Brownian excursions in a corridor and related Parisian options

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Abstract

In this paper, we study the excursion time of a Brownian motion with drift inside a corridor by using a four states semi-Markov model. In mathematical finance, these results have an important application in the valuation of options whose prices depend on the time their underlying assets prices spend between two different values. In this paper, we introduce the Parisian corridor option and obtain an explicit expression for the Laplace transform of its price formula.

Keywords: excursion time, four states Semi-Markov model, Parisian corridor options, Laplace transform.

1 Introduction

The concept of Parisian options was first introduced by Chesney, Jeanblanc-Picque and Yor [6]. It is a special case of path dependent options. The owner of a Parisian option will either gain the right or lose the right to exercise the option upon the price reaching a predetermined barrier level L and staying above or below the level for a predetermined time D before the maturity date T. More precisely, the owner of a *Parisian down-and-out option* loses the option if the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than D. For a *Parisian down-and-in option* the same event gives the owner the right to exercise the option. For details on the pricing of Parisian options see [6], [14], [16] and [13].

The Parisian corridor options replace the barrier by a corridor. Instead of considering the excursion above or below a barrier, we consider the excursions inside a corridor. For example, the owner of a *Parisian corridor in option* gains the option if the underlying asset price process S has an excursion in the corridor for longer than d before the maturity of the option. For the pricing of the Parisian options whose prices depends on the excursion outside a corridor see [10]. Later on, we will give the Laplace transforms which can be used to price this type of options.

In this paper, we are going to use the same definition for the excursion as in [6] and [7]. Let S be a stochastic process and l_1, l_2, $l_1 > l_2$ be the level of two barriers forming the corridor. We define

$$g^S_{l_i,t} = \sup\{s \leq t \mid S_s = l_i\}, \qquad d^S_{l_i,t} = \inf\{s \geq t \mid S_s = l_i\}, \qquad i = 1, 2, \quad (1)$$

with the usual conventions, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. Assuming $d_i > 0$, $i = 1, 2, 3, 4$, we now define

$$\tau^S_1 = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t > l_1\}}(t - g^S_{l_1,t}) \geq d_1\right\}, \quad (2)$$

$$\tau^S_2 = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\left\{g^S_{l_1,t} > g^S_{l_2,t}\right\}}(t - g^S_{l_1,t}) \geq d_2\right\}, \quad (3)$$

$$\tau^S_3 = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\left\{g^S_{l_1,t} < g^S_{l_2,t}\right\}}(t - g^S_{l_2,t}) \geq d_3\right\}, \quad (4)$$

$$\tau^S_4 = \inf\{t > 0 \mid \mathbf{1}_{\{S_t < l_2\}}(t - g^S_{l_2,t}) \geq d_4\}, \quad (5)$$

$$\tau^S = \tau^S_2 \wedge \tau^S_3. \quad (6)$$

We can see that τ^S_2 is the first time that the length of the excursion in the corridor reaches the given level d_2, given that this excursion starts from the upper barrier l_1; τ^S_3 corresponds to the one in the corridor with the given level d_3 starting from the lower barrier l_2; and τ^S is the smaller of τ^S_2 and τ^S_3. When we take $d_2 = d_3 = d$, τ^S is actually the the first time that the length of the excursion inside the corridor reaches given level d, which is what we want to study later on.

We can also see that τ^S_1 is the first time that the length of the excursion of process S above the barrier l_1 reaches given level d_1; τ^S_4 corresponds to the one below l_2 with required length d_4. Although τ^S_1 and τ^S_4 are not of our interest in this paper (see [10] for the pricing of the Parisian options depend on τ^S_1 and τ^S_4), we need to use these two stopping times to define our four states semi-Markov model.

Now assume r is the risk-free rate, T is the term of the option, S_t is the price of its underlying asset, K is the strike price, Q is risk neutral measure. If we have a Parisian corridor out-call option with the barrier l_1 and l_2, its price can be expressed as:

$$PC_{out-call} = e^{-rT}E_Q\left(\mathbf{1}_{\{\tau^S > T\}}\,(S_T - K)^+\right);$$

and the price of a Parisian corridor in-put option is:

$$PC_{in-put} = e^{-rT}E_Q\left(\mathbf{1}_{\{\tau^S < T\}}\,(K - S_T)^+\right).$$

In this paper, we are going to study the excursion time inside the corridor using a semi-Markov model consisting of four states. By applying the model to a Brownian motion, we can get the explicit form of the Laplace transform for the price of Parisian corridor options. One can then invert using techniques as in [14].

In Section 2 we introduce the four states semi-Markov model as well as a new process, the doubly perturbed Brownian motion, which has the same behavior as a Brownian motion except that each time it hits one of the two barriers, it moves towards the other side of the barrier by a jump of size ϵ. In Section 3 we obtain the martingale to which we can apply the optional sampling theorem and get the Laplace transform that we can use for pricing later. We give our main results applied to Brownian motion in Section 4, including the Laplace transforms for the stopping times we defined by (6) for both a Brownian motion with drift, i.e. $S = W^{\mu}$, and a standard Brownian motion, i.e. $S = W$. In Section 5 we focus on pricing the Parisian corridor options.

2 Definitions

From the description above, it is clear that we are actually considering four states, the state when the stochastic process is above the barrier l_1 the state when it is below l_2 and two states when it is between l_1 and l_2 depending on whether it comes into the corridor through l_1 or l_2. For each state, we are interested in the time the process spends in it. We therefore introduce a new process

$$Z_t^S = \begin{cases} 1, & \text{if } S_t > l_1 \\ 2, & \text{if } l_1 > S_t > l_2 \text{ and } g^S_{l_1,t} > g^S_{l_2,t} \\ 3, & \text{if } l_1 > S_t > l_2 \text{ and } g^S_{l_1,t} < g^S_{l_2,t} \\ 4, & \text{if } S_t < l_2 \end{cases} .$$

We can now express the variables defined above in terms of Z_t:

$$g^S_{l_i,t} = \sup\left\{s \leq t \mid Z^S_s \neq Z_t\right\}, \tag{7}$$

$$d^S_{l_i,t} = \inf\left\{s \geq t \mid Z^S_s \neq Z_t\right\}, \tag{8}$$

$$\tau_1^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=1\}}\left(t - g^S_{l_1,t}\right) \geq d_1\right\}, \tag{9}$$

$$\tau_2^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=2\}}\left(t - g^S_{l_1,t}\right) \geq d_2\right\}, \tag{10}$$

$$\tau_3^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=3\}}\left(t - g^S_{l_2,t}\right) \geq d_3\right\}, \tag{11}$$

$$\tau_4^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z_t^S=4\}}\left(t - g^S_{l_2,t}\right) \geq d_4\right\}. \tag{12}$$

We then define

$$V_t^S = t - \max\left(g^S_{l_1,t}, g^S_{l_2,t}\right), \tag{13}$$

the time Z_t^S has spent in the current state. It is easy to see that (Z_t^S, V_t^S) is a Markov process. Z_t^S is therefore a semi-Markov process with the state space $\{1, 2, 3, 4\}$, where 1 stands for the state when the stochastic process S is above the barrier l_1; 4 corresponds to the state below the barrier l_2; 2 and 3 represent the state when S is in the corridor given that it comes in through l_1 and l_2 respectively.

For Z_t^S the transition intensities $\lambda_{ij}(u)$ satisfy

$$P\left(Z_{t+\Delta t}^S = j, i \neq j \mid Z_t^S = i, V_t^S = u\right) = \lambda_{ij}(u)\Delta t + o\left(\Delta t\right), \qquad (14)$$

$$P\left(Z_{t+\Delta t}^S = i \mid Z_t^S = i, V_t^S = u\right) = 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o\left(\Delta t\right). \qquad (15)$$

Define

$$\bar{P}_i(\mu) = \exp\left\{-\int_0^\mu \sum_{i\neq j} \lambda_{ij}(v)\mathrm{d}v\right\}, \quad p_{ij}(\mu) = \lambda_{ij}(\mu)\bar{P}_i(\mu).$$

Notice that

$$P_i(\mu) = 1 - \bar{P}_i(\mu)$$

is the distribution function of the excursion time in state i, which is a random variable U_i defined as

$$U_i = \inf_{s>0} \left\{Z_s^S \neq i \mid Z_0^S = i, V_0^S = 0\right\}.$$

Note that because the process is time homogeneous this has the same distribution as

$$\inf_{s>0} \left\{Z_{t+s}^S \neq i \mid Z_t^S = i, V_t^S = 0\right\}$$

for any time t. We have therefore

$$p_{ij}(\mu) = \lim_{\triangle\mu \to 0} \frac{P\left(U_i \in (\mu, \mu + \triangle\mu), Z_{U_i}^S = j\right)}{\triangle\mu}.$$

Moreover, in the definition of Z^S, we deliberately ignore the situation when $S_t = l_i$, $i = 1, 2$. The reason is that we only consider the processes, which

$$\int_0^t \mathbf{1}_{\{S_u = l_i\}}\mathrm{d}u = 0, \ i = 1, 2.$$

Also, when l_1 and l_2 are the regular points of the process (see [5] for definition), we have to deal with the degeneration of p_{ij}. Let us take a Brownian Motion as an example. Assume $W_t^\mu = \mu t + W_t$ with $\mu \geq 0$, where W_t is a standard Brownian Motion. Setting x_0 to be its starting point, we know its density for the first hitting time of level l_i, $i = 1, 2$ is

$$p_{x_0} = \frac{|l_i - x_0|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(l_i - x_0 - \mu t)^2}{2t}\right\}$$

(see [4]). According to the definition of transition density, $p_{12}(t) = p_{21}(t) = p_{l_1}(t) = 0$ and $p_{34}(t) = p_{43}(t) = p_{l_2}(t) = 0$, for $t > 0$.

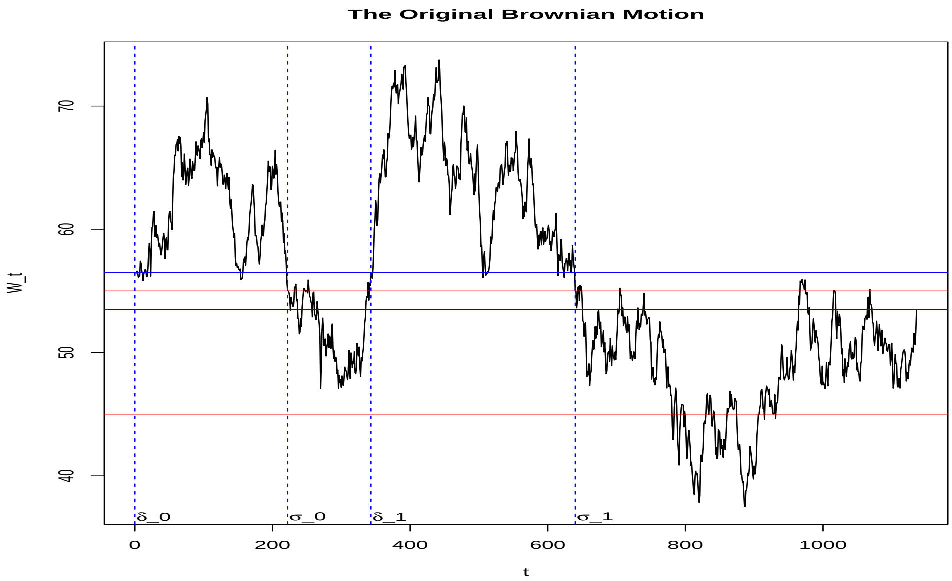

Figure 1: A Sample Path of $W_t^{(\epsilon)}$

In [9] in order to solve the similar problem, we introduced the perturbed Brownian motion $X_t^{(\epsilon)}$ with respect to the barrier we are interested in. We apply the same idea here, and construct a new process *double perturbed Brownian motion*, $Y_t^{(\epsilon)}$, $\epsilon > 0$, with respect to barriers l_1 and l_2. Assume $W_0^\mu = l_1 + \epsilon$. Define a sequence of stopping times

$$\begin{aligned} \delta_0 &= 0, \\ \sigma_n &= \inf\{t > \delta_n \mid W_t^\mu = l_1\}, \\ \delta_{n+1} &= \inf\{t > \sigma_n \mid W_t^\mu = l_1 + \epsilon\}, \end{aligned}$$

where $n = 0, 1, \cdots$ (see Figure 1). Now define

$$\begin{cases} X_t^{(\epsilon)} = W_t^\mu & \text{if} \quad \delta_n \leq t < \sigma_n \\ X_t^{(\epsilon)} = W_t^\mu - \epsilon & \text{if} \quad \sigma_n \leq t < \delta_{n+1} \end{cases}.$$

Similarly, we then define another sequence of stopping times with respect to process $X_t^{(\epsilon)}$ and barrier l_2

$$\begin{aligned} \zeta_0 &= 0, \\ \eta_n &= \inf\{t > \zeta_n \,|\, X_t^{(\epsilon)} = l_2\}, \\ \zeta_{n+1} &= \inf\{t > \eta_n \,|\, X_t^{(\epsilon)} = l_2 + \epsilon\}, \end{aligned}$$

where $n = 0, 1, \cdots$ (see Figure 2). Then define

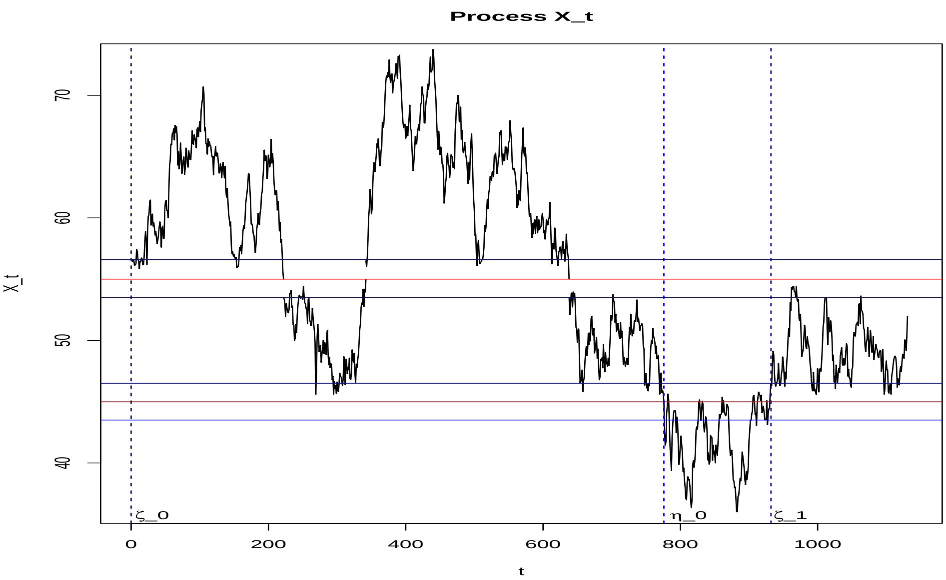

Figure 2: A Sample Path of $X_t^{(\epsilon)}$

$$\begin{cases} Y_t^{(\epsilon)} = X_t^{(\epsilon)} & \text{if} \quad \zeta_n \leq t < \eta_n \\ Y_t^{(\epsilon)} = X_t^{(\epsilon)} - \epsilon & \text{if} \quad \eta_n \leq t < \zeta_{n+1} \end{cases} .$$

The process $Y_t^{(\epsilon)}$ is actually a process which starts from $l_1 + \epsilon$ and has the same behavior as the related Brownian Motion expect that each time when it hits the barrier l_1 or l_2, it will have a jump towards the opposite side of the barrier with size ϵ (see Figure 3).

From the definition, it is clear that l_1 and l_2 become irregular points for $Y_t^{(\epsilon)}$. Also $Y_t^{(\epsilon)}$ converges to W_t^μ with $W_0^\mu = l_1$ almost surely for all t. Therefore as we prove in [9], the Laplace transforms of the variables defined based on $Y_t^{(\epsilon)}$ converge to those based on W_t^μ. As a result, we can obtain the results for the Brownian Motion by carrying out the calculation for $Y_t^{(\epsilon)}$ and take the limit as $\epsilon \rightarrow 0$.

For $Y_t^{(\epsilon)}$, we can define Z^Y, τ_1^Y, τ_2^Y and τ^Y as above (we suppress (ϵ) on the

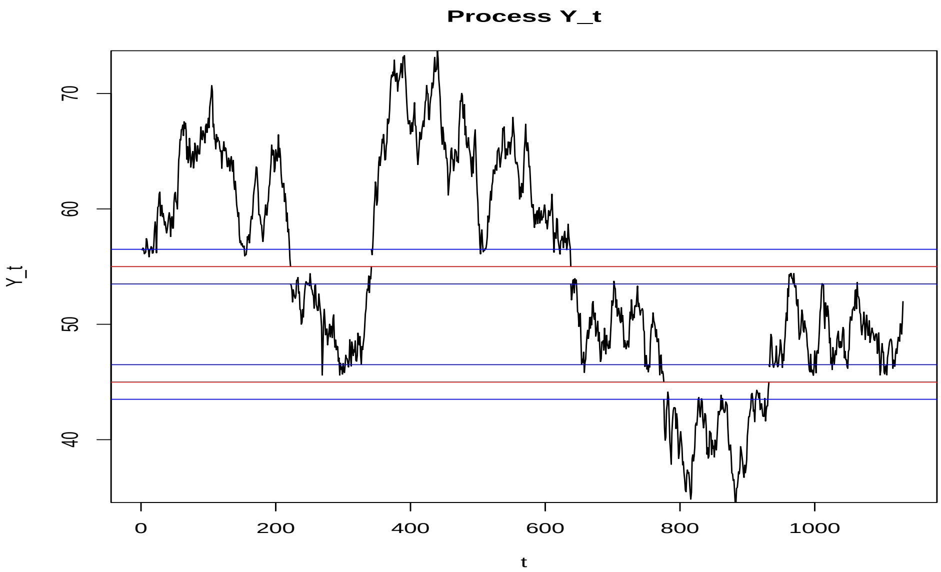

Figure 3: A Sample Path of $Y_t^{(\epsilon)}$

superscript). For Z^Y, we have the transition densities (see [4])

$$p_{12}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon+\mu t)^2}{2t}\right\}, \tag{16}$$

$$p_{21}(t) = \exp\left\{\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t\left(l_1 - l_2 - \epsilon, l_1 - l_2\right), \tag{17}$$

$$p_{24}(t) = \exp\left\{-\mu\left(l_1 - l_2 - \epsilon\right) - \frac{\mu^2 t}{2}\right\} ss_t\left(\epsilon, l_1 - l_2\right), \tag{18}$$

$$p_{31}(t) = \exp\left\{\mu\left(l_1 - l_2 - \epsilon\right) - \frac{\mu^2 t}{2}\right\} ss_t\left(\epsilon, l_1 - l_2\right), \tag{19}$$

$$p_{34}(t) = \exp\left\{-\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t\left(l_1 - l_2 - \epsilon, l_1 - l_2\right), \tag{20}$$

$$p_{43}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon-\mu t)^2}{2t}\right\}, \tag{21}$$

where

$$ss_t(x,y) = \sum_{k=-\infty}^{\infty} \frac{(2k+1)y - x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{\left((2k+1)y - x\right)^2}{2t}\right\}.$$

Also we know that

$$p_{23}(t) = p_{32}(t) = p_{14}(t) = p_{41}(t) = 0. \tag{22}$$

Clearly, all the arguments above apply to the standard Brownian motion, which is a special case of W_t^{μ} when $\mu = 0$.

3 Results for the semi-Markov model

In §2 we have introduced the Markov process $\left(Z_t^S, V_t^S\right)$. Now we apply the same definition to the doubly perturbed Brownian motion $Y_t^{(\epsilon)}$; therefore we have $\left(Z_t^Y, V_t^Y\right)$, where Z_t^Y is the current state of $Y_t^{(\epsilon)}$, taking value from state space $\{1, 2, 3, 4\}$ and V_t^Y is the time $Y_t^{(\epsilon)}$ has spent in current state. V_t^Y is also a stochastic process. Now we consider a function of the form

$$f\left(V_t^Y, Z_t^Y, t\right) = f_{Z_t^Y}\left(V_t^Y, t\right),$$

where f_i, $i = 1, 2, 3, 4$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. The generator $\mathscr{A}$ is defined as an operator such that

$$f\left(V_t^Y, Z_t^Y, t\right) - \int_0^t \mathscr{A} f\left(V_s^Y, Z_s^Y, s\right) \mathrm{d}s$$

is a martingale (see [11], chapter 2). Therefore solving

$$\mathscr{A} f = 0$$

subject to certain conditions will provide us with martingales of the form $f\left(V_t^Y, Z_t^Y, t\right)$ to which we can apply the optional stopping theorem to obtain the Laplace transform we are interested in. More precisely, we will have

$$\begin{cases} \mathscr{A} f_1(u,t) &= \dfrac{\partial f_1(u,t)}{\partial t} + \dfrac{\partial f_1(u,t)}{\partial u} + \lambda_{12}(u)(f_2(0,t) - f_1(u,t)) \\ \mathscr{A} f_2(u,t) &= \dfrac{\partial f_2(u,t)}{\partial t} + \dfrac{\partial f_2(u,t)}{\partial u} + \lambda_{21}(u)(f_1(0,t) - f_2(u,t)) + \lambda_{24}(u)(f_4(0,t) - f_2(u,t)) \\ \mathscr{A} f_3(u,t) &= \dfrac{\partial f_3(u,t)}{\partial t} + \dfrac{\partial f_3(u,t)}{\partial u} + \lambda_{31}(u)(f_1(0,t) - f_3(u,t)) + \lambda_{34}(u)(f_4(0,t) - f_3(u,t)) \\ \mathscr{A} f_4(u,t) &= \dfrac{\partial f_4(u,t)}{\partial t} + \dfrac{\partial f_4(u,t)}{\partial u} + \lambda_{43}(u)(f_4(0,t) - f_3(u,t)) \end{cases},$$

Assume f_i has the form

$$f_i(u,t) = e^{-\beta t} g_i(u).$$

By solving the equation $\mathscr{A} f=0$, i.e. $\left\{\begin{array}{lll}\mathscr{A} f_1 & = & 0 \\ \mathscr{A} f_2 & = & 0 \\ \mathscr{A} f_3 & = & 0 \\ \mathscr{A} f_4 & = & 0\end{array}\right.$ subject to $\left\{\begin{array}{lll}g_1(d_1) & = & \alpha_1 \\ g_2(d_2) & = & \alpha_2 \\ g_3(d_2) & = & \alpha_3 \\ g_4(d_2) & = & \alpha_4\end{array}\right.$

we can get

$$
\begin{aligned}
g_i(u) \quad = \quad & \alpha_i \exp\left\{-\int_u^{d_i}\left(\beta+\sum_{j\neq i}\lambda_{ij}(v)\right)\mathrm{d}v\right\} \qquad (23)\\
& +\sum_{j\neq i} g_j(0)\int_u^{d_i}\lambda_{ij}(s)\exp\left\{-\int_u^{s}\left(\beta+\sum_{k\neq i}\lambda_{ik}(v)\right)\mathrm{d}v\right\}\mathrm{d}s.
\end{aligned}
$$

In our case, we are only interested in the excursion inside the corridor. Hence, we set d_1 and d_4 to be ∞. Also $\lim_{d_1\to\infty} g_1(d_1)=\lim_{d_4\to\infty} g_4(d_4)=0$ gives $\alpha_1=\alpha_4=0$. Therefore, we have

$$
\begin{aligned}
g_2(0) &= \alpha_2 e^{-\beta d_2}\bar{P}_2(d_2)+g_2(0)\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)+g_3(0)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta), \qquad (24)\\
g_3(0) &= \alpha_3 e^{-\beta d_3}\bar{P}_3(d_3)+g_2(0)\hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)+g_3(0)\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta). \qquad (25)
\end{aligned}
$$

Solving (24) and (25) gives

$$
g_2(0)=\frac{\alpha_2 e^{-\beta d_2}\bar{P}_2(d_2)\left(1-\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)\right)+\alpha_3 e^{-\beta d_3}\bar{P}_3(d_3)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}{1-\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)-\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)+\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)-\hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}, \qquad (26)
$$

$$
g_3(0)=\frac{\alpha_3 e^{-\beta d_3}\bar{P}_3(d_3)\left(1-\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\right)+\alpha_2 e^{-\beta d_2}\bar{P}_2(d_2)\hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)}{1-\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)-\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)+\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)-\hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}. \qquad (27)
$$

where

$$
\hat{P}_{ij}(\beta)=\int_0^\infty e^{-\beta s}p_{ij}(s)\mathrm{d}s, \qquad (28)
$$

$$
\tilde{P}_{ij}(\beta)=\int_0^{d_i} e^{-\beta s}p_{ij}(s)\mathrm{d}s. \qquad (29)
$$

As a result, we have obtained the martingale

$$
M_t=f\left(V_t^Y,t\right)=e^{-\beta t}g_{Z_t^Y}\left(V_t^Y\right),\quad i=1,2,3,4. \qquad (30)
$$

We now can apply the optional stopping theorem to M_t with the stopping time $\tau^Y\wedge t$, where τ^Y is the stopping time defined by (6):

$$
E\left(M_{\tau^Y\wedge t}\right)=E\left(M_0\right). \qquad (31)
$$

The right hand side of (31) is

$$
E\left(M_{\tau^Y\wedge t}\right)=E\left(M_{\tau^Y}\mathbf{1}_{\{\tau^Y<t\}}\right)+E\left(M_t\mathbf{1}_{\{\tau^Y>t\}}\right).
$$

Furthermore,

$$
\begin{aligned}
& E\left(M_{\tau^Y}\mathbf{1}_{\{\tau^Y<t\}}\right) \\
= \ & E\left(M_{\tau^Y}\mathbf{1}_{\{\tau_2^Y<\tau_3^Y\}}\mathbf{1}_{\{\tau_2^Y<t\}}\right)+E\left(M_{\tau^Y}\mathbf{1}_{\{\tau_2^Y>\tau_3^Y\}}\mathbf{1}_{\{\tau_3^Y<t\}}\right) \\
= \ & E\left(e^{-\beta\tau^Y}g_2\left(d_2\right)\mathbf{1}_{\{\tau_2^Y<\tau_3^Y\}}\mathbf{1}_{\{\tau_2^Y<t\}}\right)+E\left(e^{-\beta\tau^Y}g_3\left(d_3\right)\mathbf{1}_{\{\tau_2^Y>\tau_3^Y\}}\mathbf{1}_{\{\tau_3^Y<t\}}\right) \\
= \ & \alpha_2 E\left(e^{-\beta\tau^Y}\mathbf{1}_{\{\tau_2^Y<\tau_3^Y\}}\mathbf{1}_{\{\tau_2^Y<t\}}\right)+\alpha_3 E\left(e^{-\beta\tau^Y}\mathbf{1}_{\{\tau_2^Y>\tau_3^Y\}}\mathbf{1}_{\{\tau_3^Y<t\}}\right).
\end{aligned}
$$

We also have

$$E\left(M_t\mathbf{1}_{\{\tau^Y>t\}}\right)=e^{-\beta t}E\left(g_{Z_t^Y}\left(V_t^Y\right)\mathbf{1}_{\{\tau^Y>t\}}\right),$$

where Z_t^Y can take values $1,2,3$ or 4.

When $Z_t^Y=2$ or 3, since $\tau^Y>t$, we have $0\le V_t^Y<d_2\wedge d_3$. According to the definition of $g_i(\mu)$ in (23), we have $g_2\left(V_t^Y\right)$ and $g_3\left(V_t^Y\right)$ are bounded.

When $Z_t^Y=1$ or 4, since $\lim_{d_1\to\infty}g_1\left(d_1\right)=\lim_{d_4\to\infty}g_4\left(d_4\right)=0$ and looking at (23) with d_1 and d_4 replaced by ∞ we have that $g_1\left(V_t^Y\right)$ and $g_4\left(V_t^Y\right)$ are bounded.

Therefore

$$\lim_{t\to\infty}E\left(M_t\mathbf{1}_{\{\tau^Y>t\}}\right)=0.$$

The left hand side of (31) gives

$$\lim_{t\to\infty}E\left(M_0\right)=E\left(M_0\right)=\begin{cases} g_2(0), & Y_0^{(\epsilon)}=l_1+\epsilon \\ g_3(0), & Y_0^{(\epsilon)}=l_2-\epsilon \end{cases}.$$

By taking $\alpha_2=\alpha_3=1$ and $d_2=d_3=d$, we will have when $Y_0^{(\epsilon)}=l_1+\epsilon$

$$
\begin{aligned}
& E\left(e^{-\beta\tau^Y}\right) \qquad (32)\\
= \ & \frac{e^{-\beta d}\bar{P}_2(d)\left(1-\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)\right)+e^{-\beta d}\bar{P}_3(d)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}{1-\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)-\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)+\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)-\hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)},
\end{aligned}
$$

when $Y_0^{(\epsilon)}=l_2-\epsilon$

$$
\begin{aligned}
& E\left(e^{-\beta\tau^Y}\right) \qquad (33)\\
= \ & \frac{e^{-\beta d}\bar{P}_2(d)\hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)+e^{-\beta d}\bar{P}_3(d)\left(1-\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\right)}{1-\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)-\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)+\hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{34}(\beta)-\hat{P}_{12}(\beta)\tilde{P}_{31}(\beta)\hat{P}_{43}(\beta)\tilde{P}_{24}(\beta)}.
\end{aligned}
$$

4 Main Results

In §2 we have stated that the main difficulty with the Brownian Motion is that its origin point is regular, i.e. the probability that W_t^μ will return to the origin at arbitrarily small time is 1. We have therefore introduced the new processes $Y_t^{(\epsilon)}$ and (Z_t^Y,V_t^Y) with transition densities for Z_t^Y defined in (16) to (22).

Theorem 1 *For a Brownian Motion W_t^μ, τ^{W^μ} defined as in (6) with $S_t = W_t^\mu$, we have following Laplace transforms:*

when $W_0^\mu = l_1$,

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = e^{-\beta d}\frac{e^{-\mu l}F_2(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right)-G_2^2\left(\beta+\frac{\mu^2}{2}\right)}; \quad (34)$$

when $W_0^\mu = l_2$,

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = e^{-\beta d}\frac{e^{\mu l}F_1(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right)-F_2(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right)-G_2^2\left(\beta+\frac{\mu^2}{2}\right)}; \quad (35)$$

where

$$l = l_1 - l_2; \quad (36)$$

$$\begin{aligned} F_1(x) &= \sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty} e^{-2|x|lk}\left\{\exp\left\{-\frac{1}{2}\left(\frac{2lk}{\sqrt{d}}-|x|\sqrt{d}\right)^2\right\}\right. \quad (37)\\ &\left.-e^{-(|x|+x)l}\exp\left\{-\frac{1}{2}\left(\frac{l(2k+1)}{\sqrt{d}}-|x|\sqrt{d}\right)^2\right\}\right\},\\ &+2|x|\sum_{k=-\infty}^{\infty} e^{-2|x|lk}\left\{e^{-(|x|+x)l}\mathcal{N}\left(\frac{l(2k+1)}{\sqrt{d}}-|x|\sqrt{d}\right)-\mathcal{N}\left(\frac{2lk}{\sqrt{d}}-|x|\sqrt{d}\right)\right\} \end{aligned}$$

$$\begin{aligned} F_2(x) &= \sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty} e^{-2|x|lk}\left\{\exp\left\{-\frac{1}{2}\left(\frac{2lk}{\sqrt{d}}-|x|\sqrt{d}\right)^2\right\}\right. \quad (38)\\ &\left.-e^{-(|x|-x)l}\exp\left\{-\frac{1}{2}\left(\frac{l(2k+1)}{\sqrt{d}}-|x|\sqrt{d}\right)^2\right\}\right\}\\ &+2|x|\sum_{k=-\infty}^{\infty} e^{-2|x|lk}\left\{e^{-(|x|-x)l}\mathcal{N}\left(\frac{l(2k+1)}{\sqrt{d}}-|x|\sqrt{d}\right)-\mathcal{N}\left(\frac{2lk}{\sqrt{d}}-|x|\sqrt{d}\right)\right\}, \end{aligned}$$

$$\begin{aligned} G_1(x) &= \frac{-2\sqrt{2x}}{1-e^{-2l\sqrt{2x}}}+2\sqrt{2x}\sum_{k=-\infty}^{\infty} e^{-2l\sqrt{2x}k}\mathcal{N}\left(\frac{2lk}{\sqrt{d}}-\sqrt{2xd}\right) \quad (39)\\ &-\sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty} e^{-2l\sqrt{2x}k}\exp\left\{-\frac{1}{2}\left(\frac{2lk}{\sqrt{d}}-\sqrt{2xd}\right)^2\right\}, \end{aligned}$$

$$\begin{aligned} G_2(x) &= \frac{2\sqrt{2x}e^{-l\sqrt{2x}}}{1-e^{-2l\sqrt{2x}}}-2\sqrt{2x}\sum_{k=-\infty}^{\infty} e^{-l\sqrt{2x}(2k+1)}\mathcal{N}\left(\frac{l(2k+1)}{\sqrt{d}}-\sqrt{2xd}\right)\\ &+\sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty} e^{-l\sqrt{2x}(2k+1)}\exp\left\{-\frac{1}{2}\left(\frac{l(2k+1)}{\sqrt{d}}-\sqrt{2xd}\right)^2\right\}. \quad (40) \end{aligned}$$

Proof: We apply the transition densities in (16) to (22) to the results in (32) and (33) and taking the limit $\epsilon \to 0$. According to the definition of $Y^{(\epsilon)}$, we know that

$$Y_t^{(\epsilon)} \xrightarrow{a.s} W_t^{\mu}, \quad \text{for all } t.$$

As we saw in [9] when $Y_t^{(\epsilon)} \xrightarrow{a.s} W_t^{\mu}$, for all t, by taking the limit $\epsilon \to 0$, the quantities defined based on $Y_t^{(\epsilon)}$ converge to those based on Brownian motion with drift. Therefore we will get the results shown by (34) and (35). □

Corollary 1.1 *For a standard Brownian Motion ($\mu = 0$), we have for both cases (i.e. when $W_0 = l_1$ and when $W_0 = l_2$)*

$$E\left(e^{-\beta\tau^W}\right) = e^{-\beta d}\frac{h(0)}{h(\beta)}; \tag{41}$$

where

$$\begin{aligned} h(\beta) = {} & \sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty} e^{-2l\sqrt{2\beta}k}\left\{e^{-l\sqrt{2\beta}}\exp\left\{-\frac{1}{2}\left(\frac{l(2k+1)}{\sqrt{d}}-\sqrt{2\beta d}\right)^2\right\}\right. \\ & \left. -\exp\left\{-\frac{1}{2}\left(\frac{2lk}{\sqrt{d}}-\sqrt{2\beta d}\right)^2\right\}\right\} - \frac{2\sqrt{2\beta}}{1+e^{-l\sqrt{2\beta}}} \\ & +2\sqrt{2\beta}\sum_{k=-\infty}^{\infty} e^{-2l\sqrt{2\beta}k}\left\{\mathcal{N}\left(\frac{2lk}{\sqrt{d}}-\sqrt{2\beta d}\right) - e^{-l\sqrt{2\beta}}\mathcal{N}\left(\frac{l(2k+1)}{\sqrt{d}}-\sqrt{2\beta d}\right)\right\} \end{aligned} \tag{42}$$

We are also interested in the cases when a Brownian Motion starts from the point other than l_1 and l_2. The results are shown in the following corollary.

Corollary 1.2 *For a Brownian Motion W_t^{μ}, $\tau^{W^{\mu}}$ defined as in (6) with $S_t = W_t^{\mu}$, we have the following Laplace transforms:*
when $W_0^{\mu} = x_0$, $x_0 \geq l_1$,

$$\begin{aligned} E\left(e^{-\beta\tau^{W^{\mu}}}\right) = {} & \exp\left\{-\left(\mu+\sqrt{2\beta+\mu^2}\right)(x_0-l_1)-\beta d\right\} \\ & \frac{e^{-\mu l}F_2(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right)-G_2^2\left(\beta+\frac{\mu^2}{2}\right)}; \end{aligned} \tag{43}$$

when $W_0^{\mu} = x_0$, $x_0 \leq l_2$,

$$\begin{aligned} E\left(e^{-\beta\tau^{W^{\mu}}}\right) = {} & \exp\left\{\left(\mu-\sqrt{2\beta+\mu^2}\right)(l_2-x_0)-\beta d\right\} \\ & \frac{e^{\mu l}F_1(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right)-F_2(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right)-G_2^2\left(\beta+\frac{\mu^2}{2}\right)}; \end{aligned} \tag{44}$$

when $W_0^\mu = x_0,\ l_2 < x_0 < l_1$,

$$
\begin{aligned}
E\left(e^{-\beta\tau^{W^\mu}}\right) \quad = \quad & e^{\mu(l_2-x_0)-\beta d}\sum_{k=-\infty}^{\infty}\left\{e^{-|\mu|(2kl+x_0-l_2)}\mathscr{N}\left(-|\mu|\sqrt{d}+\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right. \\
& \left.-e^{|\mu|(2kl+x_0-l_2)}\mathscr{N}\left(-|\mu|\sqrt{d}-\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right\} \\
& +e^{\mu(l_1-x_0)-\beta d}\sum_{k=-\infty}^{\infty}\left\{e^{-|\mu|(2kl-x_0+l_1)}\mathscr{N}\left(-|\mu|\sqrt{d}+\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right. \\
& \left.-e^{|\mu|(2kl-x_0+l_1)}\mathscr{N}\left(-|\mu|\sqrt{d}-\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right\}+e^{-\beta d} \\
& -\frac{e^{-|\mu|l-\beta d}\left\{e^{\mu(l_2-x_0)}\left(e^{|\mu|(l_1-x_0)}-e^{-|\mu|(l_1-x_0)}\right)+e^{\mu(l_1-x_0)}\left(e^{|\mu|(x_0-l_2)}-e^{-|\mu|(x_0-l_2)}\right)\right\}}{1-e^{-2|\mu|l}} \\
& +\left[\frac{e^{-\sqrt{2\beta+\mu^2}l}e^{\mu(l_2-x_0)}\left(e^{\sqrt{2\beta+\mu^2}(l_1-x_0)}-e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\right)}{1-e^{-2\sqrt{2\beta+\mu^2}l}}\right. \\
& +e^{\mu(l_2-x_0)}\sum_{k=-\infty}^{\infty}\left\{e^{\sqrt{2\beta+\mu^2}(2kl+x_0-l_2)}\mathscr{N}\left(-\sqrt{(2\beta+\mu^2)\,d}-\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right. \\
& \left.\left.-e^{-\sqrt{2\beta+\mu^2}(2kl+x_0-l_2)}\mathscr{N}\left(-\sqrt{(2\beta+\mu^2)\,d}+\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right\}\right] \\
& \frac{e^{-\mu l}F_2(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right)-G_2^2\left(\beta+\frac{\mu^2}{2}\right)} \\
& +\left[\frac{e^{-\sqrt{2\beta+\mu^2}l}e^{\mu(l_1-x_0)}\left(e^{\sqrt{2\beta+\mu^2}(x_0-l_2)}-e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\right)}{1-e^{-2\sqrt{2\beta+\mu^2}l}}\right. \\
& +e^{\mu(l_1-x_0)}\sum_{k=-\infty}^{\infty}\left\{e^{\sqrt{2\beta+\mu^2}(2kl-x_0+l_1)}\mathscr{N}\left(-\sqrt{(2\beta+\mu^2)\,d}-\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right. \\
& \left.\left.-e^{-\sqrt{2\beta+\mu^2}(2kl-x_0+l_1)}\mathscr{N}\left(-\sqrt{(2\beta+\mu^2)\,d}+\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right\}\right] \\
& \frac{e^{\mu l}F_1(\mu)G_2\left(\beta+\frac{\mu^2}{2}\right)-F_2(\mu)G_1\left(\beta+\frac{\mu^2}{2}\right)}{G_1^2\left(\beta+\frac{\mu^2}{2}\right)-G_2^2\left(\beta+\frac{\mu^2}{2}\right)}.
\end{aligned}
\tag{45}
$$

Proof: We will prove the case when $x_0 \geq l_1$ at first. Defined $T = \inf\{t \mid W_t^\mu = l_1\}$, i.e. the first time W_t^μ hits l_1. By definition, we have $\tau^{W^\mu} = T + \tau^{\widetilde{W}^\mu}$, where $\widetilde{W}^\mu$ here stands for a Brownian motion with drift started from l_1. By the strong Markov property of the Brownian motion, we therefore have

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = E\left(e^{-\beta T}\right)E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right).$$

$E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right)$ has been calculate in Theorem 1 (34). According to [4], we have

$$E\left(e^{-\beta T}\right)=\exp\left\{-\left(\mu+\sqrt{2\beta+\mu^2}\right)(x_0-l_1)\right\}.$$

For the case when $x_0\le l_2$, we can apply the same argument.

When $l_2<x_0<l_1$, by definition, we have $\tau^{W^\mu}=d$, if $T\ge d$; $\tau^{W^\mu}=T+\tau^{\widetilde{W}^\mu}$, if $T<d$, and $W_T^\mu=l_1$ where $\widetilde{W}^\mu$ here stands for a Brownian motion with drift started from l_1; $\tau^{W^\mu}=T+\tau^{\underline{W}^\mu}$, if $T<d$, and $W_T^\mu=l_2$ where $\underline{W}^\mu$ here stands for a Brownian motion with drift started from l_2. As a result

$$\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right)\\
= & E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T\ge d\}}\right)+E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_1\}}\right)+E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_2\}}\right)\\
= & e^{-\beta d}P\left(T\ge d\right)+E\left(e^{-\beta T}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_1\}}\right)E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right)\\
& +E\left(e^{-\beta T}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_2\}}\right)E\left(e^{-\beta\tau^{\underline{W}^\mu}}\right)
\end{aligned}$$

$E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right)$ and $E\left(e^{-\beta\tau^{\underline{W}^\mu}}\right)$ have been calculated in Theorem 1 (see (34) and (35)). The density for T is given in [4] as

$$p_{x_0}(t)=e^{\mu(l_2-x_0)-\frac{\mu^2t}{2}}ss_t\left(l_1-x_0,l\right)+e^{\mu(l_1-x_0)-\frac{\mu^2t}{2}}ss_t\left(x_0-l_2,l\right).$$

We can therefore calculate

$$\begin{aligned}
& P\left(T\ge d\right)\\
= & 1-\frac{e^{-|\mu|l}\left\{e^{\mu(l_2-x_0)}\left(e^{|\mu|(l_1-x_0)}-e^{-|\mu|(l_1-x_0)}\right)+e^{\mu(l_1-x_0)}\left(e^{|\mu|(x_0-l_2)}-e^{-|\mu|(x_0-l_2)}\right)\right\}}{1-e^{-2|\mu|l}}\\
& +e^{\mu(l_2-x_0)}\sum_{k=-\infty}^{\infty}\left\{e^{-|\mu|(2kl+x_0-l_2)}\mathcal{N}\left(-|\mu|\sqrt{d}+\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right.\\
& \left.-e^{|\mu|(2kl+x_0-l_2)}\mathcal{N}\left(-|\mu|\sqrt{d}-\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right\}\\
& +e^{\mu(l_1-x_0)}\sum_{k=-\infty}^{\infty}\left\{e^{-|\mu|(2kl-x_0+l_1)}\mathcal{N}\left(-|\mu|\sqrt{d}+\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right.\\
& \left.-e^{|\mu|(2kl-x_0+l_1)}\mathcal{N}\left(-|\mu|\sqrt{d}-\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right\}.
\end{aligned}$$

$$E\left(e^{-\beta T}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_1\}}\right)$$
$$= \frac{e^{-\sqrt{2\beta+\mu^2}l}e^{\mu(l_2-x_0)}\left(e^{\sqrt{2\beta+\mu^2}(l_1-x_0)}-e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\right)}{1-e^{-2\sqrt{2\beta+\mu^2}l}}$$
$$+e^{\mu(l_2-x_0)}\sum_{k=-\infty}^{\infty}\left\{e^{\sqrt{2\beta+\mu^2}(2kl+x_0-l_2)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}-\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right.$$
$$\left.-e^{-\sqrt{2\beta+\mu^2}(2kl+x_0-l_2)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}+\frac{2kl+x_0-l_2}{\sqrt{d}}\right)\right\}$$

$$E\left(e^{-\beta T}\mathbf{1}_{\{T<d\}}\mathbf{1}_{\{W_T^\mu=l_2\}}\right)$$
$$= \frac{e^{-\sqrt{2\beta+\mu^2}l}e^{\mu(l_1-x_0)}\left(e^{\sqrt{2\beta+\mu^2}(x_0-l_2)}-e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\right)}{1-e^{-2\sqrt{2\beta+\mu^2}l}}$$
$$+e^{\mu(l_1-x_0)}\sum_{k=-\infty}^{\infty}\left\{e^{\sqrt{2\beta+\mu^2}(2kl-x_0+l_1)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}-\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right.$$
$$\left.-e^{-\sqrt{2\beta+\mu^2}(2kl-x_0+l_1)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d}+\frac{2kl-x_0+l_1}{\sqrt{d}}\right)\right\}.$$

We therefore get the result in (45). □

Notice that for a Brownian motion with drift, it is possible that τ^{W^μ} will never be achieved. Take the case when $\mu>0$ and $x_0>l_1$ as an example. We obtain the following result by taking $\beta=0$ in (43).

Corollary 1.3 *For a Brownian motion with positive drift, W^μ with $\mu>0$ and $x_0>l_1$ we have that,*

$$P\left(\tau^{W^\mu}<\infty\right)=\exp\left\{-2\mu\left(x_0-l_1\right)\right\}\frac{e^{-\mu l}F_2(\mu)G_2\left(\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\frac{\mu^2}{2}\right)}{G_1^2\left(\frac{\mu^2}{2}\right)-G_2^2\left(\frac{\mu^2}{2}\right)}. \quad (46)$$

Remark 1: As a result, for a Brownian motion with positive drift and $x_0>l_1$, with probability

$$1-\exp\left\{-2\mu\left(x_0-l_1\right)\right\}\frac{e^{-\mu l}F_2(\mu)G_2\left(\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\frac{\mu^2}{2}\right)}{G_1^2\left(\frac{\mu^2}{2}\right)-G_2^2\left(\frac{\mu^2}{2}\right)}$$

that it will never achieved a excursion in the corridor (l_2,l_1) with length equal or greater than d.

Remark 2: For a Brownian motion with negative drift and $x_0>l_1$, taking $\beta=0$ in (43) gives that with probability

$$1-\frac{e^{-\mu l}F_2(\mu)G_2\left(\frac{\mu^2}{2}\right)-F_1(\mu)G_1\left(\frac{\mu^2}{2}\right)}{G_1^2\left(\frac{\mu^2}{2}\right)-G_2^2\left(\frac{\mu^2}{2}\right)}$$

that it will never achieved a excursion in the corridor (l_2, l_1) with length equal or greater than d.

Remark 3: For a standard Brownian motion, we can carry out a similar calculation to (41), from which we can easily that the result that

$$P\left(\tau^W < \infty\right) = 1.$$

We will now extent Corollary 1.2 to obtain the joint distribution of W_t and τ^W at an exponential time. This is an application of (43), (44) and Girsanov's theorem.

Theorem 2 *For a standard Brownian Motion W_t with $W_0 = x_0$ and τ^W defined as in (6) with $S_t = W_t$, we have the following result:*

For the case $x_0 \geq l_1$ and $x \geq l_1$,

$$\begin{aligned} & P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) & (47) \\ = \; & \gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\} \frac{G_2(\gamma)\left(u_1\left(x - l_1\right) - u_2(x - l_2)\right) - G_1(\gamma)\left(u_1\left(x - l_2\right) - u_2(x - l_1)\right)}{G_1(\gamma)^2 - G_2(\gamma)^2}, \end{aligned}$$

for the case $x_0 \geq l_1$ and $l_2 < x < l_1$,

$$\begin{aligned} & P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) & (48) \\ = \; & \gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\} \frac{G_2(\gamma)\left(u_3\left(x - l_1\right) - u_2(x - l_2)\right) - G_1(\gamma)\left(u_1\left(x - l_2\right) - u_4(x - l_1)\right)}{G_1(\gamma)^2 - G_2(\gamma)^2}, \end{aligned}$$

for the case $x_0 \geq l_1$ and $x \leq l_2$,

$$\begin{aligned} & P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) & (49) \\ = \; & \gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\} \frac{G_2(\gamma)\left(u_3\left(x - l_1\right) - u_4(x - l_2)\right) - G_1(\gamma)\left(u_3\left(x - l_2\right) - u_4(x - l_1)\right)}{G_1(\gamma)^2 - G_2(\gamma)^2}, \end{aligned}$$

for the case $x \leq l_2$ and $x \geq l_1$,

$$\begin{aligned} & P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) & (50) \\ = \; & \gamma \exp\left\{-\sqrt{2\gamma}\,(l_2 - x_0)\right\} \frac{G_2(\gamma)\left(u_1\left(x - l_2\right) - u_2(x - l_1)\right) - G_1(\gamma)\left(u_1\left(x - l_1\right) - u_2(x - l_2)\right)}{G_1(\gamma)^2 - G_2(\gamma)^2}, \end{aligned}$$

for the case $x \leq l_2$ and $l_2 < x < l_1$,

$$\begin{aligned} & P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) & (51) \\ = \; & \gamma \exp\left\{-\sqrt{2\gamma}\,(l_2 - x_0)\right\} \frac{G_2(\gamma)\left(u_1\left(x - l_2\right) - u_4(x - l_1)\right) - G_1(\gamma)\left(u_3\left(x - l_1\right) - u_2(x - l_2)\right)}{G_1(\gamma)^2 - G_2(\gamma)^2}, \end{aligned}$$

for the case $x \leq l_2$ and $x \leq l_2$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) \tag{52}$$
$$= \gamma \exp\left\{-\sqrt{2\gamma}\,(l_2 - x_0)\right\} \frac{G_2(\gamma)\,(u_3\,(x-l_2) - u_4(x-l_1)) - G_1(\gamma)\,(u_3\,(x-l_1) - u_4(x-l_2))}{G_1(\gamma)^2 - G_2(\gamma)^2},$$

where $\tilde{T}$ is a random variable with an exponential distribution of parameter γ that is independent of W_t and

$$u_1(x) = e^{-\sqrt{2\gamma}x} a_1\left(-\sqrt{2\gamma}\right), \tag{53}$$

$$u_2(x) = e^{-\sqrt{2\gamma}x} a_2\left(-\sqrt{2\gamma}\right), \tag{54}$$

$$u_3(x) = 2\sum_{k=-\infty}^{\infty}\left[\exp\left\{-\sqrt{2\gamma}\,((2k+1)l + x)\right\}\mathcal{N}\left(\frac{x+(2k+1)l-\sqrt{2\gamma}d}{\sqrt{d}}\right)\right. \tag{55}$$
$$\left. + \exp\left\{\sqrt{2\gamma}\,((2k+1)l + x)\right\}\mathcal{N}\left(\frac{x+(2k+1)l+\sqrt{2\gamma}d}{\sqrt{d}}\right)\right] - e^{\sqrt{2\gamma}x} a_1\left(\sqrt{2\gamma}\right),$$

$$u_4(x) = 2\sum_{k=-\infty}^{\infty}\left[\exp\left\{-\sqrt{2\gamma}\,(2kl + x)\right\}\mathcal{N}\left(\frac{x+2kl-\sqrt{2\gamma}d}{\sqrt{d}}\right)\right. \tag{56}$$
$$\left. + \exp\left\{\sqrt{2\gamma}\,(2kl + x)\right\}\mathcal{N}\left(\frac{x+2kl+\sqrt{2\gamma}d}{\sqrt{d}}\right)\right] - e^{\sqrt{2\gamma}x} a_2\left(\sqrt{2\gamma}\right),$$

$$a_1(x) = 2\sum_{k=-\infty}^{\infty}\exp\left\{x(2k+1)l\right\}\mathcal{N}\left(\frac{(2k+1)l + xd}{\sqrt{d}}\right) + \frac{e^{-\gamma d}}{x}\sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty}\exp\left\{-\frac{(2k+1)^2 l^2}{2d}\right\}, \tag{57}$$

$$a_2(x) = 2\sum_{k=-\infty}^{\infty}\exp\left\{2xkl\right\}\mathcal{N}\left(\frac{2kl + xd}{\sqrt{d}}\right) + \frac{e^{-\gamma d}}{x}\sqrt{\frac{2}{\pi d}}\sum_{k=-\infty}^{\infty}\exp\left\{-\frac{2k^2 l^2}{d}\right\}. \tag{58}$$

Proof: see appendix. □

5 Pricing Parisian Corridor Options

We want to price a Parisian corridor in-call option with the current price of its underlying asset to be x, $x > L_1$, the owner of which will obtain the right to exercise it when the length of the excursion inside the corridor formed by the barriers L_1 and L_2 ($L_1 > L_2$) reaches d before T. Its price formula is given by

$$PC_{in-call} = e^{-rT} E_Q\left((S_T - K)^+ \mathbf{1}_{\{\tau^S < T\}}\right),$$

where S is the underlying stock price, Q denotes the risk neutral measure, τ^S is defined with the respect to L_1 and L_2. We assume S is a geometric Brownian motion:

$$\mathrm{d}S_t = rS_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad S_0 = x,$$

where $x > L_1$, r is the risk free rate, W_t with $W_0 = 0$ is a standard Brownian motion under Q. Set

$$m = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right), \quad b = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right), \quad B_t = mt + W_t,$$

$$l_1 = \frac{1}{\sigma}\ln\left(\frac{L_1}{x}\right), \quad l_2 = \frac{1}{\sigma}\ln\left(\frac{L_2}{x}\right).$$

We have

$$S_t = x\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} = x\exp\{\sigma(mt + W_t)\} = xe^{\sigma B_t}.$$

By applying Girsanov's Theorem, we have

$$PC_{in-call} = e^{-\left(r+\frac{1}{2}m^2\right)T} E_P\left[\left(xe^{\sigma B_T} - K\right)^+ e^{mB_T} \mathbf{1}_{\{\tau^B < T\}}\right],$$

where P is a new measure, under which B_t is a standard Brownian motion with $B_0 = 0$, and τ^B is the stopping time defined with respect to barrier l_1, l_2. And we define

$$PC^*_{in-call} = e^{\left(r+\frac{1}{2}m^2\right)T} PC_{in-call}.$$

We are going to show that we can obtain the Laplace transform of $PC^*_{in-call}$ w.r.t T, denoted by $\mathscr{L}_T$.

Firstly, assuming $\tilde{T}$ is a random variable with an exponential distribution of parameter γ that is independent of W_t, we have

$$\begin{aligned}
& E_P\left[\left(xe^{\sigma B_{\tilde{T}}} - K\right)^+ e^{mB_{\tilde{T}}} \mathbf{1}_{\{\tau^B < \tilde{T}\}}\right] \\
= & \int_b^\infty \left(xe^{\sigma y} - K\right) e^{my} P\left(B_{\tilde{T}} \in \mathrm{d}y, \tau^B < \tilde{T}\right) \\
= & \int_0^\infty \gamma e^{-\gamma T} \int_b^\infty \left(xe^{\sigma y} - K\right) e^{my} P\left(B_T \in \mathrm{d}y, \tau^B < T\right) \mathrm{d}T \\
= & \gamma \int_0^\infty e^{-\gamma T} E_P\left[\left(xe^{\sigma B_T} - K\right)^+ e^{mB_T} \mathbf{1}_{\{\tau^B < T\}}\right] \mathrm{d}T \\
= & \gamma \mathscr{L}_T
\end{aligned}$$

Hence we have

$$\mathscr{L}_T = \frac{1}{\gamma}\int_b^\infty \left(xe^{\sigma y} - K\right) e^{my} P\left(B_{\tilde{T}} \in \mathrm{d}y, \tau^B < \tilde{T}\right).$$

By using the results in Theorem 3, this Laplace transform can be calculated explicitly.

When $b \geq l_1$, i.e. $K \geq L_1$, we have

$$\mathscr{L}_T = \frac{x}{\gamma} F_1(\sigma + m) - \frac{K}{\gamma} F_1(m), \tag{59}$$

when $l_2 < b < l_1$, i.e. $L_2 < K < L_1$, we have

$$\mathscr{L}_T = \frac{x}{\gamma} F_2(\sigma + m) - \frac{K}{\gamma} F_2(m), \tag{60}$$

when $b \leq l_2$, i.e. $K \leq L_2$, we have

$$\mathscr{L}_T = \frac{x}{\gamma} F_3(\sigma + m) - \frac{K}{\gamma} F_3(m), \tag{61}$$

where

$$\begin{aligned} F_1(x) \quad = \quad & \frac{\gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\}}{G_1(\gamma)^2 - G_2(\gamma)^2} \left[G_2(\gamma) \left\{q_1(x, b, l_1) - q_2(x, b, l_2)\right\}\right. \\ & \left. - G_1(\gamma) \left\{q_1(x, b, l_2) - q_2(x, b, l_1)\right\}\right], \end{aligned} \tag{62}$$

$$\begin{aligned} F_2(x) \quad = \quad & \frac{\gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\}}{G_1(\gamma)^2 - G_2(\gamma)^2} \left[G_2(\gamma) \left\{q_1(x, l_1, l_1) - q_2(x, b, l_2) + q_3(x, l_1, l_1) - q_3(x, b, l_1)\right\}\right. \\ & \left. - G_1(\gamma) \left\{q_1(x, b, l_2) - q_2(x, l_1, l_1) - q_4(x, l_1, l_1) + q_4(x, b, l_1)\right\}\right], \end{aligned} \tag{63}$$

$$\begin{aligned} F_3(x) \quad = \quad & \frac{\gamma \exp\left\{-\sqrt{2\gamma}\,(x_0 - l_1)\right\}}{G_1(\gamma)^2 - G_2(\gamma)^2} \left[G_2(\gamma) \left\{q_1(x, l_1, l_1) - q_2(x, l_2, l_2) + q_3(x, l_1, l_1) - q_3(x, b, l_1)\right.\right. \\ & \left. + q_4(x, b, l_2) - q_4(x, l_2, l_2)\right\} - G_1(\gamma) \left\{q_1(x, l_2, l_2) - q_2(x, l_1, l_1) + q_3(x, l_2, l_2)\right. \\ & \left.\left. - q_3(x, b, l_2) - q_4(x, l_1, l_1) + q_4(x, b, l_1)\right\}\right], \end{aligned} \tag{64}$$

$$q_1(x, y, z) = \frac{e^{\left(x - \sqrt{2\gamma}\right) y + \sqrt{2\gamma} z}}{\sqrt{2\gamma} - x} a_1\left(-\sqrt{2\gamma}\right), \tag{65}$$

$$q_2(x, y, z) = \frac{e^{\left(x - \sqrt{2\gamma}\right) y + \sqrt{2\gamma} z}}{\sqrt{2\gamma} - x} a_2\left(-\sqrt{2\gamma}\right), \tag{66}$$

$$
\begin{aligned}
q_3(x,y,z) \quad = \quad & 2\sum_{k=-\infty}^{\infty} \frac{\exp\left\{-\sqrt{2\gamma}((2k+1)l-z)\right\}}{x-\sqrt{2\gamma}} \left[e^{\left(x-\sqrt{2\gamma}\right)y} \mathscr{N}\left(\frac{y-z+(2k+1)l-\sqrt{2\gamma}d}{\sqrt{d}} \right) \right. \\
& - \exp\left\{ \left(x-\sqrt{2\gamma}\right)\left(z-(2k+1)l+\sqrt{2\gamma}d\right) + \frac{\left(x-\sqrt{2\gamma}\right)^2 d}{2} \right\} \\
& \left. \mathscr{N}\left(\frac{y-z+(2k+1)l-xd}{\sqrt{d}} \right) \right] \\
& +2\sum_{k=-\infty}^{\infty} \frac{\exp\left\{\sqrt{2\gamma}((2k+1)l-z)\right\}}{x+\sqrt{2\gamma}} \left[e^{\left(x+\sqrt{2\gamma}\right)y} \mathscr{N}\left(\frac{y-z+(2k+1)l+\sqrt{2\gamma}d}{\sqrt{d}} \right) \right. \\
& - \exp\left\{ \left(x+\sqrt{2\gamma}\right)\left(z-(2k+1)l-\sqrt{2\gamma}d\right) + \frac{\left(x+\sqrt{2\gamma}\right)^2 d}{2} \right\} \\
& \left. \mathscr{N}\left(\frac{y-z+(2k+1)l-xd}{\sqrt{d}} \right) \right] - \frac{e^{\left(\sqrt{2\gamma}+x\right)y-\sqrt{2\gamma}z}}{x+\sqrt{2\gamma}} a_1\left(\sqrt{2\gamma}\right), \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
q_4(x,y,z) \quad = \quad & 2\sum_{k=-\infty}^{\infty} \frac{\exp\left\{-\sqrt{2\gamma}(2kl-z)\right\}}{x-\sqrt{2\gamma}} \left[e^{\left(x-\sqrt{2\gamma}\right)y} \mathscr{N}\left(\frac{y-z+2kl-\sqrt{2\gamma}d}{\sqrt{d}} \right) \right. \\
& \left. - \exp\left\{ \left(x-\sqrt{2\gamma}\right)\left(z-2kl+\sqrt{2\gamma}d\right) + \frac{\left(x-\sqrt{2\gamma}\right)^2 d}{2} \right\} \mathscr{N}\left(\frac{y-z+2kl-xd}{\sqrt{d}} \right) \right] \\
& +2\sum_{k=-\infty}^{\infty} \frac{\exp\left\{\sqrt{2\gamma}(2kl-z)\right\}}{x+\sqrt{2\gamma}} \left[e^{\left(x+\sqrt{2\gamma}\right)y} \mathscr{N}\left(\frac{y-z+(2k+1)l+\sqrt{2\gamma}d}{\sqrt{d}} \right) \right. \\
& \left. - \exp\left\{ \left(x+\sqrt{2\gamma}\right)\left(z-2kl-\sqrt{2\gamma}d\right) + \frac{\left(x+\sqrt{2\gamma}\right)^2 d}{2} \right\} \mathscr{N}\left(\frac{y-z+2kl-xd}{\sqrt{d}} \right) \right] \\
& - \frac{e^{\left(\sqrt{2\gamma}+x\right)y-\sqrt{2\gamma}z}}{x+\sqrt{2\gamma}} a_2\left(\sqrt{2\gamma}\right). \qquad (68)
\end{aligned}
$$

Remark: The price can be calculated by numerical inversion of the Laplace transform.

So far, we have shown how to obtain the Laplace transform of

$$PC^{*}_{in-call} = e^{\left(r+\frac{1}{2}m^2\right)T} P_{in-call}.$$

For

$$PC_{out-call} = e^{-rT} E_Q\left((S_T-K)^{+} \mathbf{1}_{\{\tau^S > T\}} \right),$$

we can get the result from the relationship that

$$PC_{out-call} = e^{-rT} E_Q\left\{ (S_T-K)^{+} \right\} - PC_{in-call}.$$

6 Appendix: Proof of Theorem 2

Let T be the final time. According to the definition of $\Psi(x)$, we have

$$\Psi(x) = 2\sqrt{\pi}x\mathscr{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2} = \sqrt{\pi}x - \sqrt{\pi}x\mathrm{Erfc}\,(x) + e^{-x^2}.$$

It is not difficult to show that

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = E\left(\int_0^\infty \beta e^{-\beta T}\mathbf{1}_{\left\{\tau^{W^\mu}<T\right\}}\mathrm{d}T\right).$$

By Girsanov's theorem, this is equal to

$$\int_0^\infty \beta e^{-(\beta+\frac{1}{2}\mu^2)T-\mu x_0}E\left(e^{\mu W_T}\mathbf{1}_{\{\tau^W<T\}}\right)\mathrm{d}T.$$

Setting $\gamma = \beta + \frac{1}{2}\mu^2$ gives

$$\begin{aligned} E\left(e^{-\beta\tau^{W^\mu}}\right) &= \int_0^\infty (\gamma - \frac{1}{2}\mu^2)e^{-\gamma T-\mu x_0}E\left(e^{\mu W_T}\mathbf{1}_{\{\tau^W<T\}}\right)\mathrm{d}T \\ &= \frac{\gamma-\frac{1}{2}\mu^2}{\gamma}e^{-\mu x_0}E\left(e^{\mu W_{\tilde{T}}}\mathbf{1}_{\left\{\tau^W<\tilde{T}\right\}}\right), \end{aligned}$$

where $\tilde{T}$ is a random variable with an exponential distribution of parameter γ that is independent of W_t. Therefore we have

$$E\left(e^{\mu W_{\tilde{T}}}\mathbf{1}_{\{\tau^W<\tilde{T}\}}\right) = \frac{\gamma e^{\mu x_0}}{\gamma-\frac{1}{2}\mu^2}E\left(e^{-\beta\tau^{W^\mu}}\right)$$

In order to inverse the above moment generating function, we just need to inverse the following expressions:

$$\frac{\mu}{\gamma-\frac{\mu^2}{2}} = \int_0^\infty e^{\mu x}e^{-\sqrt{2\gamma}x}\mathrm{d}x - \int_{-\infty}^0 e^{\mu x}e^{\sqrt{2\gamma}x}\mathrm{d}x,$$

$$\frac{e^{\mu l_i}}{\gamma-\frac{\mu^2}{2}} = \int_{l_i}^\infty e^{\mu x}\frac{1}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}(x-l_i)}\mathrm{d}x + \int_{-\infty}^{l_i} e^{\mu x}\frac{1}{\sqrt{2\gamma}}e^{\sqrt{2\gamma}(x-l_i)}\mathrm{d}x,$$

$$e^{-n\mu l}e^{\frac{\mu^2 d}{2}+\mu l_i}\mathscr{N}\left(\frac{nl}{\sqrt{d}}-\mu\sqrt{d}\right) = \int_{-\infty}^{l_i} e^{\mu x}\frac{1}{\sqrt{2\pi d}}\exp\left\{-\frac{(x+nl-l_i)^2}{2d}\right\}.$$

Therefor the inversion of $\frac{\mu e^{-n\mu l}e^{\frac{\mu^2 d}{2}+\mu l_i}\mathscr{N}\left(\frac{nl}{\sqrt{d}}-\mu\sqrt{d}\right)}{\gamma-\frac{\mu^2}{2}}$ is as follow:

for $x \geq l_i$,

$$\int_{-\infty}^{l_i}\frac{1}{\sqrt{2\pi d}}\exp\left\{-\frac{(y+nl-l_i)^2}{2d}\right\}e^{-\sqrt{2\gamma}(x-y)}\mathrm{d}y = \exp\left\{\gamma d - \sqrt{2\gamma}\,(nl-l_i+x)\right\}\mathscr{N}\left(\frac{nl-\sqrt{2\gamma}d}{\sqrt{d}}\right);$$

for $x < l_i$,

$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi d}} \exp\left\{-\frac{(y+nl-l_i)^2}{2d}\right\} e^{-\sqrt{2\gamma}(x-y)} \mathrm{d}y$$
$$-\int_{x}^{l_i} \frac{1}{\sqrt{2\pi d}} \exp\left\{-\frac{(y+nl-l_i)^2}{2d}\right\} e^{\sqrt{2\gamma}(x-y)} \mathrm{d}y$$
$$= \exp\left\{\gamma d - \sqrt{2\gamma}\,(nl-l_i+x)\right\} \mathscr{N}\left(\frac{x+nl-l_i-\sqrt{2\gamma}d}{\sqrt{d}}\right)$$
$$-\exp\left\{\gamma d + \sqrt{2\gamma}\,(nl-l_i+x)\right\} \left\{\mathscr{N}\left(\frac{nl+\sqrt{2\gamma}d}{\sqrt{d}}\right) - \mathscr{N}\left(\frac{x+nl-l_i+\sqrt{2\gamma}d}{\sqrt{d}}\right)\right\}$$

Consequently, we can get Theorem 2.

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