CHAPTER 8

BREAKING WAVE CRITERIA; A STUDY EMPLOYING A NUMERICAL WAVE THEORY

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INTRODUCTION

Although it is well recognized that wave systems in nature are irregular, comprising a spectrum of fundamental periods, there is still a need for improving our understanding of near-breaking nonlinear wave systems which contain a single fundamental period. For example, most of the shallow water design situations and other cases including forces on small diameter structures in which drag forces predominate are more directly treated in terms of a "design wave" rather than a wave spectrum. This situation is contrasted to many important engineering design problems in which the dynamics of the system are paramount; for example, in the case of a moored drilling vessel. Finally, one may reasonably expect that accurate solutions to the problem of nonlinear wave systems with a single fundamental period will lend insight regarding productive approaches to the more realistic problem of a spectrum of nonlinear waves.

This paper investigates the applicability of the stream function wave theory[1] for the representation of breaking and near-breaking waves. This particular problem has received little attention, although considerable progress has occurred on two related problems:

1. The development of wave theories covering a wide range of relative water depths and wave heights, and

2. The development of wave theories which apply <u>at</u> breaking conditions. In general, although these theories may be applicable for the limiting wave conditions, their basis of derivation is such that they cannot be extended to non-breaking waves.

The purpose of the present investigation, then, is to establish whether or not the stream function wave theory can be applied to span the range extending up to breaking conditions.

BACKGROUND

A great deal of effort[2,3,4,5,6,7] has been devoted to development of non-breaking wave theories based on mathematical approaches selected to cover various ranges of relative water depths and wave heights. Recently, Dean[8] has compared the relative validities of eight wave theories based on the agreement of these theories to the boundary conditions included in their formulation. This comparison showed that the stream function wave theory provided a best fit to the boundary conditions over the range $0.05 < h/T^2 < 10.0$ ft/sec^2 and a range of wave heights encompassing most conditions encountered in engineering design problems.

A second problem area, that of conditions at wave breaking, has also received considerable attention[9,10,11,12,13,14]. These investigations generally predict an upper limit for wave stability, and, although the various results differ somewhat in numerical values, they are in approximate agreement. These results will be discussed in greater detail later. Aspects of these limiting wave theories that are important to the goals of the present investigation include:

1. The limiting wave theories cannot be applied to highly nonlinear, but non-breaking conditions.

2. No results are available to compare the results of existing non-breaking theories with the limiting wave theories for conditions at or near breaking.

3. All of the limiting wave theories require an *a priori* assumption concerning the shape of the wave crest. This assumption raises questions of the validity of the stability limit and of the wave kinematics and dynamics at breaking.

ANALYTICAL CONSIDERATIONS OF PRESENT INVESTIGATION

In the present investigation, two stability parameters will be defined as possible mechanisms limiting the heights that waves can attain without breaking. These parameters are defined to equal unity if their respective breaking mechanisms are fulfilled. Employing the stream function wave theory, the variations of these parameters are then investigated as the wave height is increased. The calculations are continued with increasing wave height until one of the stability parameters equals unity. The wave conditions and the values of the stability parameters at breaking are compared with previous results.

Formulation of Water Wave Problem

The system considered here will be that of a two-dimensional periodic water wave propagating over a horizontal bottom. Since the basis of the formulation includes the inherent assumption that the wave travels without change of form, it is possible to select a coordinate system moving with the wave celerity, C, thereby reducing the problem viewed in this reference system to one of steady motion. The formulation for this problem and the inherent assumptions have been presented and discussed thoroughly elsewhere[1]. Therefore, the governing equations will be simply set forth below without detailed discussion.

The Laplace differential equation expressed in terms of the velocity potential, ϕ, or stream function, ψ, must be satisfied throughout the region formed by the wave system. (See Figure 1 for a description of the wave system and notation employed.)

Formulating the problem in terms of a stream function, the associated differential equation is

$$\nabla^2\psi = 0 \tag{1}$$

The kinematic boundary conditions on the bottom and free surfaces ensure that no flow occurs normal to these surfaces, that is, on the bottom,

$$w = \frac{\partial\psi}{\partial x} = 0 \ , \ z = - h \tag{2}$$

and on the free surface,

$$\frac{\partial\eta}{\partial x} = \frac{w}{u-C} \ , \ z = \eta \tag{3}$$

The remaining (dynamic) boundary condition on the free surface ensures that the pressure on the free surface is uniform

$$\eta + \frac{1}{2g} \left[(u-C)^2 + w^2\right] = Q \ , \ z = \eta \tag{4}$$

where Q is a constant. The problem formulation is completed by requiring that all of the variables be periodic in the horizontal space coordinate, x.

If an analytical representation could be found that would satisfy the differential equation and boundary conditions, the representation would be exact within the limitations of the formulation.

The Stream Function Solution

The stream function representation for highly nonlinear waves has been explored in previous papers; hence, it will be presented here only in outline form. The stream function solution is written as

$$\psi(x,z) = \frac{L}{T} z + \sum_{n=1}^{N} A(n) \sinh \left[\frac{2\pi n}{L} (h+z)\right] \cos \frac{2\pi n}{L} x \tag{5}$$

and an expression for the free surface displacement, $\eta(x)$, is determined in implicit form by setting $z = \eta$ in Eq. (5), that is,

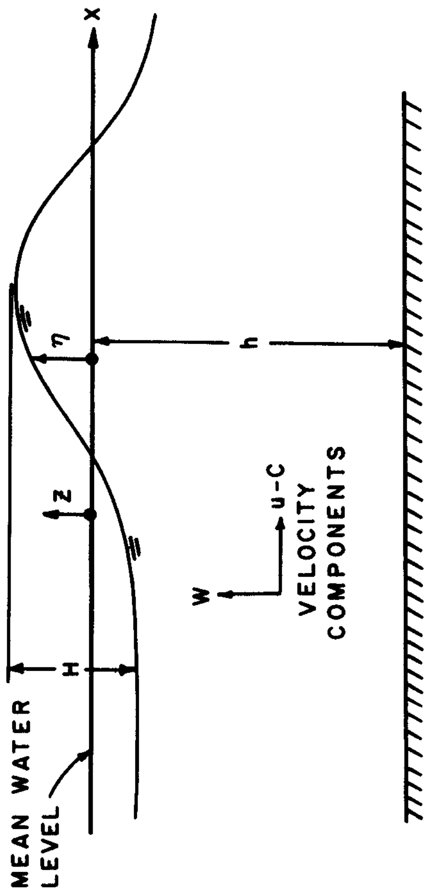

FIGURE I. DEFINITION SKETCH, STATIONARY WAVE SYSTEM

$$\eta = \frac{\psi_\eta}{C} - \frac{T}{L} \sum_{n=1}^{N} A(n) \sinh \left[\frac{2\pi n}{L}(h+\eta)\right] \cos \frac{2\pi n}{L} x \qquad (6)$$

where ψ_η is the (constant) value of the stream function on the free surface.

It can be shown that, for arbitrary A(n) and L values, the stream function solution expressed in Eqs. (5) and (6) satisfies the formulation exactly except for the dynamic free surface boundary condition, Eq. (4). The problem is now posed, for given values of wave height, H, wave period, T, and water depth, h, of determining the wave length, L, and stream function coefficients, A(n) such that the dynamic free surface boundary condition is best satisfied. (It should be noted that the stream function value at the free surface is determined by requiring that the mean of the water surface displacement, η, be zero.) A numerical iterative procedure has been established to successively improve on the values of A(n) and L in accordance with the requirement of uniform Bernoulli constant, see reference 1 for details. During the present investigation, a property of the A(n) coefficients was found which appears to be general and which facilitates their determination.

STABILITY CRITERIA

In previous investigations of limiting forms of progressive and standing waves, two possible mechanisms for wave breaking have been proposed.

Kinematic Stability Parameter (KSP)

The KSP is a measure of the maximum horizontal water particle velocity, u_m, (which occurs at the crest) relative to the wave phase speed, C,

$$\text{KSP:} \quad \frac{u_m}{C} \qquad (7)$$

This criterion is proposed to be limiting if the water particle velocity equals the celerity, that is, $u_m/C = 1$. For conditions of the KSP equal to unity, it is sometimes argued that the crest particles will travel faster than the wave form, and the wave will become asymmetric and topple over. The soundness of this argument is not clear to this author.

Dynamic Stability Parameter (DSP)

The DSP is a measure of the total maximum vertical acceleration, Dw/Dt, relative to the acceleration of gravity, g, and is defined as

$$\text{DSP:} \quad -\frac{1}{g}\frac{Dw}{Dt} \qquad (8)$$

The physical interpretation of this limiting condition is evident; if this parameter exceeds unity, it can be shown that the pressure gradient at the crest is zero and further increases in the wave height would cause water particles to leave the crest of the wave in a vertical direction. It is clear that this must be the breaking mechanism in standing waves. The KSP is generally regarded to be limiting for progressive waves. The pertinence of the DSP to limiting progressive waves is believed to be unresolved by published accounts, although Laitone[13] and Kinsman[15] have indicated that the DSP is governing for progressive waves.

Finally, it is emphasized that both the KSP and DSP are defined such that a value of unity of either of these quantities would indicate breaking due to their respective mechanisms.

RESULTS

Three wave conditions, spanning the range from relatively shallow to deep water, were chosen for further examination. The wave periods and water depths of these three cases are shown in Table I.

TABLE I

WAVE CHARACTERISTICS SELECTED FOR BREAKING STUDY

Case	Relative Depth	Water Depth (ft)	Period (sec)
A	Shallow	20.0	20.0
B	Intermediate	100.0	10.0
C	Deep	1000.0	10.0

In the initial calculations, the numerical stream function approach described earlier[1] was applied to determine the stream function coefficients, A(n), individually. It was found, by examination, that these coefficients vary in a semi-logarithmic manner with the index, n,

$$A(n) = A(1)\, e^{-bn} \tag{9}$$

An example is shown in Figure 2 for a 17th order Case A wave. It is emphasized that these coefficients were not constrained to a semi-logarithmic variation, but this form of variation appears to provide the best fit to the dynamic free surface boundary condition (DFSBC). Although the last few coefficients shown in Figure 2 deviate from the semi-logarithmic form, it was found that repeated iterations of the numerical procedure would improve this distribution. Calculations for Cases B and C and other relative depths

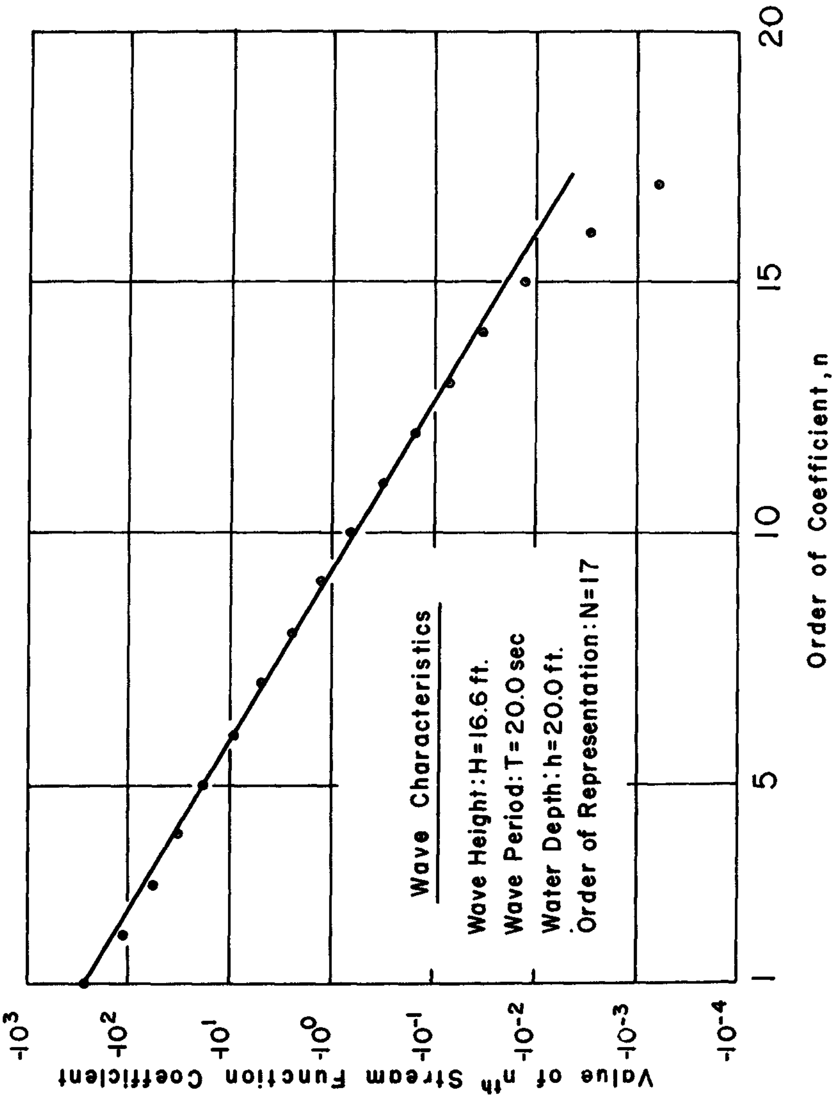

FIGURE 2. SEMI-LOGARITHMIC VARIATION OF STREAM FUNCTION COEFFICIENTS

indicated that the observed semi-logarithmic variation could be regarded as a property of these coefficients when a good fit to the DFSBC was attained, the reason for the semi-logarithmic variation is not known. The simplifications associated with the semi-logarithmic variation are substantial; for example, the number of independent wave parameters is 3 for a 20th order wave assuming a semi-logarithmic variation as compared to 21 if no a priori relationship is recognized for the A(n) coefficients. This feature was accounted for in the numerical scheme, thereby greatly reducing the computational time required, especially for near-breaking conditions, which require a high order representation.

Case A - Shallow Water Wave (h/T^2 = 0.05 ft/sec^2)

The kinematic and dynamic stability criteria for this case are plotted as a function of H/h in Figure 3. Note that breaking is predicted at a ratio H/h = 1.0 while the usually referenced breaking limit is H/h = 0.78; the range reported by previous investigators is $0.73 < H/h < 0.87$. For the results obtained here, it is clear that the kinematic criterion governs breaking, and it is surprising at first that the dynamic parameter is zero when breaking occurs. This point will be discussed later in greater detail.

The wave forms for a near-breaking Case A wave are presented in Figure 4. Note that the enclosed angle of the wave form is in approximate agreement with the previously determined value of 120 degrees.

The amplitude spectrum, determined by a harmonic analysis of the wave form, is also shown in Figure 4. As expected, the higher order terms are quite significant for the case of a near-breaking shallow water wave for which the crests are high and peaked and the troughs low and broad. As a reference for later comparison, the 10th order term (9th harmonic) is about 25 per cent of the fundamental.

Case B - Intermediate Depth Water Wave (h/T^2 = 1.0 ft/sec^2)

For the sake of brevity, the graphical results pertaining to this case are not shown here. The stability parameter results were the same as for Case A, that is, the KSP governed breaking and the DSP equals zero at breaking. The ratio H/T^2 at breaking was determined to be 0.69 ft/sec^2 which, as will be shown later, is in good agreement with previously referenced results.

Case C - Deep Water Wave (h/T^2 = 10.0 ft/sec^2)

The kinematic and dynamic stability parameters are shown in Figure 5 and indicate, as in Cases A and B, that the KSP governs and the DSP equals zero at breaking. The value of H/T^2 at breaking is 1.06 ft/sec^2 compared with the usually referenced value of 0.873 ft/sec^2. The wave form and amplitude spectrum for a near-breaking Case C wave are shown in Figure 6. Of interest is the rapid decrease in amplitude coefficients for large n. The ratio B(10)/B(1) equals 0.02 for Case C, whereas for Case A the corresponding ratio is 0.25. The angle enclosed by the wave in the vicinity of the crest is approximately 120 degrees for Case C.

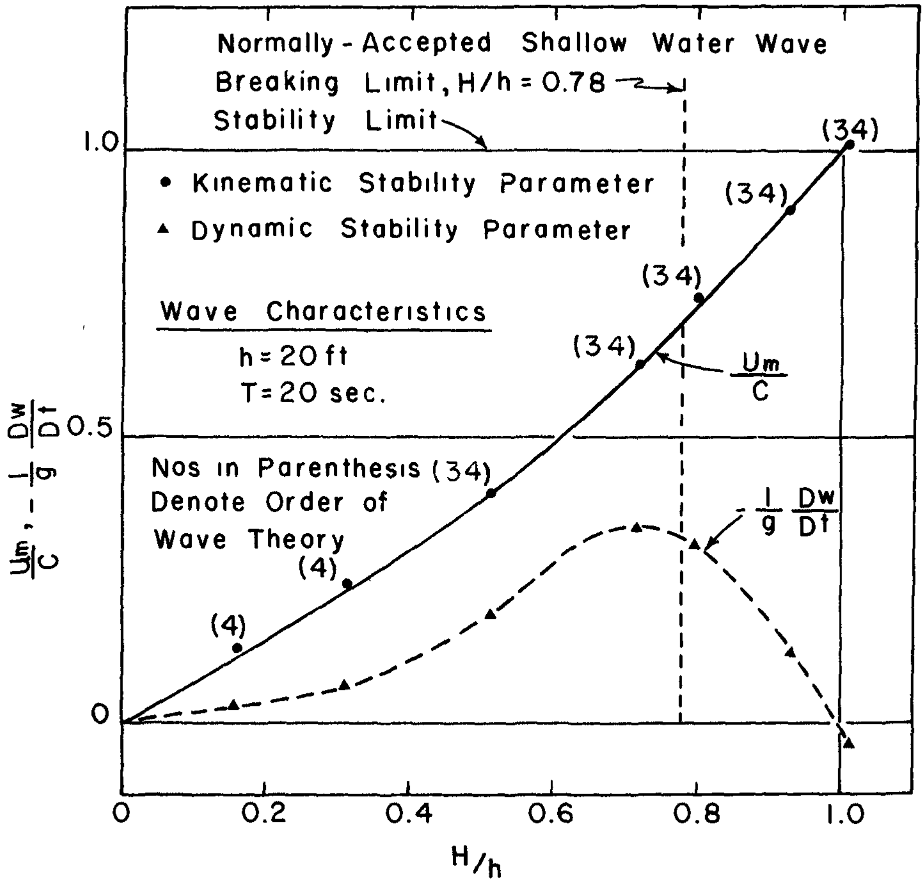

FIGURE 3. VARIATION OF STABILITY PARAMETERS WITH H/h; SHALLOW WATER WAVE

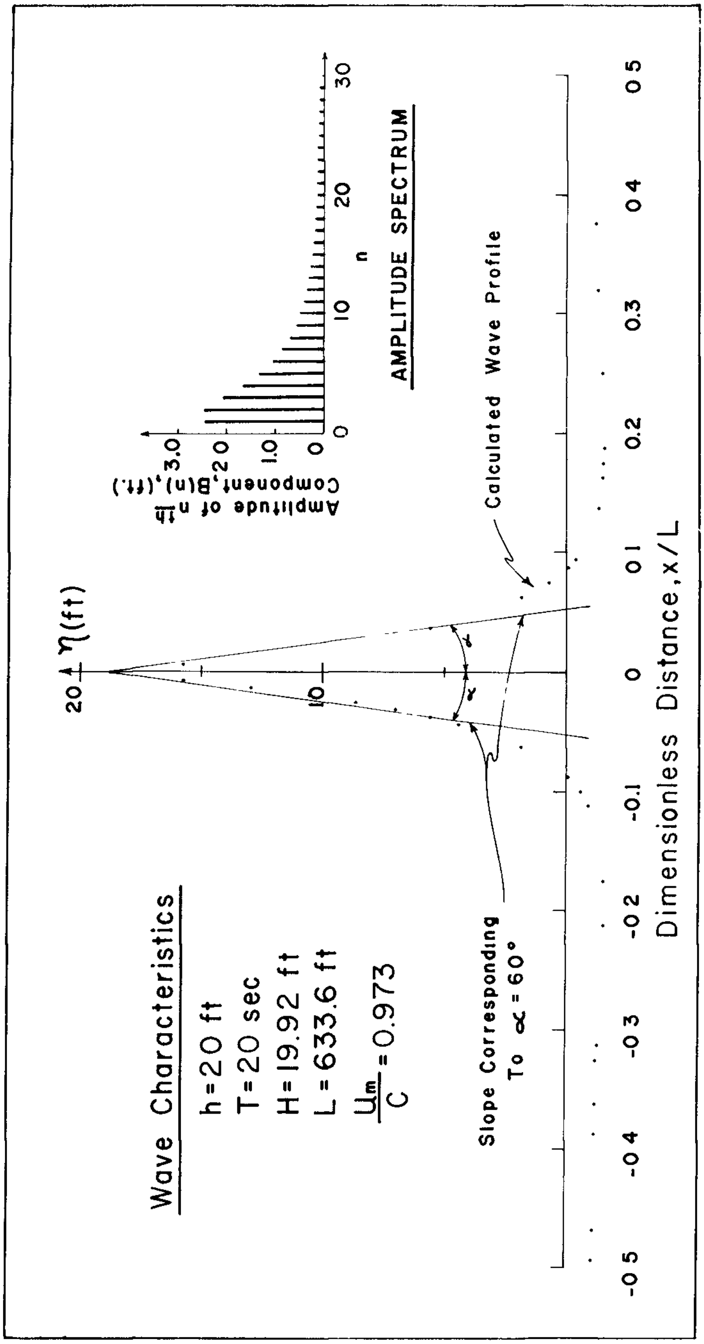

FIGURE 4 CALCULATED PROFILE AND AMPLITUDE SPECTRUM FOR NEAR-BREAKING SHALLOW WATER WAVE

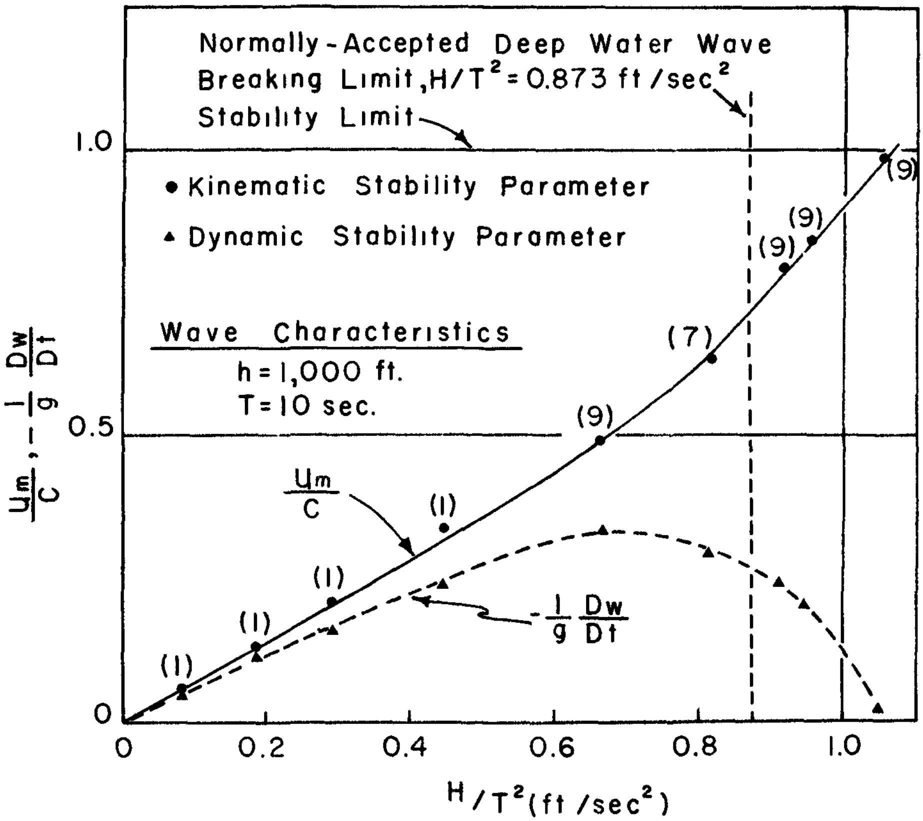

FIGURE 5. VARIATION OF STABILITY PARAMETERS WITH H/T^2; DEEP WATER WAVE

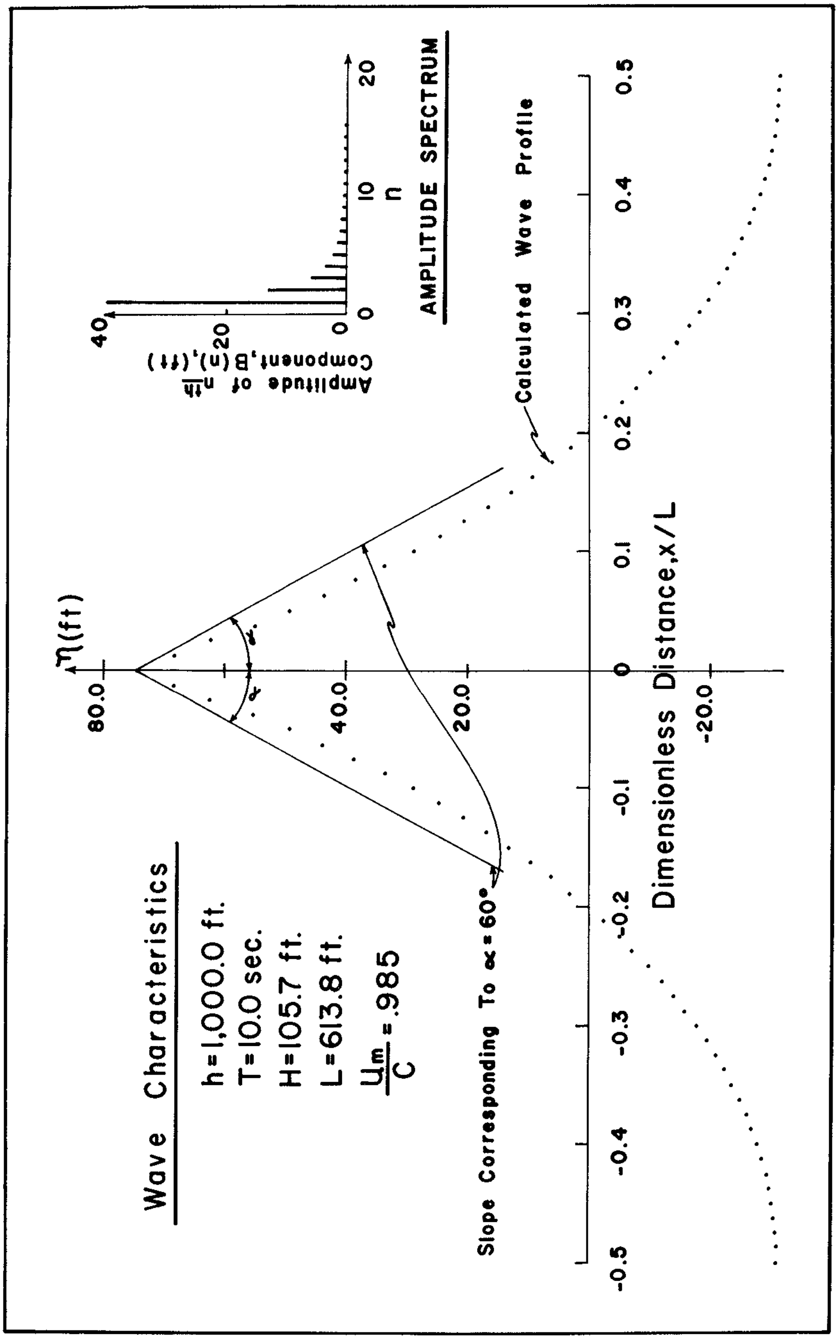

FIGURE 6. CALCULATED PROFILE AND AMPLITUDE SPECTRUM FOR NEAR-BREAKING DEEP WATER WAVE

Breaking Index

The breaking wave parameters for the three cases examined here are compared with the usually referenced breaking index in Figure 7. It is seen that the results obtained in the present investigation for Cases A, B, and C are 28 per cent higher, 0 per cent higher, and 21 per cent higher, respectively, than the index shown. Possible reasons for this difference are discussed in the next section.

CONCLUSIONS AND DISCUSSION

Conclusions

The results of this investigation indicate that:

1. For progressive waves, the kinematic stability parameter, $\frac{u_m}{C}$ rather than the dynamic parameter, $-\frac{1}{g}\frac{Dw}{Dt}$, governs breaking. The dynamic criterion was found to equal zero at breaking; this implies that immediately under the crest, hydrostatic conditions prevail.
2. The enclosed crest angle associated with the limiting wave is approximately equal to 120 degrees, a value determined by previous investigators. Earlier investigators, however, required the _a priori_ assumption that the crest form be an angle which is a submultiple of 360 degrees.
3. The breaking wave heights determined here are somewhat higher (0 to 28 per cent) than those usually referenced.
4. The stream function wave theory appears well suited for representing the geometry, kinematics, and dynamics of periodic water wave systems up to breaking conditions.

Discussion

Probably the most significant results of this study are that: (1) the dynamic parameter, as defined, is zero at breaking rather than one-half as usually referenced, and (2) the breaking wave heights determined in this study are somewhat larger than those determined in earlier investigations. It is important, when considering these differences, to recall that _all_ but one of the previous studies has required the _a priori_ assumption that the limiting wave crest at breaking be a sharp angle which is a submultiple of 360 degrees. The present investigation required no _a priori_ assumptions, but was based on determining the representation which best satisfies _all_ of the defining equations.

With regard to the question of the value of the dynamic stability parameter at breaking, it is instructive to expand the expression for this parameter at breaking for the case of a stationary wave system

$$\text{DSP:} \quad -\frac{1}{g}\frac{Dw}{Dt} = -\frac{1}{g}\left[(u-C)\frac{\partial w}{\partial x} + \frac{1}{2}\frac{\partial w^2}{\partial z}\right]_{crest} \qquad (10)$$

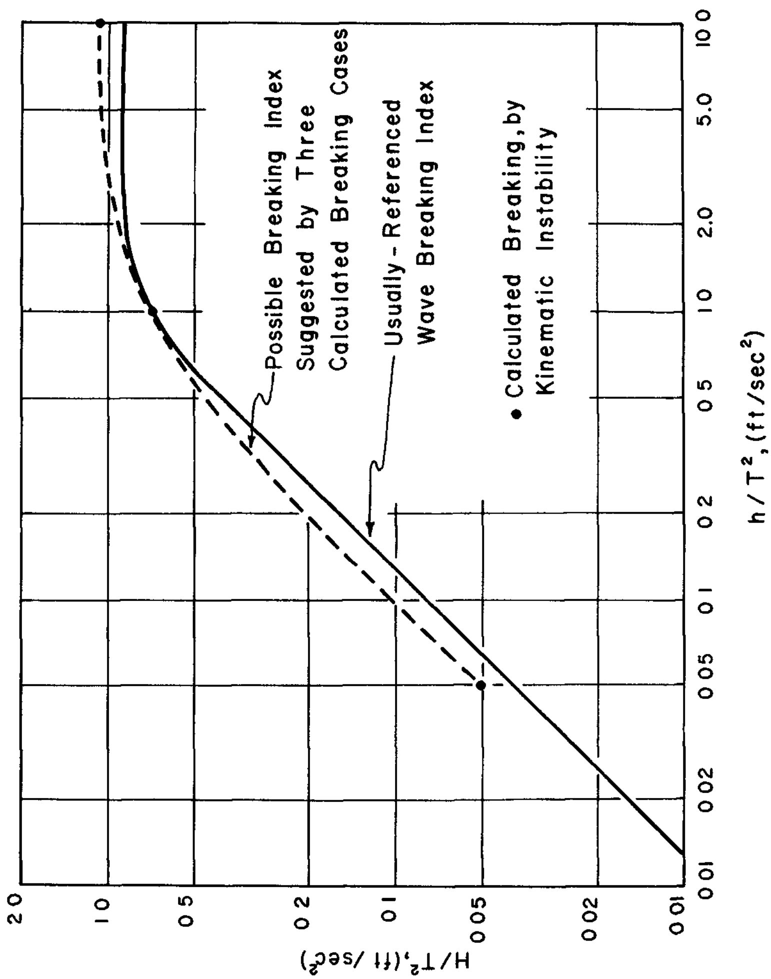

FIGURE 7. COMPARISON OF CALCULATED BREAKING WAVE CHARACTERISTICS WITH USUALLY-REFERENCED INDEX

Consider the two terms enclosed by the brackets. The second term is identically zero at the crest phase position where instability would be initiated. The first term contains the kinematic stability parameter; and since this is zero at breaking, therefore, it is seen that the dynamic stability parameter must be zero at breaking unless $\partial w/\partial x = \infty$. This will only occur if $\partial \eta/\partial x = \infty$ at the crest which is precisely the feature embodied in other investigations with the a priori assumption that the crest form a sharp angle. The interpretation here is that the DSP is zero at breaking and that the term $\partial w/\partial x$ is finite up to breaking. There is always the possibility that the term $\partial w/\partial x$ would approach infinity as the number of terms in the stream function expression increases without limit. To test this possibility, higher order representations were checked to see whether the additional terms contributed substantially to the wave form or to $\partial w/\partial x$. The conclusion reached from these calculations was that the observed result was not an artifice of the finite number of terms in the series representation. Other possible extraneous causes, that could be associated with the numerical procedure employed, are being examined.

The final question to be considered is whether the breaking values determined here or those in previous investigations are more nearly correct. Although this question cannot be absolutely resolved, the writer concludes that because the approach presented here requires no a priori assumption concerning the wave form (as discussed above) the results of the present investigation may be more valid. Further work, including breaking computations for a greater number of relative depths, is warranted to provide additional results on this facet of the problem.

ACKNOWLEDGMENTS

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REFERENCES

1. Dean, R. G., "Stream Function Representation of Nonlinear Ocean Waves," Journal of Geophysical Research, 70(18), pp. 4561-4576, September, 1965.

2. Stokes, G. G., "On the Theory of Oscillatory Waves," Transactions, Cambridge Philosophical Society, Vol. 8, and Supplement, Scientific Papers, VII, 1847.

3. McCowan, J., "On the Solitary Wave," London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, Vol. 32(5), 1891.

4. Gerstner, F., "Theorie der Wellen," Abhandlung en der Koniglichen Bohmischen Gesellschaft der Wissenschaften, Prague, 1802; also, Gilberts Annalen der Physik XXXII, pp. 412-445, 1809 (in English as Technical Report Series 3, Issue 339, University of California Institute of Engineering Research, Wave Research Laboratory, 1952, Translated by R. M. Kay).

5. Borgman, L. E. and J. E. Chappelear, "The Use of the Stokes-Struik Approximation for Waves of Finite Height," Proceedings, Sixth Conference on Coastal Engineering, Chap. 16, 1958.

6. Laitone, E. V., "The Second Approximation to Cnoidal and Solitary Waves," Journal of Fluid Mechanics, Vol. 9, Part 3, pp. 430-444, November, 1960.

7. Skjelbreia, L. and J. A. Hendrickson, "Fifth Order Gravity Wave Theory," Proceedings, Seventh Conference on Coastal Engineering, Chap. 10, pp. 184-196, 1961.

8. Dean, R. G., "Relative Validities of Water Wave Theories," Proceedings, ASCE Specialty Conference on Civil Engineering in the Oceans, San Francisco, pp. 1-30, 1968.

9. Michell, J. H., "On the Highest Waves in Water," Philosophical Magazine, 36(5), pp. 430-435, 1893.

10. McCowan, J., "On the Highest Wave of Permanent Type," London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, Vol. 38, 1894.

11. Yamada, H., "Highest Waves of Permanent Type on the Surface of Deep Water," Reports Research Institute of Applied Mechanics, Vol. 5, No. 18, pp. 37-52, 1957.

12. Chappelear, J. E., "On the Theory of the Highest Waves," Beach Erosion Board, TM No. 116, July, 1959.

13. Laitone, E. V., "Limiting Conditions for Cnoidal and Stokes Waves," Journal of Geophysical Research, 67(4), pp. 1555-1564, April, 1962.

14. Lenau, C. W., "The Solitary Wave of Maximum Amplitude," Journal of Fluid Mechanics, Vol. 26, Part 2, pp. 309-320, 1966.

15. Kinsman, B., "Wind Waves," Prentice Hall, Inc., Englewood Cliffs, New Jersey, p. 273, 1965.