Brick manifolds and toric varieties of brick polytopes

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Abstract

In type A, Bott-Samelson varieties are posets in which ascending chains are flags of vector spaces. They come equipped with a map into the flag variety G/B. These varieties are mostly studied in the case in which the map into G/B is birational to the image. In this paper we study Bott-Samelsons for general types, more precisely, we study the combinatorics a fiber of the map into G/B when it is not birational. In order to do so we use the moment map of a Bott-Samelson variety to translate this problem into one in terms of the "subword complexes" of Knutson and Miller. Pilaud and Stump realized certain subword complexes as the dual of the boundary of a polytope which generalizes the brick polytope defined by Pilaud and Santos. For a nice family of words, the brick polytope is the generalized associahedron realized by Hohlweg, Lange and Thomas. These stories connect in a nice way: we show that the moment polytope of the brick manifold is the brick polytope. In particular, we give a nice description of the toric variety of the associahedron. We give each brick manifold a stratification dual to the subword complex. In addition, we relate brick manifolds to Brion's resolutions of Richardon varieties.

Keywords: Brick polytopes; Bott-Samelson varieties; subword complexes

1 Introduction

The Bott-Samelson varieties were first defined by Bott and Samelson in [3]. Bott-Samelson varieties are a twisted product of $\mathbb{CP}^1$'s with a map into the flag variety G/B. These varieties have been studied mostly in the case in which the map into G/B is birational. In this paper we study some fibers of this map when it is not birational to the image. We show that for some Bott-Samelson varieties this fiber is a toric variety. In order to do so we translate this problem into a purely combinatorial one in terms of subword complexes.

These simplicial complexes $\Delta(Q, w)$ depend on a word Q in the generators of the Weyl group W of G and an element $w \in W$. They were defined by Knutson and Miller in [11] to describe the geometry of determinantal ideals and Schubert polynomials. In [4], Ceballos, Labbé and Stump connect subword complexes to cluster algebras of finite type. Fomin and Zelevinski introduced cluster algebras and classified the finite type cluster algebras, see [5, 6]. Ceballos, Labbé and Stump showed that for a nice family of words, subword complexes are isomorphic to the cluster complexes arising from the cluster algebra of the corresponding type. In [14], Pilaud and Stump defined the brick polytope and realized certain subword complexes as the boundary of a polytope dual to the brick polytope. For the family of subword complexes related to the cluster complexes, they obtained that the brick polytopes are the generalized associahedra of Hohlweg, Lange, and Thomas in [9]. The normal fans of these associahedra are the Cambrian fans of Reading and Speyer [15]. In Theorem 23 we prove that for the words Pilaud and Stump define as "root independent", a fiber of the Bott-Samelson map is the toric variety of the brick polytope. In particular, this provides a description of the toric variety of a generalized associahedron, which in type A we interpret in terms of flags arranged in a poset.

Actually the toric case is just a shadow of a more general situation. We prove in Theorem 21 that for any word Q and its Demazure product $w \in W$ the brick polytope is the moment polytope of a fiber of the Bott-Samelson variety. This motivates us to define the brick manifold as the fiber studied here. In this paper we show a very nice connection between subword complexes, brick polytopes and brick manifolds. In Theorem 23 we classify the toric brick manifolds. We end the paper with two results about brick manifolds: we exhibit a stratification of the brick manifolds dual to the subword complex in Theorem 24 and following [2], show that brick manifolds provide resolutions for Richardson varieties in Theorem 26.

2 Some definitions

2.1 Subword complexes

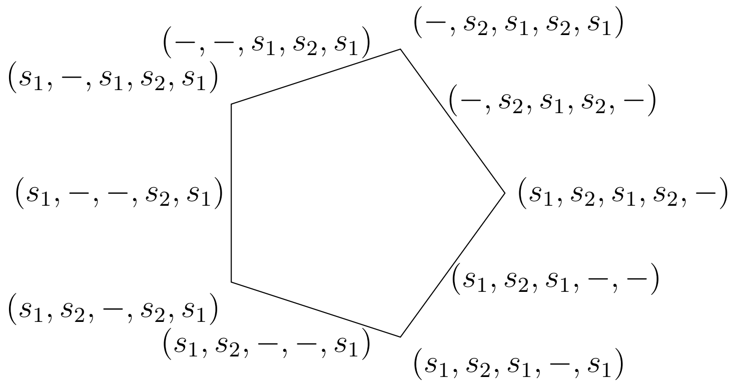
Let W be the Weyl group of a complex Lie group G with respect to a torus T, i.e., W is a crystallographic Coxeter group, and let $S = \{s_i : i \in I\}$ denote its generators.

Let $Q = (q_1, \ldots, q_m)$ be a *word* in S, i.e. an ordered sequence of elements of S. A *subword* J of Q is a subset of positions in Q and we represent it by replacing the letters of its complement by $-$. There are a total of $2^{|Q|}$ subwords of Q. Given a subword $J = (r_1, \ldots, r_m)$, we denote by $Q \setminus J$ the subword with k-th entry equal to $-$ if $r_k \neq -$ and equal to k otherwise for $k = 1, \ldots, m$. For example, $J = (1, -, 3, -, 5)$ is a subword of $Q = (s_1, s_2, s_3, s_1, s_2)$ and $Q \setminus J = (-, 2, -, 4, -)$. Given a subword J we denote by $J_{(k)}$ the product $q_{r_1} q_{r_2} \cdots q_{r_k}$ with q_- behaving as the identity, if $k \geqslant 1$, and $J_{(0)} = 1$.

Definition 1. Let $Q = (q_1, \ldots, q_m)$ be a word in S and $w \in W$. The *subword complex* $\Delta(Q, w)$ is the simplicial complex on the vertex set the set of positions of letters in Q whose facets are the subwords F of Q such that the product $(Q \setminus F)_{(m)}$ is a reduced expression for w.

In this paper, we will only consider spherical subword complexes and Q a non reduced expression.

Example 2. Let $W = \mathfrak{S}_3$, $Q = (s_1, s_2, s_1, s_2, s_1)$, then the simplicial complex $\Delta(Q, s_1 s_2 s_1)$ is



In order to make the reduced expression more explicit, we are labeling the face corresponding to J by the letters in Q that are in $Q \setminus J$.

Definition 3. We define the *Demazure product* of a word Q inductively as follows:

- $\mathrm{Dem}(\varepsilon) = \mathrm{id}$
- $\mathrm{Dem}((Q, s)) = \begin{cases} \mathrm{Dem}(Q) \cdot s & \text{if } \ell(\mathrm{Dem}(Q)s) > \ell(\mathrm{Dem}(Q)) \\ \mathrm{Dem}(Q) & \text{if } \ell(\mathrm{Dem}(Q)s) < \ell(\mathrm{Dem}(Q)), \end{cases}$

where ε denotes the empty word.

Remark 4. In [11] the authors prove that $\Delta(Q, w)$ is a sphere if and only if $\mathrm{Dem}(Q) = w$. In this paper we only consider such pairs. If in addition we assume Q is reduced, then $\Delta(Q, w) = \{\emptyset\}$, the (-1)-sphere, so we will not consider reduced Q in this paper.

We end the discussion about subword complexes with a historical note about the realization of the type A associahedron in terms of subword complexes. In [18], Woo uses the pipe dream complex for the permutation $w = s_1 \cdots s_n w_0 = [1, n, n-1, \ldots, 1]$, where w_0 is the longest element of A_{n-1}, to show that the Schubert polynomial of w specializes to the Catalan number C_n. It was first noted by Pilaud and Pocchiola in [12] that the type A associahedron can be realized using pseudoline arrangements, which are a way to describe subword complexes. The works of Stump [17] and Serrano-Stump [16] describe the type A associahedron explicitly in terms of the subword complexes.

2.2 Brick polytopes

Let $\Delta(W) := \{\alpha_s : s \in S\}$ be the simple roots of W and let $\nabla(W) := \{\omega_i : s_i \in S\}$ be its fundamental weights. We denote by $\Phi^+(W)$ the positive roots of W and by $\Phi^-(W)$ the negative roots. Pilaud and Santos [13] define type A brick polytopes; Pilaud and Stump define brick polytopes for arbitrary type and study their properties in [14]. For them, the brick polytope is the convex hull of some conjugates of the fundamental weights of the Weyl group, one per each facet of the subword complex. Our definitions in this section are based on theirs, however we define the brick polytope be the convex hull of the brick vectors corresponding to all the faces in the subword complex such that the product of the complement is w. It turns out that the two definitions are equivalent as the proof of Theorem 21 exhibits.

The following functions, used to define brick polytopes, were defined and studied in [4] and in [14]. Given a subword complex $\Delta(Q, w)$ with $|Q| = m$ define the *root function*

$$\mathsf{r}(J, \cdot) : \{\text{subwords of } Q\} \to \Phi^+(W) \cup \Phi^-(W)$$

$$\mathsf{r}(J, k) := (Q \setminus J)_{(k-1)}(\alpha_{q_k}) \tag{1}$$

and the *weight function*

$$\mathsf{w}(J, \cdot) : \{\text{subwords of } Q\} \to \text{weights of } W$$

$$\mathsf{w}(J, k) := (Q \setminus J)_{(k-1)}(\omega_{q_k}). \tag{2}$$

Definition 5. The *brick vector* of a face J of $\Delta(Q, w)$ is defined by

$$B(J) := \sum_{k \in [m]} \mathsf{w}(J, k),$$

and the *brick polytope* is the convex hull of the brick vectors of some faces of $\Delta(Q, w)$

$$B(Q, w) := \operatorname{conv}\{B(J) : J \in \Delta(Q, w) \text{ and } (Q \setminus J)_{(m)} = w\}.$$

Definition 6. A word Q is *root independent* if for some vertex $B(J)$ of $B(Q, w)$ (or all vertices) we have that the multiset $r(J) := \{\{\mathsf{r}(J, i) : i \in J\}\}$ is linearly independent.

Pilaud and Stump in [14] show that if Q is root independent, then the brick polytope is dual to the subword complex. One of the main theorems of this paper states that the brick manifold of a word Q is toric with respect to a maximal torus of the Lie group when Q is root independent.

3 Brick manifolds for $SL_n(\mathbb{C})$

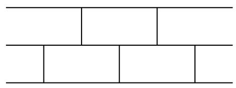
We start with the case $G = SL_n(\mathbb{C})$ as a motivation to the general complex semi-simple Lie group case.

3.1 Brick polytopes in the $SL_n(\mathbb{C})$ case

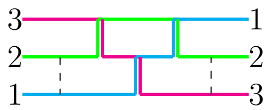
This section is based on the interpretation of type A subword complexes as pseudoline arrangements given in [13]. The *sorting network* $\mathcal{N}_Q$ of a word $Q = (q_1, \ldots, q_m)$ consists of n horizontal lines (called the *levels*) and m vertical segments (called the *commutators*) drawn from left to right so that each commutator joins consecutive levels, no two commutators share a common endpoint, and if $q_k = s_i$ then the k-th commutator connects levels i and $i+1$. A *brick* of $\mathcal{N}_Q$ is a connected component of its complement, bounded on the left by a commutator.

A *pseudoline* supported by $\mathcal{N}_Q$ is a path on $\mathcal{N}_Q$ traveling monotonically from left to right. A commutator of $\mathcal{N}_Q$ is called a *crossing* between two pseudolines if it is crossed by the two pseudolines and it is called a *contact* otherwise. A *pseudoline arrangement* on $\mathcal{N}_Q$ is a collection of n pseudolines such that each two have at most one crossing and no other intersection.

Example 7. Let $Q = (s_1, s_2, s_1, s_2, s_1)$ then the sorting network $\mathcal{N}_Q$ is



and

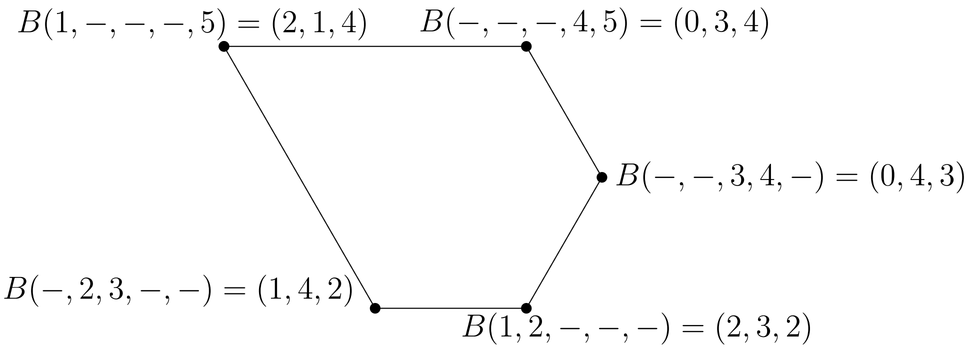


is a pseudoline arrangement on $\mathcal{N}_Q$.

Given a pseudoline arrangement supported by $\mathcal{N}_Q$, if we let $J = (r_1, \ldots, r_m)$ be the subword of Q with $r_i \neq -$ precisely when there is a contact at the i-th commutator, then the product $w = (Q \setminus J)_{(m)}$ is an element of W and the pseudoline ending on the right at level i will start on the left at level $w(i)$. We call such an arrangement a *w-pseudoline arrangement*. There is a one-to-one correspondence between faces J of $\Delta(Q, w)$ and w-pseudoline arrangements supported by $\mathcal{N}_Q$. The pseudoline arrangement in the previous example corresponds to the subword $J = (1, -, -, -, 5)$. In this setup, we have that $\mathsf{w}(J, j)$ is the characteristic vector of the pseudolines passing below the j-th brick of $\mathcal{N}_Q$. Moreover, the i-th coordinate of the *brick vector* $B(J)$ is the number of bricks in $\mathcal{N}_Q$ that lie above the i-th pseudoline with contacts J, and the brick polytope $B(Q, w)$ is the following convex hull:

$$B(Q, w) := \operatorname{conv}\{B(J) : J \in \Delta(Q, w) \text{ and } (Q \setminus J)_{(m)} = w\}.$$

Example 8. Let $Q = (s_1, s_2, s_1, s_2, s_1)$, the pseudoline arrangement corresponding to the subword $J = (1, -, -, -, 5)$ gives the vector $B(J) = (2, 1, 4)$ obtained by counting bricks above each line. The brick polytope $B(Q, w)$ is pictured below.



For more pictures of brick polytopes of various Q and w, see [14].

A purpose of this paper is to assign geometry to these polytopes. To do so, we use the Bott-Samelson varieties which we define in the following section.

3.2 Definition of Bott-Samelson varieties for $SL_n(\mathbb{C})$

Let $G = SL_n(\mathbb{C})$ and fix an ordered basis for $\mathbb{C}^n$. Let B be the subgroup of $SL_n(\mathbb{C})$ consisting of upper triangular matrices with respect to this basis. We then get an ascending flag of B-invariant vector spaces

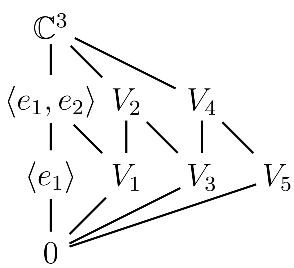
$$\langle e_1 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle,$$

which we refer to as the *base flag*. Let T be the subgroup consisting of all diagonal matrices in G, so T is a maximal torus contained in B. Let P_i be the minimal parabolic subgroup consisting of all matrices that are upper triangular except possibly at the position $(i+1, i)$. The quotient G/B is the flag variety, that is, the space of flags $\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ where each V_i is an i-dimensional vector space. Moreover, the Weyl group of G is $W = A_{n-1}$ with generators $S = \{s_1, \ldots, s_{n-1}\}$. The fundamental weights are $\nabla(W) = \{\omega_i : i = 1, \ldots, n-1\}$ where the first i entries of ω_i are 1 and the rest are 0.

We begin the definition of BS^Q with an example.

Example 9. Let $G = SL_3(\mathbb{C})$ and $Q = (s_1, s_2, s_1, s_2, s_1)$. Then the Bott-Samelson variety BS^Q is constructed by starting with the base flag and then iteratively reading the word from left to right: if the k-th letter of Q is s_i, we have an i-th dimensional vector space V_k such that $V_{k-1} \subset V_k \subset V_{k+1}$. In this example we have that

$$BS^Q = \{(V_1, V_2, V_3, V_4, V_5) : \text{ the diagram below holds}\}$$

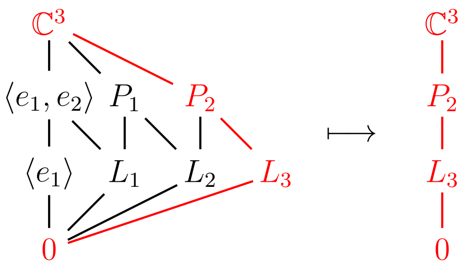


More generally, if $Q = (q_1, \ldots, q_m)$ then BS^Q consists of a list of $m+1$ flags where the zeroth one is the base flag and such that the k-th one agrees with the previous one except possibly on the k-th subspace V_k. We can give a point in BS^Q by giving the subspaces $(V_1, \ldots, V_m)$ such that the incidence relations given by the flags hold. This carries a B-action, and the map $BS^Q \xrightarrow{m_Q} G/B$ mapping the list to the last flag is B-equivariant.

Example 10. Continuing with the previous example, we have that

$$m_Q : BS^{(s_1,s_2,s_1,s_2,s_1)} \to G/B$$

is the map

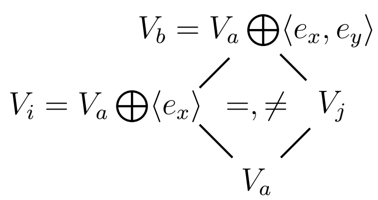


We now define the main object of study in this paper.

Definition 11. Let $Q = (q_1, \ldots, q_m)$ be a word in the generators of W with $\mathrm{Dem}(Q) = w$, then the *brick manifold* is the fiber $m_Q^{-1}(wB/B)$.

The group $SL_n(\mathbb{C})$ acts on $\mathbb{C}^n$ by multiplying matrices in $SL_n(\mathbb{C})$ times vectors in $\mathbb{C}^n$. This induces an action of $SL_n(\mathbb{C})$ on the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$ of k-dimensional subspaces of $\mathbb{C}^n$, where if $M \in SL_n(\mathbb{C})$ and $\langle b_1, \cdots, b_k\rangle \in \mathrm{Gr}(k, \mathbb{C}^n)$, then $M \cdot \langle b_1, \cdots, b_k\rangle := \langle M \cdot b_1, \cdots, M \cdot b_k\rangle$. This induces an action of $SL_n(\mathbb{C})$ on BS^Q, namely if $M \in SL_n(\mathbb{C})$ and $(V_1, \ldots, V_m) \in BS^Q$ then $M \cdot (V_1, \ldots, V_m) := (M \cdot V_1, \ldots, M \cdot V_m)$. We can restrict this action to any subgroup of $SL_n(\mathbb{C})$, in particular to B and T.

Notice that V is a T-fixed point of $\mathrm{Gr}(k, \mathbb{C}^n)$ if and only if V is spanned by a subset of $\{e_1, \ldots, e_n\}$ and the same is true for elements of BS^Q. The T-fixed point $p(J) \in BS^Q$ *corresponding to the subword* $J = (r_1, \ldots, r_m)$ is determined by deciding between $=$ and $\neq$ in each diamond



using the rule: for $Q = (q_1, \ldots, q_m)$, we pick "$=$" if $r_j \neq -$ and "$\neq$" if $r_j = -$. This gives a 1-1 correspondence between T-fixed points on BS^Q and subwords J of Q such that if $p(J)$ is the T-fixed point corresponding to J then $m_Q(p(J)) = (Q \setminus J)_{(m)} B/B \in G/B$. We illustrate this correspondence by an example.

Example 12. The subword $J = (-, 2, -, -, 5)$ of $Q = (s_1, s_2, s_1, s_2, s_1)$ corresponds to the coordinate flags

$$\begin{tikzcd}
\mathbb{C}^3 \arrow[d, dash] \arrow[drr, dash] \arrow[drrrr, dash] \\
\langle e_1, e_2\rangle \arrow[d, dash] \arrow[drr, dash] & = & \langle e_1, e_2\rangle \arrow[d, dash] \arrow[drr, dash] & \neq & \langle e_1, e_3\rangle \arrow[d, dash] \arrow[drr, dash] \\
\langle e_1\rangle \arrow[d, dash] & \neq & \langle e_2\rangle & \neq & \langle e_1\rangle & = & \langle e_1\rangle \\
0 \arrow[urr, dash] \arrow[urrrr, dash] \arrow[urrrrrr, dash]
\end{tikzcd}$$

and its image under $m_Q : BS^Q \to G/B$ is $(Q \setminus J)_{(m)}B = (s_1s_1s_2)B = (s_2)B$.

This correspondence motivates the relation between fibers of the map m_Q and subword complexes. The main tool connecting brick polytopes with fibers of Bott-Samelson varieties will be moment maps of symplectic manifolds. We will discuss the symplectic manifold structure on general BS^Q in Section 4.1. Namely, we will show that Bott-Samelson varieties are Hamiltonian symplectic manifolds with respect to the torus action described above. Therefore, a Bott-Samelson variety comes equipped with a *moment map* associated to the torus action. The image of this map is the *moment polytope* and it equals the convex hull of the images of the T-fixed points. Every toric variety is a Hamiltonian symplectic manifold with respect to the torus action. Moreover, if X is the toric variety associated to a Delzant polytope P then the image of the moment map is the polytope P.

In order to motivate latter sections and, more importantly, to be able to state the theorem connecting Bott-Samelson varieties and brick polytopes, we now describe the moment map of BS^Q for the current case of interest, $G = SL_n(\mathbb{C})$. The moment map is a map

$$\mu : BS^Q \longrightarrow \mathbb{R}\langle\nabla(W)\rangle,$$

where $\mathbb{R}\langle\nabla(W)\rangle$ is the real span of the fundamental weights of W. Let $\pi_V : \mathbb{C}^n \to V$ denote the orthogonal projection onto V and let P_V be the corresponding matrix with respect to the basis $e_1, \ldots, e_n$. Given $p = (V_1, \ldots, V_m) \in BS^Q$ the moment map is

$$\begin{aligned} BS^Q &\xrightarrow{\mu} \mathbb{R}^n \\ (V_1, \ldots, V_m) &\xmapsto{\mu} \sum_{i=1}^{m} \operatorname{diag}(P_{V_i}), \end{aligned}$$

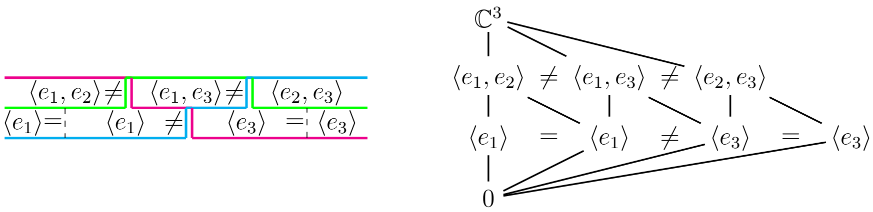
where $\operatorname{diag}(P_{V_i})$ is the vector with entries the diagonal entries of P_{V_i}.

In the following section we give a precise statement about the relation between brick polytopes and Bott-Samelson varieties.

3.3 Toric varieties for brick polytopes in the $SL_n(\mathbb{C})$ case

Recall from section 3.2 that subwords J of Q are in bijective correspondence with T-fixed points of BS^Q, and that if $p(J)$ is the point corresponding to J, as defined in the previous section, then $m_Q(p(J)) = (Q \setminus J)_{(m)} B/B \in G/B$, where $m = |Q|$. This means that the rightmost flag of the configuration $p(J)$ is the flag corresponding to $(Q \setminus J)_{(m)} \in W$ and so, for J a facet of $\Delta(Q, w_0)$ the pseudoline arrangement corresponding to J is an $(Q \setminus J)_{(m)}$-arrangement. In this section, we will abuse notation and use the term *w-pseudoline arrangement* to refer to a collection of n pseudolines where two can have more than one crossing, but they can't overlap on any levels. The following example shows the correspondence.

Example 13. The pseudoline arrangement for the subword $J = (1, -, -, -, 5)$ gives a T-fixed point of $BS^{(s_1,s_2,s_1,s_2,s_1)}$. The diagram below exhibits this correspondence. Each brick of the sorting network corresponds to a coordinate subspace of a point in the Bott-Samelson variety. Given a pseudoline arrangement supported in the sorting network of Q, the j-th subspace corresponding to the j-th brick is the coordinate subspace spanned by the e_i where i ranges over those pseudolines passing below the j-th brick. Note then that two bricks share a contact if and only if the corresponding coordinate spaces are equal. This will be proven in the theorem that follows.



Theorem 14. *Suppose $w \leqslant Dem(Q)$ in Bruhat order. There is a bijective correspondence between w-pseudoline arrangements supported by $\mathcal{N}_Q$ and T-fixed points of $m_Q^{-1}(wB/B)$. Moreover, this correspondence makes the composite map*

$$m_Q^{-1}(wB/B)^T \hookrightarrow m_Q^{-1}(wB/B) \xrightarrow{\mu} \mathbb{R}^n$$

be equivalent to the mapping

$$B : \{w\text{-pseudoline arrangements supported by } \mathcal{N}_Q\} \longrightarrow \mathbb{R}^n$$

given in [13].

Proof. The first part of the proposition is proven in the first paragraph of Section 3.3. We prove the second part of this theorem using induction on $|Q| = m$ to prove that $\mu(p(J)) = \mathsf{B}(J)$ for all subwords J, where $p(J) = (V_1, \ldots, V_m)$ is the point in BS^Q corresponding to J. We are proving this correspondence of ll subwords of Q, not only those for which $(Q \setminus J)_m = w$. Let $Q = (q_1, \ldots, q_{m+1})$. Recall that the rightmost flag of the fixed point $p(J)$ corresponding to the subword J is

$$\begin{array}{c} \langle e_{w(1)}, \ldots, e_{w(n)} \rangle = \mathbb{C}^n \\ | \\ \vdots \\ | \\ \langle e_{w(1)}, e_{w(2)} \rangle \\ | \\ \langle e_{w(1)} \rangle \\ | \\ 0 \end{array}$$

where $w = (Q \setminus J)_{(m+1)}$. Let J be a subword of Q and consider the words $Q' = (q_1, \ldots, q_m)$ and $J' = (j_1, \ldots, j_m)$. By induction we have that $\mu(p(J')) = \mathsf{B}(J')$. Now notice that

$$\begin{aligned} \mu(p(J)) &= \mu(p(J')) + (\dim_{e_1}(V_{k+1}), \ldots, \dim_{e_n}(V_{k+1})) \\ &= \mu(p(J')) + w \cdot (1, \ldots, 1, 0, \ldots, 0), \end{aligned}$$

where the 0-1 vector has as many ones as $\dim(V_{k+1})$. The vector $w \cdot (1, \ldots, 1, 0, \ldots, 0)$ adds one to the i-th coordinate if and only if the brick corresponding to the commutator q_{k+1} is above the i-th pseudoline. $\square$

Theorem 15. *Let $w = Dem(Q)$. The fiber $m_Q^{-1}(wB/B)$ is a toric variety with respect to the torus T if and only if Q is root independent and $\ell(w) < |Q| \leqslant \ell(w) + \dim(T)$. Moreover, $m_Q^{-1}(wB/B)$ is the toric variety associated to the polytope $B(Q, w)$.*

We have proved the if part of this theorem; however the only if part will follow from Theorem 23. The following corollary follows from the work of Pilaud and Santos in [13]. We define a *Coxeter element c* to be the product of all simple reflections in some order using each reflection only once. Define the ***c**-sorting word* of w to be the lexicographically first subword of $\mathbf{c}^\infty$ that is a reduced expression for w.

Corollary 16. *If Q is the concatenation of a word **c** representing a Coxeter element c and the **c**-sorting word for w_0, then $m_Q^{-1}(w_0B/B)$ is the toric variety of the associahedron as realized in [9] and in [13].*

Example 17. The toric variety of the pentagon from example 8, i.e. the associahedron corresponding to the Coxeter element $c = (s_1, s_2)$, is

$$m_Q^{-1}(wB/B) = \{(V_1, V_2, V_3) : \text{ the diagram below holds}\}$$

$$\begin{tikzcd}[column sep=small] \mathbb{C}^3 \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\ \langle e_1, e_2 \rangle \arrow[d, dash] \arrow[dr, dash] & V_2 \arrow[d, dash] \arrow[dr, dash] & \langle e_2, e_3 \rangle \arrow[d, dash] \arrow[dr, dash] & \\ \langle e_1 \rangle \arrow[d, dash] & V_1 \arrow[dl, dash] & V_3 \arrow[dll, dash] & \langle e_3 \rangle \arrow[dlll, dash] \\ 0 & & & \end{tikzcd}$$

4 Brick manifolds in the general case

Let G be a complex semisimple Lie group, let B be a Borel subgroup of G, i.e., a maximal solvable subgroup, and T be the maximal torus contained in B. Let W be the Weyl group of G with generators $S = \{s_1, \ldots, s_n\}$, which correspond to the simple roots $\Delta(W) = \{\alpha_1, \ldots, \alpha_n\}$. Let P be a parabolic subgroup of G, i.e., a subgroup of G for which the quotient P/B is a projective algebraic variety; this condition is equivalent to P contains B. We denote by P_i the minimal parabolic subgroup corresponding to s_i, we then have that $P_i/B \cong \mathbb{CP}^1$. The torus T acts on this quotient and this action has exactly two T-fixed points: one corresponding to the identity element and one corresponding to the generator s_i.

Definition 18. Let $Q = (s_{i_1}, \ldots, s_{i_m})$ be a word in the generators of W. Then the product $P_{i_1} \times \cdots \times P_{i_m}$ has an action of B^m given by:

$$(b_1, \ldots, b_m) \cdot (p_1, \ldots, p_m) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{m-1}^{-1} p_m b_m)$$

The *Bott-Samelson variety* of Q is the quotient of the product of the P_i's by this action

$$BS^Q := (P_{i_1} \times \cdots \times P_{i_m})/B^m.$$

Bott-Samelson varieties are smooth, irreducible and $|Q|$-dimensional algebraic varieties. They have a B action given by

$$b \cdot (p_1, p_2, \ldots, p_m) = (b \cdot p_1, p_2, \ldots, p_m).$$

and they come equipped with a natural B-equivariant map

$$\begin{aligned} BS^Q &\xrightarrow{\mathrm{m_Q}} G/B \\ (p_1, \ldots, p_m) &\longmapsto (p_1 \cdots p_m)B/B. \end{aligned}$$

The image of this map is the opposite Schubert variety $X^w := \overline{BwB/B}$, where $w = \mathrm{Dem}(Q)$. In the case in which Q is reduced, this map is a resolution of singularities for X^w, however in this paper we are studying cases in which Q is not reduced.

Definition 19. Let $Q = (q_1, \ldots, q_m)$ be a word in the generators of W and $w = \mathrm{Dem}(Q)$, then the *brick manifold* is the fiber $m_Q^{-1}(wB/B)$.

Theorem 20. *Brick manifolds are smooth, irreducible and* $\dim(m_Q^{-1}(wB/B)) = |Q| - \ell(w)$.

Proof. We can write the fiber as the fibered product $(wB/B) \times_{X^w} BS^Q$, so by Kleiman's transversality theorem, see [10], we have that this fiber is a smooth variety of the desired dimension. Let N be the unipotent subgroup corresponding to B and N_- the opposite unipotent subgroup. A consequence of the Bruhat decomposition of G/B is that if $N_w := N \cap wN_-w^{-1}$, then $N_w \cdot wB/B$ is a free dense orbit in X^w. Since BS^Q maps B-equivariantly to X^w, the preimage of $N_w \cdot wB/B$ is isomorphic to $m_Q^{-1}(wB/B) \times N_w$. Since BS^Q is irreducible, it follows that the brick manifold is irreducible. ☐

4.1 Symplectic structure on Bott-Samelson varieties and brick manifolds

A reference for toric moment maps of coadjoint orbits is Chapter 5 of [7]. Let $P_{\hat{i}}$ be the maximal parabolic subgroup of G corresponding to the subset of generators $S_{\hat{i}} := \{s_1, \dots, \hat{s}_i, \dots, s_n\}$. Note that for $G = SL_n(\mathbb{C})$ each quotient $G/P_{\hat{i}}$ is a Grassmannian. Let K be the maximal compact subgroup of G. Then we can view $G/P_{\hat{i}}$ as a coadjoint orbit, i.e., a K-orbit through the fundamental weight $\omega_i \in \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K. This interpretation gives us a symplectic structure on $G/P_{\hat{i}}$ with respect to the action of K such that the inclusion

$$G/P_{\hat{i}} \hookrightarrow \mathfrak{k}^*$$

is a moment map for the K-action. Then the composition

$$G/P_{\hat{i}} \hookrightarrow \mathfrak{k}^* \longrightarrow \mathfrak{t}^*$$

is the moment map of $G/P_{\hat{i}}$ with respect to the torus action, where $\mathfrak{t}$ is the Lie algebra of the torus. Moreover, the moment map for the diagonal T-action on a product $\prod G/P_{\hat{i}}$ is the sum of the moment maps $G/P_{\hat{i}} \longrightarrow \mathfrak{t}^*$.

Let T act on BS^Q by

$$t \cdot (p_1, p_2, \dots, p_m) = (t \cdot p_1, p_2, \dots, p_m).$$

Given $Q = (q_1, \dots, q_m)$ we have a T-equivariant inclusion

$$BS^Q \overset{\varphi}{\hookrightarrow} \prod_{i:s_i \in Q} G/P_{\hat{i}}$$

where $\varphi = (\varphi_1, \dots, \varphi_m)$ and the k-th component is

$$\begin{aligned} BS^Q &\xrightarrow{\varphi_k} G/P_{\hat{k}} \\ (p_1, \dots, p_m) &\longmapsto (\prod_{i<j} p_i) P_{\hat{k}}. \end{aligned}$$

This map makes BS^Q a symplectic submanifold. The composition

$$BS^Q \overset{\varphi}{\hookrightarrow} \prod_{i:s_i \in Q} G/P_{\hat{i}} \longrightarrow \mathfrak{t}^*$$

gives us a moment map for this Bott-Samelson variety with respect to the T-action. Thus Bott-Samelson varieties are Hamiltonian symplectic manifolds with respect to this torus action. The image of this map is the *moment polytope* and by Atiyah [1], Guillemin-Sternberg [8], it equals the convex hull of the images of the T-fixed points. Recall the correspondence between T-fixed points on BS^Q and subwords J of Q: if $p(J)$ is the T-fixed point corresponding to J then

$$m_Q(p(J)) = (Q \setminus J)_{(m)} B/B \in G/B.$$

This correspondence motivates the relation between fibers of the map $m_Q : BS^Q \longrightarrow G/B$ and subword complexes.

We now describe the image of the T-fixed points under the moment map. For each k we have the moment map

$$\mu_k : G/P_{\hat{k}} \longrightarrow \mathfrak{t}^*,$$

where $\mu_k(P_{\hat{k}}) = \omega_k$, the fundamental weight corresponding to s_k, and it maps a general element to a Weyl conjugate of this fundamental weight. Before we finish describing the maps μ_k, we note that the moment map of BS^Q is then

$$\sum_{k=1}^{m} \varphi_k \circ \mu_k.$$

Consider the fixed point $(p_1, \dots, p_m)$ in BS^Q corresponding to the subword J of Q then under the moment map μ_k each p_j corresponds to either the reflection s_{i_j} if $q_j \in J$ or to the identity in W. In other words, p_j corresponds to s_{i_j} if $p_j \notin B$ and to the identity in W otherwise. In conclusion we have that for J subword of Q and

$$p_J = \text{ the fixed point corresponding to } J$$

$$\begin{aligned} BS^Q &\overset{\varphi_k \circ \mu_k}{\longrightarrow} \mathfrak{t}^* \\ p_J &\longmapsto (J)_{(k-1)}(\omega_k). \end{aligned}$$

It then follows that

$$BS^Q \overset{\mu}{\longrightarrow} \mathfrak{t}^* \tag{3}$$

$$p_J \longmapsto \sum_{k=1}^{m} (J)_{(k-1)}(\omega_k) \tag{4}$$

4.2 Moment polytopes of brick manifolds

We now state and prove the main results of the paper.

Theorem 21. *Let $w = Dem(Q)$. The image of $m_Q^{-1}(wB/B)$ under the moment map is the brick polytope $B(Q, w)$.*

To prove this theorem we will use the following lemma.

Lemma 22. *Let $i < j$, we have that $q_1 \cdots q_{i-1}(\alpha_{q_i}) = \pm q_1 \cdots q_{j-1}(\alpha_{q_j})$ if and only if*

$$q_1 \cdots \widehat{q_i} \cdots q_m = q_1 \cdots \widehat{q_j} \cdots q_m, \tag{5}$$

where all the $q_1, \dots, q_m \in S$. Moreover, if there is no $k \in (i, j)$ such that $q_1 \cdots \widehat{q_k} \cdots q_m$ equals to the product in equation (5) then $q_1 \cdots q_{i-1}(\alpha_{q_i}) = -q_1 \cdots q_{j-1}(\alpha_{q_j})$. We do not assume that any of these expressions are reduced.

Proof. This follows from induction on $|(i,j)|$ and Lemma 3.3 of [4]. $\square$

Proof of Theorem 21. Theorem 3.7 of [4] states that $\Delta(Q,w)$ is isomorphic to $\Delta(Q',w_0)$ where Q' is obtained from Q by appending to Q any reduced expression for $\mathrm{Dem}(Q)^{-1}w_0$. Therefore, we are not losing generality by assuming that $\mathrm{Dem}(Q) = w_0$.

The T-fixed points of BS^Q are in 1-1 correspondence with subwords J of Q. This induces a 1-1 correspondence between T-fixed points of $m_Q^{-1}(w_0B/B)$ and the subwords J of Q with $(Q \setminus J)_{(m)} = w_0$, where $\mathrm{Dem}(Q) = w_0$. If the subword J is not a facet of the subword complex $\Delta(Q,w_0)$ then it gives a non reduced product $(Q \setminus J)_{(m)}$. We will show that for any such J, the cone of the moment polytope around p_J contains a line and thus it cannot be a vertex. This cone is spanned by the T-weights on the tangent space $T_{p_J}(m_Q^{-1}(w_0B/B))$, which is equal to the multiset difference of the T-weights on the tangent spaces $T_{p_J}(BS^Q)$ and $T_{p_J}(X^{w_0})$. We have that

$$\{\{r(J,i) : i = 1,\dots,m\}\},\ T\text{-weights of the tangent space } T_{p_J}(BS^Q),$$
$$\Phi^+(W) \cap w_0\Phi^-(W) = \Phi^+(W),\ T\text{-weights of the tangent space } T_{p_J}(X^{w_0}).$$

Since $(Q \setminus J)_{(m)} = w_0$ is not reduced, then $|J| \leqslant \ell(w_0) - 2$. Take j to be the first index such that $r(J,j) \in \Phi^-(W)$, we then have that $(Q \setminus J)_{(j-1)}$ is reduced and there exists $i < j$ such that $(Q \setminus J \cup \{q_i, q_j\})_{(j)} = (Q \setminus J)_{(j)}$. By Lemma 22 we have that $r(J,i) = -r(J,j) \in \Phi^+(W)$. By the exchange condition we have that there exists $k > j$ such that $(Q \setminus J \cup \{q_j, q_k\})_{(j)} = (Q \setminus J)_{(j)}$ and if we pick k to be the first that satisfies this property then by Lemma 22 we have that $r(J,i) = r(J,k)$. Therefore, the cone of the moment polytope around p_J contains the cone spanned by $r(J,i), r(J,j)$, which is a line. $\square$

Note that this theorem does not assume that the fiber is a toric variety so the relation between brick polytopes and brick manifolds is quite strong. The following theorem classifies toric brick manifolds.

Theorem 23. *Let $w = Dem(Q)$. The fiber $m_Q^{-1}(wB/B)$ is a toric variety with respect to the torus T if and only if Q is root independent and $\ell(w) < |Q| \leqslant \ell(w) + \dim(T)$. Moreover, $m_Q^{-1}(wB/B)$ is the toric variety associated to the polytope $B(Q,w)$.*

Proof. Note that $\dim(m_Q^{-1}(wB/B)) \leqslant \dim(T)$. However, if we have $<$ then we can make the torus smaller and so without loss of generality we can assume the dimensions are equal. It suffices to show that T doesn't have generic stabilizer of positive dimension. This is true if and only if $\mu(m_Q^{-1}(wB/B))$ spans $\mathbb{R}^n$ and this happens precisely when Q is root independent. $\square$

4.3 Stratification of the brick manifold

We give a stratification whose dual, in some sense, is the subword complex. We now introduce and recall some notation. Consider a complex semisimple Lie group G with upper and lower Borel subgroups $B = B^+$ and B^-, and Weyl group W. For $u \in W$ we have the *Schubert cell* $\mathring{X}_v := B^-uB/B$ and the *opposite Schubert cell* $\mathring{X}^v := B^+uB/B$. The Schubert variety X^v and opposite Schubert variety X_u are the closure of $\mathring{X}_u$ and $\mathring{X}^u$, respectively. Given $u, v \in W$, the *open Richardson variety* is $\mathring{X}_u^v := \mathring{X}^v \cap \mathring{X}_u$. The *Richardon variety* X_u^v is the closure of $\mathring{X}_u^v$ This variety is nonempty if and only if $u \leqslant v$ in the Bruhat order, and its dimension is $\ell(v) - \ell(u)$. Then $X_u^v = \coprod_{u \leqslant x < y \leqslant v} \mathring{X}_x^y$ is a stratification.

Given a Bott-Samelson variety $BS^Q := (P_{i_1} \times \cdots \times P_{i_m})/B^m$ and a subword R of Q, we can realize BS^R inside BS^Q by

$$BS^R = \{(p_1, \ldots, p_m) : p_{i_j} = id \text{ if } i_j \notin R\};$$

note that $BS^R \cap BS^S = BS^{R \cap S}$. Let $\mathring{BS}_u^R := BS^R \cap m_Q^{-1}(\mathring{X}_u)$ then these subvarieties yield a stratification of BS^Q, where R ranges over all subwords of Q and $u \in W$. We have that $BS_u^R \neq \emptyset$ if and only if $\mathrm{Dem}(R) \geqslant u$. Moreover, $BS_u^R \subseteq BS_v^S$ if and only if R is a subword of S and $u \geqslant v$ in Bruhat order. This induces a stratification of $m_Q^{-1}(wB/B)$, described in the following theorem, that is dual to the subword complex $\Delta(Q, \mathrm{Dem}(Q))$.

Theorem 24. *Let $w = Dem(Q)$. Brick manifolds have the stratification*

$$m_Q^{-1}(wB/B) = \coprod_R \mathring{BS}_w^R,$$

where R ranges over all subwords of Q with $Dem(R) = w$. This stratifications satisfies the nice property that the intersection of any two strata, when nonempty, is again a stratum (instead of a union of strata).

Proof. If $p \in \mathring{BS}_w^Q$, then $m_Q(p) \in X_w^w = \{wB/B\}$ and so $m_Q^{-1}(wB/B)$ is a stratum of BS^Q. Moreover, if $\mathring{BS}_w^R \subset m_Q^{-1}(wB/B)$ is nonempty then R is a subword and $\mathrm{Dem}(R) \geqslant u \geqslant w$ but then $\mathrm{Dem}(R) = u = w$. Therefore, the stratification of the Bott-Samelson variety restricts to a stratification of the brick manifold and $\mathring{BS}_w^R \cap \mathring{BS}_w^S = \mathring{BS}_w^{R \cap S}$. $\square$

4.4 Brick manifolds and Richardson varieties

A subfamily of brick varieties were used before by Brion in [2] in the proof of Theorem 4.2.1 as a resolution of singularities for Richardson varieties. Given a word $Q = (q_1, \ldots, q_m)$, the *opposite Bott-Samelson variety* BS_Q is defined analogously to BS^Q. More precisely,

$$BS_Q := (P_{i_1}^- \times \cdots \times P_{i_m}^-)/(B^-)^m,$$

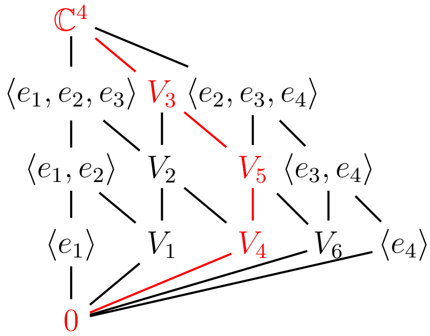
where B^- is the opposite Borel and the P_i^- are the *opposite minimal parabolics.* The natural map to the flag variety is

$$\begin{aligned} BS_Q &\xrightarrow{\mathrm{m_Q}} G/B \\ (p_1, \ldots, p_m) &\longmapsto (p_1 \cdots p_m w_0)B/B. \end{aligned}$$

Given an element $u \in W$ the opposite Bott-Samelson variety BS_Q is a resolution of the Schubert variety X_u, where Q is a reduced word for uw_0. Given $u, v \in W$, let R be a reduced word for v and T be a reduced word word for uw_0, then the fibered product $BS^R \times_{G/B} BS_T$ with the map induced by m_R is Brion's resolution of the Richardson variety X_u^v. We will prove that this fibered product is a brick manifold.

Given $u, v \in W$, let R be a reduced word for v and S be a reduced word for $u^{-1}w_0$, where w_0 is the longest word in W. Now, if $Q = R + S$, i.e. Q is the concatenation of R and S, and $u \leqslant v$ then $\mathrm{Dem}(Q) = w_0$. Moreover, the brick manifold $m_Q^{-1}(w_0B/B)$ together with the map to the flag in the middle gives a resolution of the Richardson variety X_u^v.

Example 25. Let $R = (s_1, s_2, s_3, s_1, s_2)$ and $S = (s_3, s_1, s_2, s_1)$. Then $m_Q^{-1}(w_0B/B)$ together with the map given by the red flag is a resolution of singularities for X_u^v with $v = s_1s_2s_3s_1s_2$ and $u = s_1s_2$.



Theorem 26. *Let $u \leqslant v$ and $Q = R + S$, where R is a reduced word for v and S is a reduced word for $u^{-1}w_0$. The brick manifold $m_Q^{-1}(w_0B/B)$ together with the map $m_R : BS_w^R \to G/B$ is a resolution of the singularities of the Richardson variety X_u^v.*

Proof. Let T be the reduced word for uw_0 obtained by taking S^{-1} and conjugating each letter by w_0. The result follows from identifying the fibered product $BS^R \times_{G/B} BS_T$ with the brick manifold $m_Q^{-1}(w_0B/B)$. If $R = (q_1, \ldots, q_{|R|})$, then then the points in BS^R consist of lists of $m + 1$ flags in G/B

$$(F_0 = B/B, F_1, \ldots, F_{|R|}) \in BS^R$$

such that the k-th flag agrees with the previous one except possibly on the subspace corresponding to q_k, and if $F_{k-1} = gB/B$ and $F_k = hB/B$, then $h^{-1}gB/B \in X^{q_k}$. Similarly, if $T = (q_1, \ldots, q_{|T|})$, then then the points in $BS_{S^{-1}}$ consist of lists of $m+1$ flags in G/B

$$(E_0 = w_0B/B, E_1, \ldots, E_{|T|}) \in BS_T$$

such that the k-th flag agrees with the previous one except possibly on the subspace corresponding to q_k, and if $E_{k-1} = gB/B$ and $E_k = hB/B$, then $h^{-1}gB/B \in X^{w_0^{-1}q_kw_0}$. Therefore, the fibered product $BS^R \times_{G/B} BS_T$ consists of the lists of flags of the form

$$(F_0 = B/B, F_1, \ldots, F_{|R|} = E_{|T|}, E_{|T|-1}, \ldots, E_0 = w_0B/B)$$

such that consecutive flags agree in the described way, together with the maps

$$BS^R \xrightarrow{\mathrm{m_R}} G/B \text{ and } BS_T \xrightarrow{\mathrm{m_T}} G/B$$

that map the list of flags to $F_{|R|} = E_{|T|}$. This is precisely the brick manifold $m_Q^{-1}(w_0B/B)$. □

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References

[1] M. F. Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14(1):1–15, 1982.

[2] Michel Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 33–85. Birkhäuser, Basel, 2005.

[3] R. Bott and H. Samelson. The cohomology ring of G/T. *Proc. Nat. Acad. Sci. U. S. A.*, 41:490–493, 1955.

[4] Cesar Ceballos, Jean-Philippe Labbé, and Christian Stump. Subword complexes, cluster complexes, and generalized multi-associahedra. *J. Algebraic Combin.*, 39(1):17–51, 2014.

[5] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[6] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.

[7] Victor Guillemin, Eugene Lerman, and Shlomo Sternberg. *Symplectic fibrations and multiplicity diagrams.* Cambridge University Press, Cambridge, 1996.

[8] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, 67(3):491–513, 1982.

[9] Christophe Hohlweg, Carsten E. M. C. Lange, and Hugh Thomas. Permutahedra and generalized associahedra. *Adv. Math.*, 226(1):608–640, 2011.

[10] Steven L. Kleiman. The transversality of a general translate. *Compositio Math.*, 28:287–297, 1974.

[11] Allen Knutson and Ezra Miller. Subword complexes in Coxeter groups. *Adv. Math.*, 184(1):161–176, 2004.

[12] Vincent Pilaud and Michel Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. *Discrete Comput. Geom.*, 48(1):142–191, 2012.

[13] Vincent Pilaud and Francisco Santos. The brick polytope of a sorting network. *European J. Combin.*, 33(4):632–662, 2012.

[14] Vincent Pilaud and Christian Stump. Brick polytopes of spherical subword complexes and generalized associahedra. *Adv. Math.*, 276:1–61, 2015.

[15] Nathan Reading and David E. Speyer. Cambrian fans. *J. Eur. Math. Soc. (JEMS)*, 11(2):407–447, 2009.

[16] Luis Serrano and Christian Stump. Maximal fillings of moon polyominoes, simplicial complexes, and Schubert polynomials. *Electron. J. Combin.*, 19(1):Paper 16, 18, 2012.

[17] Christian Stump. A new perspective on k-triangulations. *J. Combin. Theory Ser. A*, 118(6):1794–1800, 2011.

[18] Alexander Woo. Catalan numbers and schubert polynomials for $w = 1(n+1)\ldots 2$. 2004. arXiv:math/0407160.