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Brownian Models of Open Queueing Networks with Homogeneous Customer Populations — Source link

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BROWNIAN MODELS OF OPEN QUEUEING NETWORKS WITH HOMOGENEOUS CUSTOMER POPULATIONS

BY

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AND

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Contents

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Bold letters **B**, **I**, **F**, **M** should be typeset as script letters.
The bold letter **R** should be typeset as a special roman R.

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Abstract

We consider a family of multidimensional diffusion processes that arise as heavy traffic approximations for open queueing networks. More precisely, the diffusion processes considered here arise as approximate models of open queueing networks with homogeneous customer populations, which means that customers occupying any given node or station of the network are essentially indistinguishable from one another. The classical queueing network model of J. R. Jackson fits this description, as do other more general types of systems, but multiclass network models do not.

The objectives of this paper are (a) to explain in concrete terms how one approximates a conventional queueing model or a real physical system by a corresponding Brownian model, and (b) to state and prove some new results regarding stationary distributions of such Brownian models. The part of the paper aimed at objective (a) is largely a recapitulation of previous work on weak convergence theorems, with the emphasis placed on modeling intuition. With respect to objective (b), several important foundational issues are resolved here and under certain conditions we are able to express the stationary distribution and related performance measures in explicit formulas. More specifically, it is shown that the stationary distribution of the Brownian model has a separable (product form) density if and only if its data satisfy a certain condition, in which case the stationary density is exponential, and all relevant performance measures can be written out in explicit formulas.

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1. Introduction.

This paper is devoted to the study of some multidimensional diffusion processes that play an important role in queueing theory. The diffusions in question are of a type most often referred to as *reflected Brownian motions* [4, 12, 13, 14, 21, 24, 25, 26, 30]. It has been suggested in [10] that *regulated* Brownian motion is actually a better name, and here we shall use the neutral acronym RBM. We are concerned here with a special class of RBM's that arise as diffusion models of open queueing networks. The queueing theoretic interpretation of these processes will be reviewed, and then a systematic study of their stationary distributions will be undertaken.

In the title of the paper we have emphasized the fact that the processes studied here correspond to queueing networks with *homogeneous customer populations*, which means that customers occupying any given node or station of the network are essentially indistinguishable from one another. (The exact meaning of this phrase will become apparent shortly.) The classical model of J. R. Jackson [16, 17] fits this description, as do other more general systems to be described later. In contrast, Baskett et. al. [2], Kelly [19] and others have considered networks in which several distinct customer classes are served at one or more stations. Such a *multiclass queueing network* gives rise to a substantially more complicated type of Brownian system model, the study of which is only just beginning [11, 22]. When we refer to queueing network models hereafter, the restriction to homogeneous customer populations is implicit.

Our objectives here are (a) to explain in concrete terms how one approximates a conventional queueing model or a real physical system by a corresponding RBM, and (b) to state and prove some new results regarding stationary distributions of RBM's. The part of the paper aimed at objective (a) is largely a recapitulation of Reiman's [23] heavy traffic limit theorem, which has also been discussed at some length in the survey papers of Lemoine [20], Flores [9], and Coffman-Reiman [6]. Limit theorems of this type provide a rigorous justification for the use of Brownian system models under heavy traffic conditions (the meaning of this phrase will be explained later), but their intuitive content tends to be obscured by the technical apparatus employed. Thus, in the current discussion of approximation procedures, we shall emphasize modeling intuition and operational content rather than mathematical rigor. With respect to (b), it must be said that no general formula has yet been found for the stationary distribution of an RBM, nor do we give a general procedure for its computation. However, the issues of existence and uniqueness are resolved here, and determination of the stationary distribution is reduced to solution of a concrete analytical problem. Moreover, building on results proved earlier in [14, 30], we obtain necessary and sufficient conditions on the data of the RBM for its stationary distribution to have a separable density function. Under these conditions, the stationary density is exponential and all standard performance measures can be written out in explicit formulas.

By putting into one place all the information described in the previous paragraph, we hope to give a clear picture of the useful quantitative results that are already available for analysis of queueing networks by means of Brownian system models. In addition, we shall identify the important open problems whose solution would yield a truly comprehensive theory of performance modeling via RBM. The remainder of this paper consists of sections dealing with conventional system models, the Brownian system models that correspond to them, stationary distributions for those Brownian models, a summary of performance analysis procedures based on results for Brownian system models, and some miscellaneous concluding remarks, including a list of important open problems.

We conclude this introduction with an account of some notational and terminological conventions used throughout the paper. The letter e will often be used to denote a vector of ones, and the dimension of this vector will always be clear from context. Vectors are understood to be column vectors unless something is said to the contrary, and transposes are denoted by primes. Vector (in)equalities are to be interpreted componentwise and a vector-valued function is non-decreasing (or non-increasing) if and only if each component has this property. The inner product between two k-dimensional vectors v and w will be denoted by $v \cdot w$ $(= v'w$ for column vectors). For a vector v, diag(v) will denote the diagonal matrix whose diagonal entries are given by the components of v, and for a square matrix A, diag(A) will denote the diagonal matrix with the same diagonal entries as A. We shall use $D^K[0, \infty)$ to denote the space of functions $x : [0, \infty) \to \mathbf{R}^K$ that are right continuous on $[0, \infty)$ and have finite left limits on $(0, \infty)$. Consider this space endowed with the Skorohod topology [3, 8]. Weak convergence of probability measures on $D^K[0, \infty)$ is defined as in Billingsley [3, Chapter 1] or Ethier-Kurtz [8, Chapter 3]: a sequence of probability measures $\{P_n\}$ converges weakly to a probability measure P on $D^K[0, \infty)$ if and only if for each real-valued bounded continuous function f on $D^K[0, \infty)$, we have $\lim_{n \to \infty} \int f \, dP_n = \int f \, dP$. Weak convergence of stochastic processes whose sample paths lie in $D^K[0, \infty)$ is equivalent to weak convergence of the probability measures they induce on $D^K[0, \infty)$.

The following enumeration scheme is used in this paper. Within each section, equations, theorems, lemmas, propositions and corollaries are labelled according to a single numbering system. The designation (n) refers to the item labelled (n) in the current section, whereas in referring to other sections the notation (m.n) is used to indicate the entity with label (n) in section m.

2. An Open Network Model of Conventional Type.

We consider a network of K single-server stations in which customers occupying any given station at any given time are essentially indistinguishable from one another. The term *class k customers* is used to mean customers occupying station k, either waiting or being served. At each station customers are served one at a time, and the server remains busy so long as there are customers of the relevant class to be worked on. Upon completing service a customer either moves on to a different station or else departs the system. No particular assumption is made about the order in which customers are served, but it is easiest to think in terms of a first-in-first-out (FIFO) discipline at each station. Our concern here is with open networks, in which customers are generated by exogenous arrival processes and remain in the system for just a finite amount of time. One focus of study is the K-dimensional *queue length process* $Q = \{Q(t), t \geqslant 0\}$ whose k^{th} component $Q_k(t)$ gives the number of class k customers present at time t. We are also concerned with the K-dimensional *busy time process* $B = \{B(t), t \geqslant 0\}$ whose k^{th} component $B_k(t)$ gives the total amount of time that server k is busy within the interval $[0, t]$. It is assumed for convenience that $Q(0) = 0$, but the modifications required to accommodate an arbitrary initial state will be obvious.

Our first task is to give a precise mathematical description of the processes Q and B. The usual procedure in constructing such processes is to start with a probability space on which are defined mutually independent sequences of interarrival times, service times and routing indicators [23]. In the construction to follow, however, higher level quantities are taken as primitive model elements, which makes for a more general and more economical treatment. In particular, each station of the network is characterized by a vector flow process, or input-output process, whose distribution depends on both service time and routing characteristics. The other primitive element in the construction is a vector arrival process. The probabilistic assumptions imposed here are very weak by conventional standards: it is assumed that each of the vector flow processes underlying the model, including the arrival process, satisfies a functional central limit theorem (FCLT), and that these processes are mutually independent. It is the mean vectors and covariance matrices appearing in the FCLT's that we view as the fundamental system parameters. Later in this section, just after equation (19), our setup will be related to the more restrictive requirements of standard queueing network models.

For a precise mathematical definition of the queue length and busy time processes, we take as primitive a family of K-dimensional *flow processes* $F^j = \{F^j(t), t \geqslant 0\}$ indexed by $j = 0, 1, \ldots, K$. The k^{th} component $(k = 1, \ldots, K)$ of the vector process $F^j(t)$ is denoted $F_k^j(t)$. We interpret $F_k^0(t)$ as the total number of class k customers who have arrived from outside the system up to time t. For $j, k = 1, \ldots, K$ we interpret $F_k^j(t)$ as the total flow out of class k (expressed as a number of customers) resulting from the first t units of busy time at station j, with negative values signifying inflow. Thus $F_j^j(t)$ is simply the number of services completed in the first t units of busy time at station j, and the other components of the vector $F^j(t)$ are non-positive integers whose absolute values tell how many of those class j services have resulted in immediate transitions to the various other classes. Formally, our assumptions regarding the primitive flow processes are the following. (At this point readers may wish to review the notational and terminological conventions listed at the end of section 1.)

(1) Each component of F^j is integer valued and right-continuous, and $F^j(0) = 0$ $(j = 0, 1, \ldots, K)$.

(2) Each component of the vector arrival process F^0 is non-decreasing.

(3) For each $j = 1, \ldots, K$ the process F^j_j is non-decreasing and has unit jumps, and all other components of F^j are non-increasing. Moreover, $e \cdot (F^j(t) - F^j(t-)) \geqslant 0$ for all $t > 0$ and all $j = 1, \ldots, K$.

(4) The vector flow processes $F^0, F^1, \ldots, F^K$ are mutually independent.

(5) For each $j = 0, 1, \ldots, K$ there exists a K-vector α^j and a $K \times K$ covariance matrix Γ^j such that

$$E[F^j(t)] \sim \alpha^j t \quad \text{and} \quad \mathrm{Cov}[F^j(t)] \sim \Gamma^j t \ \text{ as } t \to \infty.$$

(6) Define the centered flow processes $\xi^j(t) = F^j(t) - \alpha^j t$. For each $j = 0, 1, \ldots, K$ the scaled processes $n^{-1/2}\xi^j(nt)$, $t \geqslant 0$ $(n = 1, 2, \ldots)$, viewed as random elements of $D^K[0, \infty)$, converge weakly as $n \to \infty$ to a $(0, \Gamma^j)$ Brownian motion.

Assumptions (1)-(3) concern the path structure of our primitive flow processes, and each of these restrictions is natural in light of the interpretations given earlier. Assumptions (4)-(6), which concern the distributional properties of the flow processes, will be discussed further below. Hereafter α^j and Γ^j will be called the *mean vector* and *covariance matrix* respectively of F^j, although that terminology is somewhat imprecise. (The terms *asymptotic mean vector* and *asymptotic covariance matrix* would be more accurate.) Using language that is now standard in probability theory [3], we paraphrase (6) by saying that each of our vector flow processes satisfies a *functional central limit theorem.*

Thus far there is nothing in our assumptions to distinguish the system as an *open* queueing network. Even before the necessary additional assumptions are introduced, however, we can formally define the processes of interest here. Using only the path structure assumptions (1)-(3), one can show by induction that there exists a unique pair of K-dimensional processes Q and B that jointly satisfy

$$(7)\quad Q(t) = F^0(t) - \sum_{j=1}^{K} F^j(B_j(t)), \quad \text{for all } t \geqslant 0, \text{ and}$$

$$(8)\quad B_k(t) = \int_0^t 1_{\{Q_k(s)>0\}}\, ds \quad \text{for all } t \geqslant 0 \text{ and } k = 1, \ldots, K.$$

Relationships analogous to (7) and (8) have been used previously by Reiman [23] in constructing the queue length process for an open network, but the notational system used here is more efficient than Reiman's. A definitional system similar to that employed here is used in [11] to treat the more general case of multiclass networks. Readers should satisfy themselves that these formal definitions are indeed consistent with the informal interpretations advanced earlier, and that the process Q defined by (7) and (8) satisfies

$$(9)\quad Q(t) \geqslant 0, \ t \geqslant 0.$$

To compactify notation, we define a $K \times K$ *input-output matrix* $R = (R_{kj})$ via

$$(10)\quad R_{kj} = \alpha_k^j \ \text{for} \ j, k = 1, \ldots, K.$$

Observe that R_{kj} represents the average rate at which server j depletes the stock of class k customers when he is working, with negative depletion interpreted as augmentation. To connect with the notation that is customary in queueing theory, we also define K-vectors $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ via

$$(11)\quad \lambda_k = \alpha_k^0 \ \text{and} \ \mu_k = \alpha_k^k \ \text{for} \ k = 1, \ldots, K.$$

In light of (5), we call λ_k and μ_k the long run average *arrival rate* (from outside the system) and long run average *service rate*, respectively, for class k customers. We assume that $\mu_1, \ldots, \mu_K$ are strictly positive, and that units have been chosen so that the service rates and non-zero arrival rates are all moderate in size (of order 1). Next, let us define a $K \times K$ matrix $P = (P_{jk})$ via

(12) $$P_{jk} = \begin{cases} 0 & \text{if } j = k \\ -\alpha_k^j/\mu_j & \text{otherwise.} \end{cases}$$

From (5) and the interpretation of F^j given earlier, it follows that P_{jk} represents the long-run fraction of customers who, upon completing service at station j, go next to station k. Thus we call P the *transition frequency matrix*, or *switching matrix*, for our queueing network model. In general, however, customer routing is not assumed to be Markovian, as we shall discuss later. It follows from (3), (5) and (11) that P is non-negative and substochastic. As the final definitive properties of the *open* network model, the following two properties are assumed to hold.

(13) At least one component of λ is strictly positive.

(14) The substochastic matrix P is transient (spectral radius less than one).

The meaning of (13) is obvious, and (14) is interpreted to mean that the average number of services required to complete the processing of any given initial population is finite.

Before the assumptions of the open network model are discussed further, it will be useful to record some basic relationships. First, defining the $K \times K$ diagonal matrix

(15) $D = \text{diag}(\mu)$,

one can use (11) and (12) to write (10) in matrix form as

(16) $R = (I - P')D$.

From (14) and (16) it follows that R is invertible and

(17) $R^{-1} = D^{-1}(I+P+P^2+ \ldots)'$.

Thus there exists a unique K-vector $\rho = (\rho_k)$ satisfying the system of linear equations

(18) $\lambda = R\rho$,

and moreover the solution $\rho = R^{-1}\lambda$ is non-negative by (17). From (5), (10), (11) and (18) it follows that, if each server k works an average of ρ_k hours per hour of elapsed time over the long run, then the overall average rates of flow into and out of each class will just balance. Thus one might describe ρ as the vector of average activity rates required to maintain material balance. It is customary in queueing theory to call ρ_k the *traffic intensity* at station k. One naturally expects that long-run stability will be achievable if and only if $\rho_k < 1$ for all k, and we shall assume hereafter that this is the case. This assumption can be expressed in vector form as

(19) $\rho < e$.

Let us consider now the probabilistic assumptions (4)-(6) that underlie our system model. Assumption (4) says that operations at the various nodes of the network are independent of one another, except for such dependencies as may be induced by the unavailability of work. This assumption is implicit in all standard models of queueing networks. Assumptions (5) and (6) are also satisfied by the standard models, but developers of conventional theory have no particular motivation to comment on that fact. To put the discussion of (5) and (6) on a concrete footing, let us consider the *generalized Jackson network* treated by Reiman [23] and others. In this model (a) the components of the vector arrival process F^0 are independent renewal processes whose interarrival times have finite second moments, (b) service times at each station are IID random variables with finite second moments, and (c) customer routing is Markovian. (The last phrase means that a customer completing service at station j goes next to station k with probability P_{jk}, *independent of all previous history.*) A *Jackson network* is characterized by the additional assumptions that all arrival processes are Poisson and all service time distributions are exponential. For a generalized Jackson network, Reiman [23] shows how to calculate the mean vectors α^j and covariance matrices Γ^j that appear in (5) and (6). Those calculations, which involve only the switching probabilities and the first two moments of the interarrival and service time distributions, are repeated in section 4 of [11], where the notational system is similar to that used here. Using the functional central limit theorem for renewal processes [3], Reiman [23] also shows that (6) holds for a generalized Jackson network.

Although the assumptions of the generalized Jackson network are mild by conventional standards, there are several good reasons for focusing on the broader class of system models satisfying (1)-(6). First, as Iglehart and Whitt [15] emphasized in their pioneering work, it is (4)-(6) that express the essential distributional properties of the queueing system model for those interested in Brownian approximations. To minimize reader disorientation, most authors prefer to deduce properties like (5) and (6) from stronger probabilistic assumptions of a more familiar type, but any such approach has the ultimate effect of obscuring what is essential with irrelevant special structure. Second, if one is interested in Brownian system models, the essential data characterizing any station j is the pair (α^j, Γ^j) appearing in (5) and (6), and it is unnatural to express these data in terms of more elementary quantities, even when such an expression is possible. Moreover, if one starts with a real physical system and wishes to fit a Brownian model, the economical procedure is to determine (α^j, Γ^j) directly from gross input-output measurements, rather than computing these values from more detailed quantities. Finally, there exist interesting and important types of queueing networks that satisfy (1)-(6) but are not generalized Jackson networks, as that term was defined in the preceding paragraph. This point was emphasized by Reiman [23] in the final section of his paper. Consider, for example, a three-station network where all customers enter at station 1 and are served on a first-in-first-out basis, even numbered customers go to station 2 for a second service, odd numbered customers go to station 3 for a second service, and all customers depart after completing two services. If interarrival times and service times at the three stations are mutually independent IID sequences with finite second moments, then assumptions (1)-(6) are all satisfied, but this is obviously not a system with Markovian routing. The switching matrix P is given in this case by $P_{12} = P_{13} = 1/2$ and $P_{jk} = 0$ otherwise. The difference between this system and one with Markovian switching at the same average frequencies shows up in the covariance matrix Γ^1, as readers may verify for themselves. Other types of models that do not fit the description of a generalized Jackson network but may still satisfy (1)-(6) involve batch arrivals, dependencies between arrival streams, server breakdown and repair, and routing that is correlated with service time. See section 6 of Reiman [23] and section 10 of Harrison [11] for further discussion.

For future reference, it will be useful to express the K-dimensional queue length process $Q(t)$ in terms of the centered flow processes $\xi^0, \xi^1, \ldots, \xi^K$ introduced in (6). Toward that end, let us first define a K-dimensional *cumulative* *idleness process* $I = \{I(t), t \geqslant 0\}$ by setting

(20) $I_k(t) = t - B_k(t), \ t \geqslant 0, \quad k = 1, \ldots, K.$

From the fundamental identity (7) we have that

(21)
$$
\begin{aligned}
Q(t) &= F^0(t) - \sum_{j=1}^{K} F^j(B_j(t)) \\
&= [\xi^0(t) + \lambda t] - \sum_{j=1}^{K} [\xi^j(B_j(t)) + \alpha^j B_j(t)] \\
&= \xi^0(t) - \sum_{j=1}^{K} \xi^j(B_j(t)) + (\lambda - \sum_{j=1}^{K} \alpha^j)t + \sum_{j=1}^{K} \alpha^j (t - B_j(t)) \\
&= \xi^0(t) - \sum_{j=1}^{K} \xi^j(B_j(t)) + (\lambda - Re)t + RI(t).
\end{aligned}
$$

Also, in preparation for future developments, we conclude this section by stating a functional central limit theorem for the K-dimensional process

(22) $\xi(t) = \xi^0(t) - \sum_{j=1}^{K} \xi^j(\rho_j t), \quad t \geqslant 0.$

Of course $\xi^0, \xi^1, \ldots, \xi^K$ are independent by (4), and thus it follows from (5) that $\mathrm{Cov}[\xi(t)] \sim \Gamma t$ as $t \to \infty$, where

(23) $\Gamma = \Gamma^0 + \sum_{j=1}^{K} \rho_j \Gamma^j.$

Moreover, the following is immediate from (4) and (6).

(24) The processes $n^{-1/2}\xi(nt), \ t \geqslant 0, \ (n = 1, 2, \ldots)$, viewed as random elements of $D^K[0, \infty)$, converge weakly as $n \to \infty$ to a $(0, \Gamma)$ Brownian motion.

3. Rescaling Under Heavy Traffic Conditions.

For open networks, the condition required for a good Brownian approximation is that the traffic intensity be near unity for each station. This means that the total load imposed on each station by the exogenous arrival processes is approximately equal to the station's capacity, which one might describe as a condition of *balanced heavy loading.* In mathematical terms, this additional assumption can be expressed as follows:

(1) there exists a large integer n such that $\max\limits_{1 \leqslant k \leqslant K} n^{1/2} | 1 - \rho_k | \leqslant 1.$

For example, if ρ_k lies in the range between 0.9 and 1.0 for each k, one can choose $n = 100$ and satisfy (1). In all that follows, readers should think in terms of this canonical situation: the total load on each station is within 10% of capacity and $n = 100$.

In the literature of queueing theory it is customary to describe (1) as a *heavy traffic condition*, and we shall employ that terminology hereafter. Readers should recognize that the probabilistic assumptions imposed earlier in section 2 (in particular, that each of the primitive flow processes obeys a functional central limit theorem) have nothing to do with heavy traffic. Condition (1) requires that the traffic intensity be near unity for *every* station of the network, but the theory extends easily to networks where some but not all stations are heavily loaded. This will be explained in section 10.

Let us suppose hereafter that a large integer n satisfying (1) has been fixed. Using n as an essential system parameter, we now define scaled versions of various key processes. In essence, this scaling amounts to re-expressing time in multiples of n and re-expressing queue lengths as multiples of $n^{1/2}$. Consider, for example, the K-dimensional scaled queue length process

(2) $Z(t) = n^{-1/2} Q(nt), \ t \geqslant 0.$

If $n = 100$ and time is measured in hours in the original model, then $Z_k(t)$ tells us how many tens of type k customers are in the system after t hundred hours of operation. Similarly, define a scaled version of the K-dimensional cumulative idleness process I via

(3) $\quad Y(t) = n^{-1/2} I(nt).$

Combining (2) and (3) with (2.21), readers may verify that

(4) $\quad Z(t) = X(t) + RY(t),$

where

(5) $\quad X(t) = n^{-1/2}\xi^0(nt) - \sum_{j=1}^{K} n^{-1/2}\xi^j(n\beta_j(t)) + \theta t,$

(6) $\quad \beta_j(t) = n^{-1}B_j(nt)$, and

(7) $\quad \theta = n^{1/2}(\lambda - Re).$

Recall from (2.18) that $\lambda = R\rho$. Thus (7) can be rewritten

(8) $\quad \theta = n^{1/2}R(\rho - e) = R[n^{1/2}(\rho - e)],$

and it follows from this and (1) that *each component of θ is of moderate size.*

To form an approximating Brownian system model, one first uses the fact (this is not easy to prove) that $B_j(t)/t \to \rho_j$ almost surely as $t \to \infty$. Moreover, if n is large then the scaled busy time process $\beta_j(t)$ behaves approximately as the deterministic process $\rho_j t$. If one simply substitutes $\rho_j t$ for $\beta_j(t)$ in the definition (5) of $X(t)$, the right-hand side reduces to $n^{-1/2}\xi(nt) + \theta t$, and (2.24) says that $n^{-1/2}\xi(nt)$ is distributed approximately as a $(0, \Gamma)$ Brownian motion when n is large. Thus we are led to approximate $X(t)$ by a (θ, Γ) Brownian motion, and more generally, to approximate the scaled triple (X, Y, Z) by the Brownian system model (X, Y, Z) to be defined in the next section. (By using exactly the same notation in defining the Brownian system model, we intend to emphasize the interpretation of its constituents in terms of the original queuing network, hoping that this re-use of notation will avoid rather than create confusion.) For a rigorous justification of this approximation, one may invoke the limit theorem proved by Reiman [23], but readers should note one difference between Reiman's treatment and ours. Reiman approximates the scaled busy time process $\beta_j(t)$ by

t, which leads to an approximating Brownian system model (X, Y, Z) with a slightly different covariance matrix: the factors ρ_j that appear in our formula (2.23) for the covariance matrix Γ are replaced by ones in Reiman's formula. Because we consider only the case where ρ_j is near one for each station j, the difference between the two formulas is small (asymptotically negligible in heavy traffic), but we feel that our proposed approximation represents a refinement of that advanced by Reiman. See section 10 for further discussion that tends to confirm this view.

Before turning to a discussion of the Brownian system model, let us observe that, by (8), the anticipated stability condition $\rho < e$ can be equivalently expressed as

(9) $R^{-1}\theta < 0.$

The data for our Brownian system model will be the input-output matrix R, drift vector θ, and covariance matrix Γ defined in this section and its predecessor. Based on the meaning of those quantities in the queueing network setting, we expect the Brownian model to be asymptotically stable if and only if (9) holds.

4. The Brownian System Model.

Let θ, Γ and R be defined in terms of queueing system data as in sections 2 and 3. Recall that Γ is a non-degenerate covariance matrix, and that $R = (I-P)'D$, where P is a transient substochastic matrix with zeros on the diagonal and D is a diagonal matrix with strictly positive diagonal elements. (This special structure of R is critical for the development that follows.) Let $X = \{X(t), t \geqslant 0\}$ be a (θ, Γ) Brownian motion with $X(0) = 0$. Harrison and Reiman [12] have shown that there exists a unique pair of K-dimensional processes Y and Z that jointly satisfy

(1) $Z(t) = X(t) + RY(t) \geqslant 0$ for all $t \geqslant 0$,

(2) Y_k is continuous and non-decreasing with $Y_k(0) = 0$ for $k = 1, \ldots, K$, and

(3) $\int_0^\infty Z_k(t)\,dY_k(t) = 0$ for $k = 1, \ldots, K$.

We paraphrase (3) by saying that Y_k increases only when $Z_k = 0$. In the current context, the triple (X, Y, Z) will be referred to as a *Brownian system model.*

The process Z has state space $S = \mathbf{R}_+^K$ and it behaves like a (θ, Γ) Brownian motion on the interior of S. That is, the increments of Z are the same as those of X while Z is in the interior of S. When the boundary is hit, some component of Y increases, which causes an instantaneous displacement in accordance with the basic system equation (1). Specifically, if the boundary face $\{z_k = 0\}$ is hit, it is Y_k that increases, the direction of displacement is given by the k^{th} column of R, and the magnitude of the displacement is the minimal amount required to keep Z_k non-negative. As stated earlier, Reiman [23] has shown that, given the heavy traffic condition (3.1), the Brownian system model (X, Y, Z) is distributed approximately as the triple of scaled queueing processes denoted by the same symbols in section 3. (This statement is formally expressed by a weak convergence theorem involving a sequence of queueing systems.) Readers should check that the features of the Brownian system model described in this paragraph are all consistent with the queueing network interpretations of the processes X, Y and Z.

For ease of exposition, in the preceding discussion of the queueing network model and the approximating Brownian system model, we have confined ourselves to the case $Z(0) = X(0) = 0$. However, all of the results (including the heavy traffic limit theorems) can be generalised to allow an arbitrary initial state in $S = \mathbf{R}_+^K$. In particular, as described more precisely in section 5, by allowing the initial state of the Brownian system model to run over all states in S, one obtains a diffusion process Z satisfying (1)-(3). Following the terminology of Harrison-Reiman [12], we shall describe Z as an RBM with state space S, drift θ, covariance matrix Γ, and *reflection matrix* R. It is the stationary distribution of this diffusion that is studied in sections 6 through 9, where the major results are as follows. First, defining

(4) $\gamma = -R^{-1}\theta$,

we show that, starting from the origin, $Z(t)$ converges in distribution as $t \to \infty$ if and only if

(5) $\gamma > 0$,

which is precisely the stability condition (3.9) anticipated on the basis of queueing theoretic considerations. When (5) holds, the limit distribution is also the unique stationary distribution of Z, as one would expect. When (5) fails, at least one component of $Z(t)$ diverges almost surely as $t \to \infty$, and there is no stationary distribution.

In the case where (5) holds, it will be shown that any stationary distribution of Z must satisfy a certain *basic adjoint relation* (BAR). We do not know how to solve (BAR) in general but, motivated by the important role of product form stationary distributions in the classical theory of queueing networks [2, 16, 17, 19], we shall determine conditions under which the stationary distribution has a separable density function (in the usual Cartesian coordinates). Specifically, given (5), it will be shown that the stationary distribution has a separable density if and only if the covariance matrix Γ satisfies

(6) $$2\Gamma_{jk} = -(P_{kj}\Gamma_{kk} + P_{jk}\Gamma_{jj}) \text{ for } j \neq k.$$

In this case it will be shown that the stationary distribution of Z has density function

(7) $$p(z) = \prod_{k=1}^{K} \eta_k \exp(-\eta_k z_k), \quad z \in S,$$

where

(8) $$\eta_k = 2\mu_k\gamma_k/\Gamma_{kk} \quad (k = 1, \ldots, K).$$

(In the specification of our Brownian system model, the parameters $\mu_1, \ldots, \mu_K$ enter as the diagonal elements of the matrix R.) That is, if (5) and (6) hold, then the stationary distribution of Z has independent, exponentially distributed components with means $\Gamma_{11}/2\mu_1\gamma_1, \ldots, \Gamma_{KK}/2\mu_K\gamma_K$ respectively. The significance of these results for stationary performance analysis of open queueing networks will be summarized in section 10.

5. Additional Notation and Preliminaries.

In this section, we introduce some additional notation and preliminary results that will be used in sections 6-9.

For each $k \in \{1, \ldots, K\}$, let $F_k \equiv \{x \in \mathbf{R}_+^K : x_k = 0\}$ denote the k$^{\text{th}}$ face of $\mathbf{R}_+^K$. For each non-negative integer n and $G \subset \mathbf{R}^d$ $(d \geqslant 1)$, let $C^n(G)$ denote the set of real-valued functions that are n-times continuously differentiable in some domain containing G. Let $C_b^n(G)$ denote the set of functions in $C^n(G)$ that together with their partial derivatives up to and including those of order n are bounded on G. Let $C_c^n(G)$ denote those functions in $C_b^n(G)$ that have compact support in $\mathbf{R}^d$. If $n = 0$, the superscript n will be omitted. Define the differential operators

$$(1) \qquad L = \frac{1}{2} \sum_{j,k=1}^{K} \Gamma_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{K} \theta_j \frac{\partial}{\partial x_j}$$

and

$$(2) \qquad D_k = \mu_k \left(\frac{\partial}{\partial x_k} - \sum_{j \neq k} P_{kj} \frac{\partial}{\partial x_j} \right) \equiv v_k \cdot \nabla,$$

where v_k is the k$^{\text{th}}$ column of the matrix R.

To establish a rigorous foundation for the proofs to follow, we now give a more precise description of the diffusion process Z introduced in section 4. Suppose $X = \{X(t), t \geqslant 0\}$ is a continuous d-dimensional process defined on some measurable space $(\Omega, \mathbf{F})$ with an associated family of probability measures $\{P_x, x \in S\}$ such that $X(0, \omega) \in S \equiv \mathbf{R}_+^K$ for all $\omega \in \Omega$ and for each $x \in S$, X is a (θ, Γ) Brownian motion on $(\Omega, \mathbf{F}, P_x)$ satisfying $X(0) = x$ P_x-a.s. Let C_0 denote the space of continuous functions $x : [0, \infty) \to \mathbf{R}^K$ satisfying $x(0) \in S$, and let C_S denote the space of continuous functions $x : [0, \infty) \to S$. In [12], Harrison and Reiman proved the following path-to-path mapping result: there is a unique pair of functions $(\Phi, \Psi) : C_0 \to C_S \times C_S$ such that for each $x \in C_0$, $y = \Phi(x)$ and $z = \Psi(x)$ satisfy the following three conditions:

(3) $\quad z(t) = x(t) + Ry(t) \in S$ for all $t \geqslant 0$,

(4) y_k is continuous and non−decreasing with $y_k(0) = 0$, for $k = 1, \ldots, K$,

(5) $\int_0^\infty z_k(t)\, dy_k(t) = 0$ for $k = 1, \ldots, K$.

Thus, by setting $Y = \Phi(X)$ and $Z = \Psi(X)$, we obtain the unique pair of (pathwise) solutions to (4.1)-(4.3). It follows from the results proved in [12] that Y and Z are adapted to X, and Z (together with the family of probability measures $\{P_x, x \in S\}$) defines a continuous strong Markov process on S that has the Feller property, namely, for each $f \in C_b(S)$, $x \to E_x[f(Z(t))]$ is bounded and continuous on S. Here, as in the sequel, E_x denotes expectation with respect to P_x.

An alternative characterization of the pair (Y, Z) is afforded by a result of Reiman (see the appendix to [23]). In fact, since Z is defined from X and Y by (4.1), it suffices to characterize Y. For this, let $\mathbf{I}$ denote the set of all continuous, non-decreasing functions $y : [0, \infty) \to S$ satisfying $y(0) = 0$. Now, for each $\omega \in \Omega$, $Y(\,\cdot\,, \omega)$ is the least element of $\mathbf{I}$ such that $X(t, \omega) + RY(t, \omega) \in S$ for all $t \geqslant 0$, i.e., $Y(\,\cdot\,, \omega)$ is minimal in the sense that for any function $V(\,\cdot\,, \omega) \in \mathbf{I}$ satisfying $X(t, \omega) + RV(t, \omega) \in S$ for all $t \geqslant 0$, we have $Y(t, \omega) \leqslant V(t, \omega)$ for all $t \geqslant 0$. This alternative characterization of the pair (Y, Z) will be used in sections 6 and 8.

In the following, the σ−field of Borel subsets of the state space $S \equiv \mathbf{R}_+^K$ will be denoted by $\mathbf{B}_S$.

6. Existence of Stationary Distributions.

Definition. *A stationary distribution for Z is a probability measure π on $(S, \mathbf{B}_S)$ such that for each bounded Borel measurable function f on S,*

(1) $$\int_S E_x[f(Z(t))]\, d\pi(x) = \int_S f(x)\, d\pi(x) \quad \text{for all } t \geqslant 0.$$

(2) **Theorem.** *Z has a stationary distribution if and only if condition (4.5) holds.*

To simplify the proof of this theorem, we first perform a linear transformation of the state space. Let $Z^* = R^{-1}Z$, $X^* = R^{-1}X$ and $S^* = \{R^{-1}z : z \in S\}$. Then (4.1) is equivalent to

(3) $$Z^*(t) = X^*(t) + Y(t) \in S^* \text{ for all } t \geqslant 0.$$

Here $S^* \subset S \equiv \mathbf{R}_+^K$ since R^{-1} is given by (2.17) and the matrices P and D^{-1} have non-negative entries. Since Z and Z^* are related by an invertible linear transformation, Z has a stationary distribution if and only if Z^* has one. Preliminary to the proof of Theorem (2), we establish some lemmas on steady-state limits for Z^*. Here we use P_x^* to denote the probability measure associated with X^* starting from $x \in S^*$ (or equivalently, X starting from $Rx \in S$). Note that $P_0^* = P_0$.

The proof of the following stochastic monotonicity result uses time-reversal and the alternative characterization of the solution of (4.1)-(4.3) given in section 5. For this, recall the definition of $\mathbf{I}$ given in section 5.

(4) **Lemma.** *For each $z^* \in S^*$, $P_0^*(Z^*(t) \leqslant z^*)$ decreases monotonically as $t \to \infty$ and*

(5) $$F^*(z^*) \equiv \lim_{t\to\infty}\downarrow P_0^*(Z^*(t) \leqslant z^*)$$
$$= P_0^*\{\omega : \exists\, V(\cdot\,,\omega) \in \mathbf{I},\ z^* - X^*(t,\omega) - V(t,\omega) \in S^*\ \forall t \geqslant 0\}.$$

Proof. Fix $t \in [0, \infty)$ and define

(6) $$\tilde{X}(s) = \begin{cases} X(t) - X(t-s) & \text{for } 0 \leqslant s \leqslant t, \\ X(s) - X(0) & \text{for } s \geqslant t. \end{cases}$$

Then, under $P_0 = P_0^*$, $\tilde{X}$ is a (θ, Γ) Brownian motion starting from the origin. Let $(\tilde{Y}, \tilde{Z}) = (\Phi(\tilde{X}), \Psi(\tilde{X}))$, the solution of (4.1)-(4.3) with $\tilde{X}$ in place of X. Then under P_0, $\tilde{Z}$ is equivalent in law to Z and so under P_0^*, $\tilde{Z}^* \equiv R^{-1}\tilde{Z}$ is equivalent in law to Z^*. Thus it suffices to prove the lemma with $\tilde{Z}^*$ in place of Z^*. For this, let $\tilde{X}^* = R^{-1}\tilde{X}$. Then by the alternative pathwise characterization of $\tilde{Y}$ (see section 5), for each $\omega \in \Omega$, $\tilde{Y}(\cdot\,,\omega)$ is the least element of $\mathbf{I}$ such that

(7) $\tilde{X}'(s,\omega) + \tilde{Y}(s,\omega) \in S'$ for all $s \geqslant 0$.

Thus, for any $z' \in S^*$ and $t \geqslant 0$,

$$(8)\ \{\omega : \tilde{Z}'(t,\omega) \leqslant z^*\}$$
$$= \{\omega :\ \exists V(\cdot,\omega) \in \mathbf{I}, \tilde{X}^*(s,\omega)+V(s,\omega) \in S^*\ \forall s \geqslant 0, \tilde{X}^*(t,\omega)+V(t,\omega) \leqslant z^*\}.$$

It follows from [12] that any continuous non-decreasing function $V(\cdot,\omega)$ on $[0, t]$ satisfying $V(0,\omega) = 0$ and

(9) $\tilde{X}(s,\omega) + RV(s,\omega) \in S$

for all $s \in [0, t]$, can be extended to a member of $\mathbf{I}$ such that (9) holds for all $s \geqslant 0$. Thus, the event in (8) is equal to

$$(10)\ \{\omega :\ \exists V(\cdot,\omega) \in \mathbf{I}, \tilde{X}'(s,\omega)+V(s,\omega) \in S^*\ \forall s \in [0, t], \tilde{X}'(t,\omega)+V(t,\omega) \leqslant z^*\}$$
$$= \{\omega : \exists V(\cdot,\omega) \in \mathbf{I}, \tilde{X}'(s,\omega)+V(s,\omega) \in S'\ \forall s \in [0, t], \tilde{X}'(t,\omega)+V(t,\omega) = z'\}.$$

The last equality follows because any $V(\cdot,\omega) \in \mathbf{I}$ satisfying the conditions in the event (10) can be continuously increased to a $V(\cdot,\omega) \in \mathbf{I}$ satisfying the conditions of the last event above. Now, $\tilde{X}'(s) = X'(t) - X'(t-s)$ for $s \in [0, t]$ and $\tilde{X}'(t) = X'(t)$ P_0'–a.s. Moreover, $y \in \mathbf{I}$ if and only if $\tilde{y} \in \mathbf{I}$ where

$$\tilde{y}(s) = \begin{cases} y(t) - y(t-s) & \text{for } s \in [0, t] \\ y(s) & \text{for } s \geqslant t. \end{cases}$$

Let $\mathbf{F}'$ denote the completion of $\mathbf{F}$ with respect to P_0'. Then (10) is P_0'–a.s. equal to the following event contained in $\mathbf{F}^*$:

$$(11)\ \{\omega :\ \exists V(\cdot,\omega) \in \mathbf{I}, X'(t,\omega) - X'(t-s,\omega) + V(t,\omega) - V(t-s,\omega) \in S^*$$
$$\forall s \in [0, t], X'(t,\omega) + V(t,\omega) = z'\}$$

and by a change of variable from s to $t-s$, the above is equal to

$$\{\omega :\ \exists V(\cdot,\omega) \in \mathbf{I}, X^*(t,\omega) + V(t,\omega) - X'(s,\omega) - V(s,\omega) \in S'\ \forall s \in [0, t],$$
$$X^*(t,\omega) + V(t,\omega) = z'\}.$$

$$= \{\omega : \exists V(\cdot, \omega) \in \mathbf{I},\ z^* - X^*(s, \omega) - V(s, \omega) \in S^*\ \forall s \in [0, t],$$
$$z^* - X^*(t, \omega) - V(t, \omega) = 0\}$$

$$= \{\omega : \exists V(\cdot, \omega) \in \mathbf{I},\ z^* - X^*(s, \omega) - V(s, \omega) \in S^*\ \ \forall s \in [0, t]\}.$$

The last equality follows by similar reasoning to that used in the sentence following (10). Since the above event decreases to the event in the last line of (5) as t increases, the desired result follows. □

The next two lemmas show that the limit in (5) is non-trivial if and only if condition (4.5) holds. Here ∂S^* denotes the boundary of S^*.

(12) **Lemma.** *Suppose* $\gamma \equiv -R^{-1}\theta > 0$. *Then there is a* K*-vector* $\delta > 0$ *such that for all* $z^* \in S^*$,

$$F^*(z^*) \geqslant 1 - \sum_{k=1}^{K} \exp\left(\frac{-2\delta_k z_k}{\Gamma_{kk}}\right), \tag{13}$$

where $z_k \equiv (Rz^*)_k \geqslant 0$ *for* $k = 1, \ldots, K$.

Proof. Since $\gamma > 0$, by the continuity of R^{-1} we can choose a K-vector $\delta > 0$ sufficiently small that $R^{-1}\delta \leqslant \gamma$. Then $\alpha \equiv \gamma - R^{-1}\delta \geqslant 0$ and V^* defined by $V^*(t) = \alpha t$ for all $t \geqslant 0$ is an element of $\mathbf{I}$. Combining this with (5), we have for each $z^* \in S^*$,

$$F^*(z^*) \geqslant P_0^*(z^* - X^*(t) - V^*(t) \in S^*\ \ \forall t \geqslant 0)$$
$$= P_0(R(z^* - X^*(t) - V^*(t)) \in S\ \ \forall t \geqslant 0)$$
$$= P_0(z - X(t) - R\alpha t \geqslant 0\ \ \forall t \geqslant 0),$$

where $z = Rz^* \geqslant 0$. Now under P_0, $X(t) = W(t) + \theta t$ for all $t \geqslant 0$ where W is a $(0, \Gamma)$ Brownian motion starting from the origin in $\mathbf{R}^K$. Substituting this in the above and noting that $-\theta - R\alpha = \delta$, we obtain

$$F^*(z^*) \geqslant P_0(z - W(t) + \delta t \geqslant 0\ \ \forall t \geqslant 0)$$
$$= P_0(z_k + B_k(t) \geqslant 0\ \ \forall t \geqslant 0,\ k = 1, \ldots, K),$$

where for each k, under P_0, $B_k(t) \equiv -W_k(t) + \delta_k t$ is a (δ_k, Γ_{kk}) Brownian motion on $\mathbf{R}$ and it is transient to $+\infty$ since $\delta_k > 0$. Thus,

$$F^*(z^*) \geqslant 1 - P_0(z_k + \inf_{t \geqslant 0} B_k(t) < 0 \text{ for some } k \in \{1, \ldots, K\})$$

$$\geqslant 1 - \sum_{k=1}^{K} P_0(z_k + \inf_{t \geqslant 0} B_k(t) < 0)$$

$$= 1 - \sum_{k=1}^{K} \exp\left(\frac{-2\delta_k z_k}{\Gamma_{kk}}\right).$$

To obtain the last equality, we have used the formula for the probability [10, p. 43] that a one-dimensional Brownian motion with drift δ_k and variance Γ_{kk} does not go below the level zero. □

Remark. It follows from Lemma (12) that when $\gamma > 0$ the limit distribution of Z^* has an exponential tail and hence has finite moments of all orders.

(14) **Lemma.** *If* $\gamma \not> 0$, *then for each* $x^*, z^* \in S^*$,

(15) $$P^*_{x^*}(Z^*(t) \leqslant z^*) \to 0 \quad as \quad t \to \infty.$$

Proof. The pair (Y, Z^*) satisfies (4.1)-(4.2) with X^* and I in place of X and R, respectively. By the well known pathwise construction of a one-dimensional reflected Brownian motion [4, section 8.2], the solution $(\hat{Y}, \hat{Z})$ of (4.1)-(4.3) with data (X^*, I) in place of (X, R) is given by

$$\hat{Y}(t) \equiv \left(\max_{0 \leqslant s \leqslant t} \{-X^*(s)\}\right)^+, \quad \hat{Z}(t) \equiv X^*(t) + \hat{Y}(t), \quad \text{for all } t \geqslant 0.$$

Then by the alternative characterization of this pair given in section 5, it follows that for each $x^* \in S^*$, $P^*_{x^*}$ –a.s.:

$$\hat{Y}(t) \leqslant Y(t) \quad \text{and} \quad \hat{Z}(t) \leqslant Z^*(t) \quad \text{for all } t \geqslant 0.$$

Now suppose $\gamma_k \equiv -(R^{-1}\theta)_k \leqslant 0$ for some $k \in \{1, \ldots, K\}$. Then $\hat{Z}_k$ is a one-dimensional reflected Brownian motion on $\mathbf{R}_+$ with non-zero variance and drift $-\gamma_k$ and so is either null recurrent or transient as $\gamma_k = 0$ or $\gamma_k < 0$. Since Z^* dominates $\hat{Z}$, it follows that (15) holds for any $x^*, z^* \in S^*$. □

(16) **Corollary.** *If* $\gamma \not> 0$, *then for each* $x, z \in S$,

$$P_x(Z(t) \leqslant z) \to 0 \quad as \quad t \to \infty.$$

Proof. This follows immediately from Lemma (14) and the fact that for each $z \in S$, $\{Z(t) \leqslant z\} \subset \{Z'(t) \leqslant R^{-1}z\}$, since R^{-1} has only non-negative entries. □

Proof of Theorem (2). If $\gamma \not> 0$, then it follows immediately from (16) and (1) that Z cannot have a stationary distribution.

Conversely, suppose $\gamma > 0$. Then, by Lemmas (4) and (12), F^* corresponds to a probability measure π^* concentrated on $(S^*, \mathbf{B}_{S^*})$, which is the limit distribution of Z^* under P_0^*. Define the probability measure π on $(S, \mathbf{B}_S)$ by

$$\pi(A) = \pi'(R^{-1}A) \text{ for all } A \in \mathbf{B}_S.$$

Since Z' starting from 0 converges in distribution to π' as $t \to \infty$, it follows that Z starting from 0 converges in distribution to π as $t \to \infty$. Thus, for any $f \in C_b(S)$ and $t \geqslant 0$,

$$\begin{aligned} \int_S d\pi(x)\, f(x) &= \lim_{s \to \infty} E_0[f(Z(s+t))] \\ &= \lim_{s \to \infty} E_0[E_{Z(s)}[f(Z(t))]] \\ &= \int_S d\pi(x)\, E_x[f(Z(t))] \end{aligned}$$

where we have used the Markov property of Z to obtain the second equality and the Feller continuity in deducing the third equality. It follows by a monotone class argument that (1) holds for all bounded Borel measurable functions f on S and hence π is a stationary distribution for Z. □

7. Uniqueness of Stationary Distributions.

(1) **Theorem.** *Any stationary distribution for Z is unique.*

Before proving this theorem, we establish some preliminary lemmas.

(2) **Lemma.** *For each $x \in S$,*

(3) $$E_x\left[\int_0^\infty 1_{\partial S}(Z(s))\,ds\right] = 0.$$

Proof. It suffices to prove for each $j \in \{1, \ldots, K\}$,

(4) $$\int_0^t 1_{\{0\}}(n_j \cdot Z(s))\,ds = 0 \quad P_x\text{-a.s. for each } t \geqslant 0 \text{ and } x \in S,$$

where n_j is the inward unit normal to the j^{th} face F_j. For this, fix $x \in S$, $t \geqslant 0$ and let $\phi \in C_b^2(\mathbf{R}_+)$ such that ϕ'' is non-increasing on $\mathbf{R}_+$, $\phi''(u) = 1$ for $0 \leqslant u \leqslant \frac{1}{2}$, $\phi''(u) = 0$ for $u \geqslant 1$, $\phi'(u) = \int_1^u \phi''(v)\,dv$ and $\phi(u) = \int_1^u \phi'(v)\,dv$, so for $u \geqslant 1$, $\phi'(u) = 0$ and $\phi(u) = 0$. By applying Itô's formula to $z \to \phi(\epsilon^{-1} n_j \cdot z)$ for $\epsilon > 0$ and using the semimartingale representation (4.1) of Z, we obtain P_x-a.s.

(5) $$\begin{aligned} \phi(\epsilon^{-1} n_j \cdot Z(t)) - \phi(\epsilon^{-1} n_j \cdot x) &= \epsilon^{-1} \int_0^t \phi'(\epsilon^{-1} n_j \cdot Z(s))\, n_j \cdot dX(s) \\ &+ \epsilon^{-1} \sum_{k=1}^K n_j \cdot v_k \int_0^t \phi'(\epsilon^{-1} n_j \cdot Z(s))\, dY_k(s) \\ &+ \frac{1}{2}\epsilon^{-2} \int_0^t n_j{}'\,\Gamma\, n_j\, \phi''(\epsilon^{-1} n_j \cdot Z(s))\, ds. \end{aligned}$$

Recall that v_k is the vector given by the k^{th} column of R. Since X is a (θ, Γ) Brownian motion and ϕ, ϕ' are bounded, it follows that when multiplied by ϵ^2, all but the last term in the above equation tends to zero almost surely or in L^2 (with respect to P_x) as $\epsilon \to 0$. Hence,

(6) $$\liminf_{\epsilon \to 0} \int_0^t n_j{}'\,\Gamma\, n_j\, \phi''(\epsilon^{-1} n_j \cdot Z(s))\, ds = 0 \quad P_x\text{-a.s.}$$

The desired result (4) then follows from Fatou's lemma and the facts that $n_j{}'\,\Gamma\, n_j = \Gamma_{jj} > 0$, $\phi''(0) = 1$ and $\phi'' \geqslant 0$ on $\mathbf{R}_+$. □

(7) **Lemma.** *For each $x \in S$ and $t > 0$,*

(8) $$P_x(Z(t) \in \partial S) = 0.$$

Proof. This follows from (3) by standard arguments using Fubini's theorem and the fact that Z is a continuous strong Markov process that behaves like a (θ, Γ) Brownian motion in $S \setminus \partial S$. □

Let m denote Lebesgue measure on S.

(9) **Lemma.** *For each $x \in S, t > 0$ and Borel set A in S we have*

(10) $$P_x(Z(t) \in A) = 0 \iff m(A) = 0.$$

Proof. Since $m(\partial S) = 0$ and (8) holds, it suffices to consider $A \subset G$ where G is a compact subset of $S \setminus \partial S$. Let $\tau = \inf\{s \geqslant 0 : Z(s) \in \partial S\}$, $\sigma = \inf\{s \geqslant 0 : Z(s) \in G\}$, $\sigma_0 = 0$ and for each $n \geqslant 1$, let $\tau_n = \sigma_{n-1} + \tau \circ \theta_{\sigma_{n-1}}$ and $\sigma_n = \tau_n + \sigma \circ \theta_{\tau_n}$ where $\theta_{\cdot}$ denotes the usual shift operator for Z. Then, for each $x \in S$, $\tau_n \uparrow \infty$ P_x-a.s. as $n \to \infty$ and

(11) $$E_x\left[\int_0^\infty 1_A(Z(t))\,dt\right] = \sum_{n=1}^\infty E_x\left[\int_{\sigma_{n-1}}^{\tau_n} 1_A(Z(t))\,dt\right].$$

From Lemma (7), the strong Markov property of Z, and the fact that Z behaves like a (θ, Γ) Brownian motion until the time of its first exit from $S \setminus \partial S$, it follows that the left member of (11) is zero if and only if $m(A) = 0$. The desired result (10) is then obtained in a similar way to that in which (7) is deduced from (2). □

Two measures will be called equivalent if they are mutually absolutely continuous. The symbol $\approx$ will be used to denote equivalence of measures.

(12) **Lemma.** *Suppose π is a stationary distribution for Z. Then $\pi \approx m$.*

Proof. Since π is stationary, for any Borel set $A \subset S$,

(13) $$\pi(A) = \int_S P_x(Z(1) \in A)\,d\pi(x).$$

It follows from this and the fact that (10) holds for $t = 1$ and *every* $x \in S$ that $\pi(A) = 0$ if and only if $m(A) = 0$. □

Remark. The above result extends immediately to the case where π is a σ-finite invariant measure for Z.

Proof of Theorem (1). Suppose π is a stationary distribution for Z. Then it follows from Lemmas (9) and (12) (cf. [31, p. 389-390]) that Z is ergodic on S and for *each* $x \in S$ and $f \in C_b(S)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} E_x[f(Z(j))] = \int_S f(z)\, d\pi(z) .$$

The uniqueness of π follows immediately from this representation. □

8. A Necessary Condition for Stationary Distributions.

In this section we establish a *basic adjoint relation* (BAR) that must be satisfied by any stationary distribution for Z. Here σ_k denotes (K-1)-dimensional Lebesgue measure (surface measure) on the face F_k.

(1) **Theorem.** *Suppose π is a stationary distribution for Z. Then, for each $k \in \{1, \ldots, K\}$, there is a finite Borel measure ν_k on F_k such that $\nu_k \approx \sigma_k$ and for each bounded Borel function f on F_k and $t \geqslant 0$,*

$$E_\pi\left[\int_0^t f(Z(s))\, dY_k(s)\right] = \frac{1}{2} t \int_{F_k} f\, d\nu_k . \tag{2}$$

Moreover, for each $f \in C_b^2(S)$,

$$\int_S Lf\, d\pi + \frac{1}{2} \sum_{k=1}^{K} \int_{F_k} D_k f\, d\nu_k = 0. \tag{BAR}$$

Remark. The $\frac{1}{2}$ in (2) and (BAR) matches the $\frac{1}{2}$ in L. It has been introduced here to simplify later manipulations.

The following lemmas will be used in proving this theorem. For the proofs of these, we need to introduce the usual σ-field $\mathbf{M}$ and filtration $\{\mathbf{M}_t, t \geqslant 0\}$ associated with the strong Markov process Z. Let $\mathbf{M}^o = \sigma\{Z(s) : 0 \leqslant s < \infty\}$ and $\mathbf{M}_t^o = \sigma\{Z(s) : 0 \leqslant s \leqslant t\}$, for each $t \geqslant 0$. For each probability measure μ on $(S, \mathbf{B}_S)$ and for P_μ defined by $P_\mu = \int P_x \, d\mu(x)$, let $\mathbf{M}^\mu$ denote the P_μ-completion of $\mathbf{M}^o$ and let $\mathbf{M}_t^\mu$ be the smallest σ-field containing $\mathbf{M}_t^o$ and all of the P_μ null sets in $\mathbf{M}^\mu$. Define $\mathbf{M} = \bigcap_\mu \mathbf{M}^\mu$ and $\mathbf{M}_t = \bigcap_\mu \mathbf{M}_t^\mu$ for all $t \geqslant 0$. For the definitions of additive process/functional used below, see [5].

(3) **Lemma.** *For each $k \in \{1, \ldots, K\}$, there is a continuous additive functional $\tilde{Y}_k$ of Z such that for each $x \in S$, $\tilde{Y}_k$ is P_x-indistinguishable from Y_k. Moreover, the support of $\tilde{Y}_k$ is contained in F_k.*

Proof. For each $x \in S$, Z is a continuous P_x-semimartingale relative to the filtration generated by X (cf. (4.1)), and hence [7, VII-60, p. 268], Z is a P_x-semimartingale on $(\Omega, \mathbf{M}, \{\mathbf{M}_t\})$. It follows from [5, Theorem 3.12] that there is a continuous $\{\mathbf{M}_t\}$-adapted process that for each $x \in S$ is a version of the P_x-stochastic integral process

$$\int_0^t 1_{S \setminus \partial S}(Z(s)) \, dZ(s), \quad t \geqslant 0.$$

Let $\tilde{X}$ denote the sum of this $\{\mathbf{M}_t\}$-adapted process with $Z(0)$, so that for each $x \in S$ we have P_x-a.s. for all $t \geqslant 0$:

$$\tilde{X}(t) = Z(0) + \int_0^t 1_{S \setminus \partial S}(Z(s)) \, dZ(s).$$

In the integral above, dZ can be replaced by dX since Y increases only when Z is on ∂S (cf. (4.3)), and then, by Lemma (7.2) and the fact that X is a Brownian motion under P_x, the integrand can be replaced by 1. It follows from this that $\tilde{X}$ is P_x-indistinguishable from X for each $x \in S$. Define $\tilde{Y} = R^{-1}(Z - \tilde{X})$. Then $\tilde{Y}$ is continuous, $\{\mathbf{M}_t\}$-adapted and P_x-indistinguishable from Y for each $x \in S$.

Now $R^{-1}(Z(\cdot) - Z(0))$ is a continuous additive process relative to Z. It then follows from the decomposition of this semimartingale and [5, Theorem 3.18] that $R^{-1}(\tilde{X}(\cdot) - \tilde{X}(0))$ and $\tilde{Y}$ can also be chosen to be additive. In this case, since Y is non-decreasing, each component $\tilde{Y}_k$ of $\tilde{Y}$ will be a (non-decreasing) continuous additive functional of Z. Condition (4.3) implies that the support of $\tilde{Y}_k$ is contained in F_k for each k. □

(4) **Lemma.** *There is a constant* $C \in (0, \infty)$ *such that for each* $x \in S$, $k \in \{1, \ldots, K\}$ *and* $t \geqslant 0$,

$$E_x[Y_k(t)] \leqslant C(t+1).$$

Proof. For each $t \geqslant 0$ and $k \in \{1, \ldots, K\}$, let $M_k(t) = \{\max_{0 \leqslant s \leqslant t} (-X_k(s))\}^+$, and let

$$V(t) = R^{-1}M(t).$$

Then, $X(t) + RV(t) \in S$ for all $t \geqslant 0$ and by the alternative characterization of Y (cf. section 5), as shown in Reiman [23, p. 457], we have for each k,

$$Y_k(t) \leqslant V_k(t) \leqslant \|R^{-1}\| \max_{j=1}^{K} M_j(t), \tag{5}$$

where $\| R^{-1} \| \equiv \max_{i=1}^{K} \sum_{j=1}^{K} | (R^{-1})_{ij} |$. For each j, P_x-a.s., M_j is less than or equal to the maximum of a one-dimensional Brownian motion that starts from the origin and has drift $-\theta_j$, and so there are constants $C_j^1, C_j^2, C_j^3 \in (0,\infty)$ (not depending on x) such that

$$E_x[M_j(t)] \leqslant C_j^1\sqrt{t} + C_j^2 t \leqslant C_j^3(t+1) \quad \text{for all } t \geqslant 0. \tag{6}$$

The desired result then follows by combining (5) and (6). □

(7) **Lemma.** *For each* $k \in \{1, \ldots, K\}$ *and* $x \in S$,

$$\int_0^\infty 1_{F_j \cap F_k}(Z(s))\, dY_k(s) = 0 \quad P_x\text{-}a.s. \ \textit{for all} \ j \neq k.$$

Proof. See Reiman-Williams [24, Theorem 1]. □

Proof of Theorem 1. By Lemma (3), it suffices to prove (2) with $\tilde{Y}$ in place of Y. Moreover, by Lemmas (3) and (4), for each k, $\tilde{Y}_k$ is a continuous additive functional of Z with support in F_k and $\tilde{Y}_k(t)$ has finite expectation with respect to E_π for each $t \geqslant 0$. Hence [1], there is a finite Borel measure ν_k on F_k such that (2) holds for all bounded Borel functions f on F_k and all $t \geqslant 0$.

Fix $k \in \{1, \ldots, K\}$. To prove $\nu_k \approx \sigma_k$, it suffices to prove

$$(8) \qquad E_x\left[\int_0^t 1_A(Z(s))\, dY_k(s)\right] = 0 \quad \Leftrightarrow \quad \sigma_k(A) = 0$$

for all Borel sets $A \subset F_k$, $t > 0$ and $x \in S$. In view of Lemma (7), it suffices to prove this for sets A that are a positive distance from the other sides $F_j, j \neq k$. But this follows by localization and the analogous property for a Brownian motion reflected in the direction v_k on a half-plane containing F_k.

Now suppose $f \in C_b^2(S)$. Then by Itô's formula and the semimartingale representation (4.1) of Z, for each $x \in S$ we have P_x-a.s. for all $t \geqslant 0$:

$$(9) \quad f(Z(t)) - f(Z(0)) = \int_0^t \nabla f(Z(s)) \cdot dW(s) + \sum_{k=1}^K \int_0^t D_k f(Z(s))\, dY_k(s) + \int_0^t Lf(Z(s))\, ds$$

where $W(t) = X(t) - \theta t$ is a $(0, \Gamma)$ Brownian motion and D_k, L are given by (5.1)-(5.2). Since ∇f is bounded, the stochastic integral with respect to dW in (9) defines a martingale and so its expectation under P_x is zero. Thus, taking expectations with respect to P_x in (9) and integrating with respect to the stationary distribution π, we obtain

$$0 = \frac{1}{2} t \sum_{k=1}^K \int_{F_k} D_k f\, d\nu_k + t \int_S Lf\, d\pi,$$

where (2) was used to obtain the first integral term and Fubini's theorem together with (6.1) for the second. Dividing by $t > 0$ yields (BAR). □

9. Product Form Solutions.

Recall from Lemma (7.12) that any stationary distribution for Z is equivalent to Lebesgue measure m on S.

Definition. *We say a stationary distribution for Z is of product form (or has a separable density) if and only if*

$$(1) \qquad p(z) \equiv \frac{d\pi}{dm}(z) = \prod_{k=1}^{K} p_k(z_k), \quad z = (z_1, \ldots, z_k) \in S,$$

where p_k is a probability density function relative to Lebesgue measure on $\mathbf{R}_+$, for $k = 1, \ldots, K$.

(2) **Theorem.** *Z has a product form stationary distribution if and only if (4.5)-(4.6) hold. In this case, the density p for this distribution is given by (4.7)-(4.8).*

The proof of Theorem (2) is divided into two parts. The only if part is proved in Theorem (3) below using a Laplace transform relation derived from the necessary condition (BAR) of section 8. For the if part, proved in Theorem (23), we first perform a linear transformation to remove the correlation between the components of X and then appeal to [30] where it is verified that conditions (4.5)-(4.6) are sufficient for the transformed process to have an exponential form (a special case of product form) stationary density, and hence for the original process to have one. We note in passing that if (BAR) could be shown to be sufficient as well as necessary for π to be a stationary distribution for Z, then the proof of Theorem (3) would also yield the if part of Theorem (2).

(3) **Theorem.** *Suppose Z has a product form stationary distribution π. Then (4.5)-(4.6) must hold and the density p of π relative to m is given by (4.7)-(4.8), and for each $k \in \{1, \ldots, K\}$,*

$$(4) \qquad \frac{d\nu_k}{d\sigma_k}(z) = 2\gamma_k \prod_{j \neq k} \eta_j \exp\{-\eta_j z_j\} \quad \textit{for all} \quad z \in F_k.$$

Proof. In this proof only, for notational convenience, the symbols Q and λ will have different meanings from those assigned in section 2.

By Theorems 6.2 and 8.1, (4.5) and (BAR) must hold. Now, for $\lambda \in \mathbf{R}_+^K$, $f(z) \equiv \exp\{-\lambda \cdot z\}$ is in $C_b^2(S)$. Thus, we may substitute such functions in (BAR) to obtain the following transform relation as necessary for π to be a stationary distribution of Z:

$$(5) \qquad Q(\lambda)\pi^*(\lambda) - \sum_{k=1}^{K} L_k(\lambda)\nu_k^*(\lambda_{|k}) = 0 \ \text{ for all } \ \lambda \in \mathbf{R}_+^K,$$

where

$$(6) \qquad Q(\lambda) = \lambda' \Gamma \lambda - 2\lambda'\theta$$

$$(7) \qquad L_k(\lambda) = \mu_k \left(\lambda_k - \sum_{j \neq k} P_{kj} \lambda_j \right), \quad k = 1, \ldots, K$$

$$(8) \qquad \pi^*(\lambda) = \int_S \exp\{-\lambda \cdot z\}\, d\pi(z)$$

$$(9) \qquad \nu_k^*(\lambda_{|k}) = \int_{F_k} \exp\{-\lambda \cdot z\}\, d\nu_k(z), \quad k = 1, \ldots, K.$$

Here $\lambda_{|k}$ denotes the $(K-1)$-vector formed by deleting the k$^{\text{th}}$ entry in λ. This notation is used to emphasize the fact that $\nu_k^*(\lambda_{|k})$ does not depend on λ_k since $z_k = 0$ on F_k.

Suppose the density p of π relative to m is separable, as in (1). Then substituting this in the above, dividing (5) by $\lambda_k > 0$ for a fixed k, and letting $\lambda_k \to \infty$, we obtain

$$(10) \qquad \Gamma_{kk}\, c_k \prod_{j \neq k} \pi_j^*(\lambda_j) = \mu_k \nu_k^*(\lambda_{|k}) \ \text{ for } \ k = 1, \ldots, K,$$

where

$$(11) \qquad \pi_j^*(\lambda_j) = \int_0^\infty e^{-\lambda_j z_j} p_j(z_j)\, dz_j, \quad j \neq k,$$

$$(12) \qquad c_k = \lim_{\lambda_k \to \infty} \lambda_k \pi_k^*(\lambda_k),$$

and we have used the fact (cf. Theorem (8.1)) that ν_j does not charge $F_j \cap F_k$ for $j \neq k$ so that $\lim\limits_{\lambda_k \to \infty} \nu_j^*(\lambda_{|j}) = 0$. After substituting the expressions for the

$\nu_k^*(\lambda_{|k})$ from (10) in (5), we obtain for all $\lambda \in \mathbf{R}_+^K$,

$$(13) \qquad Q(\lambda) \prod_{j=1}^{K} \pi_j^*(\lambda_j) - \sum_{k=1}^{K} L_k(\lambda) \Gamma_{kk} c_k \mu_k^{-1} \prod_{j \neq k} \pi_j^*(\lambda_j) = 0.$$

By setting $\lambda_j = 0$ for $j \neq k$ and solving for $\pi_k^*(\lambda_k)$, we obtain

$$(14) \qquad \pi_k^*(\lambda_k) = \frac{\Gamma_{kk} c_k}{\Gamma_{kk} \lambda_k - 2\theta_k + \sum_{j \neq k} P_{jk} \Gamma_{jj} c_j} \quad \text{for all } \lambda_k > 0, \; k = 1, \; \ldots, K.$$

By the uniqueness of Laplace transforms for probability densities, it follows from (14) that

$$(15) \qquad p_k(z_k) = \eta_k \exp(-\eta_k z_k), \;\; k = 1, \; \ldots, K,$$

where

$$(16) \qquad \eta_k = (-2\theta_k + \sum_{j \neq k} P_{jk} \Gamma_{jj} \eta_j)/\Gamma_{kk}$$

and

$$(17) \qquad c_k = \eta_k > 0.$$

The equation (16) for $\eta = (\eta_1, \; \ldots, \eta_K)'$ may be rewritten in matrix form:

$$(18) \qquad (I - P')\Lambda \eta = -2\theta,$$

where Λ is a diagonal matrix with the same diagonal entries as Γ. Since $I - P' = R\,D^{-1}$ is invertible and $\gamma = -R^{-1}\theta$, equation (18) is equivalent to:

$$(19) \qquad \eta = -2\Lambda^{-1} D\, R^{-1} \theta = 2\Lambda^{-1} D \gamma.$$

Note that (15), (19) are equivalent to (4.7), (4.8), and $\eta > 0$ if and only if $\gamma > 0$. After substituting $\pi_j^*(\lambda_j) = \eta_j/(\lambda_j + \eta_j)$ in (10) and invoking the uniqueness of Laplace transforms for measures with L^1 densities relative to surface measure σ_k on F_k, we obtain (4). Thus, ν_k is a normalized trace of the measure π on F_k. Now, substituting the above expression for $\pi_j^*(\lambda_j)$ in (13) and dividing through by $\prod_{j=1}^{K} \pi_j^*(\lambda_j)$, we obtain

(20) $Q(\lambda) - \lambda'(I - P')\Lambda(\lambda + \eta) = 0.$

When (18) is substituted in (20), the latter reduces to

(21) $\lambda'\Gamma\lambda - \lambda'(I - P')\Lambda\lambda = 0.$

After symmetrizing $(I - P')\Lambda$, we see that (21) holds for all $\lambda \in \mathbf{R}_+^K$ if and only if

(22) $2\Gamma = 2\Lambda - P'\Lambda - \Lambda P$,

which is equivalent to (4.6). This completes the proof of the necessity of (4.5)-(4.8) and (4) for π to be a product form stationary distribution of Z . □

(23) **Theorem.** *Suppose (4.5)-(4.6) hold. Then Z has a product form stationary distribution and its density p relative to m is given by (4.7)-(4.8).*

Proof. To allow use of the results in [30], we first perform a linear transformation of the state space to remove the correlation between the components of X. Let U be the rotation matrix whose rows are the orthonormal eigenvectors of the covariance matrix Γ and let A be the corresponding diagonal matrix of eigenvalues such that $\Gamma = U'AU$, where $U' = U^{-1}$. Let $V = A^{-\frac{1}{2}}U$ and define $\tilde{Z} = VZ$, $\tilde{X} = VX$ and $\tilde{R} = VR$. Then $\tilde{X}$ is a $(V\theta, I)$ Brownian motion and the k$^{\text{th}}$ row of the matrix

(24) $\tilde{N} \equiv \Lambda^{-\frac{1}{2}}U'A^{\frac{1}{2}}$

is the inward unit normal to the face of the state space of $\tilde{Z}$ on which Y_k increases. Here $\Lambda = \text{diag}(\Gamma)$, as in the proof of Theorem (3). Resolving the k$^{\text{th}}$ column of $\tilde{R}$ into components that are normal and tangential to the k$^{\text{th}}$ face of the state space of $\tilde{Z}$, we obtain $\tilde{R} = (\tilde{N}' + \tilde{Q}')H$, where $\text{diag}(\tilde{N}\tilde{Q}') = 0$, $H = \text{diag}(\tilde{N}\tilde{R}) = \Lambda^{-\frac{1}{2}}D$ and

$$\tilde{Q}' = \tilde{R}H^{-1} - \tilde{N}' = A^{-\frac{1}{2}}U(I - P')\Lambda^{\frac{1}{2}} - A^{\frac{1}{2}}U\Lambda^{-\frac{1}{2}}.$$

It was shown in [30] that if the following skew symmetry condition holds

$$\tilde{N}\tilde{Q}' + \tilde{Q}\tilde{N}' = 0 , \tag{25}$$

then $\tilde{Z}$ has an invariant measure $\tilde{\pi}$ such that $d\tilde{\pi}(\tilde{z}) = \exp(-\tilde{\eta}\cdot\tilde{z})d\tilde{z}$, where $\tilde{\eta} = -2(I - \tilde{N}^{-1}\tilde{Q})^{-1}V\theta$. Moreover, when $\exp(-\tilde{\eta}\cdot\tilde{z})$ is integrable over the state space of $\tilde{Z}$, $\tilde{\pi}$ (suitably normalized) yields the unique stationary distribution for $\tilde{Z}$. Simple algebraic manipulations show that (25) holds if and only if (22) holds. Moreover, when (22) holds,

$$\begin{aligned} V'\,\tilde{\eta} &= -2V'(2I - V(2\Gamma - \Lambda + P'\Lambda)V')^{-1}V\theta \\ &= 2V'(V(I - P')\Lambda V')^{-1}V\theta \\ &= 2\Lambda^{-1}D\,R^{-1}\theta = \eta, \end{aligned}$$

where η is given by (19). Since $\tilde{Z}$ and Z are related by the invertible linear transformation V, it follows that when (22) holds and $\exp\{-(V'\,\tilde{\eta})\cdot z\}$ is integrable over S (i.e., $\eta > 0$), then Z has a product form stationary distribution with density given by (4.7)-(4.8). □

10. Summary of Performance Analysis Procedures.

Consider an open queueing network of the type described earlier in section 2, assuming that ρ_k is less than one but near one for each station $k = 1, \ldots, K$. Let n be a large integer satisfying (3.1), and define a drift vector θ, covariance matrix Γ and reflection matrix R in terms of the queueing network's parameters as in sections 2 and 3. Finally, let Z be a K-dimensional RBM defined in terms of θ, Γ and R as in section 4. Consider Z starting from the origin and let $Q(\infty)$ and $Z(\infty)$ denote the weak limits of $Q(t)$ and $Z(t)$, respectively, as $t \to \infty$. (Note that the necessary and sufficient condition (4.5) for Z to have a limit distribution is satisfied since this is equivalent to the condition $\rho < e$.) The heavy traffic limit theorem alluded to in section 3 shows that the scaled queue length process $\{n^{-1/2}Q(nt), t \geqslant 0\}$ is distributed approximately as Z, and thus we are led to approximate (in the distributional sense) $n^{-1/2}Q(\infty)$ by $Z(\infty)$. Equivalently, the heavy traffic limit theory leads us to approximate $Q(\infty)$ by $n^{1/2}Z(\infty)$. Unfortunately, the distribution of $Z(\infty)$ is not known for general values of θ, Γ and R, but if Γ and R jointly satisfy condition (4.6), equations (4.7) and (4.8) say that $Z_1(\infty), \ldots, Z_K(\infty)$ are independent and exponentially distributed with $E[Z_k(\infty)] = \Gamma_{kk}/2\mu_k\gamma_k$. Thus, given that Γ and R jointly satisfy (4.6), we are led to the approximation $E[Q_k(\infty)] \simeq n^{1/2}\Gamma_{kk}/2\mu_k\gamma_k$. By (4.4) and (3.8), this is equivalent to

$$(1) \qquad E[Q_k(\infty)] \simeq \Gamma_{kk}/2\mu_k(1-\rho_k) \quad \text{for} \quad k = 1, \ldots, K.$$

A similar but somewhat different approximation scheme has been proposed by Whitt [28] for analysis of generalized Jackson networks, and that scheme is implemented in the software package called QNA.

We have seen that formula (1) flows directly out of heavy traffic limit theory in the case where (4.6) holds, and the courageous might venture to apply (1) regardless of whether the network data satisfy (4.6). A virtue of formula (1), like the QNA approximation scheme, is that it uses both first and second moment information. In contrast, many practically oriented users of performance analysis methodology simply apply the formulas derived by Jackson [16, 17] when they wish to evaluate a proposed network configuration. These formulas are based on the assumption of Poisson arrivals and exponential service time distributions, and to apply them one need only estimate the mean arrival rates and mean service times at the various stations of the network. The result is that second moment information is either ignored or never gathered. It may be, incidentally, that the QNA approximation for $E[Q_k(\infty)]$ is nearly equivalent to (1) when (4.6) holds and ρ_k is near one for all k, but we have not investigated this matter.

Let us consider now a network where ρ_k is less than one but close to one for some stations k, whereas ρ_k is substantially less than one for all other stations. Hereafter the former class will be referred to as *critical stations*, or the *bottleneck subnetwork*, and the latter class will be referred to as non-critical stations. Intuitively, one feels that the congestion and delay associated with non-critical stations will be negligible compared with those arising in the bottleneck subnetwork, and the formal results of Iglehart-Whitt [15] and Johnson [18] strongly confirm that view. More specifically, these results show that, for certain restricted classes of networks, one can simply delete non-critical stations from one's model, as if passage through such stations were instantaneous. (In the reduced model, of course, the flow processes F^j must describe statistically the flow of customers between successive *critical* stations.) The papers referenced above justify this procedure only for special classes of networks, but the extension to the general class of networks described in section 2 is presumably straightforward.

11. Concluding Remarks and Open Problems.

As a consistency check on the results presented in this paper, readers might wish to consider the case of an open Jackson network, characterized by Poisson arrival processes, exponential service time distributions, and Markovian routing. In that case, it is known that $Q(\infty)$ has a product form distribution, so one would expect the approximating RBM to have a product form stationary distribution as well, which is equivalent to saying that (4.6) holds. A tedious verification shows that the covariance matrix Γ and reflection matrix R associated with a Jackson network do indeed satisfy (4.6). This would not be true, incidentally, if one used the expression for Γ advanced in Reiman's [23] original treatment; with Reiman's formula for Γ, (4.6) holds for Jackson networks only in the limit as $\rho_k \to 1$ for all stations k. We feel that this tends to confirm the view, expressed earlier in section 3, that our formula (2.23) represents a refinement of the formula for Γ advanced by Reiman.

We conclude the paper with some conjectures and open research problems, all of which involve stationary distributions for the class of RBM's defined in section 4. Given that a stationary distribution π exists, we would like to establish that π has a density function p on S such that the boundary measure ν_k has density $\mu_k^{-1}\Gamma_{kk}p$ on F_k $(k = 1, \ldots, K)$. It has been shown that this is the case when (4.6) holds (see section 9), and we conjecture that it is true generally. If so, then the basic adjoint relation (BAR) of section 8 takes the form

$$\text{(BAR)} \quad \int_S Lf \; p \; dx \; + \; \frac{1}{2}\sum_{k=1}^{K} \mu_k^{-1}\Gamma_{kk} \int_{F_k} D_k f \; p \; d\sigma_k \; = \; 0 \quad \text{for all } f \in C_b^2(S).$$

where σ_k denotes $(K-1)$-dimensional Lebesgue measure on F_k. Having justified the expression of (BAR) in this simplified form, another open problem is to show that a probability density function p is the stationary density of the RBM under study if and only if it satisfies (BAR). Only the necessity has been established in section 8. The final open problem, of course, is to solve (BAR), which presumably means developing efficient numerical methods for computing important performance measures associated with the stationary density p, such as the means of the marginal distributions.

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