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BROWDER AND WEYL SPECTRA OF UPPER TRIANGULAR OPERATOR MATRICES

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Abstract

Let $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(\mathcal{X} \oplus \mathcal{X})$ be an upper triangulat Banach space operator. The relationship between the spectra of M_C and M_0, and their various distinguished parts, has been studied by a large number of authors in the recent past. This paper brings forth the important role played by SVEP, the *single–valued extension property*, in the study of some of these relations. Operators M_C and M_0 satisfying Browder's, or a-Browder's, theorem are characterized, and we prove necessary and sufficient conditions for implications of the type "M_0 satisfies a-Browder's (or a-Weyl's) theorem $\Longleftrightarrow$ M_C satisfies a-Browder's (resp., a-Weyl's) theorem" to hold.

1. Introduction

Let $B(\mathcal{X})$ denote the algebra of operators (equivalently, bounded linear transformations) on a Banach space $\mathcal{X}$. For $A, B, C \in B(\mathcal{X})$, let M_C denote the upper triangular operator $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and let $M_0 = A \oplus B$. The spectrum, and certain distinguished parts thereof, of the operators M_C and M_0 has been studied by a number of authors in the recent past; see references. Of particular interest here is the relationship between the spectral, the Fredholm, the Browder and the Weyl properties.

Given a Banach space operator T, let $\sigma(T)$, $\sigma_a(T)$, $\sigma_b(T)$, $\sigma_e(T)$, $\sigma_w(T)$, $\sigma_{ab}(T)$ and $\sigma_{aw}(T)$ denote (respectively) the spectrum, the approximate point spectrum, the Browder spectrum, the (Fredholm) essential spectrum, the Weyl spectrum, the essential Browder approximate point spectrum and the essential Weyl approximate point spectrum of T. Recall that T is said to have SVEP, *the single–valued extension property*, at a point λ (in the complex plane $\mathbf{C}$) if for every neighbourhood $\mathcal{O}_\lambda$ of λ the only analytic function f from $\mathcal{O}_\lambda$ into the Banach space satisfying $(T -$

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$\mu)f(\mu) = 0$ is the function $f \equiv 0$; *T has SVEP if it has SVEP at every λ.* Let $\Xi(T) = \{\lambda \in \sigma(T) : T \text{ does not have SVEP at } \lambda\}$. It is known that

$$\begin{aligned}
&\sigma_x(M_0) = \sigma_x(A) \cup \sigma_x(B) = \sigma_x(M_C) \cup \{\sigma_x(A) \cap \sigma_x(B)\} \\
&\text{where } \sigma_x = \sigma \text{ or } \sigma_b \text{ or } \sigma_e; \\
&\sigma_w(M_0) \subseteq \sigma_w(A) \cup \sigma_w(B) = \sigma_w(M_C) \cup \{\sigma_w(A) \cap \sigma_w(B)\}; \\
&\text{if } \sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B), \text{ then } \sigma(M_C) = \sigma(M_0), \text{ and} \\
&\sigma_{aw}(M_0) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) = \sigma_{aw}(M_C) \cup \{\Xi(A) \cup \Xi(A^*)\}
\end{aligned}$$

(see [4, 9, 10, 11, 19, 20]). Again, letting $SP(T)$ denote the spectral picture of T, $\mathcal{P}_0(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < dim(T-\lambda)^{-1}(0) < \infty\}$, $\mathcal{P}_0^a(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < dim(T-\lambda)^{-1}(0) < \infty\}$, it is known that: if either $SP(A)$ or $SP(B)$ has no pseudo holes, then $acc\sigma(M_0) \subseteq \sigma_w(M_0) \Longrightarrow acc\sigma(M_C) \subseteq \sigma_w(M_C)$ [19, Theorem 2.3]; if additionally the isolated points of $\sigma(A)$ are eigenvalues of A and $\sigma(A) \setminus \sigma_w(A) = \mathcal{P}_0(A)$, then $\sigma(M_0) \setminus \sigma_w(M_0) = \mathcal{P}_0(M_0) \Longrightarrow \sigma(M_C) \setminus \sigma_w(M_C) = \mathcal{P}_0(M_C)$ [19, Theorem 2.4]. If $\{\Xi(A) \cap \Xi(B^*)\} \cup \Xi(A^*) = \emptyset$, then $acc\sigma(M_0) \subseteq \sigma_w(M_0) \Longrightarrow acc\sigma(M_C) \subseteq \sigma_w(M_C)$; and $acc\sigma_a(M_0) \subseteq \sigma_{aw}(M_0) \Longrightarrow acc\sigma_a(M_C) \subseteq \sigma_{aw}(M_C)$ [11, Proposition 4.1]. Again, if $\sigma_a(A^*)$ has empty interior, isolated points of $\sigma_a(A)$ are eigenvalues of A and $\sigma_a(A) \setminus \sigma_{aw}(A) = \mathcal{P}_0^a(A)$, then $\sigma_a(M_0) \setminus \sigma_{aw}(M_0) = \mathcal{P}_0^a(M_0) \Longrightarrow \sigma_a(M_C) \setminus \sigma_{aw}(M_C) = \mathcal{P}_0^a(M_C)$ [6, Theorem 3.3].

In current terminology, an operator T satisfying $acc\sigma(T) \subseteq \sigma_w(T)$ (resp., $acc\sigma_a(T) \subseteq \sigma_{aw}(T)$) is said to satisfy Browder's theorem, or Bt (resp., a-Browder's theorem, or $a-Bt$); if T satisfies $\sigma(T) \setminus \sigma_w(T) = \mathcal{P}_0(T)$ (resp., $\sigma_a(T) \setminus \sigma_{aw}(T) = \mathcal{P}_0^a(T)$), then T said to satisfy Weyl's theorem, or Wt (resp., a-Weyl's theorem, or $a-Wt$). In this paper, we introduce most of our notation and terminology in Section 2, Section 3 is devoted to proving a number of complementary results, and Sections 4 and 5 are devoted to proving our main results. We start Section 4 by characterizing operators M_0 and M_C satisfying Bt: much of the work here is an extension of the characterizations known to hold for single linear operators T satisfying $acc\sigma(T) \subseteq \sigma_w(T)$. A natural progression here leads us to consider conditions under which M_0 satisfies $Bt \Longrightarrow M_C$ satisfies Bt, and *vice versa*. Next, we characterize operators M_0 and M_C satisfying $a-Bt$, and this is followed by a consideration of conditions ensuring M_0 satisfies $a-Bt \Longrightarrow M_C$ satisfies $a-Bt$ (and *vice versa*). We consider Wt and $a-Wt$ for the operators M_0 and M_C in Section 5. Here we prove a necessary and sufficient for the equivalence M_0 *satisfies* $Wt \Longleftrightarrow M_C$ *satisfies* Wt for operators M_C such that $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, which is then applied to deduce a number of known results. For operators M_0 and M_C such that $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$, we prove a sufficient condition for the implications M_0 *satisfies* $a-Wt \Longrightarrow M_C$ *satisfies* $a-Wt$ and M_C *satisfies* $a-Wt \Longrightarrow M_0$ *satisfies* $a-Wt$, once again applying the ensuing theorem to deducing some known results.

Almost all our results extend, with evident minor changes, to the case in which A is a Banach space operator in $B(\mathcal{X})$, B is a Banach space operator in $B(\mathcal{Y})$ and C is a Banach space operator in $B(\mathcal{Y}, \mathcal{X})$. We shall, however, restrict ourselves to operators M_0 and $M_C \in B(\mathcal{X} \oplus \mathcal{X})$.

2. Notation and terminology

In the following, the diagonal operator M_0 and the upper triangular operator M_C will be defined as in the introduction, and $T \in B(\mathcal{Y})$ shall denote a general Banach space operator. With $\mathbf{C}$ denoting the complex plane, $\alpha(T) = \dim(T^{-1}(0))$, $\beta(T) = \dim(\mathcal{Y}/T\mathcal{Y})$ and $\operatorname{ind}(T) = \alpha(T) - \beta(T)$, let

$$\begin{aligned}
&\Phi_+(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is upper semi–Fredholm}\},\\
&\Phi_+^-(T) = \{\lambda \in \Phi_+(T) : \operatorname{ind}(T - \lambda) \le 0\},\\
&\Phi_-(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is lower semi–Fredholm}\},\\
&\Phi_-^+(T) = \{\lambda \in \Phi_-(T) : \operatorname{ind}(T - \lambda) \ge 0\},\\
&\Phi(T) = \Phi_+(T) \cap \Phi_-(T), \text{ and}\\
&\Phi^0(T) = \{\lambda \in \Phi(T) : \operatorname{ind}(T - \lambda) = 0\}.
\end{aligned}$$

Then the *upper semi–Fredholm spectrum* $\sigma_{SF_+}(T)$, the *lower semi–Fredholm spectrum* $\sigma_{SF_-}(T)$, the *(Fredholm) essential spectrum* $\sigma_e(T)$, the *Weyl spectrum* $\sigma_w(T)$, the *Weyl essential approximate point spectrum* $\sigma_{aw}(T)$ and the *Weyl essential surjectivity spectrum* $\sigma_{sw}(T)$ of T are the sets

$$\begin{aligned}
&\sigma_{SF_+}(T) = \{\lambda \in \sigma(T) : \lambda \notin \Phi_+(T)\}, \ \ \sigma_{SF_-}(T) = \{\lambda \in \sigma(T) : \lambda \notin \Phi_-(T)\},\\
&\sigma_e(T) = \{\lambda \in \sigma(T) : \lambda \notin \Phi(T)\}, \ \ \sigma_w(T) = \{\lambda \in \sigma(T) : \lambda \notin \Phi^0(T)\},\\
&\sigma_{aw}(T) = \{\lambda \in \sigma_a(T) : \lambda \notin \Phi_+^-(T)\}, \ \ \sigma_{sw}(T) = \{\lambda \in \sigma_s(T) : \lambda \notin \Phi_-^+(T)\}.
\end{aligned}$$

Here $\sigma_a(T)$ and $\sigma_s(T)$ denote the approximate point spectrum and the surjectivity spectrum of T, respectively. The ascent $\operatorname{asc}(T)$ of T and the descent $\operatorname{dsc}(T)$ of T are, respectively, the least non–negative integers n and m such that $T^{-n}(0) = T^{-(n+1)}(0)$ and $T^m\mathcal{Y} = T^{m+1}\mathcal{Y}$; if no such integer n (resp., m) exists, then $\operatorname{asc}(T) = \infty$ (resp., $\operatorname{dsc}(T) = \infty$). It is easily verified, see [22, Exercise 7, Page 293], that

$$\begin{aligned}
&\operatorname{asc}(A - \lambda) \le \operatorname{asc}(M_C - \lambda) \le \operatorname{asc}(A - \lambda) + \operatorname{asc}(B - \lambda);\\
&\operatorname{dsc}(B - \lambda) \le \operatorname{dsc}(M_C - \lambda) \le \operatorname{dsc}(A - \lambda) + \operatorname{dsc}(B - \lambda)
\end{aligned}$$

for every $\lambda \in \mathbf{C}$. The *Browder spectrum* $\sigma_b(T)$ and the *Browder essential approximate point spectrum* $\sigma_{ab}(T)$ of T are the sets

$$\begin{aligned}
\sigma_b(T) &= \{\lambda \in \sigma(T) : \lambda \notin \Phi(T) \text{ or one of } \operatorname{asc}(T - \lambda) \text{ and } \operatorname{dsc}(T - \lambda) \text{ is infinite}\};\\
\sigma_{ab}(T) &= \{\lambda \in \sigma_a(T) : \lambda \notin \Phi_+(T) \text{ or } \operatorname{asc}(T - \lambda) \text{ is infinite}\}.
\end{aligned}$$

Let $\sigma_x(T)$ denote $\sigma(T)$ or a distinguished part thereof; let $\operatorname{acc}\sigma_x(T)$, $\operatorname{iso}\sigma_x(T)$, $\mathcal{R}_0(T)$ and $\mathcal{P}_0(T)$ denote the accumulation points of $\sigma_x(T)$, the isolated points of $\sigma_x(T)$, the finite rank poles of (the resolvent of) T and the isolated points of $\sigma(T)$ which are eigenvalues of T of finite multiplicity, respectively. (Recall that $\lambda \in \operatorname{iso}\sigma(T)$ is a pole if and only if $\operatorname{asc}(T - \lambda) = \operatorname{dsc}(T - \lambda) < \infty$.) Let

$$\mathcal{R}_0^a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : \lambda \in \Phi_+(T), \operatorname{asc}(T - \lambda) < \infty\}$$

and

$$\mathcal{P}_0^a(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T-\lambda) < \infty\}.$$

In keeping with current terminology, [1, 11, 14, 15, 19], we say that T satisfies :

Browder's theorem, or Bt, if $\text{acc}\sigma(T) \subseteq \sigma_w(T)$;

a-Browder's theorem, or $a - Bt$, if $\text{acc}\sigma_a(T) \subseteq \sigma_{aw}(T)$;

Weyl's theorem, or Wt, if $\sigma(T) \setminus \sigma_w(T) = \mathcal{P}_0(T)$;

a-Weyl's theorem, or $a - Wt$, if $\sigma_a(T) \setminus \sigma_{aw}(T) = \mathcal{P}_0^a(T)$.

Remark 2.1. Calling the conditions above "theorems" is a bit of a misnomer: it would be more appropriate to call these conditions "Browder's condition", "a-Browder's condition" etc.

It is well known, [1, 2, 14, 15], that T satisfies $Bt \Longleftrightarrow T^*$ satisfies Bt, T satisfies $a - Bt \Longrightarrow T$ satisfies Bt, T satisfies $a - Wt \Longrightarrow T$ satisfies $Wt \Longrightarrow T$ satisfies Bt and T satisfies $a - Wt \Longrightarrow T$ satisfies $a - Bt$. The one sided implications here are strict (in the sense that the reverse implication in general fails). A necessary and sufficient condition for T to satisfy Bt (resp., $a - Bt$) is that T has SVEP on $\{\lambda \in \sigma(T) : \lambda \notin \sigma_w(T)\}$ (resp., on $\{\lambda \in \sigma_a(T) : \lambda \notin \sigma_{aw}(T)\}$); a necessary and sufficient condition for T to satisfy Wt (resp., $a - Wt$) is that T satisfies Bt and $\mathcal{P}_0(T) \subseteq \mathcal{R}_0(T)$ (resp., T satisfies $a - Bt$ and $\mathcal{P}_0^a(T) \subseteq \mathcal{R}_0^a(T)$). (See [2, 12, 13, 1].)

The *quasinilpotent part* $H_0(T-\lambda)$ and the *analytic core* $K(T-\lambda)$ of $(T-\lambda)$ are defined by

$$H_0(T-\lambda) = \{y \in \mathcal{Y} : \lim_{n \longrightarrow \infty} ||(T-\lambda)^n y||^{\frac{1}{n}} = 0\}$$

and

$$\begin{aligned} K(T-\lambda) = \{y \in \mathcal{Y} : &\text{there exists a sequence } \{y_n\} \subset \mathcal{Y} \text{ and } \delta > 0 \\ &\text{for which } y = y_0, (T-\lambda)(y_{n+1}) = y_n \text{ and } \|y_n\| \leq \delta^n \|y\| \text{ for all } n = 1, 2, ...\}. \end{aligned}$$

$H_0(T-\lambda)$ and $K(T-\lambda)$ are (generally) non-closed hyperinvariant subspaces of T such that $(T-\lambda)^{-q}(0) \subseteq H_0(T-\lambda)$ for all $q = 0, 1, 2, ...$, and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$; also, if $\lambda \in \text{iso}\sigma(T)$, then $\mathcal{Y} = H_0(T-\lambda) \oplus K(T-\lambda)$ [21].

Given a subset $\sigma_{xt}(T)$ of $\sigma_x(T)$, we shall denote the complement of $\sigma_{xt}(T)$ in $\sigma_x(T)$ by $\sigma_{xt}(T)^{\mathcal{C}}$: thus $\sigma_w(T)^{\mathcal{C}} = \sigma(T) \setminus \sigma_w(T)$ and $\sigma_{aw}(T)^{\mathcal{C}} = \sigma_a(T) \setminus \sigma_{aw}(T)$. Any further notation, incidental or otherwise, will be introduced on an as and when required basis.

3. Some Complementary Results

We start by gathering together some technical results, all known, which will be used in the sequel, often without further reference. The following implications hold [17, Chapter IV, Article 38]: $\text{asc}(T-\lambda) < \infty \Longrightarrow \alpha(T-\lambda) \leq \beta(T-\lambda)$; $\text{dsc}(T-\lambda) < \infty \Longrightarrow \beta(T-\lambda) \leq \alpha(T-\lambda)$; if $\alpha(T-\lambda) = \beta(T-\lambda)$, then either of $\text{asc}(T-\lambda) < \infty$ and $\text{dsc}(T-\lambda) < \infty \Longrightarrow \text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$. If $\lambda \in \Phi_{\pm}(T)$, then T has SVEP at $\lambda \Longleftrightarrow \text{asc}(T-\lambda) < \infty$ and T^* has SVEP at $\lambda \Longleftrightarrow \text{dsc}(T-\lambda) < \infty$ [1, Theorems 3.16, 3.17]. From this it follows that if both T and T^* have SVEP at $\lambda \in \Phi_{\pm}(T)$, then $\lambda \in \Phi^0(T)$ and $\lambda \in \mathcal{R}_0(T)$. If $\lambda \in \Phi^0(T)$ and either of $\text{asc}(T-\lambda)$ and $\text{dsc}(T-\lambda)$ is finite (equivalently, either T or T^* has SVEP at λ), then $\lambda \in \mathcal{R}_0(T)$. Again, if $\lambda \in \Phi_{\pm}(T)$ and T has SVEP at λ, then $\lambda \in \mathcal{R}_0^a(T)$ [1, Theorem 3.23]. If $H_0(T-\lambda)$ is closed, or $H_0(T-\lambda) \cap K(T-\lambda)$ is closed, then ($H_0(T-\lambda) \cap K(T-\lambda) = \{0\}$ and) T has SVEP at λ [1, Theorem 2.31]; analogously, if $H_0(T-\lambda) + K(T-\lambda)$ is norm dense in $\mathcal{Y}$, then T^* has SVEP at λ [1, Theorem 2.32]. If $\lambda \in \text{iso}\sigma(T)$, then $\mathcal{Y} = H_0(T-\lambda) \oplus K(T-\lambda)$ [21]; hence, if $\lambda \in \text{iso}\sigma(T)$, then $\dim H_0(T-\lambda) < \infty \Longleftrightarrow \text{codim} K(T-\lambda) < \infty$. If $\lambda \in \Phi_{\pm}(T)$, then $T-\lambda$ is essentially semi–regular (i.e., there exist two T-invariant subspaces M and N of $\mathcal{Y}$ such that $\mathcal{Y} = M \oplus N$, M is finite dimensional and $(T-\lambda)|_M$ is nilpotent) and T (resp., T^*) has SVEP at λ if and only if $H_0(T-\lambda) = M$ (resp., $K(T-\lambda) = N$) [1, Theorems 3.14, 3.15].

For an operator $P \in B(\mathcal{Y})$ and $\sigma_x(T)$ a subset of $\sigma(T)$, let

$$S_{\sigma_x(T)}(P) = \{\lambda \in \sigma(T) \setminus \sigma_x(T) : P \text{ does not have SVEP at } \lambda\};$$

let

$$S(T) = \{\lambda \in \sigma(T) : T \text{ does not have SVEP at } \lambda\}.$$

A straightforward argument then proves that the following relations hold:

(I). $\sigma(M_0) = \sigma(A) \cup \sigma(B) = \sigma(M_C) \cup \{\sigma(A) \cap \sigma(B)\} = \sigma(M_C) \cup \{S_{\sigma_a(A)}(A^*) \cap S_{\sigma_s(B)}(B)\}$.

(II). $\sigma_e(M_0) = \sigma_e(A) \cup \sigma_e(B) = \sigma_e(M_C) \cup \{\sigma_e(A) \cap \sigma_e(B)\}$, and if $S_{\sigma_e(M_C)}(A^*) \cap S_{\sigma_e(M_C)}(B) = \emptyset$, then $\sigma_e(M_0) = \sigma_e(M_C)$.

(III). $\sigma_b(M_0) = \sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_C) \cup \{\sigma_b(A) \cap \sigma_b(B))\} = \sigma_b(M_C) \cup \{S_{\sigma_b(M_C)}(A^*) \cap S_{\sigma_b(M_C)}(B)\}$.

(IV). $\sigma_w(M_0) \subseteq \sigma_w(A) \cup \sigma_w(B) = \sigma_w(M_C) \cup \{\sigma_w(A) \cap \sigma_w(B)\} = \sigma_w(M_0) \cup \{\sigma_w(A) \cap \sigma_w(B)\}$.

(V). $\sigma_w(A) \cup \sigma_w(B)$ is a subset of the sets $\sigma_w(M_C) \cup \{S_{\sigma_w(M_C)}(A) \cup S_{\sigma_w(M_C)}(A^*)\}$, $\sigma_w(M_C) \cup \{S_{\sigma_w(M_C)}(B) \cup S_{\sigma_w(M_C)}(B^*)\}$, $\sigma_w(M_C) \cup \{S_{\sigma_w(M_C)}(A) \cup S_{\sigma_w(M_C)}(B)\}$, and $\sigma_w(M_C) \cup \{S_{\sigma_w(M_C)}(A^*) \cup S_{\sigma_w(M_C)}(B^*)\}$.

Remark 3.1. (i) Let $r(T)$ and $r_w(T)$ denote the spectral radius $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ and the Weyl spectral radius $r_w(T) = \sup\{|\lambda| : \lambda \in \sigma_w(T)\}$. Then (I) taken alongwith the equality part of (IV) implies that $r(M_C) = r(M_0)$ and $r_w(M_C) = r_w(M_0)$. Suppose now that $\mathcal{R}_0(M_0) = \emptyset$. Let ∂F denote the boundary of $F \subset \mathbf{C}$. Then $\lambda \in \partial\sigma(M_0)$ implies $\lambda \in \partial\sigma_w(M_C)$. To see this, start by observing that both M_0 and M_C have SVEP at λ; in particular, both A and B have SVEP at λ. If $\lambda \notin \partial\sigma_w(M_C)$, then $\lambda \in \Phi^0(A) \cap \Phi^0(B)$. Thus, $\lambda \in \Phi^0(M_0)$; since M_0 has SVEP at λ, $\lambda \in \mathcal{R}_0(M_0)$ – a contradiction since $\mathcal{R}_0(M_0) = \emptyset$. Hence $\partial\sigma(M_0) = \partial\sigma_w(M_C)$, which implies that $r(M_0) = r_w(M_C)$. We have proved the following improved version of [20, Corollary 6]: *if* $\mathcal{R}_0(M_0) = \emptyset$, *then* $r_w(M_C) = r(M_C) = r(M_0) = r_w(M_0)$.
(ii) If either of the sets $S_{\sigma_w(M_C)}(A) \cup S_{\sigma_w(M_C)}(A^*)$, $S_{\sigma_w(M_C)}(A) \cup S_{\sigma_w(M_C)}(B)$, $S_{\sigma_w(M_C)}(A^*) \cup S_{\sigma_w(M_C)}(B^*)$ and $S_{\sigma_w(M_C)}(B) \cup S_{\sigma_w(M_C)}(B^*)$ is the empty set, then (V) implies that $\sigma_w(M_0) = \sigma_w(A) \cup \sigma_w(B) = \sigma_w(M_C)$. In particular, if one of A and B is either polynomially compact or a Riesz operator or has countable spectrum or spectrum with empty interior, then $\sigma_w(M_0) = \sigma_w(A) \cup \sigma_w(B) = \sigma_w(M_C)$; *cf.* [20, Corollary 5]. Later on we shall prove further conditions which imply the equality of these spectra.

The equality $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$ has a bearing on $\sigma(M_C)$.

Proposition 3.2. *If* $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, *or* $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, *then* $\sigma(M_C) = \sigma(A) \cup \sigma(B)$.

Proof. Since $\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B)$, it would suffice to prove the reverse inclusion. Let $\lambda \notin \sigma(M_C)$. Then the invertibility of $M_C - \lambda$ implies that $A - \lambda$ is left invertible, $B - \lambda$ is right invertible and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) = 0$ (which, since $\alpha(A - \lambda) = \beta(B - \lambda) = 0$ implies that $\beta(A - \lambda) = \alpha(B - \lambda)$). Assume, to start with, that $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$. If $\beta(A - \lambda) \neq 0$, then $\lambda \in \sigma_w(A) \cap \sigma_w(B) \subseteq \sigma(M_C)$, a contradiction. Hence $\beta(A - \lambda) = \alpha(B - \lambda) = 0$, which implies that $\lambda \notin \sigma(A) \cup \sigma(B)$. Assume now that $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. If $\beta(A - \lambda) = \alpha(B - \lambda) \neq 0$, then $\text{ind}(A - \lambda) < 0$ and $\text{ind}(B - \lambda) > 0$. This, since already $\lambda \in \Phi_+(A) \cap \Phi_-(B)$, implies that $\lambda \in \Phi_+^-(A) \cap \Phi_-^+(B)$. Observe that if (also) $\lambda \notin \sigma_{aw}(B)$, then $\lambda \in \Phi^0(A) \cap \Phi^0(B) \Longrightarrow \beta(A - \lambda) = \alpha(B - \lambda) = 0$. Consequently, $\lambda \in \sigma_{aw}(B)$. But then $\lambda \in \sigma(M_C)$ – once again a contradiction. Hence $\beta(A - \lambda) = \alpha(B - \lambda) = 0$, and so $\lambda \notin \sigma(A) \cup \sigma(B)$. □

Proposition 3.2 extends [4, Proposition 3] and [11, Corollary 6].

It is easily seen that if $\lambda \in \sigma_{aw}(M_0)^{\mathcal{C}}$ $(= \sigma_a(M_0) \setminus \sigma_{aw}(M_0))$, then $\lambda \in \Phi_+(A) \cap \Phi_+(B)$ and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) \leq 0$. Apparently, $\sigma_{aw}(M_0) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B)$. A relation of type (IV) between $\sigma_{aw}(M_C)$ and $\sigma_{aw}(M_0)$ is seemingly not possible. Indeed:

Proposition 3.3. *[8, Theorem 4.6] If* $\lambda \in \sigma_{aw}(M_C)^{\mathcal{C}}$, *then*
(a) $\lambda \in \Phi_+(A)$ *and* $ind(A - \lambda) + ind(B - \lambda) \leq 0$;
(b) either $\lambda \in \Phi_+(B)$, *or the range of* $B - \lambda$ *is not closed, or the essential embedding* $(A^* - \lambda I^*)^{-1}(0) \prec \mathcal{X}^*/(B^* - \lambda I^*)\mathcal{X}^*$ *does not hold.*

Here, for a pair of Banach spaces $\mathcal{Y}$ and $\mathcal{Z}$, $\mathcal{Y} \prec \mathcal{Z}$ denotes "$\mathcal{Y}$ can be essentially embedded in $\mathcal{Z}$", where we say that $\mathcal{Y}$ can be embedded in $\mathcal{Z}$, $\mathcal{Y} \preceq \mathcal{Z}$, if there exists a left invertible operator $J : \mathcal{Y} \longrightarrow \mathcal{Z}$, and that $\mathcal{Y} \prec \mathcal{Z}$ if $\mathcal{Y} \preceq \mathcal{Z}$ and $\mathcal{Z}/W\mathcal{Y}$ is an infinite dimensional linear space for every $W \in B(\mathcal{Y}, \mathcal{Z})$.

Evidently, $\sigma_a(M_0) = \sigma_a(A) \cup \sigma_a(B)$ and $\sigma_a(A) \subseteq \sigma_a(M_C) \subseteq \sigma_a(A) \cup \sigma_a(B)$.The following proposition gives a sufficient condition for $\sigma_a(M_C) = \sigma_a(M_0)$.

Proposition 3.4. *If $S_{\sigma_a(M_C)}(A^*) = \emptyset$, then $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B)$.*

Proof. If $\lambda \in \sigma_a(M_C)^{\mathcal{C}}$, then $A - \lambda$ is left invertible. Hence $A^* - \lambda I^*$ is right invertible. Since a surjective operator has SVEP at 0 if and only if it is injective [1, Corollary 2.24], $A^* - \lambda I^*$ (and so also $A - \lambda$)) is invertible. But then $B - \lambda$ is left invertible. Hence $\lambda \in \sigma_a(M_C)^{\mathcal{C}}$ implies $\lambda \notin \sigma_a(A) \cup \sigma_a(B)$. □

Let

$$\gamma(T) = \inf_{x \notin T^{-1}(0)} \frac{||Tx||}{\text{dist}(x, T^{-1}(0))}$$

denote the *reduced minimum modulus* of T. Then $\gamma(T) = \gamma(T^*) > 0$ if and only if $T\mathcal{Y}$ is closed. The following technical lemma will be required in our next result.

Lemma 3.5. *If $\lambda \in iso\sigma_a(T)$, or $\lambda \in iso\sigma(T)$, then the mapping $\lambda \longrightarrow \gamma(T - \lambda)$ is not continuous if and only if $\gamma(T - \lambda) > 0$.*

Proof. We consider the case $\lambda \in \text{iso}\sigma_a(T)$; the proof for the other case is similar. Let $\lambda \in \text{iso}\sigma_a(T)$. Then there exists an ϵ-neighbourhood $\mathcal{O}_\epsilon$ of λ such that $T - \mu$ is left invertible for every $(\lambda \neq)\mu \in \mathcal{O}_\epsilon$. Let $x \in (T - \lambda)^{-1}(0)$. Then

$$\begin{aligned} \gamma(T - \mu) &\leq \frac{||(T - \mu)x||}{\text{dist}(x, (T - \mu)^{-1}(0))} = \frac{||(T - \mu)x||}{||x||} \\ &= \frac{||(T - \mu)x - (T - \lambda)x||}{||x||} = |\lambda - \mu|, \end{aligned}$$

which implies that $\gamma(T - \mu) \longrightarrow 0$ as $\mu \longrightarrow \lambda$. Hence $\gamma(T - \lambda) > 0$ precisely when $\lambda \longrightarrow \gamma(T - \lambda)$ is not continuous. □

Observe that $\mathcal{R}_0(T) \cap \sigma_b(T) = \emptyset$ for every operator T; hence $\mathcal{R}_0(T) \cap \sigma_w(T) = \emptyset$ for every operator T.

Proposition 3.6. *If $\lambda \in \mathcal{R}_0(M_C)$, then the mapping $\lambda \longrightarrow \gamma(M_C - \lambda)$ is not continuous if and only if the mappings $\lambda \longrightarrow \gamma(A - \lambda)$ and $\lambda \longrightarrow \gamma(B^* - \lambda I^*)$ are not continuous.*

Proof. Let $\lambda \in \mathcal{R}_0(M_C)$, and assume that the mappings $\lambda \longrightarrow \gamma(A - \lambda)$ and $\lambda \longrightarrow \gamma(B^* - \lambda I^*)$ are not continuous. Then $\gamma(A - \lambda)$ and $\gamma(B - \lambda)$ are > 0. Since

$$\begin{aligned} \gamma(M_C - \lambda) &\geq \gamma \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \min\{1, \gamma(A - \lambda), \gamma(B - \lambda), \gamma(A - \lambda)\gamma(B - \lambda)\} \\ &\geq \frac{1}{1 + ||C||} \min\{1, \gamma(A - \lambda), \gamma(B - \lambda), \gamma(A - \lambda)\gamma(B - \lambda)\} \\ &> 0, \end{aligned}$$

Lemma 3.5 implies that the mapping $\lambda \longrightarrow \gamma(M_C-\lambda)$ is not continuous. Conversely, assume that the mapping $\lambda \longrightarrow \gamma(M_C-\lambda)$ is not continuous at every $\lambda \in \mathcal{R}_0(M_C)$. Since $\mathcal{R}_0(M_C)\cap\sigma_{aw}(M_C)=\emptyset$, $\lambda\in\sigma_{aw}(M_C)^{\mathcal{C}}$. The hypothesis $\lambda\in\mathcal{R}_0(M_C)$ also implies that $\mathrm{asc}(A-\lambda)$ and $\mathrm{dsc}(B-\lambda)$ are finite; hence $\lambda\in\sigma_{aw}(M_C)^{\mathcal{C}}$ implies that $\lambda\in\Phi_+^-(A)\cap\Phi_-^+(B)$ (so that $\gamma(A-\lambda)$ and $\gamma(B^*-\lambda I^*)$ are >0). Since A and B^* have SVEP at λ, $\lambda\in\mathrm{iso}\sigma_a(A)\cup\mathrm{iso}\sigma_a(B^*)$. Lemma 3.5 applies, and we conclude that the mappings $\lambda\longrightarrow\gamma(A-\lambda)$ and $\lambda\longrightarrow\gamma(B^*-\lambda I^*)$ are not continuous. □

If we let $\mathcal{R}(T)$ denote the set of poles (of the resolvent) of T, and $\rho(T)$ the resolvent set of T, then $\mathcal{R}(M_0)=\mathcal{R}(A)\cup\mathcal{R}(B)$ $(=\{\mathcal{R}(A)\cap\rho(B)\}\cup\{\mathcal{R}(A)\cap\mathcal{R}(B)\}\cup\{\rho(A)\cap\mathcal{R}(B)\})$. The following proposition is an immediate consequence of the observations that:

$$\begin{aligned}&(M_0-\lambda)^{-1}(0)=\{(A-\lambda)^{-1}(0)\cap\rho(B-\lambda)\}\cup\{(A-\lambda)^{-1}(0)\cap(B-\lambda)^{-1}(0)\}\\&\cup\{\rho(A-\lambda)\cap(B-\lambda)^{-1}(0)\};\\&\alpha(M_0-\lambda)<\infty\Longrightarrow\alpha(A-\lambda)<\infty,\alpha(B-\lambda)<\infty,\\&\text{and }\ \lambda\in\mathcal{R}(M_0)\Longleftrightarrow\lambda\in\mathcal{R}(A)\cup\mathcal{R}(B).\end{aligned}$$

Proposition 3.7. $\lambda\in\mathcal{R}_0(M_0)\Longleftrightarrow\lambda\in\mathcal{R}_0(A)\cup\mathcal{R}_0(B)$.

If $\lambda\in\mathcal{R}(A)\cup\mathcal{R}(B)$, then the inequalities $\mathrm{asc}(M_C-\lambda)\le\mathrm{asc}(A-\lambda)+\mathrm{asc}(B-\lambda)$ and $\mathrm{dsc}(M_C-\lambda)\le\mathrm{dsc}(A-\lambda)+\mathrm{dsc}(B-\lambda)$ imply that $\lambda\in\mathcal{R}(M_C)$. Also, since $\alpha(M_C-\lambda)\le\alpha(A-\lambda)+\alpha(B-\lambda)$, $\lambda\in\mathcal{R}_0(A)\cup\mathcal{R}_0(B)\Longrightarrow\lambda\in\mathcal{R}_0(M_C)$; *cf.* [4, Theorem 1]. Observe that if $\lambda\in\mathcal{R}_0(A)\cup\mathcal{R}_0(B)$, then A, A^*, B and B^* have SVEP at λ.

Proposition 3.8. *(i). If either A^* or B has SVEP on $\mathcal{R}_0(M_C)$, then $\lambda\in\mathcal{R}_0(M_C)$ if and only if $\lambda\in\mathcal{R}_0(A)\cup\mathcal{R}_0(B)$.*
(ii). If $\sigma(M_C)=\sigma(A)\cup\sigma(B)$, then $\lambda\in\mathcal{R}_0(M_C)$ if and only if $\lambda\in\mathcal{R}_0(A)\cup\mathcal{R}_0(B)$.

Proof. (i). We have to prove that $\lambda\in\mathcal{R}_0(M_C)$ implies $\lambda\in\mathcal{R}_0(A)\cup\mathcal{R}_0(B)$. Since $\lambda\in\mathcal{R}_0(M_C)$ implies M_C and $M_C{}^*$ have SVEP at λ, A and B^* have SVEP at λ. Again, since $\mathcal{R}_0(M_C)\cap\sigma_w(M_C)=\emptyset$, $\lambda\in\mathcal{R}_0(M_C)$ implies $\lambda\in\Phi_+^-(A)\cap\Phi_-^+(B)$ and $\mathrm{ind}(A-\lambda)+\mathrm{ind}(B-\lambda)=0$. Consequently, if A^* has SVEP at λ, then $\lambda\in\Phi^0(A)$, which forces $\lambda\in\Phi^0(B)$; again, if B has SVEP at λ, then $\lambda\in\Phi^0(B)$, and this forces $\lambda\in\Phi^0(A)$. The proof now follows from the fact that A and B^* have SVEP at λ in the first case, and A and B have SVEP at λ in the second case.
(ii). If $\lambda\in\mathcal{R}_0(M_C)$ and $\sigma(M_C)=\sigma(A)\cup\sigma(B)$, then $\lambda\in\mathrm{iso}\sigma(A)\cup\mathrm{iso}\sigma(B)$ $(=\{\mathrm{iso}\sigma(A)\cap\rho(B)\}\cup\{\mathrm{iso}\sigma(A)\cap\mathrm{iso}\sigma(B)\}\cup\{\rho(A)\cap\mathrm{iso}\sigma(B)\})$, $\alpha(A-\lambda)<\infty$, $\beta(B-\lambda)<\infty$, $\mathrm{asc}(A-\lambda)<\infty$ and $\mathrm{dsc}(B-\lambda)<\infty$. The conclusions that $\lambda\in\mathrm{iso}\sigma(B)\cup\rho(B)$, $\beta(B-\lambda)<\infty$ and $\mathrm{dsc}(B-\lambda)<\infty$ imply that $\lambda\in\mathcal{R}_0(B)\cup\rho(B)$ [1, Theorem 3.81]. Evidently, $\lambda\in\Phi_+(A)$. Hence $\lambda\in\mathrm{iso}\sigma(A^*)\cup\rho(B)$, $\mathrm{dsc}(A^*-\lambda I^*)<\infty$ and $\beta(A^*-\lambda I^*)<\infty$. But then, [1, Theorem 3.81], $\lambda\in\mathcal{R}_0(A^*)\cup\rho(B)\Longrightarrow\lambda\in\mathcal{R}_0(A)\cup\rho(B)$. □

Proposition 3.8(ii) had earlier been proved by Barnes [4, Theorem 2] using a different technique.

4. Bt and $a-Bt$

In the following, alongwith considering necessary and (/or) sufficient conditions for M_0 and M_C to satisfy Bt or $a-Bt$, we consider conditions for the implications M_0 (or M_C) satisfies $Bt \Rightarrow M_C$ (resp., M_0) satisfies Bt and M_0 (or M_C) satisfies $a-Bt \Rightarrow M_C$ (resp., M_0) satisfies $a-Bt$. We start by characterizing operators M_0 satisfying Bt: many of these conditions are known to be equivalent for a single operator satisfying Bt (see, for example, [2]). Recall that T satisfies Bt if $\mathrm{acc}\sigma(T) \subseteq \sigma_w(T)$.

Theorem 4.1. *The following conditions are equivalent:*
(i). M_0 satisfies Bt.
(ii). $S_{\sigma_w(M_0)}(M_0)=\emptyset$.
(iii). $S_{\sigma_w(M_0)}(A)=S_{\sigma_w(M_0)}(B)=\emptyset$.
(iv). $\lambda \in \Phi^0(A)\cap\Phi^0(B)$ and $\lambda \in iso\sigma(A)\cup iso\sigma(B)$ for every $\lambda\in\sigma_w(M_0)^{\mathcal{C}}$.
(v). $\sigma_w(M_0)=\sigma_b(M_0)$.
(vi). $\sigma_w(M_0)^{\mathcal{C}}=\mathcal{R}_0(M_0)=\mathcal{R}_0(A)\cup\mathcal{R}_0(B)$.
(vii). The mappings $\lambda\longrightarrow\gamma(A-\lambda)$ and $\lambda\longrightarrow\gamma(B-\lambda)$ are not continuous on $\mathcal{R}_0(M_0)$.
(viii). $dimH_0(A-\lambda)$ and $dimH_0(B-\lambda)$ are finite on $\sigma_w(M_0)^{\mathcal{C}}$.
(ix). $codimK(A-\lambda)$ and $codimK(B-\lambda)$ are finite on $\sigma_w(M_0)^{\mathcal{C}}$.
(x). $asc(A-\lambda)$ and $asc(B-\lambda)$, or $dsc(A-\lambda)$ and $dsc(B-\lambda)$, are finite on $\sigma_w(M_0)^{\mathcal{C}}$.
(xi). ${M_0}^*$ satisfies Bt.

Proof. The equivalence of the conditions (i), (v), (vi) and (xi), for every Banach space operator, is well known [2]; we prove the equivalence of the remaining conditions to (i).

$(i)\Longrightarrow(ii)\Longrightarrow(iii)$. If (i) is satisfied, then $\sigma(M_0)=\mathrm{acc}\sigma(M_0)\cup\mathrm{iso}\sigma(M_0)\subseteq\sigma_w(M_0)\cup\mathrm{iso}\sigma(M_0)\subseteq\sigma(M_0)$ implies that $\sigma_w(M_0)^{\mathcal{C}}\subseteq\mathrm{iso}\sigma(M_0)\Longrightarrow S_{\sigma_w(M_0)}(M_0)=\emptyset$. Hence $(i)\Longrightarrow(ii)$. Since M_0 has SVEP at a point if and only if A and B have SVEP at the point, $(ii)\Longrightarrow(iii)$.

$(iii)\Longrightarrow(iv)$. If (iii) is satisfied, then $\lambda\in\sigma_w(M_0)^{\mathcal{C}}\Longrightarrow\lambda\in\Phi(A)\cap\Phi(B)$, $\mathrm{ind}(A-\lambda)+\mathrm{ind}(B-\lambda)=0$, and both A and B have SVEP at λ ($\Rightarrow \mathrm{ind}(A-\lambda)$ and $\mathrm{ind}(B-\lambda)$ are ≤ 0). Hence (iii) implies that $\lambda\in\Phi^0(A)\cap\Phi^0(B)$, $\mathrm{asc}(A-\lambda)<\infty$ and $\mathrm{asc}(B-\lambda)<\infty$, and this in turn implies that $\lambda\in\Phi^0(A)\cap\Phi^0(B)$ and $\lambda\in\mathrm{iso}\sigma(A)\cup\mathrm{iso}\sigma(B)$, for every $\lambda\in\sigma_w(M_0)^{\mathcal{C}}$.

$(iv)\Rightarrow(vii)$. Evidently, (iv) implies that $\lambda\in\mathcal{R}_0(A)\cup\mathcal{R}_0(B)=\mathcal{R}_0(M_0)$ for every $\lambda\in\sigma_w(M_0)^{\mathcal{C}}$. Now apply (a slightly modified) Proposition 3.6.

$(vii)\Rightarrow(viii)$. It is clear from (vii) that the range of $M_0-\lambda$ is closed at points $\lambda\in\mathcal{R}_0(M_0)$. Since $\alpha(M_0-\lambda)<\infty$ (so that $\lambda\in\Phi_+(M_0)$), and M_0 and ${M_0}^*$ have SVEP at points $\lambda\in\mathcal{R}_0(M_0)$, we conclude that $\lambda\in\mathcal{R}_0(M_0)\Longleftrightarrow\lambda\in\sigma_w(M_0)^{\mathcal{C}}$ and $\mathrm{dim}H_0(M_0-\lambda)=\mathrm{dim}H_0(A-\lambda)+\mathrm{dim}H_0(B-\lambda)<\infty$ at points $\lambda\in\mathcal{R}_0(M_0)$ [1, Theorem 3.18].

$(viii)\Rightarrow(ix)$. Let $\lambda\in\sigma_w(M_0)^{\mathcal{C}}$. If $(viii)$ is satisfied, then $\lambda\in\Phi^0(M_0)$ and $\mathrm{dim}H_0(M_0-\lambda)=\mathrm{dim}H_0(A-\lambda)+\mathrm{dim}H_0(B-\lambda)<\infty$. Hence $\lambda\in\mathrm{iso}\sigma(M_0)=\mathrm{iso}\sigma(A)\cup\mathrm{iso}\sigma(B)$. But then $\mathcal{X}=H_0(A-\lambda)\oplus K(A-\lambda)=H_0(B-\lambda)\oplus K(B-\lambda)$,

which implies that $\mathrm{codim}K(A-\lambda)$ and $\mathrm{codim}K(B-\lambda)$ are finite on $\sigma_w(M_0)^{\mathcal{C}}$.
$(ix) \Rightarrow (x)$. If (ix) is satisfied, then the following implications hold:

$$\begin{aligned}
\lambda \in \sigma_w(M_0)^{\mathcal{C}} \quad &\Leftrightarrow \quad \lambda \in \Phi(A)\cap\Phi(B), \mathrm{ind}(A-\lambda)+\mathrm{ind}(B-\lambda)=0,\\
&\qquad \mathrm{codim}K(A-\lambda) \text{ and } \mathrm{codim}K(B-\lambda) \text{ are finite}\\
&\Rightarrow \quad \lambda \in \Phi(A)\cap\Phi(B), \mathrm{ind}(A-\lambda)+\mathrm{ind}(B-\lambda)=0,\\
&\qquad \text{and } A, B^* \text{ have SVEP at } \lambda\\
&\Rightarrow \quad \lambda \in \Phi^0(A)\cap\Phi^0(B) \text{ and } A, B^* \text{ have SVEP at } \lambda\\
&\Rightarrow \quad \lambda \in \mathcal{R}_0(A)\cup\mathcal{R}_0(B)\\
&\Rightarrow \quad \mathrm{asc}(A-\lambda)=\mathrm{dsc}(A-\lambda)<\infty, \mathrm{asc}(B-\lambda)=\mathrm{dsc}(B-\lambda)<\infty.
\end{aligned}$$

$(x) \Rightarrow (i)$. If (x) holds, then we have

$$\begin{aligned}
\lambda \in \sigma_w(M_0)^{\mathcal{C}} \quad &\Rightarrow \quad \lambda \in \Phi^0(A)\cap\Phi^0(B), A \text{ and } B (\text{or } A^* \text{ and } B^*) \text{ have SVEP at } \lambda\\
&\Rightarrow \quad \lambda \in \mathrm{iso}\sigma(A)\cup\mathrm{iso}\sigma(B)\\
&\Rightarrow \quad \lambda \in \mathrm{iso}\sigma(M_0).
\end{aligned}$$

This completes the proof. □

Remark 4.2. Here, we note for future reference the following easy consequence of condition (v) of Theorem 4.1: if M_0 satisfies Bt, then $\sigma_w(M_0)=\sigma_w(A)\cup\sigma_w(B)$.

It is easily seen, argue as for a single linear operator [15, 2, 12], that the following implications hold: (i) M_C satisfies $Bt \Longleftrightarrow (ii)$ $S_{\sigma_w(M_C)}(M_C)=\emptyset \Longleftrightarrow (iii)$ $\sigma_w(M_C)=\sigma_b(M_C) \Longleftrightarrow (iv)$ $\sigma_w(M_C)^{\mathcal{C}}=\mathcal{R}_0(M_C) \Longleftrightarrow (v)$ ${M_C}^*$ satisfies Bt. Furthermore, these conditions are equivalent to the condition that: (vi) the mappings $\lambda \longrightarrow \gamma(A-\lambda)$ and $\lambda \longrightarrow \gamma(B^*-\lambda I^*)$ are not continuous on $\mathcal{R}_0(M_C)$: this follows from a combination of Proposition 3.6 and the following lemma.

Lemma 4.3. *If the mappings $\lambda \longrightarrow \gamma(A-\lambda)$ and $\lambda \longrightarrow \gamma(B^*-\lambda I^*)$ are not continuous on $\mathcal{R}_0(M_C)$ (resp., $\mathcal{R}_0^a(M_C)$), then $\sigma_w(M_C)^{\mathcal{C}}=\mathcal{R}_0(M_C)$ (resp., $\sigma_{aw}(M_C)^{\mathcal{C}}=\mathcal{R}_0^a(M_C)$).*

Proof. The proof in both the cases is very similar: we consider the case $\sigma_{aw}(M_C)^{\mathcal{C}}=\mathcal{R}_0^a(M_C)$. It is easily seen that the discontinuity of the mappings $\lambda \longrightarrow \gamma(A-\lambda)$ and $\lambda \longrightarrow \gamma(B^*-\lambda I^*)$ on $\mathcal{R}_0^a(M_C)$ implies that $\mathcal{R}_0^a(M_C)\subseteq\sigma_{aw}(M_C)^{\mathcal{C}}$. For the reverse inclusion, let $\mu_0\in\sigma_{aw}(M_C)^{\mathcal{C}}$. Since $\alpha(M_C-\mu_0)=0$ implies $\mu_0\notin\sigma_a(M_C)$, we may assume that $\alpha(M_C-\mu_0)>0$. There exists an ϵ-neighbourhood $\mathcal{O}_\epsilon$ of μ_0 such that $\mu\in\Phi_+(M_C)$, $\mathrm{ind}(M_C-\mu_0)=\mathrm{ind}(M_C-\mu)$ and $\alpha(M_C-\mu)<\alpha(M_C-\mu_0)$ remains constant for every $\mu\in\mathcal{O}_\epsilon\setminus\{\mu_0\}$. Suppose that $\alpha(M_C-\mu)>0$. Then there exists a $\mu_1\in\mathcal{O}_\epsilon\setminus\{\mu_0,\mu\}$ such that $\mu_1\in\Phi_+(M_C)$, $\mathrm{ind}(M_C-\mu)=\mathrm{ind}(M_C-\mu_1)$ and $\alpha(M_C-\mu_1)<\alpha(M_C-\mu)$. Since this is a contradiction, we must have $\alpha(M_C-\mu)=0$ for all $\mu\in\mathcal{O}_\epsilon\setminus\{\mu_0\}$. But then $M_C-\mu$ is left invertible; hence $\mu_0\in\mathrm{iso}\sigma_a(M_C)\cap\Phi_+(M_C)$ and $\mathrm{ind}(M_C-\mu_0)\le 0$, i.e., $\mu_0\in\mathcal{R}_0^a(M_C)$. Thus $\sigma_{aw}(M_C)^{\mathcal{C}}\subseteq\mathcal{R}_0^a(M_C)$. □

The example of the unitary operator $\begin{pmatrix} U & 1-UU^* \\ 0 & U^* \end{pmatrix}$, where U is the forward unilateral shift on a Hilbert space, shows that conditions of type (iv), $(viii)$, (ix) and (x) of Theorem 4.1 are not necessary for M_C to satisfy Bt. However, if $\sigma_w(M_C) = \sigma_w(A)\cup\sigma_w(B)$, then the corresponding conditions are both necessary and sufficient.

Theorem 4.4. *If $\sigma_w(M_C) = \sigma_w(A)\cup\sigma_w(B)$, then the following implications hold:*

$$\begin{aligned}
& M_C \ \textit{satisfies}\ Bt \\
\Leftrightarrow\ & \lambda \in \Phi(A)\cap\Phi(B) \ \textit{and}\ \lambda \in iso\sigma(A)\cup iso\sigma(B) \ \textit{for}\ \lambda\in\sigma_w(M_C)^{\mathcal{C}} \\
\Leftrightarrow\ & dimH_0(A-\lambda) \ \textit{and}\ dimH_0(B-\lambda) \ \textit{are finite for}\ \lambda\in\sigma_w(M_C)^{\mathcal{C}} \\
\Leftrightarrow\ & codimK(A-\lambda) \ \textit{and}\ codimK(B-\lambda) \ \textit{are finite for}\ \lambda\in\sigma_w(M_C)^{\mathcal{C}} \\
\Leftrightarrow\ & asc(A-\lambda) \ \textit{and}\ asc(B-\lambda), \ \textit{or}\ dsc(A-\lambda) \ \textit{and}\ dsc(B-\lambda), \\
& \textit{are finite for}\ \lambda\in\sigma_w(M_C)^{\mathcal{C}}.
\end{aligned}$$

Proof. The hypothesis $\sigma_w(M_C) = \sigma_w(A)\cup\sigma_w(B)$ implies that $\sigma_w(M_C) = \sigma_w(M_0)$ and $\sigma(M_C) = \sigma(M_0)$ (see Proposition 3.2). Recall that M_C satisfies Bt if and only if M_C has SVEP on $\sigma_w(M_C)^{\mathcal{C}}$. We prove that M_C has SVEP on $\sigma_w(M_C)^{\mathcal{C}}$ if and only if M_0 has SVEP on $\sigma_w(M_C)^{\mathcal{C}}$; this, by Theorem 4.1, would then imply the equivalence of the implications of the theorem. Evidently, M_0 has SVEP at a point λ if and only if A and B have SVEP at λ; hence M_0 has SVEP on $\sigma_w(M_C)^{\mathcal{C}}$ implies M_C has SVEP on $\sigma_w(M_C)^{\mathcal{C}}$. For the reverse implication, M_C has SVEP at $\lambda\in\sigma_w(M_C)^{\mathcal{C}}$ implies $\lambda\in \text{iso}\sigma(M_C) \Longrightarrow \lambda\in\text{iso}\sigma(A)\cup\text{iso}\sigma(B)$. Hence both A and B have SVEP at $\lambda \Longrightarrow M_0$ has SVEP at λ. □

The conclusion that $\sigma_w(M_C) = \sigma_w(A)\cup\sigma_w(B)$ may be obtained in a variety of ways; the following proposition lists some sufficient conditions.

Proposition 4.5. *If either (i) A and A^*, or (ii) A and B, or (iii) A^* and B^*, or (iv) B and B^* have SVEP on $\sigma_w(M_C)^{\mathcal{C}}$, then $\sigma_w(M_C) = \sigma_w(A)\cup\sigma_w(B)$.*

Proof. Let $\lambda\in\sigma_w(M_C)^{\mathcal{C}}$; then $\lambda\in\Phi_+(A)\cap\Phi_-(B)$ and $\text{ind}(A-\lambda)+\text{ind}(B-\lambda)=0$. If A and A^* have SVEP at λ, then $\text{ind}(A-\lambda)=0$, and so $\text{ind}(B-\lambda)=0$ and $\lambda\in\Phi^0(A)\cap\Phi^0(B)$; if A and B have SVEP at λ, then $\text{ind}(A-\lambda)$ and $\text{ind}(B-\lambda)$ are ≤ 0, so that $\text{ind}(A-\lambda)=\text{ind}(B-\lambda)=0$ and $\lambda\in\Phi^0(A)\cap\Phi^0(B)$; if A^* and B^* have SVEP at λ, then $\text{ind}(A-\lambda)$ and $\text{ind}(B-\lambda)$ are ≥ 0, so that $\text{ind}(A-\lambda)=\text{ind}(B-\lambda)=0$ and $\lambda\in\Phi^0(A)\cap\Phi^0(B)$; finally, if B and B^* have SVEP at λ, then $\text{ind}(B-\lambda)=0$, and so (also) $\text{ind}(A-\lambda)=0$ and (once again) $\lambda\in\Phi^0(A)\cap\Phi^0(B)$. In either case, $\sigma_w(M_C)\supseteq\sigma_w(A)\cup\sigma_w(B)$. Since $\sigma_w(M_C)\subseteq\sigma_w(A)\cup\sigma_w(B)$ always, the equality follows. □

Since M_0 satisfies Bt if and only if ${M_0}^*$ satisfies Bt, the hypothesis M_0 satisfies Bt implies that A, A^*, B and B^* all have SVEP on $\sigma_w(M_0)^{\mathcal{C}}$. This implies that if either of the pairs (i) to (iv) of Proposition 4.5 have SVEP on $\sigma_w(M_0)\backslash\sigma_w(M_C)$ and M_0 satisfies Bt, then $\sigma_w(M_C) = \sigma_w(A)\cup\sigma_w(B)$ and M_C satisfies Bt. Conversely:

Proposition 4.6. *If M_C satisfies Bt, and if either A^* or B has SVEP on $\sigma_w(M_C)^{\mathcal{C}}$, then M_0 satisfies Bt.*

Proof. If M_C satisfies Bt, and $\lambda \in \sigma_w(M_C)^{\mathcal{C}}$, then ($A$ and B^* have SVEP at λ,) $\lambda \in \Phi_+^-(A) \cap \Phi_-^+(B)$ and $\text{ind}(A-\lambda)+\text{ind}(B-\lambda) = 0$. Thus either of the hypotheses A^* has SVEP on $\lambda \in \sigma_w(M_C)^{\mathcal{C}}$ and B has SVEP on $\lambda \in \sigma_w(M_C)^{\mathcal{C}}$ implies that $\lambda \in \Phi^0(A) \cap \Phi^0(B)$. Hence $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$. To complete the proof, argue as in the proof of Theorem 4.4 to prove that M_0 has SVEP on $\sigma_w(M_C)^{\mathcal{C}} = \sigma_w(M_0)^{\mathcal{C}}$. □

Our next result leads to a number of conditions for M_0 satisfies Bt to imply M_C satisfies Bt.

Proposition 4.7. *If M_C has SVEP on $\sigma_w(M_0) \setminus \sigma_w(M_C)$, then M_0 satisfies Bt implies M_C satisfies Bt.*

Proof. Suppose that M_0 satisfies Bt. Then A and B have SVEP on $\sigma_w(M_0)^{\mathcal{C}}$; hence M_C has SVEP on $\sigma_w(M_0)^{\mathcal{C}}$. The hypothesis M_C has SVEP on $\sigma_w(M_0) \setminus \sigma_w(M_C)$ now implies that M_C has SVEP on $\sigma_w(M_C)^{\mathcal{C}}$; hence M_C satisfies Bt. □

The hypothesis M_C has SVEP on $\sigma_w(M_0) \setminus \sigma_w(M_C)$ in the proposition above may be satisfied in a variety of ways: for example, if A and A^* have SVEP on $\sigma_w(M_0) \setminus \sigma_{SF_+}(A)$, or B and B^* have SVEP on $\sigma_w(M_0) \setminus \sigma_{SF_-}(B)$, or A has SVEP on $\sigma_w(M_0) \setminus \sigma_{SF_+}(A)$ and B has SVEP on $\sigma_w(M_0) \setminus \sigma_{SF_-}(B)$, or A^* has SVEP on $\sigma_w(M_0) \setminus \sigma_{SF_+}(A)$ and B^* has SVEP on $\sigma_w(M_0) \setminus \sigma_{SF_-}(B)$; *cf.* [14, Theorem 3.1(a)], [11, Proposition 4.1] and [7, Theorem 3.2].

Theorem 4.8. *If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then M_0 satisfies Bt if and only if M_C satisfies Bt.*

Proof. Evidently $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B) = \sigma_w(M_0)$, $\sigma(M_C) = \sigma(A) \cup \sigma(B) = \sigma(M_0)$, and the following implications hold:

$$\begin{aligned} & M_0 \ \textit{satisfies} \ Bt \Leftrightarrow A, B \ \textit{have SVEP on} \ \sigma_w(M_0)^{\mathcal{C}} \\ \Rightarrow \ & M_C \ \textit{has SVEP on} \ \sigma_w(M_C)^{\mathcal{C}} \Rightarrow M_C \ \textit{satisfies} \ Bt \end{aligned}$$

and

$$\begin{aligned} & M_C \ \textit{satisfies} \ Bt \Leftrightarrow \sigma(M_C) \setminus \sigma_w(M_C) = \mathcal{R}_0(M_C) = \sigma_w(M_C)^{\mathcal{C}} \\ \Rightarrow \ & \textit{every} \ \lambda \in \sigma_w(M_C)^{\mathcal{C}} \ \textit{is isolated in} \ \sigma(M_C) = \sigma(M_0) \\ \Rightarrow \ & M_0 \ \textit{has SVEP on} \ \sigma_w(M_0)^{\mathcal{C}} = \sigma_w(M_C)^{\mathcal{C}} \Rightarrow M_0 \ \textit{satisfies} \ Bt. \end{aligned}$$

This completes the proof. □

Next, we consider $a-Bt$ for operators M_0 and M_C. Recall that T satisfies $a-Bt$ if and only if $acc\sigma_a(T) \subseteq \sigma_{aw}(T)$.

Theorem 4.9. *The following conditions are equivalent:*
(i). M_0 satisfies $a-Bt$.
(ii). $\sigma_a(M_0) = \sigma_{aw}(M_0) \cup \{iso\sigma_a(A) \cup iso\sigma_a(B)\}$.
(iii). $\sigma_{aw}(M_0) = \sigma_{ab}(M_0)$.

(iv). A and B have SVEP on $\sigma_{aw}(M_0)^{\mathcal{C}}$.
(v). $asc(A-\lambda)$ *and* $asc(B-\lambda)$ *are finite on* $\sigma_{aw}(M_0)^{\mathcal{C}}$.
(vi), $dim H_0(A-\lambda)$ *and* $dim H_0(B-\lambda)$ *are finite on* $\sigma_{aw}(M_0)^{\mathcal{C}}$.
(vii). $H_0(A-\lambda)$ *and* $H_0(B-\lambda)$ *are closed on* $\sigma_{aw}(M_0)^{\mathcal{C}}$.
(viii). Points $\lambda \in \sigma_{aw}(M_0)^{\mathcal{C}}$ *are isolated in* $\sigma_a(M_0)$.
(ix). The mappings $\lambda \longrightarrow \gamma(A-\lambda)$ *and* $\lambda \longrightarrow \gamma(B-\lambda)$ *are not continuous on* $\mathcal{R}_0^a(M_0)$.
(x). $\sigma_{aw}(M_0)^{\mathcal{C}} = \mathcal{R}_0^a(A) \cup \mathcal{R}_0^a(B) = \mathcal{R}_0^a(M_0)$.

Proof. $(i) \Rightarrow (ii)$. Since $\sigma_a(M_0) = \sigma_a(A) \cup \sigma_a(B)$, $\lambda \in \text{iso}\sigma_a(M_0) \Longrightarrow \lambda \in \text{iso}\sigma_a(A) \cup \text{iso}\sigma_a(B)$. Thus, if $acc\sigma_a(M_0) \subseteq \sigma_{aw}(M_0)$, then $\sigma_a(M_0) = acc\sigma_a(M_0) \cup \text{iso}\sigma_a(M_0) \subseteq \sigma_{aw}(M_0) \cup \text{iso}\sigma_a(M_0) \subseteq \sigma_a(M_0)$.
$(ii) \Rightarrow (iii)$. Since $\sigma_{aw}(T) \subseteq \sigma_{ab}(T)$ for every operator T, we have to prove $\sigma_{ab}(M_0) \subseteq \sigma_{aw}(M_0)$. If $\lambda \in \sigma_{aw}(M_0)^{\mathcal{C}}$, then $\lambda \in \Phi_+(A) \cap \Phi_+(B)$ and $\text{ind}(A-\lambda) + \text{ind}(B-\lambda) \leq 0$. Since (ii) implies that A and B have SVEP at λ, $\lambda \in \Phi_+^-(A) \cap \Phi_+^-(B)$, and both $\text{asc}(A-\lambda)$ and $\text{asc}(B-\lambda)$ are finite. Hence $\lambda \notin \sigma_{ab}(A) \cup \sigma_{ab}(B) = \sigma_{ab}(M_0)$.
The implications $(iii) \Rightarrow (iv) \Rightarrow (v)$ are evident.
$(v) \Rightarrow (vi)$. If $\text{asc}(A-\lambda)$ and $\text{asc}(B-\lambda)$ are finite at $\lambda \in \sigma_{aw}(M_0)^{\mathcal{C}}$, then $\lambda \in \Phi_+(A) \cap \Phi_+(B)$ and $\lambda \in \text{iso}\sigma_a(A) \cup \text{iso}\sigma_a(B)$. This, [1, Theorem 3.78], implies that $H_0(A-\lambda)$ and $H_0(B-\lambda)$ are finite dimensional.
$(vi) \Rightarrow (vii)$. Evident.
$(vii) \Rightarrow (viii)$. It is clear from (vii) that $H_0(M_0-\lambda) = H_0(A-\lambda) \oplus H_0(B-\lambda)$ is closed on $\sigma_{aw}(M_0)^{\mathcal{C}}$. Thus M_0 has SVEP on $\sigma_{aw}(M_0)^{\mathcal{C}}$ [1, Theorem 3.14]; hence $\lambda \in \text{iso}\sigma_a(M_0)$ for every $\lambda \in \sigma_{aw}(M_0)^{\mathcal{C}}$ [1, Theorem 3.14].
$(viii) \Rightarrow (x)$. If $(viii)$ is satisfied, then $\lambda \in \sigma_{aw}(M_0)^{\mathcal{C}}$ implies $\lambda \in \Phi_+(A) \cap \Phi_+(B)$, $\text{ind}(A-\lambda)+\text{ind}(B-\lambda) \leq 0$ and A, B have SVEP at λ. Hence $\lambda \in \mathcal{R}_0^a(A) \cup \mathcal{R}_0^a(B) = \mathcal{R}_0^a(M_0)$. Since $\mathcal{R}_0^a(M_0) \cap \sigma_{aw}(M_0)^{\mathcal{C}} = \emptyset$, the proof follows.
$(x) \Leftrightarrow (ix)$. The implication $(ix) \Rightarrow (x)$ follows from Lemma 4.3, and the implication $(x) \Rightarrow (ix)$ follows from Lemma 3.5 (and the fact that $\gamma(M_0-\lambda) \geq min\{1, \gamma(A-\lambda), \gamma(B-\lambda)\}$).
The implication $(x) \Rightarrow (i)$ being evident, the proof is complete. □

If M_0 satisfies $a-Bt$, then $\sigma_{aw}(M_0) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. This fails for M_C, as follows from a consideration of (the earlier considered) operator $\begin{pmatrix} U & 1-UU^* \\ 0 & U^* \end{pmatrix}$. What makes Theorem 4.9 possible is the information on A and B one is able to extract from M_0 at points in $\sigma_{aw}(M_0)^{\mathcal{C}}$; a similar situation does not prevail for M_C (see Proposition 3.3). One does however have the following:

Proposition 4.10. *The following conditions are equivalent: (i)* M_C *satisfies* $a-Bt$*; (ii)* $\sigma_a(M_C) = \sigma_{aw}(M_C) \cup iso\sigma_a(M_C)$*; (iii)* M_C *has SVEP on* $\sigma_{aw}(M_C)^{\mathcal{C}}$*; (iv)* $\sigma_a(M_C) \setminus \sigma_{aw}(M_C) = \mathcal{R}_0^a(M_C)$.

Proof. The proof of the proposition is the same as that for a single linear operator; see[2]. □

Remark 4.11. Lemma 4.3 implies that the discontinuity of the maps $\lambda \longrightarrow \gamma(A-\lambda)$ and $\lambda \longrightarrow \gamma(B-\lambda)$ is a sufficient condition for M_C to satisfy $a-Bt$: is this condition necessary too?

We consider now sufficient conditions for M_C satisfies $a-Bt$ to imply M_0 satisfies $a-Bt$, and *vice versa*. As one would expect, M_0 satisfies $a-Bt$ does not imply M_C satisfies $a-Bt$. For example, if $A, B, C \in B(\ell^2 \oplus \ell^2)$ are the operators $A = U \otimes 1$, $B = U^* \otimes 1$ and C is the diagonal operator with entries $(0, 1-UU^*, 1-UU^*, ...)$, where $U \in B(\ell^2)$ is the forward unilateral shift, then $\sigma_a(M_0) = \sigma_{aw}(M_0)$, $\mathcal{R}_0^a(M_0) = \emptyset$, and M_0 satisfies $a-Bt$; however, $\sigma(M_C)$ is the closed unit disc $\mathbf{D}$, $\sigma_w(M_C)$ is the boundary $\partial\mathbf{D}$ of $\mathbf{D}$, $\mathcal{R}_0(M_C) = \emptyset$, and M_C does not satisfy Bt (much less $a-Bt$). Conversely, M_C satisfies $a-Bt$ does not imply M_0 satisfies $a-Bt$, as the example of the operator $\begin{pmatrix} U & 1-UU^* \\ 0 & U^* \end{pmatrix}$ shows. Recall, however, that M_0 satisfies $a-Bt$ if and only if A and B have SVEP on $\sigma_{aw}(M_0)^{\mathcal{C}}$; hence, if M_C has SVEP on $\sigma_{aw}(M_0)\setminus\sigma_{aw}(M_C)$, then, since M_0 satisfies $a-Bt$ implies M_C has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, M_C satisfies $a-Bt$. The following theorem shows that this happens in a variety of ways.

Theorem 4.12. *(i). If $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, then M_0 satisfies $a-Bt$ implies M_C satisfies $a-Bt$. If, additionally, either A^* or B has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, then M_C satisfies $a-Bt$ if and only if M_0 satisfies $a-Bt$.*
(ii). If A and A^*, or A^* and B^*, have SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, then M_C satisfies $a-Bt$ if and only if M_0 satisfies $a-Bt$.
(iii). If A and A^* have SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}} \setminus \sigma_{SF_+}(A)$, or A^* has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}} \setminus \sigma_{SF_+}(A)$ and B^* has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}} \setminus \sigma_{SF_+}(B)$, then M_C satisfies $a-Bt$ if and only if M_0 satisfies $a-Bt$.

Proof. (i). The hypothesis $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ implies that $\sigma_{aw}(M_0) = \sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. If M_0 satisfies $a-Bt$, then A and B have SVEP on $\sigma_{aw}(M_0)^{\mathcal{C}}$ implies that M_C has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, and so M_C satisfies $a-Bt$. Assume now that M_C satisfies $a-Bt$, $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, and either A^* or B has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$. Since M_C satisfies $a-Bt$ implies A has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, if B has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, then M_0 has SVEP on $\sigma_{aw}(M_0)^{\mathcal{C}} = \sigma_{aw}(M_C)^{\mathcal{C}}$, and so M_0 satisfies $a-Bt$. Assume now that A^* has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$: we prove that $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B) = \sigma_a(M_0)$. If $\mu \notin \sigma_a(M_C)$, then $M_C - \mu$ and $A-\mu$ are left invertible, $\mu \in \sigma_{aw}(M_C)^{\mathcal{C}}$. The left invertibility of $A-\mu$ implies the right invertibility of $A^* - \mu I^*$; hence, since A^* has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, $A^* - \mu I^*$ is invertibile. But then the invertibility of $A-\mu$, taken alongwith the left invertibility of $M_C - \mu$, implies that $B-\mu$ is left invertible. Hence $\mu \notin \sigma_a(A) \cup \sigma_a(B)$. Since $\sigma_a(M_C) \subseteq \sigma_a(A) \cup \sigma_a(B)$ always, $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B) = \sigma_a(M_0)$. Assume now that M_C satisfies $a-Bt$. Then $\lambda \in \sigma_{aw}(M_C)^{\mathcal{C}}$ implies that $\lambda \in \mathrm{iso}\sigma_a(M_C) = \mathrm{iso}\sigma_a(A) \cup \mathrm{iso}\sigma_a(B)$; hence ($A$ and B have SVEP on $\sigma_{aw}(M_0)^{\mathcal{C}} = \sigma_{aw}(M_C)^{\mathcal{C}}$ implies) M_0 has SVEP on $\sigma_{aw}(M_0)^{\mathcal{C}}$, and so M_0 satisfies $a-Bt$.
(ii). Let $\lambda \in \sigma_{aw}(M_C)^{\mathcal{C}}$. Then the hypothesis that A and A^* have SVEP on

$\sigma_{aw}(M_C)^{\mathcal{C}}$ implies that $\lambda \in \Phi^0(A) \cap \Phi_+^-(B) \subseteq \Phi_+^-(A) \cap \Phi_+^-(B)$. Consequently, $\sigma_{aw}(M_C) \supseteq \sigma_{aw}(A) \cup \sigma_{aw}(B)$; hence $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Again, if A^* and B^* have SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, then $\lambda \in \sigma_{aw}(M_C)^{\mathcal{C}} \Longrightarrow \lambda \in \Phi_+(A), \text{ind}(A - \lambda) + \text{ind}(B - \lambda) \leq 0, \beta(A - \lambda) \leq \alpha(A - \lambda), \beta(B - \lambda) \leq \alpha(B - \lambda)$. Hence, in view of Proposition 3.3, $\lambda \in \Phi^0(A) \cap \Phi^0(B) \subseteq \Phi_+^-(A) \cap \Phi_+^-(B)$, which (once again) leads to the conclusion that $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Applying part ($i$), the proof follows.

(iii). Let $\lambda \in \sigma_{aw}(M_C)^{\mathcal{C}}$. Then $\lambda \in \Phi_+(A)$ and $\text{ind}(A-\lambda)+\text{ind}(B-\lambda) \leq 0$. If A and A^* have SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}} \setminus \sigma_{SF_+}(A)$, then $\lambda \in \Phi^0(A)$ (is isolated in $\sigma_a(A)$), and this forces $\lambda \in \Phi_+^-(B)$. Hence $\lambda \notin \sigma_{aw}(A) \cup \sigma_{aw}(B)$, which leads us to the equality $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Again, if A^* has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}} \setminus \sigma_{SF_+}(A)$, then $\lambda \in \Phi_+^+(A)$ ($\Longrightarrow \lambda \in \Phi_+(B)$); thus, if B^* has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}} \setminus \sigma_{SF_+}(B)$, then $\lambda \in \Phi_+^+(B)$, which forces $\lambda \in \Phi^0(A) \cap \Phi^0(B)$ and $\lambda \in \text{iso}\sigma(A) \cup \text{iso}\sigma(B)$. Once again, we conclude that $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. The proof now follows from an application of part (ii) (since both A and A^* have SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$). □

5. Wt and $a - Wt$

The problem that we consider in this section is that of finding necessary and (/or) sufficient conditions for the implications M_0 satisfies $Wt \Longleftrightarrow M_C$ satisfies Wt and M_0 satisfies $a - Wt \Longleftrightarrow M_C$ satisfies $a - Wt$ to hold. Recall that T satisfies Wt (resp., $a - Wt$) if and only if $\sigma(T) \setminus \sigma_w(T) = \mathcal{P}_0(T)$ (resp., $\sigma_a(T) \setminus \sigma_{aw}(T) = \mathcal{P}_0^a(T)$).

Theorem 5.1. *If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then the equivalence*

$$M_0 \text{ satisfies } Wt \Longleftrightarrow M_C \text{ satisfies } Wt$$

holds if and only if $\mathcal{P}_0(M_0) = \mathcal{P}_0(M_C)$.

Proof. The hypothesis $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$ implies that $\sigma_w(M_0) = \sigma_w(M_C)$ and $\sigma(M_0) = \sigma(M_C)$ (see Proposition 3.2). Suppose that M_0 satisfies Wt; then M_0 satisfies Bt, and so

$$\sigma(M_C) \setminus \sigma_w(M_C) = \sigma(M_0) \setminus \sigma_w(M_0) = \mathcal{P}_0(M_0) = \mathcal{R}_0(M_0) = \mathcal{R}_0(M_C) \subseteq \mathcal{P}_0(M_C),$$

where the equality $\mathcal{R}_0(M_0) = \mathcal{R}_0(M_C)$ follows from Proposition 3.8. Again, if M_C satisfies Wt, then (M_C satisfies Bt and)

$$\sigma(M_0) \setminus \sigma_w(M_0) = \sigma(M_C) \setminus \sigma_w(M_C) = \mathcal{P}_0(M_C) = \mathcal{R}_0(M_C) = \mathcal{R}_0(M_0) \subseteq \mathcal{P}_0(M_0),$$

where (once again) the equality $\mathcal{R}_0(M_C) = \mathcal{R}_0(M_0)$ follows from Proposition 3.8. Thus, the statements of the theorem are equivalent if and only if $\mathcal{P}_0(M_0) = \mathcal{P}_0(M_C)$. □

The theorem has a number of consequences: some of these are listed below. Recall that the *spectral picture* $SP(T)$ of T is the set $\sigma_e(T)$, the holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes. Recall from

[19, Lemma 2.2] that if the entries A and B in M_C are Hilbert space operators (with C correspondingly defined), then the hypothesis $SP(A)$ or $SP(B)$ has no pseudoholes holes implies the equality $\sigma_w(M_0) = \sigma_w(M_C)$. The operator T is said to be *isoloid* if the isolated points of the spectrum of T are eigenvalues of T.

Corollary 5.2. *[19, Theorem 2.4] If $\sigma_w(M_0) = \sigma_w(M_C)$, A is isoloid and satisfies Wt, then M_0 satisfies Wt implies M_C satisfies Wt.*

Proof. We prove that if the hypotheses of the corollary are satisfied, then $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$ and $\mathcal{P}_0(M_0) = \mathcal{P}_0(M_C)$; the proof of the corollary would them follow from Theorem 5.1. The hypothesis A satisfies Wt implies that $\sigma(A) \setminus \sigma_w(A) = \mathcal{R}_0(A) = \mathcal{P}_0(A)$ (so that both A and A^* have SVEP on $\sigma_w(A)^{\mathcal{C}}$). If $\lambda \in \sigma_w(M_0)^{\mathcal{C}}$, then M_0 satisfies Wt implies that $\lambda \in \mathcal{P}_0(M_0)$. Hence $\lambda \in \text{iso}\sigma(A) \cup \rho(A)$ and $\alpha(A-\lambda) < \infty$. By hypothesis, A is isoloid; hence $\lambda \in \mathcal{P}_0(A)$, which implies that both A and A^* have SVEP on $\sigma_w(M_0)^{\mathcal{C}}$. Since $\lambda \in \sigma_w(M_0)^{\mathcal{C}}$, and A and A^* have SVEP at λ, implies $\lambda \in \Phi^0(A) \cap \Phi^0(B)$, it follows that $\sigma_w(M_0) = \sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$ (which, see Proposition 3.2, implies that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$). Again, since A and A^* have SVEP on $\sigma_w(M_0)^{\mathcal{C}} = \sigma_w(M_C)^{\mathcal{C}}$, $\mathcal{R}_0(M_0) = \mathcal{R}_0(M_C)$ (see Proposition 3.8). Hence $\mathcal{P}_0(M_0) = \mathcal{R}_0(M_0) = \mathcal{R}_0(M_C) \subseteq \mathcal{P}_0(M_C)$. Finally, since $\text{iso}\sigma(M_C) = \text{iso}\sigma(A) \cup \text{iso}\sigma(B)$, $\lambda \in \mathcal{P}_0(M_C) \Longrightarrow \lambda \in \mathcal{P}_0(A) \cup \mathcal{P}_0(B) = \mathcal{P}_0(M_0)$. Hence $\mathcal{P}_0(M_0) = \mathcal{P}_0(M_C)$. □

Corollary 5.3. *[4, Theorem 4] If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, A and B satisfy Bt, $\mu \in iso\sigma(A)$ implies $\mu \in \mathcal{R}_0(A)$ and $\nu \in iso\sigma(B)$ implies $\nu \in \mathcal{R}_0(B)$, then M_C satisfies Wt.*

Proof. Evidently, $\sigma_w(M_0) = \sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$ ($\Longrightarrow \sigma(M_C) = \sigma(A) \cup \sigma(B)$). The hypothesis $\mu \in \text{iso}\sigma(A) \Longrightarrow \mu \in \mathcal{R}_0(A)$ implies that $\mathcal{P}_0(A) \subseteq \mathcal{R}_0(A)$, and the hypotheses $\mu \in \text{iso}\sigma(A) \Longrightarrow \mu \in \mathcal{R}_0(A)$ and $\mu \in \text{iso}\sigma(B) \Longrightarrow \mu \in \mathcal{R}_0(B)$ imply that $\mathcal{P}_0(M_0) \subseteq \mathcal{R}_0(M_0)$; hence, since A and B satisfy Bt implies M_0 (has SVEP on $\sigma_w(M_0)^{\mathcal{C}} = \sigma_w(A)^{\mathcal{C}} \cap \sigma_w(B)^{\mathcal{C}} = \mathcal{R}_0(A) \cap \mathcal{R}_0(B)$ implies M_0) satisfies Bt, A and M_0 satisfy Wt. Since A is evidently isoloid, the proof follows from Corollary 5.2. □

Corollary 5.4. *[11, Theorem 4.2] If A and A^*, or A^* and B^* have SVEP, and A is isoloid and satisfies Wt, then M_0 satisfies Wt implies M_C satisfies Wt.*

Proof. Apply Remark 2.1(ii) and Corollary 5.2. □

The following corollary generalizes [7, Theorem 3.3].

Corollary 5.5. *If either $\sigma_{ea}(A) = \sigma_{SF_+}(B)$, or $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B) = \emptyset$, and A is isoloid and satisfies Wt, then M_0 satisfies Wt implies M_C satisfies Wt.*

Proof. Either of the hypotheses $\sigma_{ea}(A) = \sigma_{SF_+}(B)$ and $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B) = \emptyset$ implies that $\sigma_w(M_C) = \sigma_w(M_0)$ (see [14, Corollary 3.3(b)] or [7, Corollary 2.2 and Theorem 3.2]); hence, if M_0 satisfies Bt, then $\sigma_w(M_C) = \sigma_w(M_0) = \sigma_w(A) \cup \sigma_w(B)$. Now argue as above. □

The implications

$$\begin{aligned}&\lambda \notin \sigma_b(A)\cup\sigma_b(B) \Longleftrightarrow \lambda\in\sigma_b(A)^{\mathcal{C}}\cap\sigma_b(B)^{\mathcal{C}}\\ &\Longleftrightarrow \lambda\in\Phi^0(A)\cap\Phi^0(B), \mathrm{asc}(A-\lambda)=\mathrm{dsc}(A-\lambda)<\infty,\\ &\qquad \mathrm{asc}(B-\lambda)=\mathrm{dsc}(B-\lambda)<\infty\\ &\Longleftrightarrow \lambda\in\Phi(M_C), \mathrm{asc}(M_C-\lambda)=\mathrm{dsc}(M_C-\lambda)<\infty, A^* \text{ has SVEP at}\\ &\lambda \text{ or } B \text{ has SVEP at } \lambda\end{aligned}$$

imply that

$$\sigma_b(A)\cup\sigma_b(B)=\sigma_b(M_C)\cup\{S_{\sigma_b(M_C)}(A^*)\cap S_{\sigma_b(M_C)}(B)\}.$$

Again, the implications

$$\begin{aligned}&\lambda\notin\sigma_b(M_C)\cup\{\sigma_{ab}(A^*)\cap\sigma_{ab}(B)\}\\ \Longrightarrow\ &\lambda\in\Phi_+^-(A)\cap\Phi_-^+(B), \mathrm{asc}(A-\lambda)<\infty \text{ and } \mathrm{ind}(A-\lambda)+\mathrm{ind}(B-\lambda)=0,\\ &\mathrm{dsc}(B-\lambda)<\infty, \text{ and either } \lambda\in\Phi_+^-(A^*) \text{ and}\\ &\mathrm{asc}(A^*-\lambda I^*)<\infty \text{ or } \lambda\in\Phi_+^-(B) \text{ and } \mathrm{asc}(B-\lambda)<\infty\\ \Longrightarrow\ &\lambda\in\sigma_b(A)^{\mathcal{C}}\cap\sigma_b(B)^{\mathcal{C}}\end{aligned}$$

imply that

$$\sigma_b(A)\cup\sigma_b(B)\subseteq\sigma_b(M_C)\cup\{\sigma_{ab}(A^*)\cap\sigma_{ab}(B)\};$$

cf. [5, Theorem 2.7]. The following Corollary generalizes [5, Theorem 2.9].

Corollary 5.6. *If $\sigma_{ab}(A^*)\cap\sigma_{ab}(B)=\emptyset$, A is isoloid and satisfies Wt, then M_0 satisfies Wt implies M_C satisfies Wt.*

Proof. The hypothesis M_0 satisfies Wt implies that A and B have SVEP on $\{\lambda\in\Phi(A)\cap\Phi(B):\mathrm{ind}(A-\lambda)+\mathrm{ind}(B-\lambda)=0\}$ and $\sigma_b(M_0)=\sigma_w(M_0)=\sigma_w(A)\cup\sigma_w(B)$. Since $\lambda\in\sigma_w(M_C)^{\mathcal{C}}\Longrightarrow\lambda\in\Phi_+(A)\cap\Phi_-(B)$ and $\mathrm{ind}(A-\lambda)+\mathrm{ind}(B-\lambda)=0$, it follows that if (also) $\sigma_{ab}(A^*)\cap\sigma_{ab}(B)=\emptyset$, then $\lambda\in\Phi(A)\cap\Phi(B)$ and $\mathrm{ind}(A-\lambda)+\mathrm{ind}(B-\lambda)=0$. Thus $\sigma_w(M_C)\supseteq\sigma_w(M_0)$, which implies that $\sigma_w(M_C)=\sigma_w(M_0)$. Now apply Corollary 5.2. □

Next, we prove an $a-Wt$ analog of Theorem 5.1.

Theorem 5.7. *(i). If $\sigma_{aw}(M_0)=\sigma_{aw}(M_C)$, then M_0 satisfies $a-Wt$ implies M_C satisfies $a-Wt$ if and only if $\mathcal{P}_0^a(M_C)\subseteq\mathcal{P}_0^a(M_0)$.*
(ii). Conversely, if $\sigma_{aw}(M_0)=\sigma_{aw}(M_C)$ and A^* has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, then M_C satisfies $a-Wt$ implies M_0 satisfies $a-Wt$ if and only if $\mathcal{P}_0^a(M_0)\subseteq\mathcal{P}_0^a(M_C)$.

Proof. (i). Since M_0 satisfies $a-Wt$ implies M_0 satisfies $a-Bt$, A and B have SVEP on the complement of $\sigma_{aw}(M_C)$ $(=\sigma_{aw}(M_0)=\sigma_{aw}(A)\cup\sigma_{aw}(B))$. Hence M_C satisfies $a-Bt$ (see Theorem 4.12(i)). Thus $\lambda\in\mathcal{R}_0^a(M_C)\Longleftrightarrow\lambda\in\sigma_{aw}(M_C)^{\mathcal{C}}=\sigma_{aw}(M_0)^{\mathcal{C}}=\mathcal{R}_0^a(M_0)$. Since $\mathcal{R}_0^a(M_C)\subseteq\mathcal{P}_0^a(M_C)$, it follows that

$$\sigma_a(M_C)\setminus\sigma_{aw}(M_C)=\mathcal{R}_0^a(M_C)=\mathcal{R}_0^a(M_0)=\mathcal{P}_0^a(M_0)\subseteq\mathcal{P}_0^a(M_C),$$

which proves that M_C satisfies $a-Wt$ if and only if $\mathcal{P}_0^a(M_C)\subseteq\mathcal{P}_0^a(M_0)$.

(ii). The argument of the proof of Theorem 4.12(i) shows that if $\sigma_w(M_C)=\sigma_w(M_0)$ and A^* has SVEP on $\sigma_{aw}(M_C)^{\mathcal{C}}$, then $\sigma_a(M_C)=\sigma_a(M_0)=\sigma_a(A)\cup\sigma_a(B)$. Thus, if M_C satisfies $a-Wt$, then M_0 satisfies $a-Bt$ (i.e., $\sigma_a(M_0)\setminus\sigma_{aw}(M_0)=\mathcal{R}_0^a(M_0)$) and

$$\sigma_a(M_0)\backslash\sigma_{aw}(M_0)=\sigma_a(M_C)\backslash\sigma_{aw}(M_C)=\mathcal{R}_0^a(M_C)=\mathcal{R}_0^a(M_0)=\mathcal{P}_0^a(M_0)\subseteq\mathcal{P}_0^a(M_C),$$

where the equality $\mathcal{R}_0^a(M_0)=\mathcal{R}_0^a(M_C)$ follows from the implications $\lambda\in\mathcal{R}_0^a(M_C)\Longleftrightarrow\lambda\in\sigma_{aw}(M_C)^{\mathcal{C}}=\sigma_{aw}(M_0)^{\mathcal{C}}\Longleftrightarrow\lambda\in\mathcal{R}_0^a(M_0)$. Hence M_0 satisfies $a-Wt$ if and only if $\mathcal{P}_0^a(M_C)\subseteq\mathcal{P}_0^a(M_0)$. □

The following corollary appears in [7, Theore 3.5]. Recall that T is a-isoloid if T is isoloid at every $\lambda\in\text{iso}\sigma_a(T)$.

Corollary 5.8. *If $\sigma_{aw}(A)=\sigma_{SF_+}(B)$, A is a-isoloid and satisfies $a-Wt$, then M_0 satisfies $a-Wt$ implies M_C satisfies $a-Wt$.*

Proof. Start by observing that if $\lambda\in\Phi_+^-(M_C)$ and $\text{ind}(A-\lambda)>0$, then $\lambda\in\Phi(A)\cap\Phi_+(B)$ and $\text{ind}(A-\lambda)+\text{ind}(B-\lambda)\leq 0$; if, instead, $\text{ind}(A-\lambda)\leq 0$, then $\sigma_{aw}(A)=\sigma_{SF_+}(B)$ and $\lambda\in\Phi_+^-(M_C)$ imply that $\lambda\in\Phi_+^-(A)\cap\Phi_+(B)$ and $\text{ind}(A-\lambda)+\text{ind}(B-\lambda)\leq 0$. In either case, $\lambda\in\Phi_+^-(M_C)\Longrightarrow\lambda\in\Phi_+^-(M_0)$; hence $\sigma_{aw}(M_C)=\sigma_{aw}(M_0)$. In view of Theorem 5.7, we are thus left to prove that $\mathcal{P}_0^a(M_C)\subseteq\mathcal{P}_0^a(M_0)$. If $\lambda\in\mathcal{P}_0^a(M_C)$, then $\lambda\in\text{iso}\sigma_a(A)\cup\text{iso}\sigma_a(B)$, and so $\lambda\in\mathcal{P}_0^a(A)=\sigma_{aw}(A)^{\mathcal{C}}=\sigma_{SF_+}(B)^{\mathcal{C}}$ (since A is a-isoloid, A satisfies $a-Wt$ and $\sigma_{aw}(A)=\sigma_{SF_+}(B)$). But then, since M_0 satisfies Bt implies B has SVEP at λ, $\lambda\in\mathcal{R}_0^a(B)$. Hence $\lambda\in\mathcal{R}_0^a(M_0)=\mathcal{P}_0^a(M_0)$. □

If A^* has SVEP, then $\lambda\in\sigma_{aw}(M_C)^{\mathcal{C}}$ implies $\lambda\in\Phi(A)\cap\Phi_+^-(B)$, $\text{ind}(A-\lambda)\geq 0$ and $\text{ind}(A-\lambda)+\text{ind}(B-\lambda)\leq 0$; this in turn implies that $\lambda\notin\sigma_{aw}(A)\cup\sigma_{aw}(B)$. Thus, if A^* has SVEP and M_0 satisfies $a-Bt$, then $\sigma_{aw}(M_0)=\sigma_{aw}(A)\cup\sigma_{aw}(B)=\sigma_{aw}(M_C)$.

Corollary 5.9. *If $\sigma_a(A^*)$ has empty interior, A is a-isoloid and satisfies $a-Wt$, then M_0 satisfies $a-Wt\Longrightarrow M_C$ satisfies $a-Wt$.*

Proof. Evidently, A^* has SVEP, M_0 satisfies $a-Bt$ and $\sigma_{aw}(M_0)=\sigma_{aw}(M_C)$. Now argue as in the (latter part of the) proof of Corollary 5.8. □

Corollary 5.9 generalizes [6, Theorem 3.3].

For an operator $T\in B(\mathcal{Y})$ such that T^* has SVEP, T satisfies Wt if and only if T satisfies $a-Wt$ [1, Theorem 3.108]. Thus, if A^* and B^* have SVEP, then $M_X^*={M_0}^*$ or ${M_C}^*$ has SVEP, and the (two way) implication M_X satisfies $Wt\Longleftrightarrow M_X$ satisfies $a-Wt$ holds. The following theorem, our final result, proves more.

Theorem 5.10. *If $S_{\sigma_{SF_+}(A)}(A^*)\cup S_{\sigma_{SF_+}(B)}(B^*)=\emptyset$, then M_C satisfies $Wt\Longleftrightarrow M_C$ satisfies $a-Wt$.*

Proof. The implication M_C satisfies $a-Wt \Longrightarrow M_C$ satisfies Wt being clear, we prove the reverse implication. For this, it would suffice to prove that $\sigma(M_C) = \sigma_a(M_C)$ (which would then imply $\mathcal{P}_0(M_C) = \mathcal{P}_0^a(M_C)$) and $\sigma_w(M_C) = \sigma_{aw}(M_C)$.

Evidently, $\sigma_a(M_C) \subseteq \sigma(M_C)$. Let $\lambda \notin \sigma_a(M_C)$. Then $M_C - \lambda$ and $A - \lambda$ are left invertible. The left invertibility of $A - \lambda$ implies $\lambda \in \Phi_+(A)$. Since A^* has SVEP at points $\lambda \in \Phi_+(A)$, it follows that $A - \lambda$ is invertible. But then $B - \lambda$ is left invertible, which (because B^* has SVEP at points $\lambda \in \Phi_+(B)$) implies that $B - \lambda$ is invertible. Thus, $\lambda \notin \sigma_a(M_C) \Longrightarrow \lambda \notin \sigma(A) \cup \sigma(B)$, i.e., $\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B) \subseteq \sigma_a(M_C)$. Next, we prove that $\sigma_w(M_C) \subseteq \sigma_{aw}(M_C)$: this would then imply the equality $\sigma_w(M_C) = \sigma_{aw}(M_C)$. Let $\lambda \notin \sigma_{aw}(M_C)$; then $\lambda \in \Phi_+(A)$ (and $\text{ind}(A-\lambda) + \text{ind}(B-\lambda) \leq 0$). Since A^* has SVEP at points $\lambda \in \Phi_+(A)$, it follows that $\text{ind}(A-\lambda) \geq 0 \Longrightarrow \lambda \in \Phi(A)$ (with $\text{ind}(A-\lambda) \geq 0$). Since this forces $\lambda \in \Phi_+(B)$, it follows (from the hypothesis B^* has SVEP on the set of $\lambda \in \Phi_+(B)$) that $\lambda \in \Phi(B)$ and $\text{ind}(B-\lambda) \geq 0$. Since $\text{ind}(A-\lambda) + \text{ind}(B-\lambda) \leq 0$, we conclude that $\lambda \in \Phi^0(A) \cap \Phi^0(B)$. Hence $\sigma_w(M_C) \subseteq \sigma_w(A) \cup \sigma_w(B) \subseteq \sigma_{aw}(M_C)$, and the proof is complete. □

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