Contemporary Physics, March – April 2004, volume 45, number 2, pages 147 – 157

Brownian motion and gambling: from ratchets to paradoxical games

J. M. R. PARRONDO and LUIS DINÍS

Two losing gambling games, when alternated in a periodic or random fashion, can produce a winning game. This paradox has been inspired by certain physical systems capable of rectifying fluctuations: the so-called Brownian ratchets. In this paper we review this paradox, from Brownian ratchets to the most recent studies on collective games, providing some intuitive explanations of the unexpected phenomena that we will find along the way.

1. Introduction: a noisy revolution

Imagine two simple gambling games, say A and B, in which I play against you. Each one is a losing game for me, in the sense that my average capital is a decreasing function of the number of turns we play. Once you are convinced that I lose in both games, I give you a third proposal: alternate the games following the sequence AABBAABB... If you frown, the proposal can be modified to make it less suspicious: in each run we will randomly choose the game that is played. If you accept either of these proposals you would have trusted your intuition too much, not realizing that random systems may behave in an unexpected way.

The phenomenon we have just described is known as *Parrondo's paradox* [1 – 3]. It was originally inspired by a class of physical systems: *the Brownian ratchets* [4 – 8] and lately has received the attention of scientists working on several fields, ranging from biology to economics. These are systems capable of rectifying thermal fluctuations, such as those exhibited by a Brownian particle.

Brownian motion was one of the first crucial proofs of the discreteness of matter. First observed by Jan Ingenhousz in 1785, and later rediscovered by Brown in 1828, the phenomenon consists of the erratic or fluctuating motion of a small particle when it is embedded in a fluid. At the beginning of the 20th century, Einstein realized that these fluctuations were a manifestation of the molecular nature of the fluid† and devised a method to measure Avogadro's number by using Brownian motion [9]. Since then, the study of fluctuations has been a major topic in statistical mechanics.

The theory of fluctuations helped to understand noise in electrical circuits, activation processes in chemistry, the statistical nature of the second law of thermodynamics, and the origin of critical phenomena and spontaneous symmetry breaking, to cite only a few examples. In most of these cases, the role played by thermal fluctuations or thermal noise is either to trigger some process or to act as a disturbance. However, in the past two decades, the study of fluctuations has led to models and phenomena where the effect of noise is more complex and sometimes unexpected and even counter-intuitive.

Noise can enhance the response of a nonlinear system to an external signal, a phenomenon known as *stochastic resonance* [10]. It can create spatial patterns and ordered states in spatially extended systems [11, 12], and Brownian ratchets show that noise can be rectified and used to induce a systematic motion in a Brownian particle [4 – 8]. In these new phenomena, noise has a very different role from that considered in the past: it contributes to the creation of order. This could be relevant in several fields, and specially in biology, since most biological systems manage to keep themselves in ordered states even while surrounded by noise, both thermal noise at the level of the cell and environmental fluctuations at the macroscopic level.

However, fluctuations are not only restricted to physics, chemistry or biology. The origin of the theory of probability is closely related to gambling games, social statistics and even to the efficiency of juries [13]. Statistical mechanics and probability theory have both contributed to each other and also to fields like economics. In 1900, five years before Einstein's theory of the Brownian motion, the French mathematician Louis Bachelier worked out a

Authors' address: Grupo Interdisciplinar de Sistemas Complejos (GISC) and Dept. de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid, 28040-Madrid, Spain.

†The thermal origin of Brownian motion was first proposed by Delsaux in 1877 and later on by Gouy in 1888 (see [9]).

Contemporary Physics ISSN 0010-7514 print/ISSN 1366-5812 online © 2004 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00107510310001644836

theory for the price of a stock very similar to Einstein's [14]. Recently this link between probability, statistical mechanics and economics has crystallized in a new field: *econophysics* [15].

Some of the aforementioned constructive role of noise has been observed in complex systems beyond physics. Stochastic resonance, for instance, has an increasing relevance in the study of perception and other cognitive processes [10, 16]. Similarly, we expect that other elementary stochastic phenomena such as rectification will be observed in many situations not restricted to physics.

With this idea in mind, Parrondo's paradox came up as a translation to simple gambling games of a Brownian ratchet discovered by Ajdari and Prost [4]. The ratchet was afterwards named by Astumian and Bier the *flashing ratchet* [6] and it was related to the idea proposed by Magnasco [5] that biological systems could rectify fluctuations to perform work and systematic motion.

The paradox does not make use of Brownian particles, but only of the simpler fluctuations arising in a gambling game. However, it illustrates the mechanism of rectification in a very sharp way, and for this reason we think that it could contribute to extend the 'noisy revolution', i.e. the idea that noise can create order, to those fields where stochastic dynamics is relevant.

The paper is organized as follows. In section 2 we briefly review the flashing ratchet and explain how it can rectify fluctuations. Section 3 is devoted to the original Parrondo's paradox. There we introduce the paradoxical games as a discretization of the flashing ratchet, discuss an intuitive explanation of the paradox that we have called *reorganization of trends*, and present an extension of the original paradox inspired by this idea. In section 4 we introduce several versions of the games involving a large number of players. Some interesting effects can be observed in these collective games: redistribution of capital brings wealth [17], and collective decisions taken by voting or by optimizing the returns in the next turn can lead to worse performance than purely random choices [18, 19]. Finally, in section 5 we briefly review the literature on the paradox and present our main conclusions.

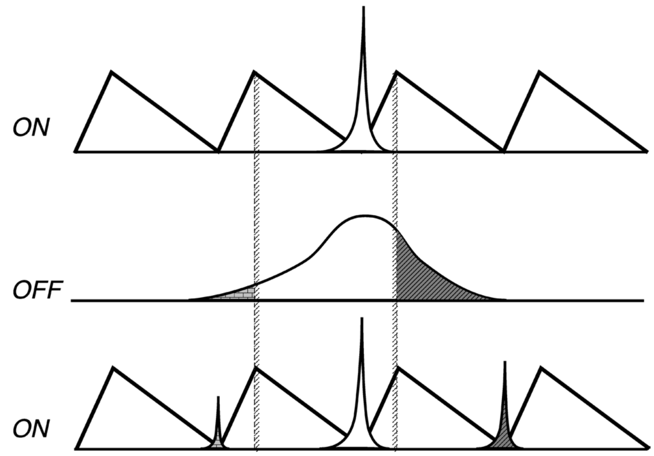

Figure 1. The flashing ratchet at work. The figure represents three snapshots of the potential and the density of particles. Initially (upper figure), the potential is on and all the particles are located around one of the minima of the potential. Then the potential is switched off and the particles diffuse freely, as shown in the centred figure, which is a snapshot of the system immediately before the potential is switched on again. Once the potential is connected again, the particles in the darker region move to the right-hand minimum whereas those within the small grey region move to the left. Due to the asymmetry of the potential, the ensemble of particles move, on average, to the right.

2. Ratchets

Here we revisit the flashing ratchet [4, 6], one of the simplest Brownian ratchets and the most closely related to the paradoxical games. We refer to the exhaustive review by Reimann on Brownian ratchets [7] or the special issue in *Applied Physics* A, edited by Linke [8], for further information on the subject.

Consider an ensemble of independent one-dimensional Brownian particles in the asymmetric sawtooth potential depicted in figure 1. It is not difficult to show that, if the potential is switched on and off periodically, the particles exhibit an average motion to the right. Let us assume that the temperature T is low enough to ensure that kT is much smaller than the maxima of the potential, and that we start with the potential switched on and with all the particles around one of its minima, as shown in the upper plot of figure 1. When the potential is switched off, the particles diffuse freely, and the density of particles spreads as depicted in the central plot of the figure. If the potential is then switched on again, each particle will move back to the initial minimum or to one of the nearest neighbouring minima, depending on its position. Particles within the dark region will move to the right-hand minimum, those within the small grey region will move to the left-hand minimum, and those within the white region will move back to their initial positions. As is apparent from the figure and due to the asymmetry of the potential, more particles fall into the right-hand minimum and thus there is a net motion of particles to the right. For this to occur, the switching can be either random or periodic, but the average period must be of the order of the time to reach the nearest barrier by free diffusion (see [4, 6] for details).

This motion can be seen as a rectification of the thermal noise associated with free diffusion. The diffusion is symmetric: some particles move to the right and some to the left, but their average position does not change. However, when the potential is switched on again, most of the particles that moved to the left are driven back to the starting position, whereas many particles that moved to the right are pushed to the right-hand minimum. The asymmetric potential acts as a rectifier: it 'kills' most of the negative fluctuations and 'promotes' most of the positive ones.

The effect remains if we add a small force toward the left, i.e. in a direction opposite to the induced motion. In this case, the ratchet still induces a motion against the force. Consequently, particles perform work and the system can be considered a Brownian motor. It can be proved that this type of motor is compatible with the second law of thermodynamics. In fact, the efficiency of such a motor is far below the limits imposed by the second law [20, 21]. However, the ratchet with a force exhibits a curious property: when the potential is permanently on or off, the Brownian particles move in the same direction as the force, whereas they move in the opposite direction when the potential is switched on and off. This is the essence of the paradoxical games: we have two dynamics; in each one a quantity, namely the position of the Brownian particle, decreases in average; however, the same quantity increases in average when the two dynamics are alternated.

3. Games

The flashing ratchet can be discretized in time and space, keeping most of its interesting features. The discretized version adopts the form of a pair of simple gambling games, which are the basis of Parrondo's paradox.

3.1. *The original paradox*

We consider two games, A and B, in which a player can make a bet of 1 euro. $X(t)$ denotes the capital of the player, where $t = 0, 1, 2\ldots$ stands for the number of turns played. Game A consists of tossing a slightly biased coin so that the player has a probability p_A of winning which is less than a half. That is, $p_A = 1/2 - \varepsilon$, where the bias ε is a small positive number.

The second game, B, is played with two biased coins, a 'bad coin' and a 'good coin'. The player must toss the bad coin if her capital $X(t)$ is a multiple of *3*, the probability of winning being $p_{\mathrm{bad}} = 1/10 - \varepsilon$. Otherwise, the good coin is tossed and the probability of winning is $p_{\mathrm{good}} = 3/4 - \varepsilon$. The rules of games A and B are represented in figure 2, in which the darkness represents the 'badness' of each coin.

For these choices of p_A, p_{good} and p_{bad}, both games are fair if $\varepsilon = 0$, in the sense that $\langle X(t)\rangle$ is constant. This is evident for game A, since the probabilities to win and lose are equal. The analysis of game B is more involved, but we will soon prove that the effect of the good and the bad coins cancel each other for $\varepsilon = 0$.

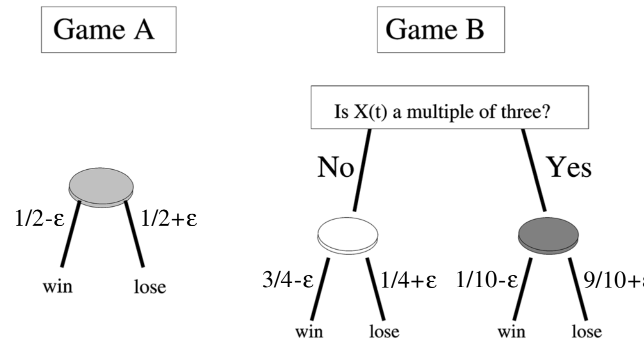

Figure 2. Rules of the paradoxical games. In game A, the player wins (her capital increases by one euro) with a probability $1/2 - \varepsilon$ and loses (her capital decreases by one euro) with a probability $1/2 + \varepsilon$, ε being a small positive number. In the figure, these probabilities are represented by a coin with two possible outcomes. In game B, the probability to win and lose depends on the capital $X(t)$ of the player: if $X(t)$ is a multiple of three, then we use a 'bad' coin, with a probability to win equal to $1/10 - \varepsilon$; if $X(t)$ is not a multiple of three, then a 'good' coin, with a probability to win equal to $3/4 - \varepsilon$, is used. In the figure the darkness of the coins represents their 'badness' for the player.

On the other hand, both games have a tendency to lose if $\varepsilon > 0$, i.e. $\langle X(t)\rangle$ decreases with the number of turns t. Surprisingly enough, if the player randomly chooses the game to play in each turn, or plays them following some predefined periodic sequence such as ABBABB..., then her average capital $\langle X(t)\rangle$ is an increasing function of t, as can be seen in figure 3.

The paradox is closely related to the flashing ratchet. If we visualize the capital $X(t)$ as the position of a Brownian particle in a one-dimensional lattice, game A, for $\varepsilon = 0$, is a discretization of the free diffusion, whereas game B resembles the motion of the particle under the action of the asymmetric sawtooth potential. Figure 4 shows this spatial representation for game B compared with the ratchet potential. When the particle is on a dark site, the bad coin is used and the probability to win is very low, whereas on the white sites the most likely move is to the right. The sawtooth potential has a short spatial interval in which the force is negative and a long interval with a positive force. Equivalently, game B uses a bad coin on a 'short interval', i.e. on one site of every three on the lattice, and a good coin on a 'long interval' corresponding to two consecutive sites which are not a multiple of three (see figure 4).

As in the flashing ratchet, game B rectifies the fluctuations of game A. Suppose that we play the sequence AABBAABB... and that $X(t)$ is a multiple of three immediately after two instances of game B. Then we play game A twice, which can drive the capital back to $X(t)$ or to $X(t) \pm 2$. In the latter case, the next turn is for game B with a capital that is not a multiple of three, which means a good

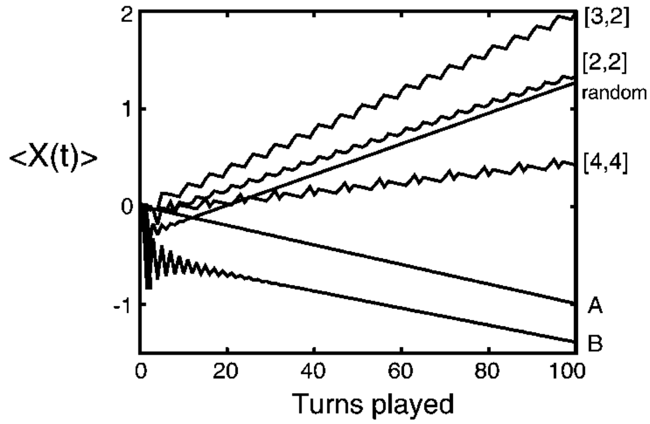

Figure 3. Average capital for 5000 players as a function of the number of turns for game A, B and their periodic and random combinations. $\varepsilon = 0.005$ and [*a*, *b*] stands for periodic sequences where A (B) is played *a* (*b*) consecutive turns.

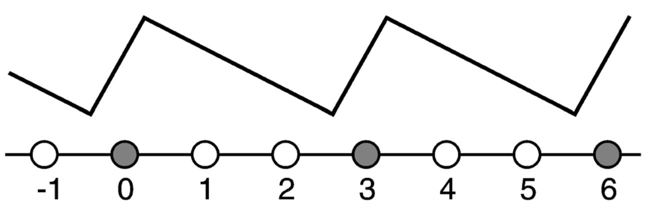

Figure 4. A random walk picture of game B compared with the ratchet potential. The bad coin (black dots) plays the role of the negative force acting on a short interval, whereas the two consecutive good coins (white dots) are the analogue of the positive force acting on the long intervals.

chance of winning. That is, game B rectifies the fluctuations that occurred in the two turns of game A. The rectification is not as neat as in the low temperature flashing ratchet, but enough to cause the paradox.

There is a more rigorous way of associating a potential to a gambling game by using a master equation [22]. However, it provides a similar picture of game B, as a random walk that is non-symmetric under inversion of the spatial coordinate and capable of rectifying fluctuations.

3.2. *Reorganization of trends*

Beside the ratchet effect, one can explain the paradox considering another interesting mechanism. Recall that game B is played with two coins: a good one, used whenever the capital of the player is not a multiple of three, and a bad one which is used when the capital is a multiple of three. Therefore, the 'profitability' of game B crucially depends on how often the bad coin is used, i.e. on the probability π_0 that the capital is a multiple of three. It turns out that, when game B is played, this probability is not 1/3, as one could naively expect, but larger. This can be reasoned from figure 4. When the capital is at a white site, its most likely move is to the right, whereas at dark sites the most likely move is to the left. The capital thus spends more time jumping forth and back between a multiple of three and its left-hand neighbour than what would happen if it moved completely at random. Consequently, the probability π_0 is larger than 1/3. On the other hand, under game A the capital does move in a random way. Therefore, playing game A in some turns shifts π_0 towards 1/3, or, equivalently, reduces the number of times the bad coin of game B is used. In other words, game A, although losing, boosts the effect of the good coin in B, giving the overall game a winning tendency. We have named this mechanism *reorganization of trends*, since game A reinforces the positive trend already present in game B.

In this section, we formulate this argument in a quantitative way. Let us first consider game B separately. The probability to win in the *t*-th turn can be calculated as

$$p_{\rm win}(t) = \pi_0(t)p_{\rm bad} + [1 - \pi_0(t)]p_{\rm good}, \qquad (1)$$

where $\pi_0(t)$ is the probability of $X(t)$ being a multiple of 3 (i.e. of using the bad coin).

One can calculate the value of $\pi_0(t)$, by using very simple techniques from the theory of Markov chains. First, we define the random process

$$Y(t) \equiv X(t) \bmod 3 \qquad (2)$$

taking on only three possible values or states, $Y(t) = 0, 1, 2$, depending on whether the capital $X(t)$ is a multiple of three, a multiple of three plus one, or a multiple of three plus two, respectively. This variable $Y(t)$ is a Markov process, i.e. the statistical properties of $Y(t + 1)$ depend only on the value taken on by $Y(t)$. This allows one to derive a *master equation* for its probability distribution.

Let $\pi_0(t)$, $\pi_1(t)$ and $\pi_2(t)$ be the probability that $Y(t)$ is equal to 0, 1 and 2, respectively. There are two possibilities for $Y(t) = 2$ to occur: either $Y(t-1) = 0$ and we lose in the tth turn (with probability $1-p_{\rm bad}$), or $Y(t-1) = 1$ and we win in the tth turn (with probability $p_{\rm good}$). Therefore

$$\pi_2(t) = (1 - p_{\rm bad})\pi_0(t-1) + p_{\rm good}\pi_1(t-1). \qquad (3)$$

Following the same type of argument, one can derive equations for $\pi_0(t)$ and $\pi_1(t)$, and the three equations can be written in matrix form as

$$\boldsymbol{\pi}(t) = \boldsymbol{\Pi}_{\rm B}\boldsymbol{\pi}(t-1), \qquad (4)$$

where

$$\boldsymbol{\pi}(t) \equiv \begin{pmatrix} \pi_0(t) \\ \pi_1(t) \\ \pi_2(t) \end{pmatrix} \qquad (5)$$

and

$$\Pi_B \equiv \begin{pmatrix} 0 & 1-p_{\mathrm{good}} & p_{\mathrm{good}} \\ p_{\mathrm{bad}} & 0 & 1-p_{\mathrm{good}} \\ 1-p_{\mathrm{bad}} & p_{\mathrm{good}} & 0 \end{pmatrix}. \qquad (6)$$

After a small number of turns of game B, $\boldsymbol{\pi}(t)$ approaches a stationary value $\boldsymbol{\pi}_{\mathrm{B}}^{\mathrm{st}}$, which is invariant under the transformation given by equation (4), i.e.

$$\boldsymbol{\pi}_{\mathrm{B}}^{\mathrm{st}} = \boldsymbol{\Pi}_{\mathrm{B}}\boldsymbol{\pi}_{\mathrm{B}}^{\mathrm{st}}. \qquad (7)$$

The first component of the solution of this equation reads

$$\pi_{0\mathrm{B}}^{\mathrm{st}} = \frac{5}{13} - \frac{440}{2197}\varepsilon + O(\varepsilon^2) \simeq 0.38 - 0.20\,\varepsilon, \qquad (8)$$

where we have used the values of the original paradox, $p_{\mathrm{bad}} = 1/10 - \varepsilon$ and $p_{\mathrm{good}} = 3/4 - \varepsilon$, and have expanded the solution up to first order of ε, to simplify the exposition.

Substituting this value in equation (1) we obtain the probability of winning for game B for sufficiently large t

$$p_{\mathrm{win,B}} = \frac{1}{2} - \frac{147}{169}\varepsilon + O(\varepsilon^2), \qquad (9)$$

which is less than 1/2 for $\varepsilon > 0$. This proves that game B is fair for $\varepsilon = 0$ and losing for $\varepsilon > 0$, as shown in figure 3.

The paradox arises when game A comes into play. Game A is always played with the same coin, regardless of the value of the capital $X(t)$, and therefore drives the probability distribution $\boldsymbol{\pi}(t)$ to a uniform distribution. Thus, game A makes $\pi_0(t)$ tend to 1/3. Since $1/3 < \pi_{0\mathrm{B}}^{\mathrm{st}}$, the effect of game A is to decrease the probability of using the bad coin in the turns where B is played.

This can be seen in a more precise way, since the random combination of games A and B can be again solved by using the master equation:

$$\boldsymbol{\pi}_{\mathrm{AB}}^{\mathrm{st}} = \frac{1}{2}[\boldsymbol{\Pi}_{\mathrm{B}} + \boldsymbol{\Pi}_{\mathrm{A}}]\boldsymbol{\pi}_{\mathrm{AB}}^{\mathrm{st}}, \qquad (10)$$

where

$$\boldsymbol{\Pi}_{\mathrm{A}} = \begin{pmatrix} 0 & 1-p_{\mathrm{A}} & p_{\mathrm{A}} \\ p_{\mathrm{A}} & 0 & 1-p_{\mathrm{A}} \\ 1-p_{\mathrm{A}} & p_{\mathrm{A}} & 0 \end{pmatrix} \qquad (11)$$

with $p_{\mathrm{A}} = 1/2 - \varepsilon$. The probability of using the bad coin decreases to

$$\pi_{0\mathrm{AB}}^{\mathrm{st}} = \frac{245}{709} - \frac{48880}{502681}\varepsilon + O(\varepsilon^2) \simeq 0.35 - 0.10\,\varepsilon. \qquad (12)$$

The probability of winning in this randomized combination of games A and B is

$$\begin{aligned} p_{\mathrm{win,AB}} &= \pi_{0\mathrm{AB}}^{\mathrm{st}}\frac{p_{\mathrm{bad}} + p_{\mathrm{A}}}{2} + [1 - \pi_{0\mathrm{AB}}^{\mathrm{st}}]\frac{p_{\mathrm{good}} + p_{\mathrm{A}}}{2} \\ &= \frac{727}{1418} - \frac{486795}{502681}\varepsilon + O(\varepsilon^2), \end{aligned} \qquad (13)$$

which is greater than 1/2 for a sufficiently small ε. This is the mechanism behind the paradox which we have termed 'reorganization of trends': although game A itself consists of a negative trend because it uses a slightly bad coin, it increases the probability of using the good coin of B, i.e. game A reinforces the positive trend already present in B enough to make the combination win.

Periodic sequences can also be studied as Markov chains and their probability of winning in a whole period can be easily computed using different combinations of the matrices $\boldsymbol{\Pi}_{\mathrm{A}}$ and $\boldsymbol{\Pi}_{\mathrm{B}}$. Finally, the slopes of the curves in figure 3 can be calculated as $\langle X(t+1)\rangle - \langle X(t)\rangle = 2p_{\mathrm{win}} - 1$.

3.3. *Capital-independent games*

The modulo rule in game B is quite natural in the original representation of the games as a Brownian ratchet. However, the rule may not suit some applications of the paradox to biology, biophysics, population genetics, evolution and economics. Thus, it would be desirable to devise new paradoxical games based on rules independent of the capital. Parrondo *et al.* introduced such a game in [23], inspired by the reorganization of trends explained in the last section.

In the new version, game A remains the same as before, but a game B′, which depends on the history of wins and losses of the player, is introduced. Game B′ is played with four coins B'_1, B'_2, B'_3, B'_4 following the history-based rules explained in table 1.

The paradox reappears, for instance, when setting $p_1 = 9/10 - \varepsilon$, $p_2 = p_3 = 1/4 - \varepsilon$ and $p_4 = 7/10 - \varepsilon$. With these numbers and for ε small and positive, B′ is a losing game, while either a random or a periodic alternation of A and B′ produces a winning result. Figure 5 shows a theoretical computation of the average capital for these history-dependent paradoxical games.

The paradox is reproduced because there are bad coins in game B′ which are played more often than in a completely random game, i.e. a quarter of the time. For the above choices of p_i, $i = 1, 2, 3, 4$, the bad coins are B'_2 and B'_3. The other two coins, B'_1 and B'_4, are good coins.

Due to the fact that game B′ rules depend on the history of wins and losses, the capital $X(t)$ is no longer a Markovian process. However, the random vector

Table 1. History-based rules for game B′.

Before last $t-2$	Last $t-1$	Coin at t	Prob. of win at t	Prob. of loss at t
Loss	Loss	B'_1	p_1	$1-p_1$
Loss	Win	B'_2	p_2	$1-p_2$
Win	Loss	B'_3	p_3	$1-p_3$
Win	Win	B'_4	p_4	$1-p_4$

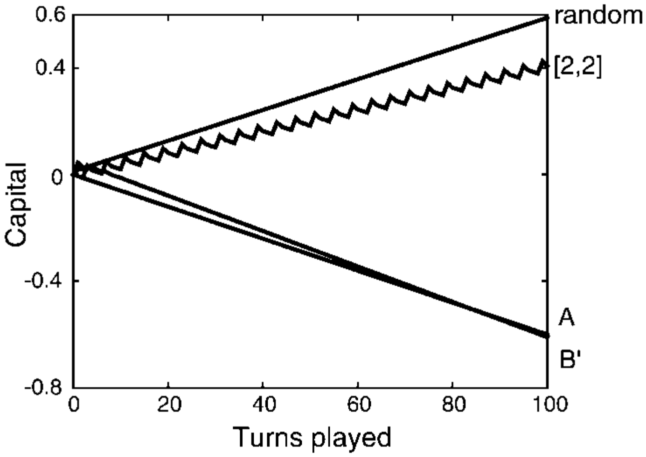

Figure 5. Average capital as a function of the number of turns in the capital independent games. We plot the result for game A and B′, as well as for the random combination and the periodic sequence AAB′B′… In all the cases, $\varepsilon = 0.003$.

$$Y(t) = \begin{pmatrix} X(t) - X(t-1) \\ X(t-1) - X(t-2) \end{pmatrix} \tag{14}$$

can take on four different values and is indeed a Markov chain. The transition probabilities are again easily obtained from the rules of game B′ and an analytical solution can be obtained following a similar argument as in section 3.2 (see however [23] for details).

We see that the mechanism that we have called reorganization of trends can be used to extend the paradox to other gambling games. It is also noteworthy that the price we must pay to eliminate the dependence on the capital in the original paradox is to consider history-dependent rules, i.e. games where the capital is no longer Markovian.

4. Collective games

In this section we analyse three different versions of paradoxical games played by a large number of individuals. The three share the feature that it can sometimes be better for the players to sacrifice short term benefits for higher returns in the future.

4.1. *Capital redistribution brings wealth.*

Reorganization of trends tells us that the essential role of game A in the paradox is to randomize the capital and make its distribution more uniform. Toral has found that a redistribution of the capital in an ensemble of players has the same effect [17].

In the new paradoxical games introduced by Toral in [17], there are N players and one of them is randomly selected in each turn. They can play two games. The first one, game A′, consists of giving a unit of his capital to another randomly chosen player in the ensemble, that is, game A′ is nothing but a redistribution of the total capital. The second one, game B, is the same as in the original paradox. Under game A′ the capital does not change, whereas game B is, as before, a losing game. The striking result is that the random combination of the two games is winning, i.e. the redistribution of capital performed in the turns where A′ is played turns the losing game B into a winning one, actually increasing the total capital available. Thus, the redistribution of capital turns out to be beneficial for everybody. This effect is shown in figure 6 where the average total capital in a simulation with 10 players and 500 realizations is depicted for games B and A′, and for their random combination. It is remarkable that the effect is still present when the capital is required to flow from richer to poorer players (see [17] for details).

The explanation of this phenomenon follows the same lines as in the original paradox.

4.2. *Dangerous choices I: the voting paradox*

Up to now, we have considered sequences of games that are 'imposed' on the player or players. Either they play game A, game B, or a periodic or random sequence, but we never allow the players to choose the game to be played in each turn. In the case of a single player this deference is quite generous, since her capital would increase on average under the following trivial choice: she selects game B if her capital is not a multiple of three and game A otherwise. This is undoubtedly the best strategy, because the best coin is always used in every turn. Moreover, it is not difficult to see that this choice strategy performs much better than any other random or periodic combination of games.

However, things change when we consider an ensemble of players. How can the ensemble choose the game to be played in each turn? There are some possibilities, such as letting them vote for the preferred game or trying to maximize the winning probability in each turn. Which is then the best choice strategy? We will see that the paradoxical games also yield some surprises in this context: the choice preferred by the majority of the ensemble turns out to be worse than a random or periodic combination of games. Even if we select the game maximizing the profit in every turn, we can end with systematic losses, as shown in the next section.

Consider a set of N players who play game A or B against a casino. In each turn, *all* of them play the *same* game. Therefore, they have to make a collective decision, choosing between game A or B in each turn. We will first use a *majority rule* to select the game, that is, the game

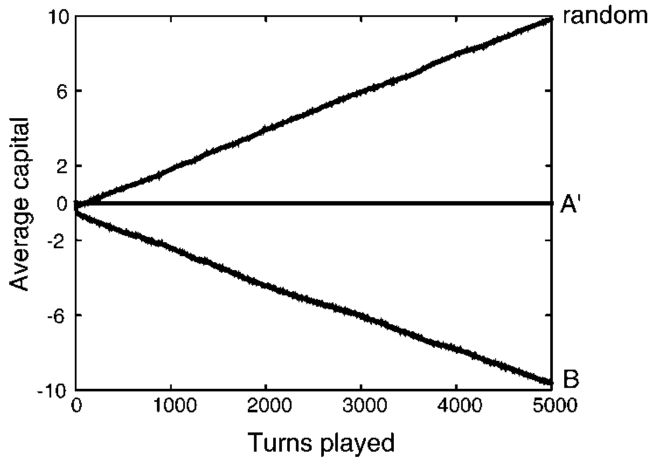

Figure 6. Average capital per player as a function of the number of turns in game B, game A′ (redistribution of capital) and the random combination. The data have been obtained for $\varepsilon = 0.01$, simulating an ensemble of 10 players and averaging over 500 realizations.

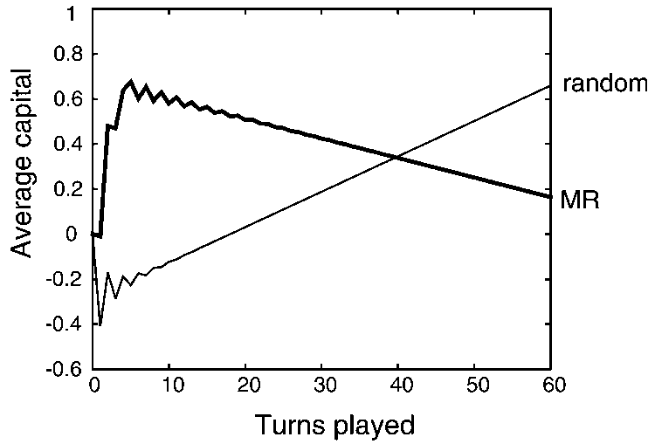

Figure 7. Average capital per player in the collective games as a function of the number of turns, when the game is selected at random and following the preference of the majority of the players (MR). Notice that, in the stationary regime, the majority rule (MR) yields systematic losses whereas the random choice wins on average. These are analytical results with $\varepsilon = 0.005$ and an infinite number of players.

which receives more votes is played by all the players simultaneously. The vote of each player will be determined by her capital, following the strategy that we have explained above for a single player. Players with capital that is a multiple of three will vote for game A, whereas the rest will vote for B.

This strategy, which is optimal for a single player, turns out to be losing if the number of players is large enough [19]. This is shown in figure 7, where we plot the average capital in an ensemble with an infinite number of players. On the other hand, if the game is selected at random the capital increases with time.

In order to explain this behaviour, we will focus on the evolution of $\pi_0(t)$, the fraction of players whose capital is a multiple of three. The selection of the game by voting can be rephrased in terms of $\pi_0(t)$. As mentioned above, every player votes for the game which offers her the higher probability of winning according to her own state. Then, every player whose capital is a multiple of three will vote for game A in order to avoid the bad coin in B. That accounts for a fraction $\pi_0(t)$ of the votes. The remaining fraction $1-\pi_0(t)$ of the players will vote for game B to play with the good coin. Since the majority rule establishes that the game which receives more votes is selected, game A will be played if $\pi_0(t) > 1/2$. Conversely, the whole set of players will play game B when $\pi_0(t)$ is below 1/2.

On the other hand, as we have seen in section 3.2, playing game B makes $\pi_0(t)$ tend to a stationary value given by equation (8), namely, $\pi_{0\mathrm{B}}^{\mathrm{st}} \simeq 0.38 - 0.2\varepsilon < 1/2$ for $\varepsilon > 0$, whereas playing game A makes π_0 tend to 1/3. This is still valid for the present model, since the N players only interact when they make the collective decision, otherwise they are completely independent.

If $\pi_0(t) > 1/2$, then the ensemble of players will select game A. The fraction $\pi_0(t)$ will decrease until it crosses this critical value 1/2. At that turn, B is the selected game and it will remain so as long as π_0 does not exceed 1/2. However, this can never happen, since game B drives π_0 closer and closer to $\pi_{0\mathrm{B}}^{\mathrm{st}}$ which is below 1/2. Hence, after a number of turns, the system gets trapped playing game B forever with π_0 asymptotically approaching $\pi_{0\mathrm{B}}^{\mathrm{st}}$. Since ε is positive, game B is a losing game (cf. section 3.2) and, therefore, the majority rule yields systematic losses, as can be seen in figure 7. We have also plotted in figure 8 the fraction $\pi_0(t)$, to check that, once $\pi_0(t)$ crosses 1/2, game B is always chosen and $\pi_0(t)$ approaches $\pi_{0\mathrm{B}}^{\mathrm{st}}$, staying far below 1/2.

On the other hand, if, instead of using the majority rule, we select the game at random or following a periodic sequence, game A will be chosen even though $\pi_0 < 1/2$. This is a bad choice for the majority of the players, since playing B would make them toss the good coin. That is, the random or periodic selection will contradict from time to time the will of the majority. Nevertheless, choosing the game at random keeps π_0 away from $\pi_{0\mathrm{B}}^{\mathrm{st}}$, as shown in figure 8, i.e. in a region where game B is winning $\left(\pi_0 < \pi_{0\mathrm{B}}^{\mathrm{st}}\right)$. Therefore, the random choice yields systematic gains, as shown in figure 7.

It is worth noting that choosing the game at random is exactly the same as if every player voted at random. Therefore, the players get a winning tendency when they vote at random whereas they lose their capital when they vote according to their own benefit in each run.

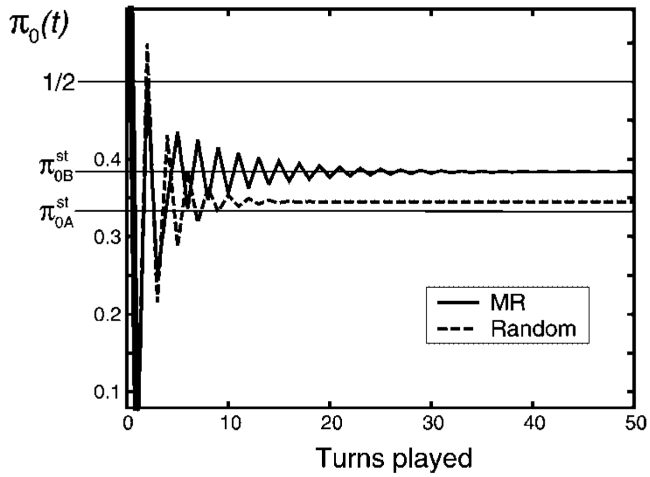

Figure 8. The fraction of players $\pi_0(t)$ with a capital multiple of three as a function of time when the game is chosen at random and following the majority rule (MR). In both cases, $\varepsilon = 0.005$ and $N = \infty$. The horizontal lines indicate the threshold value for the majority rule (1/2), and the stationary values for games A and B, π_{0A}^{st} and π_{0B}^{st}, respectively. The figure clearly shows that the random strategy keeps $\pi_0(t)$ small, whereas the majority rule, selecting B most of the time, drives $\pi_0(t)$ to a value where both game A and B are losing.

4.3. *Dangerous choices II: the risks of short-range optimization*

Yet another 'losing now to win later' effect can be observed in the collective paradoxical games with another choice strategy. As in the previous example, we consider a large set of players, but we have to add a small ingredient to achieve the desired effect: now only a randomly selected fraction γ of them play the game in each turn. Suppose we know the capital of every player so we can compute which game, A or B, will give the larger average payoff in the next turn. Again, and even more strikingly, selecting the 'most favourable game' results in systematic losses whereas choosing the game at random or following a periodic sequence steadily increases the average capital [18].

Knowledge of the capital of every player allows us to choose the game with the highest average payoff in the next turn, since this optimal game can easily be obtained from the fraction $\pi_0(t)$ of players whose capital is a multiple of three. These individuals will play the bad coin if game B is chosen and the remaining fraction $1-\pi_0(t)$ will play the good coin. Hence, the probability of winning for game B reads

$$p_{\text{winB}} = \pi_0 p_{\text{bad}} + (1-\pi_0)p_{\text{good}}. \tag{15}$$

In case game A is selected, the probability to win is $p_{\text{winA}} = p_A = 1/2-\varepsilon$ for all time t. Therefore, to choose the game with the larger payoff $\langle X(t+1)\rangle - \langle X(t)\rangle = 2p_{\text{win}}-1$ in every turn t, we must

$$\begin{aligned} &\text{play A} \quad \text{if} \quad p_{\text{winA}} \geq p_{\text{winB}}(\pi_0),\\ &\text{play B} \quad \text{if} \quad p_{\text{winA}} < p_{\text{winB}}(\pi_0), \end{aligned} \tag{16}$$

or equivalently

$$\begin{aligned} &\text{play A} \quad \text{if} \quad \pi_0(t) \geq \pi_{0c},\\ &\text{play B} \quad \text{if} \quad \pi_0(t) < \pi_{0c}, \end{aligned} \tag{17}$$

with $\pi_{0c} \equiv (p_A-p_{\text{good}})/(p_A-p_{\text{bad}}) = 5/13$. We will call this way of selecting the game *the short-range optimal strategy*. We will also consider that the game is selected following either a random or periodic sequence. These are both *blind* strategies, since they do not make any use of the information about the state of the system. However, and surprisingly enough, they turn out to be much better than the short-range optimal strategy, as shown in figure 9.

Notice that (17) is similar to the way the game is selected by the majority rule considered in the previous section, but replacing 1/2 by the new critical value $\pi_{0c} = 5/13$. Therefore, the explanation of this model goes quite along the same lines as for the voting paradox, although with some differences. Unlike 1/2, π_{0c} equals the stationary value of $\pi_0(t)$ for game B when $\varepsilon = 0$. As in the voting paradox, game A drives π_0 below π_{0c} because game A makes π_0 tend to 1/3. If $\pi_0(t) < \pi_{0c}$, then game B is played, but $\pi_0(t+1)$ will be still below π_{0c} only for γ sufficiently small. For example, if $\gamma = 1/2$ and $\varepsilon = 0$, game B is chosen forty times in a row before switching back to game A, making π_0 become approximately equal to π_{0B}^{st} at almost every turn. This behaviour is shown in figure 10. As long as π_0 is close to π_{0B}^{st}, the average capital remains approximately constant, as shown in figure 11.

In contrast, the periodic and random strategies choose game A with $\pi_0 < \pi_{0c}$. Although this does not produce earnings in that turn, it keeps π_0 away from π_{0B}^{st}. When game B is chosen again, it has a large expected payoff since π_0 is not close to π_{0B}^{st}. By keeping π_0 not too close to π_{0B}^{st}, the blind strategies perform better than the short-range optimal prescription, as can be seen in figure 11.

The introduction of $\varepsilon > 0$ has two effects. First of all, it makes π_{0B}^{st} decrease by a small amount, as indicated in equation (8). This makes it even more difficult for the short-range optimal strategy to choose game A, and after a few runs game B is always selected. Since game B is now a losing game, the short-range optimal strategy is also losing whereas periodic and random strategies keep their winning tendency, as can be seen in figure 9.

To summarize, the short-range optimal strategy chooses B most of the time, since it is the game which gives the highest returns in each turn. However, this choice drives $\pi_0(t)$ to a region in which B is no longer a winning game. On the other hand, the random strategy

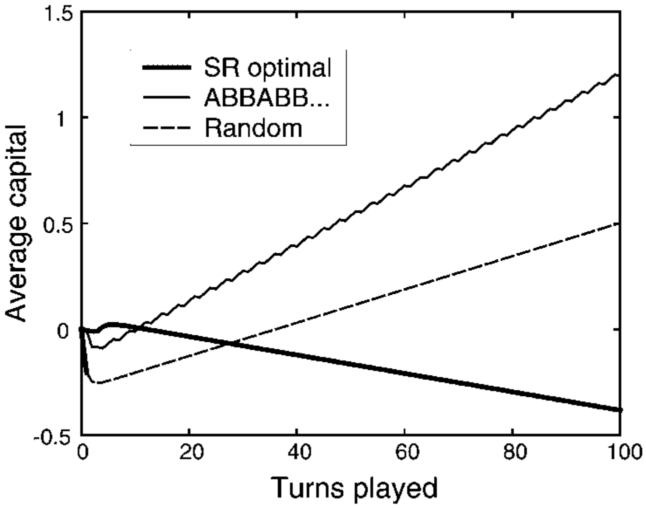

Figure 9. Average capital as a function of time for the three different strategies explained in the text, with $N = \infty$, $\gamma = 0.5$ and $\varepsilon = 0.005$. The short-range (SR) optimal strategy is losing in the stationary regime, whereas the two blind strategies, i.e. choosing the game to be played either at random or following the periodic sequence (ABBABB...), yield a systematic gain.

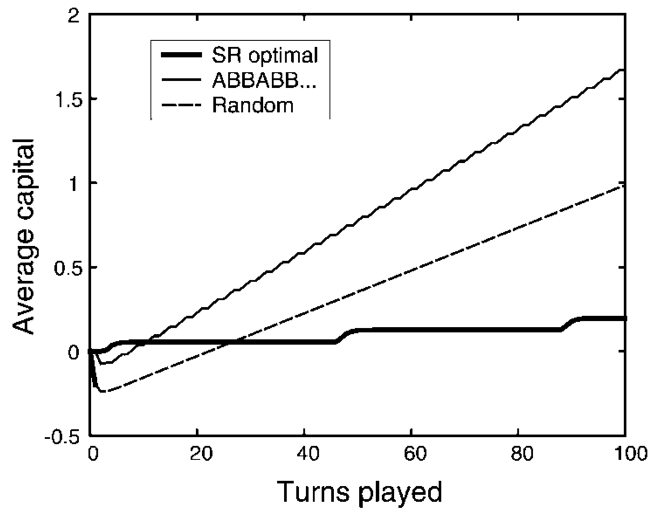

Figure 11. Average capital as a function of time for the three different strategies explained in the text and for $\varepsilon = 0$, $N = \infty$ and $\gamma = 0.5$. In this case the short-range optimal strategy is still winning, due to the small jumps coinciding with the selection of game A. However, for most of the turns game B is played with a value of $\pi_0(t)$ very close to π_{0B}^{st}.

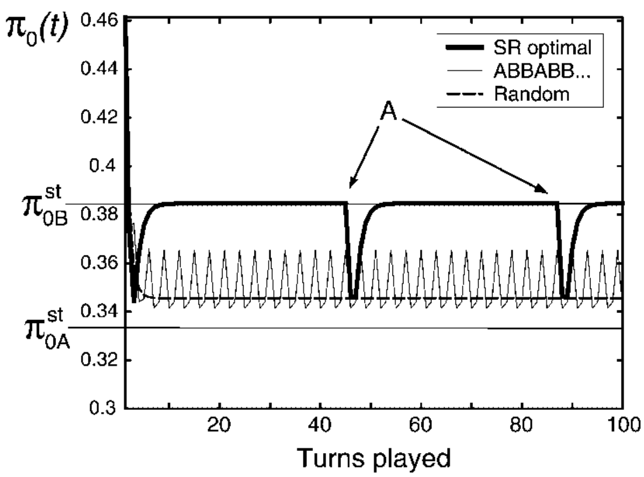

Figure 10. The fraction $\pi_0(t)$ of players with a capital multiple of three as a function of the number of turns, for $\varepsilon = 0$, $N = \infty$ and $\gamma = 0.5$. The horizontal lines show the stationary values for game A and game B (which coincides with the critical fraction π_{0c} for the short-range optimal strategy). As we have in figure 8 with the majority rule, the short-range optimal strategy drives $\pi_0(t)$ towards higher values than the other two strategies.

from time to time sacrifices the short term returns by selecting game A, but this choice keeps the system in a 'productive region'. We could say that the short-range optimal strategy is 'killing the goose that laid the golden eggs', an effect that is also present in simple deterministic systems [18].

5. Conclusions

We have presented the original Parrondo's paradox and several examples showing how the basic mechanisms underlying the paradox can yield other counter-intuitive phenomena. We finish by reviewing these mechanisms as well as the literature related with the paradox.

The first mechanism, the ratchet effect, occurs when fluctuations can help to surmount a potential barrier or a 'losing streak'. These fluctuation either come from another losing game, such as in the original paradox, from a redistribution of the capital, such as in Toral's collective games [17], or from a purely diffusive motion, such as in the flashing ratchet.

A second mechanism is the reorganization of trends, which occurs when game A reinforces a positive trend already present in game B. The same mechanism can be observed in the games with capital independent rules and it helps to understand the counter-intuitive behaviour of the collective games presented in section 4.2 and 4.3, where random choices perform better than the choice preferred by the majority or the one optimizing short-term returns. These models also prompt the question of how information can be used to design a strategy. It is a relevant question for control theory and also for statistical mechanics, since the paradox is a purely collective effect that goes away for a single player, i.e. the choices following the short-range optimal strategy and the majority rule perform much better than the random or periodic ones.

There is a third mechanism which we have not addressed in the paper, but immediately arises if we consider the games as dynamical systems: the outcome of an alternation of dynamics can always be interpreted as a stabilization of transient states. This interpretation has allowed some authors to extend the basic message of the paradox to pattern formation in spatially extended systems [24–27]. In these papers, a new mechanism of pattern formation based on the alternation of dynamics is introduced. They show how the global alternation of two dynamics, each of which leads to a homogeneous steady state, can produce stationary or oscillatory patterns upon alternation.

Another interesting application of the stabilization of transient states is presented in [28]. Two dynamics for the population of a virus are introduced with the following property: in each dynamics, the population vanishes, whereas the alternation of the two dynamics, whose origin could be the seasonal variation, induces an outbreak of the virus.

Similar effects can be seen in quantum systems. Lee *et al.* have devised a toy model in which the alternation of two decoherence dynamics can significantly decrease the decoherence rate of each separate dynamics [29]. Also in the quantum domain, the paradox has received some attention: there have been some proposals of a quantum version of the games [30, 31] closely related with the recent theory of quantum games [32], and the paradox has been reproduced in the contexts of quantum lattice gases [33] and quantum algorithms [34].

To finish this partial account of the existing literature on the paradox, we mention the work by Arena *et al.* [35], who analyse the performance of the games using chaotic instead of random sequences of choices; that of Chang and Tsong [36], who study the hidden coupling between the two games in the paradox and present several extensions even for deterministic dynamics; and the paper by Kocarev and Tasev [37], relating the paradox with Lyapunov exponents and stochastic synchronization.

In summary, Parrondo's paradox has drawn the attention of many researchers to non-trivial phenomena associated with switching between two dynamics. We have tried to reveal in this paper some of the basic mechanisms that can yield an unexpected behaviour when switching between two dynamics, and how these mechanisms work in several versions of the paradox. As mentioned in the introduction, we believe that the paradox and its extensions are contributing to a deeper understanding of stochastic dynamical systems. In the case of statistical mechanics, switching is in fact a source of non-equilibrium which is ubiquitous in nature, due to day–night or seasonal variations [28]. Nevertheless, it had not been studied in depth until the recent introduction of ratchets and paradoxical games. As the paradox suggests, we will probably see in the future new models and applications confirming that noise and switching, even between equilibrium dynamics, can be a powerful combination to create order and complexity.

Acknowledgements

We thank valuable comments on the original manuscript made by Katja Lindenberg, Javier Buceta, Martin Plenio and H. Leonardo Martínez. This work was supported by a grant from the *New Del Amo Program* (UCM) and by MCYT-Spain Grant BFM 2001-0291.

References

[1] Harmer, G. P., and Abbott, D., 1999, *Stat. Sci.*, **14,** 206.
[2] Harmer, G. P., and Abbott, D., 1999, *Nature*, **402,** 846.
[3] Harmer, G. P., and Abbott, D., 2002, *Fluct. Noise Lett.*, **2,** R71.
[4] Ajdari, A., and Prost, J., 1993, *CR Acad. Sci.* (Paris) II, **315,** 1635.
[5] Magnasco, M. O., 1993, *Phys. Rev. Lett.*, **71,** 1477.
[6] Astumian, R. D., and Bier, M., 1994, *Phys. Rev. Lett.*, **72,** 1766.
[7] Reimann, P., 2002, *Phys. Rep.*, **361,** 57.
[8] Linke, H., 2002, *Appl. Phys.* A, **75,** 167.
[9] Einstein, A., 1956, *Investigations on the Theory of Brownian Movement* (New York: Dover).
[10] Gammaitoni, L., Hanggi, P., Jung, P., and Marchesoni, F., 1998, *Rev. Mod. Phys.*, **70,** 223.
[11] Van den Broeck, C., Parrondo, J. M. R., and Toral, R., 1994, *Phys. Rev. Lett.*, **73,** 3395.
[12] García-Ojalvo, J., and Sancho, J. M., 1999, *Noise in Spatially Extended Systems* (New York: Springer-Verlag).
[13] Hacking, I., 1990, *The Taming of Chance* (Cambridge: Cambridge University Press).
[14] Bachelier, L., 1900, *Théorie de la Spéculation* (Thesis), Annales Scientifiques de l'Ecole Normale Superieure, III-17, pp. 21–86.
[15] Mantegna, R. N., and Stanley, H. E., 2000, *An Introduction to Econophysics Correlations and Complexity in Finance* (Cambridge: Cambridge University Press).
[16] Mori, T., and Kai, S., 2002, *Phys. Rev. Lett.*, **88,** 218101.
[17] Toral, R., 2002, *Fluct. Noise Lett.*, **2,** L305.
[18] Dinis, L., and Parrondo, J. M. R., 2003, *Europhys. Lett.*, **63,** 319.
[19] Dinis, L., and Parrondo, J. M. R., 2003, *Inefficiency of voting in Parrondo games*, submitted.
[20] Parrondo, J. M. R., and Cisneros, B. J., 2002, *Appl. Phys.* A, **75,** 179.
[21] Parrondo, J. M. R., Blanco, J. M., Cao, F. J., and Brito, R., 1998, *Europhys. Lett.*, **43,** 248.
[22] Toral, R., Amengual, P., and Mangioni, S., 2003, *Physica* A, **327,** 105.
[23] Parrondo, J. M. R., Harmer, G. P., and Abbott, D., 2000, *Phys. Rev. Lett.*, **84,** 5526.
[24] Buceta, J., Lindenberg, K., and Parrondo, J. M. R., 2002, *Phys. Rev. Lett.*, **88,** 024103.
[25] Buceta, J., Lindenberg, K., and Parrondo, J. M. R., 2002, *Phys. Rev.* E, **66,** 036216; 2002, *ibid.*, **66,** 069902(E).
[26] Buceta, J., Lindenberg, K., and Parrondo, J. M. R., 2002, *Fluct. Noise Lett.*, **2,** R139.
[27] Buceta, J., and Lindenberg, K., 2002, *Phys. Rev.* E, **66,** 046202.
[28] Escudero, C., Buceta, J., de la Rubia, F. J., and Lindenberg, K., 2003, cond-mat/0303596.

[29] Lee, C. F., Johnson, N. F., Rodríguez, F., and Quiroga, L., 2002, *Fluct. Noise Lett.*, **2,** L293.
[30] Flitney, A. P., Ng, J., and Abbott, D., 2002, *Physica* A, **314,** 35.
[31] Meyer, D. A., and Blumer, H., 2002, *Fluct. Noise Lett.*, **2,** L257.
[32] Eisert, J., Wilkens, M., and Lewenstein, M., 1999, *Phys. Rev. Lett.*, **83,** 3077.
[33] Meyer, D. A., and Blumer, H., 2002, *J. Stat. Phys.*, **107,** 225.
[34] Lee, C. F., and Johnson, N. F., 2002, quant-ph/0203043.
[35] Arena, P., Fazzino, S., Fortuna, L., and Maniscalco, P., 2003, *Chaos Solitons and Fractals*, **17,** 545.
[36] Chang, C. H., and Tsong, T. Y., 2003, *Phys. Rev.* E, **67** 025101.
[37] Kocarev, L., and Tasev, Z., 2002, *Phys. Rev.* E, **65,** 046215.

Juan M. R. Parrondo studied in the Universidad Complutense de Madrid (Spain) where he obtained both his Major (1987) and his PhD (1992) in Physics. He worked as a postdoc at the University of California, San Diego (USA) and at the Limburgs Universitair Centrum (Belgium). Since 1996 he has been a professor at the Universidad Complutense de Madrid. His main research interests are the study of fluctuations in non-equilibrium statistical physics and the foundations of statistical mechanics.

Luis Dinís obtained his Major in Physics (2000) and his Master in Complex Systems (2002) at the Universidad Complutense de Madrid. His main research interests lie in the area of statistical physics and biophysics.