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Brill–Noether theory and non-special scrolls

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Abstract In this paper we study the Brill–Noether theory of invertible subsheaves of a general, stable rank-two vector bundle on a curve C with general moduli. We relate this theory to the geometry of unisecant curves on smooth, non-special scrolls with hyperplane sections isomorphic to C. Most of our results are based on degeneration techniques.

Keywords Vector bundles on curves · Brill–Noether theory · Ruled surfaces · Hilbert schemes of scrolls · Moduli · Embedded degenerations

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1 Introduction

The classical Brill–Noether theory aims at the description of all families $G^r_d(C)$ of linear series of fixed degree d and dimension r on a given curve C of genus g. Equivalently, one

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can consider the image via the Abel–Jacobi map $W^r_d(C) \subseteq \mathrm{Pic}^d(C)$ of $G^r_d(C)$. In such a generality, the project is certainly too ambitious. However, for C sufficiently general in $\mathcal{M}_g$ the problem has been completely solved. The main results are Griffiths–Harris' theorem (see [21]), which determines the dimensions of the families $G^r_d(C)$, and Gieseker's theorem (see [19]), proving the so called *Petri's conjecture* which refines Griffiths–Harris' result giving further important information about the local structure of $G^r_d(C)$. Recall also Fulton–Lazarsfeld's theorem (see [10]) asserting that $W^r_d(C)$ is connected, for any curve C, as soon as its dimension is positive.

There are various extensions of Brill–Noether theory involving vector bundles, one of which we consider here. Given a curve C of genus $g \geq 1$, one can consider the moduli space $U_C(d)$ of semistable, degree d, rank-two vector bundles on C, which is an irreducible, projective variety of dimension $4g - 3 + \epsilon$, where $\epsilon = 1$, if $g = 1$ and d is even (cf. [36]), $\epsilon = 0$ otherwise (cf. e.g. [30]). For any $[\mathcal{F}] \in U_C(d)$, one can consider the set

$$M_n(\mathcal{F}) := \{N \subset \mathcal{F} \mid N \text{ invertible subsheaf of } \mathcal{F},\ \deg(N) = n\}, \tag{1.1}$$

which has a natural structure of Quot-scheme. Note that $M_n(\mathcal{F})$ is isomorphic to $M_{n+2l}(\mathcal{F} \otimes L)$, for any $L \in \mathrm{Pic}^l(C)$. If $[\mathcal{F}] \in U_C(d)$ is general, then $M_n(\mathcal{F})$ is not empty if and only if $n \leq \left\lfloor \frac{d-g+1}{2} \right\rfloor =: \overline{n}$ (cf. Corollary 4.16, Remark 4.18 and [26]). The problem we consider here is to study the loci $M_n(\mathcal{F})$, for C general of genus g and $\mathcal{F}$ general in $U_C(d)$, as well as their images $W_n(\mathcal{F})$ in $\mathrm{Pic}^n(C)$. Of course, similar questions can be asked for vector bundles of any rank and in this generality they have been considered by various authors (see e.g. [5,24,26,31,32]).

As is well known, the study of vector bundles on curves is equivalent to the one of scrolls in projective space. Therefore, the above questions can be translated in terms of the geometry of scrolls. Let S be a smooth, non-special scroll of degree d and sectional genus $g \geq 0$ which is linearly normal in $\mathbb{P}^R$, $R = d - 2g + 1$. If $d \geq 2g + 3 + \min\{1, g - 1\}$, such scrolls fill up a unique component $\mathcal{H}_{d,g}$ of the Hilbert scheme of surfaces in $\mathbb{P}^R$ which dominates $\mathcal{M}_g$ (cf. Theorem 3.1 below).

Let $[S] \in \mathcal{H}_{d,g}$ be a general point, such that $S \cong \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a very ample rank-two vector bundle of degree d on C, a curve of genus g with general moduli, and S is embedded in $\mathbb{P}^R$ via the global sections of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$. In [7] we showed that, if $g \geq 1$ and S is general, then $\mathcal{F}$ is general in $U_C(d)$ (cf. [2] and [7, Theorem 5.5]). We then proved that S is a *general ruled surface* in the sense of Ghione [14], namely the scheme $\mathrm{Div}^{1,m}_S$ parametrizing unisecant curves of given degree m on S behaves as *expected* (for details, cf. [7, Def. 6.6 and Thm. 6.9]). If we put $m := d - n$, in Proposition 4.11 we prove that there is a natural isomorphism

$$\mathrm{Div}^{1,m}_S \cong M_n(\mathcal{F}).$$

This provides the translation from the vector bundle to the scroll setting. The map

$$\pi_n : M_n(\mathcal{F}) \to W_n(\mathcal{F}) \subseteq \mathrm{Pic}^n(C)$$

can also be interpreted in terms of curves on the scroll: the fibres of π_n are connected (cf. Lemma 4.21) and can be identified with linear systems of unisecant curves of degree m on S. Therefore, the map π_n can be regarded as an analogue of the Abel–Jacobi map. It is then natural to consider the subschemes $W^r_n(\mathcal{F}) \subseteq W_n(\mathcal{F})$ of points where the fibre of π_n has dimension at least r. These are analogues of the classical Brill–Noether loci.

The scheme $\mathrm{Div}^{1,m}_S$ was originally studied by C. Segre (cf. [33]), then in [14] and, by the present authors, in [7], where we used degeneration techniques. These techniques, in particular the degeneration of a general scroll in $\mathcal{H}_{d,g}$ to the union of a rational normal scroll and g quadrics (cf. Construction 3.2), are also the main tool in the present paper.

First of all, we apply the results in [7] to prove a conjecture by Oxbury asserting that $M_{\overline{n}}(\mathcal{F})$ is connected for any curve C of genus g, $[\mathcal{F}] \in U_C(d)$ general and $d-g$ even (cf. [31, Conjecture 2.8]; Oxbury's conjecture refers more generally to vector bundles of any rank).

Then, we turn to the consideration of $W^r_n(\mathcal{F})$. In order to study such loci, a basic ingredient is the contraction map

$$\mu_N : H^0(\mathcal{F} \otimes N^\vee) \otimes H^0(\mathcal{F}^\vee \otimes \omega_C \otimes N) \to H^0(\omega_C)$$

which, in accordance with the line bundle case, is called the *Petri map* of the pair $(\mathcal{F}, N)$ (cf. e.g. [31]). The case $n = \overline{n}$ is already studied in [31] (cf. Proposition 4.36 below). For $n < \overline{n}$ the situation is more complicated. In Proposition 4.39, we give some general results about the Brill–Noether filtration in the general moduli case. In particular we show that, when $[S] \in \mathcal{H}_{d,g}$ is general, one has:

(a) if $\dim(\mathrm{Div}^{1,m}_S) \geq g$ and $[\Gamma] \in \mathrm{Div}^{1,m}_S$ is general, then $\dim(|\mathcal{O}_S(\Gamma)|) = \dim(\mathrm{Div}^{1,m}_S) - g$,
(b) if $0 \leq \dim(\mathrm{Div}^{1,m}_S) < g$ and $[\Gamma] \in \mathrm{Div}^{1,m}_S$ is general, then $\dim(|\mathcal{O}_S(\Gamma)|) = 0$.

In Theorem 5.1, we concentrate on $W^1_n(\mathcal{F})$ and, when C has general moduli, we prove that each of its irreducible components has the expected dimension. We finish the paper by proving an enumerative result, i.e. Theorem 6.1, in which we compute the class of the sum of all invertible subsheaves of $\mathcal{F}$ of maximal degree, when these are finitely many and $[\mathcal{F}] \in U_C(d)$ is general.

The paper is organized as follows. In Sect. 2 we collect standard definitions and properties of scrolls and unisecant curves. In Sect. 3 we recall the results in [6] and in [7]. In Sects. 4 and 5 we prove the above-mentioned results of the Brill–Noether theory, whereas Sect. 6 contains the enumerative result.

2 Notation and preliminaries

In this section we will fix notation and general assumptions as in [7]. For terminology not recalled here, we refer the reader to [7,23,30,34].

Let C be a smooth, projective curve of genus $g \geq 0$ and let $\rho\colon F \to C$ be a *geometrically ruled surface* on C, namely $F = \mathbb{P}(\mathcal{F})$, for some rank-two vector bundle, or locally free sheaf, $\mathcal{F}$ on C. In this paper, we shall make the following:

Assumption 2.1 We assume that $h^0(C, \mathcal{F}) = R + 1$, for some $R \geq 3$, that $|\mathcal{O}_F(1)|$ is base-point-free and that the corresponding morphism $\Phi : F \to \mathbb{P}^R$ is birational to its image.

We denote by d the degree $\deg(\mathcal{F}) := \deg(\det(\mathcal{F}))$.

Definition 2.2 The surface $\Phi(F) := S \subset \mathbb{P}^R$ is called a *scroll of degree d and of (sectional) genus g*, and S is called the *scroll determined by the pair* $(\mathcal{F}, C)$. S is smooth if and only if $\mathcal{F}$ is very ample; if S is singular, then F is its minimal desingularization. For any $x \in C$, let $f_x := \rho^{-1}(x) \cong \mathbb{P}^1$. The line $l_x := \Phi(f_x)$ is called a *ruling* of S. Abusing terminology, the family $\{l_x\}_{x \in C}$ is also called the *ruling* of S.

For further details on ruled surfaces, we refer to [23, Sect. V], [2,11–14,17,18,20,26–28, 33,35]. If we denote by H a section of ρ such that $\mathcal{O}_F(H) = \mathcal{O}_F(1)$, then $\mathrm{Pic}(F) \cong$

$\mathbb{Z}[\mathcal{O}_F(H)] \oplus \rho^*(\mathrm{Pic}(C))$; if $\underline{d} \in \mathrm{Div}(C)$, we denote by $\underline{d}f$ the divisor $\rho^*(\underline{d})$ on F, where f is the general fibre of ρ. A similar notation will be used when $\underline{d} \in \mathrm{Pic}(C)$. Thus, any element of $\mathrm{Pic}(F)$ corresponds to a divisor on F of the form $nH + \underline{d}f$, for some $n \in \mathbb{Z}$ and $\underline{d} \in \mathrm{Pic}(C)$.

Definition 2.3 Any curve $B \in |H + \underline{d}f|$ is called a *unisecant curve* of F. Any irreducible unisecant curve B of F is smooth and is called a *section* of F.

There is a one-to-one correspondence between sections B of F and surjections $\mathcal{F} \twoheadrightarrow L$, with $L = L_B$ a line bundle on C (cf. [23, Sect. V, Prop. 2.6 and 2.9]). Then, one has an exact sequence

$$0 \to N \to \mathcal{F} \to L \to 0, \tag{2.4}$$

where N is a line bundle on C. If $L = \mathcal{O}_C(\underline{m})$, with $\underline{m} \in \mathrm{Div}^m(C)$, then $m = HB$ and $B \sim H + (\underline{m} - \det(\mathcal{F}))f$. One has

$$\mathcal{O}_B(B) \cong N^\vee \otimes L \tag{2.5}$$

(cf. [23, Sect. 5]). In particular,

$$B^2 = \deg(L) - \deg(N) = d - 2\deg(N) = 2m - d. \tag{2.6}$$

Similarly, if B_1 is a reducible unisecant curve of F such that $HB_1 = m$, there exists a section $B \subset F$ and an effective divisor $\underline{a} \in \mathrm{Div}(C)$, $a := \deg(\underline{a})$, such that $B_1 = B + \underline{a}f$, where $BH = m - a$. In particular there exists a line bundle $L = L_B$ on C, with $\deg(L) = m - a$, fitting in (2.4). Thus, one obtains the exact sequence

$$0 \to N \otimes \mathcal{O}_C(-\underline{a}) \to \mathcal{F} \to L \oplus \mathcal{O}_{\underline{a}} \to 0. \tag{2.7}$$

(for details, cf. [7]).

Definition 2.8 Let S be a scroll of degree d and genus g corresponding to $(\mathcal{F}, C)$ and let $B \subset F$ be a section and L as in (2.4). If $\Phi|_B$ is birational to its image, then $\Gamma := \Phi(B) \subset S$ is called a *section* of S. We will say that the pair (S, Γ) is *associated with* (2.4) and that Γ *corresponds to* L on C. If $m = \deg(L)$, then Γ is a *section of degree* m of S; moreover, $\Phi|_B : B \cong C \to \Gamma$ is determined by the linear series $\Lambda \subseteq |L|$, which is the image of the map $H^0(\mathcal{F}) \to H^0(L)$. If $B_1 \subset F$ is a (reducible) unisecant curve and $\Phi|_{B_1}$ is birational to its image, then $\Phi(B_1) = \Gamma_1$ is a *unisecant curve of degree* m of S. As above, the pair (S, Γ_1) corresponds to a sequence of type (2.7).

By Riemann–Roch, one has

$$R + 1 := h^0(\mathcal{O}_F(1)) = d - 2g + 2 + h^1(\mathcal{O}_F(1)). \tag{2.9}$$

Definition 2.10 (*cf.* [33, Sect. 3, p. 128]) We will call $h^1(\mathcal{O}_F(1))$ the *speciality* of the scroll S. A scroll S is said to be *special* if $h^1(\mathcal{O}_F(1)) > 0$, *non-special* otherwise.

For bounds and remarks on $h^1(\mathcal{O}_F(1))$, we refer the reader to e.g. [7, Lemma 3.7, Example 3.10] and to [33, pp. 144-145].

Definition 2.11 Let $\Gamma_1 \subset S$ be a unisecant curve of S of degree m such that (S, Γ_1) is associated to a sequence like (2.7). Then, Γ_1 is said to be *special*, if $h^1(C, L) > 0$, and *linearly normally embedded*, if $H^0(\mathcal{F}) \twoheadrightarrow H^0(L \oplus \mathcal{O}_{\underline{a}})$.

3 Hilbert schemes

Let S be a linearly normal, non-special scroll of degree d and genus g. When $g = 0$, S is rational and its properties are well-known (see e.g. [20]). Thus, from now on, we shall focus on the case $g \geq 1$. From (2.9), one has that $S \subset \mathbb{P}^R$ where $R = d - 2g + 1$ and $d \geq 2g + 2$, because of the condition $R \geq 3$ in Assumptions 2.1. If, in addition, we assume that S is smooth, then $d \geq 2g + 3 + k$, where $k = \min\{1, g - 1\}$ (cf. e.g. [6, Remark 4.20]). In this situation, one has the following result essentially contained in [2] (cf. also [6, Theorem 1.2] and [7, Theorem 5.4]).

Theorem 3.1 *Let $g \geq 0$ be an integer and let $k = \min\{1, g - 1\}$. If $d \geq 2g + 3 + k$, there exists a unique, irreducible component $\mathcal{H}_{d,g}$ of the Hilbert scheme of scrolls of degree d, sectional genus g in $\mathbb{P}^R$ such that the general point $[S] \in \mathcal{H}_{d,g}$ represents a smooth, non-special and linearly normal scroll S. Furthermore,*

(i) $\mathcal{H}_{d,g}$ *is generically reduced*;
(ii) $\dim(\mathcal{H}_{d,g}) = 7(g - 1) + (d - 2g + 2)^2 = 7(g - 1) + (R + 1)^2$;
(iii) $\mathcal{H}_{d,g}$ *dominates the moduli space* $\mathcal{M}_g$ *of smooth curves of genus* g.

If, moreover $g \geq 1$, let $(\mathcal{F}, C)$ be a pair which determines S, where $[C] \in \mathcal{M}_g$ is general. If $U_C(d)$ denotes the moduli space of semistable, degree d, rank-two vector bundles on C, then $[\mathcal{F}] \in U_C(d)$ is a general point. □

We recall a construction of some reducible surfaces corresponding to points in $\mathcal{H}_{d,g}$. This is one of the key ingredients of the degeneration arguments used in [7], which will also be used in this paper. The presence of points in $\mathcal{H}_{d,g}$ corresponding to reducible surfaces was already pointed out in [2]. However the reducible surfaces we need in this paper are different.

Construction 3.2 (see [7, Construction 5.11]) *Let $g \geq 1$. Then $\mathcal{H}_{d,g}$ contains points $[Y]$ such that Y is a reduced, connected, reducible surface, with global normal crossings, of the form*

$$Y := W \cup Q_1 \cup \cdots \cup Q_g, \tag{3.3}$$

where W is a rational normal scroll, corresponding to a general point of $\mathcal{H}_{d-2g,0}$, each Q_j is a smooth quadric, such that $Q_j \cap Q_k = \emptyset$, if $1 \leq j \neq k \leq g$, and $W \cap Q_j = l_{1,j} \cup l_{2,j}$, where $l_{i,j}$ are general rulings of W, for $1 \leq i \leq 2$, $1 \leq j \leq g$, and where the intersections are transverse. Furthermore, for any such Y, one has that $h^1(Y, \mathcal{N}_{Y/\mathbb{P}^R}) = 0$; in particular, $[Y]$ is a smooth point of $\mathcal{H}_{d,g}$.

We finish this section with the following definition and result.

Definition 3.4 (*see* [14, Definition 6.1]) Let C be a smooth, projective curve of genus $g \geq 0$. Let $F = \mathbb{P}(\mathcal{F})$ be a geometrically ruled surface over C and let $d = \deg(\mathcal{F})$. For any positive integer m, we denote by

$$\mathrm{Div}_F^{1,m} \tag{3.5}$$

the Hilbert scheme of unisecant curves of F, which are of degree m with respect to $\mathcal{O}_F(1)$; it has a natural structure as a Quot-scheme (cf. [22]), whose *expected dimension* is

$$d_m := \max\{-1,\ 2m - d - g + 1\}; \tag{3.6}$$

therefore $\dim(\mathrm{Div}_F^{1,m}) \geq d_m$.

In [7], we proved

Theorem 3.7 (see [7, Theorem 6.9]) *Let $g, d, \mathcal{H}_{d,g}$ be as in Theorem 3.1. If $[S] \in \mathcal{H}_{d,g}$ is a general point, then S is a general ruled surface, namely, for any $m \geq 1$:*

(i) $\dim(\mathrm{Div}_S^{1,m}) = d_m$, *for any $m \geq 1$;*
(ii) $\mathrm{Div}_S^{1,m}$ *is smooth, for any m such that $d_m \geq 0$;*
(iii) $\mathrm{Div}_S^{1,m}$ *is irreducible, for any m such that $d_m > 0$.* □

4 Brill–Noether theory

4.1 Preliminaries

Let $S \subset \mathbb{P}^R$ be a smooth, non degenerate scroll of degree d and genus g. Let $(\mathcal{F}, C)$ be a pair determining S. Let Γ be any unisecant curve of S of degree m, corresponding to the exact sequence

$$0 \to N \to \mathcal{F} \to L \oplus \mathcal{O}_{\underline{a}} \to 0, \tag{4.1}$$

where L and N are line bundles and $\underline{a} \in \mathrm{Div}^a(C)$ such that $m = \deg(L) + a$. Set

$$n := \deg(N) = d - m. \tag{4.2}$$

In this section we will study the subschemes of $\mathrm{Pic}(C)$ parametrizing the invertible subsheaves $N \subset \mathcal{F}$ as in (4.1).

Definition 4.3 Let C be a smooth, projective curve of genus $g \geq 0$ and let $\mathcal{F}$ be any rank-two vector bundle on C. The *Segre invariant* of $\mathcal{F}$ is defined as:

$$s(\mathcal{F}) := \deg(\mathcal{F}) - 2(\mathrm{Max}\ \{\deg(N)\}),$$

where the maximum is taken among all the invertible subsheaves N of $\mathcal{F}$ (cf. e.g. [24]). We denote by $M(\mathcal{F})$ the set of all invertible subsheaves of $\mathcal{F}$ of maximal degree. Notice that $M(\mathcal{F})$ has a natural structure of Quot-scheme (cf. e.g. [31]).

In other words, $s(\mathcal{F})$ is the minimum of the self-intersections of sections of $F := \mathbb{P}(\mathcal{F})$ (cf. Formula (2.5) and see e.g. [24]) and therefore, $s(\mathcal{F}) = s(\mathcal{F} \otimes L)$, where L is any line bundle. Similarly, $M(\mathcal{F})$ is isomorphic to $M(\mathcal{F} \otimes L)$. Note that the vector bundle $\mathcal{F}$ is stable (resp., semi-stable) if and only if $s(\mathcal{F}) \geq 1$ (resp., $s(\mathcal{F}) \geq 0$). In the following proposition we recall a result by Nagata, cf. [28].

Proposition 4.4 *Let C be a smooth, projective curve of genus $g \geq 0$ and let $\mathcal{F}$ be any rank-two vector bundle on C. One has*:

$$s(\mathcal{F}) \leq g. \tag{4.5}$$

Proof Let $d = \deg(\mathcal{F})$. Let Γ be a section of $F = \mathbb{P}(\mathcal{F})$, such that $\Gamma^2 = s(\mathcal{F})$. It corresponds to an exact sequence of type (4.1), with $\underline{a} = 0$. Let $m = \deg(L)$, so that $\Gamma^2 = 2m - d$ (cf. Formula (2.6)). Consider $\mathrm{Div}_F^{1,m}$. By the assumption $\Gamma^2 = s(\mathcal{F})$, then all the curves in $\mathrm{Div}_F^{1,m}$ are sections. Therefore, $\dim(\mathrm{Div}_F^{1,m}) \leq 1$. On the other hand, by (3.6), $\dim(\mathrm{Div}_F^{1,m}) \geq d_m = 2m - d - g + 1 = \Gamma^2 - g + 1$. Hence, (4.5) follows. □

The proof of Proposition 4.4 shows that invertible subsheaves N of $\mathcal{F}$ with maximal degree $\overline{n}$ correspond to sections in $\mathrm{Div}_F^{1,\overline{m}}$, with $0 \leq d_{\overline{m}} \leq 1$.

Lemma 4.6 *Let C be a curve of genus $g \geq 1$ with general moduli and let $[\mathcal{F}] \in U_C(d)$ be a general point. Then, the line bundles in $M(\mathcal{F})$ have degree*

$$\overline{n} := \left\lfloor \frac{d-g+1}{2} \right\rfloor. \tag{4.7}$$

Proof This is proved in [24, Prop. 3.1]. Here we give an alternative proof, which directly follows from what discussed up to now.

From what is recalled above on $M(\mathcal{F})$, by tensoring $\mathcal{F}$ with a sufficiently large multiple of an ample line bundle we can assume that the scroll S corresponding to the pair $(\mathcal{F}, C)$ is a general point in $\mathcal{H}_{d,g}$ as in Theorem 3.1. The assertion follows from Theorem 3.7 and from (2.4). □

Let n be any integer such that

$$n \leq \overline{n}. \tag{4.8}$$

For any such n, one can consider the set

$$M_n(\mathcal{F}) := \{N \subset \mathcal{F} \mid N \text{ invertible subsheaf of } \mathcal{F},\ \deg(N) = n\}. \tag{4.9}$$

With this notation, $M_{\overline{n}}(\mathcal{F}) = M(\mathcal{F})$ as in Definition 4.3 (cf. also [31]). As for the maximal case, any $M_n(\mathcal{F})$ has a natural structure of Quot-scheme.

For any $[N] \in M_n(\mathcal{F})$, one can define $s_N(\mathcal{F}) := \deg(\mathcal{F}) - 2\deg(N)$; observe that, as for the Segre invariant, one has $s_{N\otimes L}(\mathcal{F} \otimes L) = s_N(\mathcal{F})$, for any $L \in Pic(C)$. The proof of Lemma 4.6 shows that, in order to study the schemes $M_n(\mathcal{F})$, for C with general moduli and $[\mathcal{F}] \in U_C(d)$ general, we may assume that the pair $(\mathcal{F}, C)$ determines a general point in $\mathcal{H}_{d,g}$ as in Theorem 3.1. Then, one has the morphism

$$\psi_{m,n} : \mathrm{Div}_S^{1,m} \to M_n(\mathcal{F}), \tag{4.10}$$

with $m = d - n$ as in (4.2), defined by

$$\psi_{m,n}([\Gamma]) = [N],$$

where Γ corresponds to $L \oplus \mathcal{O}_{\underline{a}}$ on C fitting in (4.1). The morphism $\psi_{m,n}$ is bijective; in fact, given $N \hookrightarrow \mathcal{F}$ one has an exact sequence of type (4.1), which uniquely determines the corresponding unisecant curve Γ. This defines the inverse $\psi_{m,n}^{-1}$. In particular,

$$\dim(\mathrm{Div}_S^{1,m}) = \dim(M_n(\mathcal{F})).$$

Proposition 4.11 *Let $g \geq 1$ and d be integers as in Theorem 3.1. Let $n \leq \overline{n}$ and $m = d - n$ be integers. Let $[S] \in \mathcal{H}_{d,g}$ be a general point. Then*

$$\psi_{m,n} : \mathrm{Div}_S^{1,m} \to M_n(\mathcal{F})$$

is an isomorphism.

Proof Since $M_n(\mathcal{F})$ is a Quot-scheme, it is smooth at those points $[N] \in M_n(\mathcal{F})$ such that $\mathrm{Ext}^1(N, \mathcal{F}/N) = (0)$. From (4.1), $\mathrm{Ext}^1(N, \mathcal{F}/N) \cong H^1((L \oplus \mathcal{O}_{\underline{a}}) \otimes N^\vee) \cong H^1(L \otimes N^\vee)$. Let $[\Gamma_1] \in \mathrm{Div}_S^{1,m}$ be the unisecant curve as in (4.1). Γ_1 is of the form

$$\Gamma_1 = \Gamma \cup l_1 \cup \cdots \cup l_a, \quad a = \deg(\underline{a}),$$

where $[\Gamma] \in \mathrm{Div}_S^{1,m-a}$ is a section and the l_i's are lines of the ruling. From the inclusion of schemes $\Gamma \subset \Gamma_1$, we get

$$L \oplus \mathcal{O}_{\underline{a}} \twoheadrightarrow L. \tag{4.12}$$

Therefore, the section Γ corresponds to a sequence

$$0 \to N' \to \mathcal{F} \to L \to 0,$$

where N' is a line bundle on C of degree $n' = n + a$. Moreover, from (4.12), it follows that

$$N \hookrightarrow N'. \tag{4.13}$$

Since $H^1(L \otimes N^\vee) \cong H^0(\omega_C \otimes L^\vee \otimes N)^\vee$, by (4.13) we have

$$H^0(\omega_C \otimes L^\vee \otimes N) \hookrightarrow H^0(\omega_C \otimes L^\vee \otimes N'). \tag{4.14}$$

From (2.5),

$$L \otimes (N')^\vee \cong \mathcal{N}_{\Gamma/S}$$

and $h^1(\mathcal{N}_{\Gamma/S}) = 0$, for any $[\Gamma] \in \mathrm{Div}_S^{1,m-a}$ (cf. Theorem 3.7). This implies that $H^1(L \otimes N^\vee) = (0)$ so $M_n(\mathcal{F})$ is smooth. Since $\psi_{m,n}$ is bijective, it is an isomorphism (cf. [23, Exercise I, 3.3]). □

As an immediate consequence of Proposition 4.11 and of the proof of Lemma 4.6, we have the following:

Corollary 4.15 *Let C be a curve of genus $g \geq 1$ with general moduli and $[\mathcal{F}] \in U_C(d)$ be a general point. Let $n \leq \overline{n}$ and $m = d - n$ be integers. Then $M_n(\mathcal{F})$ is smooth, of dimension d_m and it is irreducible when $d_m > 0$.*

The formula $\dim(M_n(\mathcal{F})) = d_m$ is a special case of [32, Theorem 0.2].

We have also the following result (cf. [26], [24, Corollary 3.2] and [31, Proposition 1.4, Theorem 3.1, Example 3.2]).

Corollary 4.16 *Let C be a smooth, projective curve of genus $g \geq 1$ and let $\mathcal{F}$ be a rank-two vector bundle of degree d on C. One has:*

(a) *if $s(\mathcal{F}) = g$, then $\dim(M(\mathcal{F})) = 1$, $d - g$ is even and $\overline{n} = \frac{d-g}{2}$.*
(b) *if C has general moduli, $[\mathcal{F}] \in U_C(d)$ general and $s(\mathcal{F}) \leq g-1$, then $\dim(M(\mathcal{F})) = 0$, $s(\mathcal{F}) = g - 1$, $d - g$ is odd and $\overline{n} = \frac{d-g+1}{2}$.*

Proof As usual, we may assume that $(\mathcal{F}, C)$ corresponds to a point in $\mathcal{H}_{d,g}$. Let $\overline{m} := d - \overline{n}$. (a) One has $1 \geq \dim(M(\mathcal{F})) = \dim(\mathrm{Div}_F^{1,\overline{m}}) \geq d_{\overline{m}} = 1$ (cf. the proof of Proposition 4.4). The assertion follows. (b) By the generality assumptions, one has $\dim(M(\mathcal{F})) = \dim(\mathrm{Div}_F^{1,\overline{m}}) \geq d_{\overline{m}} = 0$. The assertion follows. □

The following corollary proves a particular case of [31, Conjecture 2.8].

Corollary 4.17 *Let C be any smooth, projective curve of genus $g \geq 1$. Let d be an integer such that $d - g$ is even. Let $[\mathcal{F}] \in U_C(d)$ be general. Then $M(\mathcal{F})$ is a connected curve.*

Proof By Corollary 4.15, $M(\mathcal{F})$ is a smooth and irreducible curve if C has general moduli. On the other hand, since we are in in case (a) of Corollary 4.16, then $M(\mathcal{F})$ is in any case a curve. Now, [31, Theorem 3.1] implies that the numerical equivalence class of $M(\mathcal{F})$ is independent of C. Therefore, $M(\mathcal{F})$, as a limit of a smooth, irreducible curve, is connected. □

Remark 4.18 Note that, in case (b) of Corollary 4.16, Maruyama proves more, i.e. he assumes C to be any curve, d any positive integer and $[\mathcal{F}] \in U_C(d)$ general. Furthermore, in this case, $M(\mathcal{F})$ consists of 2^g distinct elements (cf. [36, Theorem 16], [24, Corollary 3.2] and [31, Proposition 1.4, Theorem 3.1, Example 3.2]; see also Proposition 4.11 and [7, Theorem 7.1.1]). Thus, when $d - g$ is odd, one has a rational map

$$\lambda \colon U_C(d) \dashrightarrow Sym^{2^g}\left(\mathrm{Pic}^{\frac{d-g+1}{2}}(C)\right).$$

For $g = 1$, λ is everywhere defined and it is an isomorphism. This is proved in [36] and the bijectivity is proved implicitly in [3] (cf. also [7, Remark 5.5]).

As soon as $g \geq 2$, $\dim(U_C(d)) < \dim\left(Sym^{2^g}(\mathrm{Pic}^{\frac{d-g+1}{2}})\right)$. Natural questions are:

- is λ injective?
- is $d\lambda$ injective where λ is defined?

Affirmative answers would give (global and infinitesimal) Torelli type theorems.

There are several remarks, pointed out to us by the referee, which are related to the above questions. When $g = 2$, the fact that λ is generically injective follows from results in [29]. In fact, suppose $\mathcal{F}$ has four non-isomorphic maximal, invertible subsheaves N_i, $1 \leq i \leq 4$. Then $\mathcal{F}$ can be written as an extension $0 \to N_1 \to \mathcal{F} \to \mathcal{N}_2(p_2) \to 0$, for some $p_2 \in C$, and is determined by N_1, N_2 and p_2. The bundles N_1 and N_2 do not determine $\mathcal{F}$, but using similar expressions for $\mathcal{F}$ with quotients $N_3(p_3)$ and $N_4(p_4)$, one can see that the set of four bundles does determine $\mathcal{F}$.

On the other hand, for any g, if one restricts to bundles of a fixed determinant, the generic injectivity of λ is proved in [8].

4.2 The Brill–Noether loci

As in [31, Sect. 1], for any $n \leq \overline{n}$ one can consider the natural morphism

$$\pi_n : M_n(\mathcal{F}) \to \mathrm{Pic}^n(C) \tag{4.19}$$

sending any invertible subsheaf $N \subset \mathcal{F}$ of degree n to $[N] \in \mathrm{Pic}^n(C)$. We shall denote by

$$W_n(\mathcal{F}) := Im(\pi_n) \subseteq \mathrm{Pic}^n(C) \tag{4.20}$$

(cf. [15, Theorem 3], [16] and [31], where $W_{\overline{n}}(\mathcal{F})$ is denoted by $W(\mathcal{F})$). The map π_n can be viewed as an analogue of the classical Abel–Jacobi map and $M_n(\mathcal{F})$ has to be viewed as an analogue of the symmetric product of the curve C.

Lemma 4.21 *For any $[N] \in W_n(\mathcal{F})$,*

$$\pi_n^{-1}([N]) \cong \mathbb{P}(H^0(\mathcal{F} \otimes N^\vee)).$$

In particular, π_n has connected fibres.

Proof This follows from the definition of $W_n(\mathcal{F})$ (cf. [31, p. 11], for $n = \overline{n}$). Indeed, $[N] \in W_n(\mathcal{F})$ iff $[N] \in \mathrm{Pic}^n(C)$ is an invertible subsheaf of $\mathcal{F}$, equivalently, iff there exists a non-zero global section in $H^0(\mathcal{F} \otimes N^\vee)$. □

Remark 4.22 Recalling (4.10), we have the commutative diagram

$$\begin{tikzcd} \mathrm{Div}_S^{1,m} \arrow[r, "\psi_{m,n}"] \arrow[dr, "\Phi_{m,n}"'] & M_n(\mathcal{F}) \arrow[d, "\pi_n"] \\ & W_n(\mathcal{F}) \end{tikzcd} \tag{4.23}$$

For any $[N] \in W_n(\mathcal{F})$ and $[\Gamma_N] = \Phi_{m,n}^{-1}(N)$, we have

$$\mathbb{P}(H^0(\mathcal{F} \otimes N^\vee)) \cong |\mathcal{O}_S(\Gamma_N)|, \tag{4.24}$$

i.e., the fibres of π_n can be identified with linear systems of unisecant curves of degree $m = d - n$ on S.

The above setting suggests the definition of Brill–Noether type loci in $W_n(\mathcal{F})$. One proceeds as follows. For any integer $p \geq 0$, one defines the *Brill–Noether locus*

$$W_n^p(\mathcal{F}) := \{[N] \in \mathrm{Pic}^n(C) \,|\, h^0(\mathcal{F} \otimes N^\vee) \geq p + 1\}. \tag{4.25}$$

Since $[\mathcal{F}] \in U_C(d)$ is general, this is a degeneracy-locus of a suitable vector bundle map on $\mathrm{Pic}^n(C)$ and, as such, has a natural scheme structure (cf. the construction in [31, pp. 11–12], for the case $n = \overline{n}$, which extends to any $n \leq \overline{n}$). In particular, for any $n \leq \overline{n}$, $W_n(\mathcal{F}) = W_n^0(\mathcal{F})$ and there is a filtration

$$\emptyset = W_n^{k+1}(\mathcal{F}) \subset W_n^k(\mathcal{F}) \subseteq W_n^{k-1}(\mathcal{F}) \subseteq \cdots \subseteq W_n^2(\mathcal{F}) \subseteq W_n^1(\mathcal{F}) \subseteq W_n^0(\mathcal{F}) = W_n(\mathcal{F}), \tag{4.26}$$

for some $k \geq 0$ (cf. [16]). Note that, for any $p \geq 0$, $W_n^{p+1}(\mathcal{F})$ is contained in the singular locus of $W_n^p(\mathcal{F})$. Recalling Remark 4.22, we see that the pull-back via $\Phi_{m,n}$ of $W_n^p(\mathcal{F})$ is

$$\mathrm{Div}_S^{1,m}(p) := \{[\Gamma] \in \mathrm{Div}_S^{1,m} \;\;|\;\; \dim(|\mathcal{O}_S(\Gamma)|) \geq p\}, \tag{4.27}$$

which is a subscheme of $\mathrm{Div}_S^{1,m}$ (cf. [15, p. 68]). Via the isomorphism $\psi_{m,n}$, the scheme $\mathrm{Div}_S^{1,m}(p)$ can be identified with

$$M_n^p(\mathcal{F}) := \{N \subset \mathcal{F} \,|\, \deg(N) = n \;\text{ and }\; h^0(\mathcal{F} \otimes N^\vee) \geq p + 1\}, \tag{4.28}$$

which is the subscheme of $M_n(\mathcal{F})$ pull-back of $W_n^p(\mathcal{F})$ via π_n.

We recall the following proposition from [15, Theorems 2, 3], [16] (see also [31, Lemma 2.2], for the case $n = \overline{n}$):

Proposition 4.29 *Let d_m be as in (3.6). For any integer $p \geq 0$, let*

$$\tau_p(\mathcal{F}) := \max\{-1,\; g - (p+1)(p + g - d_m)\}. \tag{4.30}$$

If $W_n^p(\mathcal{F}) \neq \emptyset$, then

$$\dim(W_n^p(\mathcal{F})) \geq \;\min\{g, \tau_p(\mathcal{F})\}, \tag{4.31}$$

where the right-hand-side is the expected dimension of $W_n^p(\mathcal{F})$. In particular, with d as in Theorem 3.1, one has:

(i) *if* $0 \leq \tau_p(\mathcal{F}) < g$, *then*

$$\dim(\mathrm{Div}_S^{1,m}(p)) \geq \tau_p(\mathcal{F}) + p =: \mathrm{expdim}(\mathrm{Div}_S^{1,m}(p)),$$

whereas,

(ii) *if* $\tau_p(\mathcal{F}) = g$, *then for any* $p_0 \leq p$, *one has*

$$W_n^{p_0}(\mathcal{F}) = \mathrm{Pic}^n(C) \text{ and } \mathrm{Div}_S^{1,m}(p_0) = \mathrm{Div}_S^{1,m};$$

furthermore, the general fibre of $\Phi_{m,n}$ *has dimension* $d_m - g = 2m - d - 2g + 1$.

If, moreover, the equality in (4.31) *holds with* $0 \leq \tau_p(\mathcal{F}) < g$, *then the class in* $\mathrm{Pic}^n(C)$ *of* $W_n^p(\mathcal{F})$ *is*

$$[W_n^p(\mathcal{F})] \equiv \left(\prod_{i=0}^{p} \frac{i!}{(p+g+i-d_m)!} \right) \cdot 2^{g-\tau_p(\mathcal{F})} \cdot \theta^{g-\tau_p(\mathcal{F})},$$

where $\equiv$ *denotes the numerical equivalence of cycles and* θ *is the class of the theta divisor in* $\mathrm{Pic}^n(C)$. $\square$

Note that, since $m = d - n$, one has

$$\tau_0(\mathcal{F}) = d_m, \tag{4.32}$$

which agrees with the notion of expected dimension for $\mathrm{Div}_S^{1,m}$ (cf. Formula (3.6)). Moreover, in case (ii), for any $[\Gamma] \in \mathrm{Div}_S^{1,m}$ one has

$$\dim(|\mathcal{O}_S(\Gamma)|) \geq 2m - d - 2g + 1,$$

which agrees with Riemann–Roch theorem. Equality holds if $\mathrm{Div}_S^{1,m}$ has the expected dimension and $[\Gamma] \in \mathrm{Div}_S^{1,m}$ is general. For the proof of Proposition 4.29, see [16]. In [15, Theorems 2, 3], one finds the expression of the class of $[\mathrm{Div}_S^{1,m}(p)]$ in $\mathrm{Div}_S^{1,m}$, for S a general ruled surface (cf. Theorem 3.7). In order to study the morphism

$$\pi_n : M_n(\mathcal{F}) \to W_n(\mathcal{F}) \tag{4.33}$$

and the schemes $W_n^p(\mathcal{F})$, for $p \geq 0$, a basic ingredient is the following contraction map

$$\mu_N : H^0(\mathcal{F} \otimes N^\vee) \otimes H^0(\mathcal{F}^\vee \otimes \omega_C \otimes N) \to H^0(\omega_C) \tag{4.34}$$

defined for any $[N] \in M_n(\mathcal{F})$. In accordance with the classical case of line bundles, μ_N is called the *Petri map* of the pair $(\mathcal{F}, N)$ (cf. e.g. [31]). As in [1, Ch. IV, Sect. 1], one has (cf. [31, Prop. 2.4], for the maximal case $n = \overline{n}$):

Lemma 4.35 *For* $[N] \in W_n^p(\mathcal{F}) \setminus W_n^{p+1}(\mathcal{F})$,

$$T_{[N]}(W_n^p(\mathcal{F})) \cong Im(\mu_N)^\perp.$$

Therefore, if not empty, $W_n^p(\mathcal{F})$ *is smooth and of the expected dimension at* $[N]$ *if and only if the Petri map* μ_N *is injective.*

Therefore if the Petri map μ_N is injective for any $[N] \in W_n^p(\mathcal{F}) \setminus W_n^{p+1}(\mathcal{F})$, then the singular locus of $W_n^p(\mathcal{F})$ coincides with $W_n^{p+1}(\mathcal{F})$.

The maximal case $n = \overline{n}$ has been studied in [31]. We recall the results.

Proposition 4.36 *Let $g \geq 1$ be an integer and let C be any smooth, projective curve of genus g. For any integer d, let $[\mathcal{F}] \in U_C(d)$ be general. Then*:

(i) *the map $\pi_{\overline{n}}$ is an isomorphism; in particular, $W_{\overline{n}}(\mathcal{F})$ is smooth and strictly contained in* $\mathrm{Pic}^{\overline{n}}(C)$.

(ii) $W^p_{\overline{n}}(\mathcal{F}) = \emptyset$, *for any $p \geq 1$.*

(iii) *If d is as in Theorem 3.1 and if*

$$\overline{m} := d - \overline{n} = \left\lfloor \frac{d+g}{2} \right\rfloor, \tag{4.37}$$

then for the general $[S] \in \mathcal{H}_{d,g}$ and for any $[\Gamma] \in \mathrm{Div}^{1,\overline{m}}_S$, one has $\dim(|\mathcal{O}_S(\Gamma)|) = 0$.

Parts (i) and (ii) are contained in [31]. Part (iii) is an immediate consequence of Remark 4.22.

Remark 4.38 $W_{\overline{n}}(\mathcal{F})$ is a divisor in $\mathrm{Pic}^{\overline{n}}(C)$ when $g = 2$ and d is even (cf. [31, Remark 1.6]): up to twists, $d = 0$ so $\overline{n} = -1$; in this case, $W_{\overline{n}}(\mathcal{F})$ can be identified with the divisor $D_{\mathcal{F}} = \{M \in \mathrm{Pic}^1(C) \mid h^0(\mathcal{F} \otimes M) = 1\} \in |2\Theta|$, where Θ denotes the theta divisor in $\mathrm{Pic}^1(C)$.

For $n < \overline{n}$ the situation is more complicated. We will prove the following:

Proposition 4.39 *Let C be a smooth, projective curve of genus $g \geq 1$ with general moduli and let d be an integer. Let $[\mathcal{F}] \in U_C(d)$ be general and let $\tau_0(\mathcal{F})$ be as in (4.32). Let $n < \overline{n}$ be any integer. (a) If $\tau_0(\mathcal{F}) \geq g$, then for general $[N] \in M_n(\mathcal{F})$, $h^1(\mathcal{F} \otimes N^\vee) = 0$ and we have the filtration*

$$\emptyset \subset \cdots \subseteq W_n^{d_m-g+1}(\mathcal{F}) \subset W_n^{d_m-g}(\mathcal{F}) = \cdots = W_n^1(\mathcal{F}) = W_n(\mathcal{F}) = \mathrm{Pic}^n(C). \tag{4.40}$$

(b) If $0 \leq \tau_0(\mathcal{F}) < g$, then $W_n^0(\mathcal{F})$ is not empty, strictly contained in $\mathrm{Pic}^n(C)$ and also the inclusion $W_n^1(\mathcal{F}) \subset W_n(\mathcal{F})$ is strict. Moreover:

(i) *$W_n^0(\mathcal{F})$ is smooth, of dimension $\tau_0(\mathcal{F})$, at any $[N] \in W_n^0(\mathcal{F}) \backslash W_n^1(\mathcal{F})$.*

(ii) *$\pi_{n|} : M_n(\mathcal{F}) \backslash M_n^1(\mathcal{F}) \to W_n^0(\mathcal{F}) \backslash W_n^1(\mathcal{F})$ is an isomorphism.*

(iii) *$W_n^0(\mathcal{F})$ is irreducible when $\tau_0(\mathcal{F}) > 0$.*

Proof (a) As usual, we may assume that the pair $(\mathcal{F}, C)$ determines a general point in $\mathcal{H}_{d,g}$ as in Theorem 3.1. Consider the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{F} \otimes N^\vee \to (L \oplus \mathcal{O}_{\underline{a}}) \otimes N^\vee \to 0, \tag{4.41}$$

obtained from (4.1). One has $h^1((L \oplus \mathcal{O}_{\underline{a}}) \otimes N^\vee) = 0$ (see the proof of Proposition 4.11). Thus

$$0 \to H^0(\mathcal{O}_C) \to H^0(\mathcal{F} \otimes N^\vee) \to H^0(L \otimes N^\vee) \xrightarrow{\partial} H^1(\mathcal{O}_C) \to H^1(\mathcal{F} \otimes N^\vee) \to 0, \tag{4.42}$$

where the coboundary map ∂ can be identified with the differential of the morphism $\pi_n : M_n(\mathcal{F}) \to \mathrm{Pic}^n(C)$. Since $\tau_0(\mathcal{F}) \geq g$, the morphism π_n is surjective (cf. Proposition 4.29 - (ii)). Hence ∂ is surjective if $[N]$ is general and therefore $h^1(\mathcal{F} \otimes N^\vee) = 0$. (b) Since $\tau_0(\mathcal{F}) = d_m$, as in (4.32), then from Theorem 3.7 $\dim(\mathrm{Div}^{1,m}_S) = d_m \geq 0$. By (4.10), also $M_n(\mathcal{F}) \neq \emptyset$, so $W_n^0(\mathcal{F})$ is not empty. Since $\dim(M_n(\mathcal{F})) = d_m$, by (4.31), we have $\dim(W_n^0(\mathcal{F})) = \tau_0(\mathcal{F}) = d_m$ (cf. also [32, Theorem 0.3]). Since $\tau_0(\mathcal{F}) < g$, then $W_n^0(\mathcal{F})$ is strictly contained in $\mathrm{Pic}^n(C)$. From Proposition 4.11 and Lemma 4.21, it follows that

$\pi_n : M_n(\mathcal{F}) \to W_n(\mathcal{F})$ is birational. Since $M_n(\mathcal{F})$ is smooth (see Theorem 3.7 and Proposition 4.11), then the scheme $W_n(\mathcal{F})$ is generically smooth. This proves that the inclusion $W_n^1(\mathcal{F}) \subset W_n(\mathcal{F})$ is strict. Part (i) follows by the injectivity of the Petri map μ_N. In fact, $h^0(\mathcal{F}\otimes N^\vee) = 1$, for any $[N] \in W_n(\mathcal{F})\backslash W_n^1(\mathcal{F})$. Moreover, since $[\mathcal{F}] \in U_C(d)$ is general, then $\mathcal{F}$ is *very-stable* (cf. [25] and [31, p. 12]), which means that μ_N is injective on each factor of the tensor product.

Part (ii) follows since $\pi_n|$ is a bijective morphism between smooth varieties hence it is an isomorphism.

Part (iii) follows from Theorem 3.7 and Proposition 4.11. □

The above argument shows the following:

Corollary 4.43 *Let d and g be positive integers as in Theorem 3.1. Let C be a smooth, projective curve of genus g with general moduli. Let $[\mathcal{F}] \in U_C(d)$ be general. Let $[S] \in \mathcal{H}_{d,g}$ be determined by $(\mathcal{F}, C)$. Let $m > \overline{m}$ be any integer. Then*:

(a) *If $d_m \geq g$ and $[\Gamma] \in \mathrm{Div}_S^{1,m}$ is general, then* $\dim(|\mathcal{O}_S(\Gamma)|) = \tau_0(\mathcal{F}) - g = d_m - g$.
(b) *If $0 \leq d_m < g$, then for any unisecant curve $[\Gamma] \in \mathrm{Div}_S^{1,m}\backslash\mathrm{Div}_S^{1,m}(1)$ one has* $\dim(|\mathcal{O}_S(\Gamma)|) = 0$.

In the circle of ideas presented in this section, a natural and interesting problem would be to prove the analogue of Petri's conjecture:

Conjecture 4.44 *Let C be a smooth, projective curve of genus g with general moduli. Let $[\mathcal{F}] \in U_C(d)$ be general. Let $[N] \in M_n(\mathcal{F})$ be any point. Then, the Petri map μ_N is injective.*

As remarked above, the validity of this conjecture would imply:

(i) $W_n^p(\mathcal{F})$ has the expected dimension, i.e. $\min\{g,\ \tau_p(\mathcal{F})\}$ as in (4.31);
(ii) $W_n^p(\mathcal{F})$ is smooth off $W_n^{p+1}(\mathcal{F})$.

Statement (i) above is an analogue of the Brill–Noether Theorem. In the next section, we will prove (i) for $p = 1$ under suitable numerical assumptions.

5 Brill–Noether's theorem for $W_n^1(\mathcal{F})$

In this section we will study $W_n^1(\mathcal{F})$ and prove that it has the expected dimension $e := e_n^1(d)$, which is:

(i) -1, when $n > \frac{2d-3g}{4}$,
(ii) $2d - 4n - 3g < g$, when $\frac{d-2g}{2} < n \leq \frac{2d-3g}{4}$,
(iii) g, when $n \leq \frac{d-2g}{2}$,

(cf. (4.30), (4.31)). Case (iii) is contained in Proposition 4.39-(a). Therefore, it suffices to consider $n > \frac{d-2g}{2}$.

Theorem 5.1 *Let C be a smooth, projective curve of genus $g \geq 1$ with general moduli and d be an integer. Let $[\mathcal{F}] \in U_C(d)$ be general. Let $n > \frac{d-2g}{2}$ be any integer. Then, each irreducible component of $W_n^1(\mathcal{F})$ has the expected dimension.*

Proof As usual, we can assume that the pair $(\mathcal{F}, C)$ corresponds to a general point $[S] \in \mathcal{H}_{d,g}$. In order to prove the theorem, it suffices to show that, for $m = d - n$, one has $\dim(\mathrm{Div}_S^{1,m}(1)) = e + 1$, if $e \geq 0$, whereas $\mathrm{Div}_S^{1,m}(1)$ is empty, if $e = -1$(cf. (4.27)).

We will prove this by degeneration, studying the limit of $\mathrm{Div}_S^{1,m}(1)$ when S degenerates to a surface $Y = W \cup Q_1 \cup \cdots \cup Q_g$, where W is a general rational normal scroll of degree $d - 2g$ and $Q_1, \ldots, Q_g$ are general quadrics as in Construction 3.2, from which we keep the notation.

In order to study the limit in question, let $\mathcal{P}$ be a linear pencil of curves in $\mathrm{Div}_S^{1,m}$ and let $\mathcal{P}_0$ be the flat limit of $\mathcal{P}$ on Y. Then $\mathcal{P}_0$ consists of a collection of linear pencils $\mathcal{L}$, $\mathcal{L}_1, \ldots, \mathcal{L}_g$ of unisecant curves on $W, Q_1, \ldots, Q_g$. By the genericity of $Q_1, \ldots, Q_g$ none of these pencils contain the double lines $l_{i,j}$, where $1 \le i \le 2$, $1 \le j \le g$, in their fixed locus. Moreover, they verify the obvious matching properties along them. Let $\mu, \mu_1, \ldots, \mu_g$ be the degrees of the curves in $\mathcal{L}, \mathcal{L}_1, \ldots, \mathcal{L}_g$, respectively. We will call such a $\mathcal{P}_0$ a *limit unisecant pencil* of type $(\mu, \mu_1, \ldots, \mu_g)$. One has $m = \mu + \sum_{i=1}^g \mu_i$. We may assume that $\mu_1 = \mu_2 = \ldots = \mu_h = 1$, whereas $\mu_{h+1}, \ldots, \mu_g \ge 2$. Note that $h \ge 1$; otherwise we would have $m \ge \mu + 2g \ge \frac{d-2g}{2} + 2g = \frac{d}{2} + g$ (cf. (4.37) applied to μ and W). This reads $d \ge 2n + 2g$ which implies $\tau_1(\mathcal{F}) \ge g$ hence $e = g$, a case which we are not considering. Recall that $W \cap Q_j$ consists of the pair of lines $l_{1,j}, l_{2,j}$, $1 \le j \le g$. The Segre embedding Σ_j of $l_{1,j} \times l_{2,j}$ sits in a $\mathbb{P}^3$, whose dual we denote by Π_j. Let $\mathbb{G}$ be the grassmannian of lines in $\mathrm{Div}_W^{1,\mu}$. One has a natural rational map

$$r \colon \mathbb{G} \dashrightarrow \Pi_1 \times \cdots \times \Pi_h,$$

which is defined as follows. Let $\mathcal{L}$ be a general pencil in $\mathrm{Div}_W^{1,\mu}$; $\mathcal{L}$ cuts on the divisor $l_{1,j} + l_{2,j}$ a linear series of dimension one and degree two which can be interpreted as a curve on Σ_j, cut out by a plane corresponding to a point $\ell_j \in \Pi_j$. The map r sends $\mathcal{L}$ to the h-tuple $(\ell_1, \ldots, \ell_h)$.

Claim 5.2 *If* $e = -1$, *then* r *is not dominant.*

Proof of Claim 5.2 One has $m = \mu + h + \sum_{j=h+1}^g \mu_j \ge \mu + 2g - h$. The assumption $e = -1$ is equivalent to $m < \frac{2d+3g}{4}$; therefore one has $\frac{2d+3g}{4} > \mu + 2g - h$, i.e. $4\mu + 5g - 2d < 4h$, which implies $\dim(\mathbb{G}) = 4\mu + 4g - 2d < 3h = \dim(\Pi_1 \times \cdots \times \Pi_h)$. This proves the assertion. □

This claim settles the case $e = -1$. In fact it shows that, by the genericity of the quadrics $Q_1, \ldots, Q_h$, the pencils $\mathcal{L}_1, \ldots, \mathcal{L}_h$ cannot match any pencil $\mathcal{L}$ on W to give a limit unisecant pencil $\mathcal{P}_0$. Thus, from now on, we assume $e \ge 0$ and we study the possible components of the flat limit of $\mathrm{Div}_S^{1,m}(1)$ when S degenerates to Y. Since $\mathrm{Div}_S^{1,m}(1)$ is not empty in this case, its flat limit is not empty. Let $\mathcal{P}_0$ be a limit unisecant pencil of type $(\mu, \mu_1, \ldots, \mu_g)$ as above. By the genericity of the quadrics $Q_1, \ldots, Q_h$, the map r has to be dominant. Let Ψ be the general fibre of r. One has

$$\dim(\Psi) = \dim(\mathbb{G}) - 3h = 4m - 2d - 3g - \sum_{i=h+1}^{g} (4\mu_i - 7). \tag{5.3}$$

Now we are ready to compute the dimension of a component of limit unisecant pencils. Let $\mathbb{G}_j$ be the grassmannian of lines of $\mathrm{Div}_{Q_j}^{1,\mu_j}$, for $j = h+1, \ldots, g$. We have two rational maps

$$p \colon \Psi \dashrightarrow \Pi_{h+1} \times \cdots \times \Pi_g, \qquad q \colon \mathbb{G}_{h+1} \times \cdots \times \mathbb{G}_g \dashrightarrow \Pi_{h+1} \times \cdots \times \Pi_g$$

defined as follows. A general point of Ψ is a pencil $\mathcal{L}$ in $\mathrm{Div}_W^{1,\mu}$. It cuts a linear series of degree 2 and dimension 1 on the divisor $l_{1,j} + l_{2,j}$, $j = h+1, \ldots, g$, which, as usual, gives rise to a point $\ell_j \in \Pi_j$. The map p sends $\mathcal{L}$ to $(\ell_{h+1}, \ldots, \ell_g)$. The definition of the map q

is similar (see the proof of Claim 5.4 below). A component Z of limit unisecant pencils of type $(\mu, \mu_1, \ldots, \mu_g)$ can be interpreted as an irreducible component of the fibred product of p and q.

Claim 5.4 *All fibres of the map q have dimension $\sum_{i=h+1}^{g}(4\mu_i - 7)$.*

Proof Fix a $j = h+1, \ldots, g$. We have a map $q_j : \mathbb{G}_j \dashrightarrow \Pi_j$ and $q = q_{h+1} \times \cdots \times q_g$. It suffices to prove that all fibres of q_j have dimension $4\mu_j - 7$. Since $\mu_j > 1$, the linear system $\mathrm{Div}^{1,\mu_j}{}_{Q_j}$ of dimension $2\mu_j - 1$ cuts out a complete linear series Λ_j of dimension 3 on the divisor $l_{1,j} + l_{2,j}$. So we have a surjective projection map $s_j : \mathrm{Div}^{1,\mu_j}{}_{Q_j} \dashrightarrow \Lambda_j$, with centre a projective space of dimension $2\mu_j - 5$. This induces a map $\sigma_j : \mathbb{G}_j \dashrightarrow \overline{\mathbb{G}}_j$, where $\overline{\mathbb{G}}_j$ is the grassmannian of lines of Λ_j. All fibres of σ_j are grassmannians of dimension $4\mu_j - 8$. We have also a map $\tau_j : \overline{\mathbb{G}}_j \dashrightarrow \Pi_j$ sending, as usual, a pencil in Λ_j to a point $\ell_j \in \Pi_j$. Every fibre of τ_j has dimension 1. Indeed, a point $\ell_j \in \Pi_j$ can be interpreted as a projective transformation $\omega_j : l_{1,j} \to l_{2,j}$, and this in turn determines the quadric Ω_j described by all lines joining corresponding points on $l_{1,j}$ and $l_{2,j}$. The pairs of such corresponding points are cut out by all pencils of planes based on lines of the ruling of Ω_j to which $l_{1,j}$ and $l_{2,j}$ belong. Since $q_j = \tau_j \circ \sigma_j$, the above considerations imply the assertion. □

Putting together (5.3) and Claim 5.4, one obtains that $\dim(Z) = 2d - 4n - 3g$, which proves the theorem. □

6 Tensor product of quotient line bundles

In this section we consider the following problem. Let C be a smooth, projective curve with general moduli and d be an integer. Let $[\mathcal{F}] \in U_C(d)$ be general. Assume $d - g$ odd, and let $\overline{n} = \frac{d-g+1}{2}$ as in (4.7). Let $[N_i] \in M_{\overline{n}}(\mathcal{F})$, with $N_i \neq N_j$, for $1 \leq i \neq j \leq 2^g$, and let ν_i denote a divisor class on C such that $N_i = \mathcal{O}_C(\nu_i)$. We want to compute the equivalence class of the divisor

$$\nu := \sum_{i=1}^{2^g} \nu_i .$$

Set $L_i := \det(\mathcal{F}) \otimes N_i^{\vee}$ and let λ_i be a divisor class such that $L_i = \mathcal{O}_C(\lambda_i)$. Consider $\lambda := \sum_{i=1}^{2^g} \lambda_i$ and notice the relation $\lambda + \nu = 2^g H$, where $\det(\mathcal{F}) = \mathcal{O}_C(H)$.

Theorem 6.1 *With the above notation, one has*:

$$\nu = 2^{g-2}(2H - K_C), \text{ if } g \geq 2, \quad \text{and} \quad \nu = H, \text{ if } g = 1 \tag{6.2}$$

and

$$\lambda = 2^{g-2}(2H + K_C), \text{ if } g \geq 2, \quad \text{and} \quad \lambda = H, \text{ if } g = 1. \tag{6.3}$$

As remarked by the referee, the case $g = 1$ follows from [36] whereas the case $g = 2$ follows from the description of the 4 invertible subsheaves as in Remark 4.18 and the fact that $\mathcal{O}_C(p_2 + p_3 + p_4) \cong \omega_C \otimes \det(\mathcal{F}) \otimes (N_1^{\vee})^{\otimes 2}$.

Proof of Theorem 6.1 It suffices to show (6.2).

Claim 6.4 *There exist* $\alpha,\ \beta \in \mathbb{Z}$ *such that*

$$\nu = \alpha K_C + \beta H. \tag{6.5}$$

We first show that Claim 6.4 implies (6.2). Then, we will prove the claim.

If $g = 1$, K_C is trivial and therefore the first summand in (6.5) does not appear. Moreover, it is clear that α and β in (6.5) do not depend on H. As usual, we may assume that the pair $(\mathcal{F}, C)$ is associated to a scroll S correpsonding to a general point in $\mathcal{H}_{d,g}$. Since any N_i is a maximal invertible subsheaf of $\mathcal{F}$, then each L_i corresponds to a section in $\mathrm{Div}_S^{1,\overline{m}}$, where $\overline{m} = d - \overline{n} = \frac{d+g-1}{2}$. Consider the exact sequence

$$0 \to N_i \to \mathcal{F} \to L_i \to 0, \quad 1 \le i \le 2^g.$$

Let $p \in C$ be a general point. Twist the above sequence by $\mathcal{O}_C(p)$

$$0 \to N_i(p) \to \mathcal{F}(p) \to L_i(p) \to 0, \quad 1 \le i \le 2^g.$$

Observe that

$$H' := \det((\mathcal{F}(p)) = H \otimes \mathcal{O}_C(2p),$$

hence $\deg(\mathcal{F}(p)) = d+2$ and $\mathcal{F}(p)$ corresponds to a general point of $U_C(d+2)$. Thus, $N_i(p)$ is an invertible subsheaf of $\mathcal{F}(p)$ of maximal degree, for $1 \le i \le 2^g$. Set $N_i(p) = \mathcal{O}_C(\nu_i')$ and $\nu' = \sum_{i=1}^{2^g} \nu_i'$. One has

$$\nu' = \nu + 2^g p. \tag{6.6}$$

By Claim 6.4, there exist two integers α', β', independent of H and p, such that

$$\nu' = \alpha' K_C + \beta' H'.$$

By comparing the former relation with (6.5) and (6.6), we find

$$(\alpha - \alpha')K_C + (\beta - \beta')H + 2(2^{g-1} - \beta')p = 0.$$

Since $\alpha, \alpha', \beta, \beta' \in \mathbb{Z}$ do not depend on H and p, we deduce

$$\beta = \beta' = 2^{g-1} \quad \text{and} \quad \alpha = -2^{g-2},$$

proving (6.2) (in case $g = 1$, we simply get $\nu = H$).

We are left to prove Claim 6.4. To do this, we follow a similar argument as in [9]. Let $\mathcal{M}_g^0$ be the Zariski open subset of the moduli space $\mathcal{M}_g$, whose points correspond to equivalence classes of smooth curves of genus g without non-trivial automorphisms. By definition, $\mathcal{M}_g^0$ is a fine moduli space, i.e. we have a universal family $p : \mathcal{C} \to \mathcal{M}_g^0$, where $\mathcal{C}$ and $\mathcal{M}_g^0$ are smooth schemes and p is a smooth morphism. $\mathcal{C}$ can be identified with the Zariski open subset $\mathcal{M}_{g,1}^0$ of the moduli space $\mathcal{M}_{g,1}$ of smooth, pointed, genus g curves, whose points correspond to equivalence classes of pairs (C, x), with $x \in C$ and C a smooth curve of genus g without non-trivial automorphisms. On $\mathcal{M}_{g,1}^0$ there is again a universal family $p_1 : \mathcal{C}_1 \to \mathcal{M}_{g,1}^0$, where $\mathcal{C}_1 = \mathcal{C} \times_{\mathcal{M}_g^0} \mathcal{C}$. The family p_1 has a natural regular global section δ whose image is the diagonal. By means of δ, for any integer n, we have the universal family of Picard varieties of order n, i.e.

$$p_1^{(n)} : \mathcal{P}ic^{(n)} \to \mathcal{M}_{g,1}^0$$

(cf. [9, Sect. 2]). For any closed point $[(C, x)] \in \mathcal{M}_{g,1}^0$, its fibre via $p_1^{(n)}$ is isomorphic to $\mathrm{Pic}^{(n)}(C)$. As in [7, Theorem 5.4], let $q : \mathcal{U}_d \to \mathcal{M}_{g,1}^0$ be the relative moduli stack of degree d, rank-two semistable vector bundles; namely, the fibre of q over $[(C, x)]$ is $U_C(d)$. Thus, we have the following natural surjective map over $\mathcal{M}^0_{g,1}$

$$\mathcal{U}_d \xrightarrow{rd} \mathcal{P}ic^{(d)}, \tag{6.7}$$

which is given by the *relative determinant*; namely

$$rd((C, x, \mathcal{F})) = (C, x, \det(\mathcal{F})),$$

for any $[(C, x)] \in \mathcal{M}^0_{g,1}$ and any $[\mathcal{F}] \in U_C(d)$. Observe that the fibre of rd over any $[(C, x, H)] \in \mathcal{M}^0_{g,1}$ is $SU_C(H)$, i.e. the moduli stack of semistable, rank-two vector bundles on C with fixed determinant $H \in \mathrm{Pic}^d(C)$. From [4], we know that any $SU_C(H)$ is *stably rational*, i.e. $SU_C(H) \times \mathbb{P}^k$ is rational for some $k \geq 0$. In particular, it is unirational.

Set $a := \deg(\nu) = 2^{g-1}(d - g + 1)$. One has an obvious morphism

$$\varphi : \mathcal{U}_d \to \mathcal{P}ic^{(a)} \tag{6.8}$$

which maps $(C, x, \mathcal{F})$ to the class of ν. By the unirationality of the fibres of rd, we have a morphism ϕ which makes the following diagram commutative

$$\begin{tikzcd} \mathcal{U}_d \arrow[r, "rd"] \arrow[dr, "\varphi"'] & \mathcal{P}ic^{(d)} \arrow[d, "\phi"] \\ & \mathcal{P}ic^{(a)}. \end{tikzcd} \tag{6.9}$$

At this point, one concludes by imitating the proof of [9, Proposition (5.1)], which can be repeated almost verbatim.

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