data only for small negative values of the coupling parameter of the BD field. Another important observational bound requires $\Omega_\phi < 0.044$ at nucleosynthesis [18]. This condition can also be satisfied by appropriately choosing the matter equation of state in the very early stages of evolution of the Universe. Generally, in the present models we have $\rho_\phi > \rho_m$, and the evolution is accelerating. An accelerating expansion during the whole evolution of the Universe is not favored by the current models of structure formation. However, as shown in [19], the scalar field, which is non-minimally coupled to gravity, may undergo clustering processes, eventually forming density perturbations, which can be investigated only within a non-linear approach, and leading to the formation of some cosmic structures. On the other hand, to satisfy the actual observational constraints, a much larger value than that predicted by the inflationary scenario is needed for the constant V_0.

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