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BROWNIAN MOTION WITH A PARABOLIC DRIFT AND AIRY FUNCTIONS

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Let $\{W(t): t \geq s\}$ be Brownian motion, starting at x at time s. The densities of first passage times of the process $\{W(t) - ct^2: t \geq s\}$ are determined analytically in terms of Airy functions; the joint distribution of the maximum and the location of the maximum of this process is also expressed in terms of Airy functions. Corresponding results are given for two-sided Brownian motion. The structure of a jump process of locations of maxima of Brownian motion with respect to a family of parabolas is derived. This process plays a fundamental role in describing the limiting global behavior of certain estimators of densities and distribution functions. As a probabilistic side result the distribution of excursion integrals is obtained.

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1. Introduction. Let $\{W(t): t \geq s\}$ be one-dimensional Brownian motion in standard scale, starting at x at time s. We will study processes of the form

$$(1.1) \qquad \{W(t) - ct^2 : t \geq s\},$$

where $c > 0$, and in particular determine analytically in terms of Airy functions the joint density of the maximum and the location of the maximum of such a process (Corollary 3.1 in section 3). We will also derive analogous results for the process

$$(1.2) \qquad \{W(t) - ct^2 : t \in \mathbb{R}\},$$

where $\{W(t): t \in \mathbb{R}\}$ is two-sided Brownian motion, originating from zero.

Processes of type (1.1) and (1.2) arise in the following contexts: maximum number of infectives during a closed epidemic (Daniels (1974), Barbour (1981), Daniels & Skyrme (1984)), strength of a series-parallel system or a bundle of threads (Daniels (1945), Phoenix & Taylor (1973), Smith (1982)), estimation of the mode of a distribution (Chernoff (1964), Venter (1967)), estimation of a monotone or unimodal density or hazard rate (Prasaka Rao (1969), Groeneboom (1984)), and monotone empirical Bayes tests (van Houwelingen (1984)); this list of situations where the processes of type (1.1) and (1.2) are studied is by no means exhaustive.

As an example, it was shown by Chernoff (1964) that an intuitively appealing estimator of the mode of a distribution (based on an interval of fixed length, shifted along the line to a position where it contains the highest number of observations) converges in distribution (after standardization) to the location of the maximum of the process (1.2), with $c = 1$, as the sample size tends to infinity, if the underlying distribution satisfies certain regularity conditions. He also showed that this random variable

$$(1.3) \qquad V = \sup\{t \in \mathbb{R} : W(t) - t^2 \text{ is maximal}\}$$

has a density given by

$$(1.4) \qquad f_V(t) = \tfrac{1}{2} \lim_{x \uparrow t^2} \frac{\partial}{\partial x} u(t,x) \frac{\partial}{\partial x} u(-t,x),$$

where

$$(1.5) \qquad u(t,x) = P^{(t,x)}\{W(u) > u^2, \text{ for some } u \geq t\}$$

is the probability that Brownian motion, starting at position x at time t, will cross the parabola $g(u) = u^2$ at some time $u \geq t$. Note that we define the location of the maximum of the process (1.2) as the *last* time that the maximum is attained, although, with probability one, there will only be one such maximum; we define the location of the maximum by (1.3) to have a well-defined functional of the process also on sets of probability zero. It is also shown in Chernoff (1964) that $u(t,x)$ is a solution of the (backward) heat equation

$$(1.6) \qquad \frac{\partial}{\partial t} u(t,x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x)$$

under the boundary conditions

$$(1.7) \qquad u(t,t^2) = \lim_{x \uparrow t^2} u(t,x) = 1,$$

$$\lim_{x \to -\infty} u(t,x) = 0,$$

in the domain $\{(t,x): x < t^2\}$.

The nature of the solution of the heat equation, defined by (1.6) and (1.7), has been somewhat of a mystery. Numerical results were kindly provided to me by W.R. van Zwet, who developed a method for solving the heat equation numerically, in cooperation with M.N. Spijker.

It is shown in Theorem 3.1 of this paper that there exists in fact a solution in closed form in terms of the Airy functions Ai and Bi (for definitions, see [1]). Our basic tools are the Cameron-Martin-Girsanov formula, which is used to reduce the computations for the drifting process to computations for a (killed) process without drift, and the Feynman-Kac formula by which the Radon-Nikodym derivative of the drifting process with respect to the process without drift is further analyzed. In this way we obtain at the same time probabilistic interpretations of the analytical results; Theorem 2.1 provides an example of this parallel development. In particular, it is shown that there is a close relation between Airy functions and expectations of certain functionals of Bes(3) processes, where we denote by Bes(3) a 3-dimensional Bessel process (radial part of

3-dimensional Brownian motion).

The plan of the paper is as follows. The basic machinery is developed in section 2, where we determine the structure of the drifting process (1.1), killed when reaching a certain level (see e.g. Lemma 2.1); we also determine the densities of the first passage times of the process (1.1) (see Theorem 2.1). In section 3 we determine the joint density of the maximum and the location of the maximum of the processes (1.1) and (1.2). As a corollary we obtain the (marginal) density of the location of the maximum of the process (1.2), which is given by

$$(1.8) \qquad f_Z(t) = \tfrac{1}{2} g_c(t) g_c(-t),$$

where the function g_c has Fourier transform

$$(1.9) \qquad \hat{g}_c(s) = (2/c)^{1/3}/\mathrm{Ai}(i(2c^2)^{-1/3}s), \qquad s \in \mathbb{R},$$

see Corollary 3.3. This last result was recently derived independently, by different methods, in Daniels & Skyrme (1984).

In section 4 we derive the structure of a jump process of locations of maxima of Brownian motion with respect to a *family* of parabolas. This process plays a fundamental role in describing the limiting *global* behavior of certain "isotonic" estimators of densities and distribution functions. This is detailed in Groeneboom (1984) in the context of the estimation of a monotone density. As a probabilistic side result we obtain the distribution of

$$(1.10) \qquad \int_0^1 e(t)\,dt,$$

where $\{e(t): t \in [0,1]\}$ is a Brownian excursion on $[0,1]$ (which is, loosely speaking, a Brownian bridge "conditioned to be positive"). The Laplace transform of the density of the random variable (1.10) is given by (4.13) in Lemma 4.2.

2. First passage times of the process $\{W(t) - ct^2 : t \geq s\}$.

Let, for $s \in \mathbb{R}$, $C([s,\infty);\mathbb{R})$ be the space of continuous functions $f: [s,\infty) \to \mathbb{R}$, endowed with the topology of uniform convergence on compact sets, and let $\mathcal{F}$ be the Borel σ-field of $C([s,\infty);\mathbb{R})$. Furthermore, let, for $c > 0$, the probability measure $Q_c^{(s,x)}$ on $\mathcal{F}$ correspond to the process $\{X(t): t \geq s\}$, starting at x at time s, where $X(t) = W(t) - ct^2$ and $\{W(t): t \geq s\}$ is Brownian motion (in standard scale), starting at $x + cs^2$ at time s.

In this section we will show that the densities under $Q_c^{(s,x)}$ of the first passage times

$$(2.1) \qquad \tau_a = \inf\{t \geq s : X(t) = a\}, \qquad a > x,$$

can be written as functionals of a Bessel Bes(3) process, and we will characterize analytically these functionals in terms of Airy functions (for definitions and properties of Airy functions, see e.g. [1]).

Some of the relevant properties of a Bes(3) process are summarized below. A Bes(3) process is a one-dimensional diffusion process with transition densities

$$(2.2) \qquad p_t(x,y) = \begin{cases} 2t^{-3/2}y^2\phi(y/\sqrt{t}), & x = 0, \quad y > 0, \\ t^{-1/2}x^{-1}y\{\phi(\frac{y-x}{\sqrt{t}}) - \phi(\frac{y+x}{\sqrt{t}})\}, & x,y > 0, \end{cases}$$

where $\phi(z) = (2\pi)^{-\frac{1}{2}}\exp(-\frac{1}{2}z^2)$. The process describes the distribution of the radial part of 3-dimensional Brownian motion, see e.g. Itô and McKean (1974), section 2.3. The process can also be characterized as Brownian motion (Doob-)conditioned to hit ∞ before 0, see e.g. Williams (1974) (this last interpretation is the one which is most useful for our purposes).

The distribution of the first passage time τ_a is given in the following theorem.

Theorem 2.1. *Let, for* $c > 0$, $s, x \in \mathbb{R}$, $Q_c^{(s,x)}$ *be the probability measure on the Borel* σ*-field of* $C([s,\infty);\mathbb{R})$, *corresponding to the process* $\{X(t): t \geq s\}$, *where* $X(t) = W(t) - ct^2$ *and* $\{W(t): t \geq s\}$ *is Brownian motion, starting at* $x + cs^2$ *at time* s. *Let the first passage time* τ_a *of the process* X *be defined by* (2.1), *where, as usual, we define*

$\tau_a = \infty$, *if* $\{t \geq s: X(t) = a\} = \emptyset$. *Then*

(i) $$Q_c^{(s,x)}\{\tau_a \in dt\} =$$

$$= \exp\{-\tfrac{2}{3}c^2(t^3 - s^3) - 2cs(a-x)\}\psi_{a-x}(t-s)$$

$$\cdot E^0\{\exp(-2c\int_0^{t-s} B(u)du) \,|\, B(t-s) = a-x\}\, dt,$$

where $\{B(u): u \geq 0\}$ *is a* Bes(3) *process, starting at zero at time* 0, *with corresponding expectation* E^0, *and where* $\psi_z(u) = \{2\pi u^3\}^{-\frac{1}{2}} z\exp(-z^2/2u)$, $u, z > 0$, *is the value at* u *of the density of the time of the first passage through zero of Brownian motion, starting at* z *at time* 0.

(ii) $$Q_c^{(s,x)}\{\tau_a \in dt\} =$$

$$= \exp\{-\tfrac{2}{3}c^2(t^3 - s^3) - 2sc(a-x)\}\, h_{c,a-x}(t-s)dt,$$

where the function $h_{c,a-x}: \mathbb{R}^+ \to \mathbb{R}^+$ *has Laplace transform*

$$\hat{h}_{c,a-x}(\lambda) = \int_0^\infty e^{-\lambda u}\, h_{c,a-x}(u)du =$$

$$= Ai((4c)^{1/3}(a-x) + \xi)/Ai(\xi), \qquad \xi = (2c^2)^{-1/3}\lambda > 0,$$

and Ai *denotes the Airy function* Ai, *as defined on* p.446 *of* [1].

We will prove Theorem 2.1 by studying the structure of the process X, which is killed when reaching a. It follows from the Cameron-Martin-Girsanov formula that the transition densities of this process factorize into the transition densities of ordinary Brownian motion, killed when reaching a, and a factor involving an integral over a Brownian bridge, which is conditioned on staying below a. This factorization is given in the following lemma.

Lemma 2.1. *Let, for* $a > x, y$ *and* $s < t$*, the transition density* q^{∂} *be defined by*

$$(2.3) \qquad Q_c^{(s,x)}\{X(t) \in dy, \ \max_{s \le u \le t} X(u) < a\} = q^{\partial}(s,x;t,y)dy,$$

i.e. q^{∂} *is the transition density of the process* X*, killed when reaching* a*. Then*

$$(2.4) \qquad q^{\partial}(s,x;t,y) = (t-s)^{-\frac{1}{2}}\left\{\phi\left(\frac{x-y}{\sqrt{t-s}}\right) - \phi\left(\frac{x+y-2a}{\sqrt{t-s}}\right)\right\}$$
$$\cdot \exp\{-\tfrac{2}{3} c^2(t^3 - s^3) + 2c(ty - sx)\}$$
$$\cdot E_{\partial}^{(s,x)}\{\exp(2c\int_s^t W(u)du) \,|\, W(t) = y\},$$

where $\{W(u): u \ge s\}$ *is a Brownian motion process, starting at* x *at time* s*, and killed when reaching* a*, with corresponding expectation operator* $E_{\partial}^{(s,x)}$*, and where* $\phi(z) = (2\pi)^{-\frac{1}{2}}\exp(-\frac{1}{2}z^2)$.

Remark 2.1. Here and in the following, the index ∂ (for "cemetary") is used as a notational convention to indicate that there is a killing going on.

Proof of Lemma 2.1. Let $P^{(s,x)}$ be the probability measure on the Borel σ-field F of $C([s,\infty); \mathbb{R})$, corresponding to the Brownian motion $\{W(t): t \ge s\}$, starting at x at time s. Furthermore, let $F_t = \sigma\{W(z): s \le z \le t\}$. By the Cameron-Martin-Girsanov formula (Stroock and Varadhan (1979), section 6.4) we have

$$Q_c^{(s,x)}(A) = E^{P^{(s,x)}} 1_A \cdot Z(t), \qquad A \in F_t,$$

where 1_A is the indicator of the set A and

$$Z(t) = \exp\{-2c\int_s^t u dW(u) - \tfrac{2}{3} c^2(t^3 - s^3)\}.$$

The stochastic integral $\int_s^t u dW(u)$ can be defined by integration by parts:

$$(2.5) \qquad \int_s^t u dW(u) = tW(t) - sW(s) - \int_s^t W(u)du.$$

Now let $A = \{W(u) < a, \ s \le u \le t\}$ and define $f_\varepsilon = (2\varepsilon)^{-1} 1_{[y-\varepsilon,y+\varepsilon]}$, where $1_{[y-\varepsilon,y+\varepsilon]}$ is the indicator of the interval $[y-\varepsilon,y+\varepsilon]$.

Then, if $E^{(s,x)}$ is the expectation operator corresponding to the measure $Q_c^{(s,x)}$, we get

$$E^{(s,x)} f_\varepsilon(X(t)) 1_{\{\max_{s \le u \le t} X(u) < a\}}$$

$$= (2\varepsilon)^{-1} \int_{y-\varepsilon}^{y+\varepsilon} E^{P^{(s,x)}}\{1_A \exp(-2c \int_s^t u dW(u)) | W(t) = z\} \cdot p^\partial_{t-s}(x,z) dz,$$

where $p_u(x,z) = u^{-\frac{1}{2}}\{\phi(\frac{x-z}{\sqrt{u}}) - \phi(\frac{x+z-2a}{\sqrt{u}})\}$ is the transition density of Brownian motion, killed when reaching a. Letting $\varepsilon \downarrow 0$, we obtain

$$q^\partial(s,x;t,y) =$$

$$= p^\partial_{t-s}(x,y)\ E^{P^{(s,x)}}\{1_A \exp(-2c \int_s^t u dW(u)) | W(t) = y\} \cdot \exp\{-\tfrac{2}{3} c^2 (t^3 - s^3)\}$$

$$= p^\partial_{t-s}(x,y)\ E_\partial^{(s,x)}\{\exp(-2c \int_s^t u dW(u)) | W(t) = y\} \cdot \exp\{-\tfrac{2}{3} c^2 (t^3 - s^3)\}.$$

Relation (2.4) now follows from (2.5). □

It is well-known that, for $a > 0$, the density f_a of the first passage time $\tau_a = \inf\{t \ge 0: W(t) = a\}$ of ordinary Brownian motion (without drift) $\{W(t): t \ge 0\}$, starting at $x < a$ at time 0, satisfies

$$f_a(t) = -\frac{1}{2} \lim_{y \uparrow a} \frac{\partial}{\partial y} p^\partial_t(x,y),$$

where $p^\partial_t(x,y) = t^{-\frac{1}{2}}\{\phi(\frac{x-y}{t}) - \phi(\frac{x+y+2a}{t})\}$ is the transition density of Brownian motion, killed when reaching a. The following lemma shows that the same relation holds for the drifting process X.

Lemma 2.2. *With the notation of Lemma* 2.1 *we have, for* $s < t$ *and* $x < a$,

$$Q_c^{(s,x)}\{\tau_a \in dt\} = -\frac{1}{2} \partial_4\, q^\partial(s,x;t,a) dt,$$

where $\partial_4 q^\partial(s,x;t,a) = \lim_{y \uparrow a} \frac{\partial}{\partial y} q^\partial(s,t;t,y)$ *and* $\tau_a = \inf\{t \ge s: X(t) = a\}$.

Proof. We have, if $a > x$ and $t > s$,

$$Q_c^{(s,x)}\{\tau_a > t\} = \int_{-\infty}^{a} q^\partial(s,x;t,y) dy$$

(note that $\tau_a > t$ means that the killed process has not died before time t, and hence has a value $y < a$ at time t). Thus the density of τ_a at t, induced by the measure $Q_c^{(s,x)}$, is given by

$$-\frac{\partial}{\partial t}\int_{-\infty}^{a} q^{\partial}(s,x;t,y)\,dy.$$

Since $\{X(t): t \geq s\} = \{W(t) - ct^2: t \geq s\}$, the transition density q^{∂} satisfies the (forward) equation

$$\frac{\partial}{\partial t} q^{\partial}(s,x;t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2} q^{\partial}(s,x;t,y) + 2ct\frac{\partial}{\partial y} q^{\partial}(s,x;t,y),$$

if $t > s$, and $a > x,y$. Hence we get

$$\frac{\partial}{\partial t}\int_{-\infty}^{a} q^{\partial}(s,x;t,y)\,dy =$$

$$= \frac{1}{2}\int_{-\infty}^{a}\frac{\partial^2}{\partial y^2} q^{\partial}(s,x;t,y)\,dy + 2ct\lim_{y\uparrow a} q^{\partial}(s,x;t,y)$$

$$= \frac{1}{2}\lim_{y\uparrow a}\frac{\partial}{\partial y} q^{\partial}(s,x;t,y),$$

since $\lim_{y\uparrow a} q^{\partial}(s,x;t,y) = 0$, as is seen from the representation (2.4) of q^{∂} given in Lemma 2.1. □

Remark 2.2. The interchange of differentiation and integration, used in the proof of Lemma 2.2, can be justified in several different ways. One possibility is to use the representation of q^{∂} in terms of Airy functions, given in Corollary 2.1 below (which has a proof that is independent of Lemma 2.2).

Returning to the representation (2.4) of the transition density q^{∂}, it is seen by reflection with respect to the line $\{(t,a): t \in \mathbb{R}\}$ that we can write

(2.6) $$q^{\partial}(s,x;t,y) =$$

$$= p_{t-s}^{\partial}(a-x,a-y)\exp\{-\tfrac{2}{3}c^2(t^3 - s^3) + 2ct(a-y) - 2cs(a-x)\}$$

$$\cdot E_{\partial}^{(0,a-x)}\{\exp(-2c\int_0^{t-s} W(u)\,du) \mid W(t-s) = a-y\},$$

where, with a change of notation, $\{W(u): u \geq 0\}$ denotes Brownian motion, starting at $a - x > 0$ at time 0, killed when reaching zero, with corresponding expectation operator $E_{\partial}^{(0,a-x)}$, and where

$$(2.7) \qquad p_u(x_1,x_2) = u^{-\frac{1}{2}}\{\phi\left(\frac{x_1-x_2}{\sqrt{u}}\right) - \phi\left(\frac{x_1+x_2}{\sqrt{u}}\right)\}$$

denotes the transition density of this process. In (2.6), the time-homogeneity of Brownian motion is used to translate the origin of the process from $(s,a-x)$ to $(0,a-x)$.

Now let $\{P_t^\partial : t \geq 0\}$ be the semigroup of operators, acting on the set B of bounded Borel-measurable functions $f: (0,\infty) \to \mathbb{R}$ by

$$[P_t^\partial f](x) = E_\partial^{(0,x)} f(W(t)), \quad x > 0, \quad f \in B,$$

where $E_\partial^{(0,x)}$ and W are as in (2.6), i.e. the semigroup $\{P_t^\partial : t \geq 0\}$ corresponds to Brownian motion, starting at a value $x > 0$, and killed when reaching zero. Let $\{Q_t^\partial : t \geq 0\}$ be the semigroup, acting on B by

$$(2.8) \qquad [Q_t^\partial f](x) = E_\partial^{(0,x)} f(W(t)) \exp(-2c \int_0^t W(u)du), \quad x > 0, \quad f \in B.$$

Then,

$$(2.9) \qquad E_\partial^{(0,x)} \int_0^\infty f(W(t)) \exp(-\lambda t - 2c \int_0^t W(u)du)dt = [R_\lambda^\partial f](x),$$

where R_λ^∂ is the λ-resolvent (or λ-potential operator) associated with $\{Q_t^\partial\}$. By the Feynman-Kac formula (see e.g. Williams (1979), p.158, (39.5)), we have

$$(2.10) \qquad [R_\lambda^\partial f](x) = [\overline{R}_\lambda^\partial f](x) - [\overline{R}_\lambda^\partial (v \cdot R_\lambda^\partial f)](x),$$

where $\overline{R}_\lambda^\partial$ is the λ-resolvent associated with $\{P_t : t \geq 0\}$, and where the function $v: (0,\infty) \to \mathbb{R}$ is defined by $v(x) = 2cx$.

The following lemma will enable us to characterize analytically the transition density q^∂ of the process X, killed when reaching a, and hence, by Lemma 2.2, the density of the first passage time τ_a.

<u>Lemma 2.3</u>. *Let* $f: (0,\infty) \to \mathbb{R}$ *be a function with compact support and at most a finite number of discontinuities, and let the resolvent* R_λ^∂ *be defined by* (2.9). *Then*

(i) *The function* $R_\lambda^\partial f: (0,\infty) \to \mathbb{R}$ *is the unique continuously differentiable solution of the differential equation*

(2.11) $\frac{1}{2}y''(x) - (\lambda + 2cx)y(x) = f(x), \quad x > 0,$

(where (2.11) holds at all continuity points x *of* f *), under the boundary conditions*

(2.12) $\lim_{x \downarrow 0} y(x) = 0, \quad \lim_{x \to \infty} y(x) = 0.$

(ii) *We have, for* $x > 0$,

(2.13) $[R_\lambda^\partial f](x) = 2g_\lambda(x) \int_x^\infty h_\lambda(t)f(t)dt + 2h_\lambda(x) \int_0^x g_\lambda(t)f(t)dt,$

where, with $\xi = (2c^2)^{-1/3}\lambda > 0$,

(2.14) $g_\lambda(t) = \pi(4c)^{-1/3}Ai(\xi)^{-1}\{Ai(\xi)Bi(\xi + (4c)^{1/3}t) - Bi(\xi)Ai(\xi + (4c)^{1/3}t)\},$

(2.15) $h_\lambda(t) = Ai(\xi + (4c)^{1/3}t),$

and where Ai *and* Bi *are the Airy functions as defined e.g. in* [1], p.446.

Proof.

Ad (i). It is well-known (and easily verified) that the resolvent $\overline{R}_\lambda^\partial$ has transition density

(2.16) $\overline{r}_\lambda^\partial(x,y) = (2\lambda)^{-\frac{1}{2}}\{e^{-(2\lambda)^{\frac{1}{2}}|x-y|} - e^{-(2\lambda)^{\frac{1}{2}}(x+y)}\}, \quad x,y > 0.$

Hence we have, if f satisfies the conditions of the lemma,

(2.17) $\lambda[\overline{R}_\lambda^\partial f](x) - \frac{1}{2}\frac{d^2}{dx^2}[\overline{R}_\lambda^\partial f](x) = f(x),$

except at discontinuity points x of f (which is, of course, an expression of the fact that Brownian motion, killed when reaching zero, behaves locally as ordinary Brownian motion during its lifetime). It now follows from (2.10) and (2.17) that $R_\lambda^\partial f$ satisfies the differential equation (2.11), and it follows from (2.10) and (2.16) that $R_\lambda^\partial f$ satisfies the boundary conditions (2.12) and is continuously differentiable. Since we are dealing with the classical Sturm-Liouville problem on the interval $[0,\infty)$ (defining $y(0) = \lim_{x \downarrow 0} y(x) = 0$), there is only one continuously differentiable solution of (2.11), satisfying the boundary conditions (2.12).

Ad (ii). A pair of linearly independent solutions of the homogeneous equation

$$(2.18) \qquad \tfrac{1}{2}y''(x) - (\lambda + 2cx)y(x) = 0,$$

is given by the functions $t \to Ai(\xi + (4c)^{1/3}t)$ and $t \to Bi(\xi + (4c)^{1/3}t)$, where ξ and Ai and Bi are as in (2.14). The functions g_λ and h_λ, defined by (2.14) and (2.15), are also linearly independent solutions of (2.18), where g_λ satisfies the boundary condition $g_\lambda(0) = 0$ and h_λ satisfies the boundary condition $\lim_{x\to\infty} h_\lambda(x) = 0$. Moreover

$$g_\lambda(x)h_\lambda'(x) - g_\lambda'(x)h_\lambda(x) = 1, \qquad x \in \mathbb{R},$$

by 10.4.10, p.446 of [1]. Hence the unique continuously differentiable function y, satisfying (2.11) and (2.12) is given by the right-hand side of (2.13), and, by unicity, must be equal to $R_\lambda^\partial f$. (For a clear exposition of the Sturm-Liouville problem and its solutions, see e.g. Dieudonné (1969), section 11.7. Although he only considers functions on a fixed *bounded* interval, the treatment is not essentially different in the case we consider.) □

The main purpose of introducing the Bes(3) process (instead of limiting our considerations to killed Brownian motion) is to give an interpretation to limits of the expectations

$$E_\partial^{(0,x)}\{\exp(-2c\int_0^t W(u)du)\,|\,W(t) = y\}$$

(see e.g. (2.6)), as x or y tends to zero. This interpretation is given in the following lemma.

<u>Lemma 2.4</u>. *Let, for* $x,y > 0$ *and* $c > 0$

$$H_t(x,y) = E_\partial^{(0,x)}\{\exp(-c\int_0^t W(u)du)\,|\,W(t) = y\},$$

where $\{W(u): u \geq 0\}$ *is Brownian motion, starting at* x *at time* 0 *and killed when reaching* 0, *with corresponding expectation* $E_\partial^{(0,x)}$. *Then we have, if* $t > 0$,

(i) $$H_t(x,y) = E^x\{\exp(-c\int_0^t B(u)du) \mid B(t)=y\},$$

where $\{B(u): u \geq 0\}$ *is a* Bes(3) *process, starting at* x *at time* 0, *with expectation operator* E^x.

(ii) $$\lim_{x\downarrow 0} H_t(x,y) = E^0\{\exp(-c\int_0^t B(u)du) \mid B(t)=y\},$$

where $\{B(u): u \geq 0\}$ *is as in* (i), *but starts at* 0 *at time* 0.

(iii) $$\lim_{y\downarrow 0} H_t(x,y) = E^0\{\exp(-c\int_0^t B(u)du) \mid B(t)=x\},$$

where $\{B(u): u \geq 0\}$ *is as in* (ii).

(iv) $$\lim_{x\downarrow 0,\ y\downarrow 0} H_t(x,y) = E\ \exp(-ct^{3/2}\int_0^1 e(u)du),$$

where $\{e(u): u \in [0,1]\}$ *is a Brownian excursion on* $[0,1]$.

Proof. Ad (i). It is intuitively clear that Brownian motion, starting at $x > 0$ at time 0, killed when reaching zero, but conditioned to be equal to $y > 0$ at time $t > 0$ (so still alive at time t) has on the interval $[0,t]$ the same distribution as a Bes(3) process, starting at x at time 0 and conditioned to be equal to y at time t.

For a formal proof, note that the time-space Brownian motion

$$\{(u,B(u)) : 0 \leq u \leq t\},$$

starting at $(0,x)$, killed when reaching the boundary $\{(u,0): 0 \leq u \leq t\}$ and (Doob-)conditioned to converge to (t,y) has the transition function

$$R_u((t_1,x_1),(t_2,dx_2)) =$$

$$= \begin{cases} h(t_1,x_1)^{-1}\ p_u^{\partial}(x_1,x_2)h(t_2,x_2)dx_2, & \text{if} \quad u = t_2 - t_1 \\ 0, & \text{if} \quad u \neq t_2 - t_1 \end{cases}$$

where $0 \leq t_1 < t_2 < t$, $p_u^{\partial}(x_1,x_2)$ is the transition density defined by (2.7), and $h(s,x) = p_{t-s}^{\partial}(x,y)$ is an invariant function for the (killed) time-space process. For the concepts of Doob-conditioning and h-path transforms, see e.g. Doob (1984), section 2.VI,13, and Williams (1979), Ch.3. It is easily seen that the Bes(3) process on $[0,t]$, "h-path transformed" by the invariant function

$$\bar{h}(s,x) = p_{t-s}(x,y),$$

where $p_{t-s}(x,y)$ is defined by (2.2), has the same transition function.

Part (ii) now follows from part (i) and the fact that a Bessel process, starting at 0 at time 0, is the weak limit of Bessel processes, starting at a value $x > 0$, as $x \downarrow 0$.

Part (iii) follows from (ii) by a time reversal argument.

For part (iv), we first note that, by Brownian scaling

$$H_t(x,y) = E_\partial^{(0,x/\sqrt{t})}\{\exp(-ct^{3/2}\int_0^1 W(u)du) \mid W(1) = y/\sqrt{t}\},$$

and next that the weak limit of Brownian bridges between $(0,x/\sqrt{t})$ and $(1,y/\sqrt{t})$, conditioned to be positive on $[0,1]$, is a Brownian excursion process on $[0,1]$, as $x \downarrow 0$ and $y \downarrow 0$, see e.g. Durrett et al (1977), and Blumenthal (1983). □

As a corollary to Lemma 2.3 and Lemma 2.4.(i) we have the following characterization of the transition density q^∂ of the process X, killed when reaching a.

<u>Corollary 2.1</u>. *Let the transition density* q^∂ *be defined as in* (2.4), *Lemma* 2.1. *Then we have, for* $a > x,y$ *and* $t > s$,

(i) $$q^\partial(s,x;t,y) = \exp\{-\tfrac{2}{3}c^2(t^3 - s^3) + 2ct(a-y) - 2cs(a-x)\}$$

$$\cdot\, p^\partial_{t-s}(a-x,a-y)\; E^{a-x}\{\exp(-2c\int_0^{t-s} B(u)du) \mid B(t-s) = a-y\},$$

where $\{B(u): u \geq 0\}$ *is a* Bes(3) *process starting at* $a-x$ *at time* 0, *and* $p^\partial_u(x_1,x_2)$ *is defined by* (2.7).

(ii) *Let, for* $c,x,y > 0$, *the function* $t \longrightarrow r_c(t;x,y)$, $t \geq 0$, *be defined by*

$$r_c(t;x,y) = p^\partial_t(x,y)\; E^x\{\exp(-2c\int_0^t B(u)du) \mid B(t) = y\}, \tag{2.19}$$

where p_t^∂ *and* $\{B(u): u \geq 0\}$ *are defined as in* (i). *Then the function* $r_c(\cdot;x,y)$ *has Laplace transform*

$$\hat{r}_c(\lambda;x,y) = 2g_\lambda(x \wedge y)h_\lambda(x \vee y), \tag{2.20}$$

where $x \wedge y = \min(x,y)$, $x \vee y = \max(x,y)$, *and* g_λ *and* h_λ *are defined by* (2.14) *and* (2.15).

Proof. Part (i) of the corollary is immediate from (2.6) and Lemma 2.4.(i). We prove (ii) by using a method which is similar to that used by Shepp (1982) in his computation of the distribution of $\int_0^1 |Br(t)|dt$, where $\{Br(t): t \in [0,1]\}$ is a Brownian bridge on $[0,1]$.

Let $f_\varepsilon = (2\varepsilon)^{-1} 1_{[y-\varepsilon,y+\varepsilon]}$, where $y - \varepsilon > 0$. By Lemma 2.3.(ii), we have

$$[R_\lambda^\partial f_\varepsilon](x) = 2g_\lambda(x)\int_x^\infty h_\lambda(t)f_\varepsilon(t)dt + 2h_\lambda(x)\int_0^x g_\lambda(t)f_\varepsilon(t)dt, \quad x > 0,$$

where g_λ and h_λ are defined by (2.14) and (2.15) and R_λ^∂ is the resolvent of the semigroup $\{Q_t^\partial: t \geq 0\}$, defined by (2.8). Hence

$$\lim_{\varepsilon \downarrow 0} [R_\lambda^\partial f_\varepsilon](x) = 2g_\lambda(x \wedge y)h_\lambda(x \vee y). \tag{2.21}$$

We also have, proceeding as in the proof of Lemma 2.1 and using Lemma 2.4.(i)

$$\begin{aligned}[R_\lambda^\partial f_\varepsilon](x) &= E_\partial^{(0,x)} \int_0^\infty \exp(-\lambda t - 2c\int_0^t W(u)du)f_\varepsilon(W(t))dt \\ &\to \int_0^\infty e^{-\lambda t}\, p_t^\partial(x,y)\, E^x\{\exp(-2c\int_0^t B(u)du)\,|\,B(t) = y\}dt,\end{aligned} \tag{2.22}$$

as $\varepsilon \downarrow 0$. Part (ii) now follows from (2.21) and (2.22). □

Proof of Theorem 2.1.

Ad(i). Let $c,t > 0$ and $x > y > 0$. By Corollary 2.1.(ii) we can write

$$E^x\{\exp(-2c\int_0^t B(u)du)\,|\,B(t) = y\} = r_c(t;x,y)/p_t^\partial(x,y) \tag{2.23}$$

where $r_c(t;x,y)$ has the representation

$$(2.24)\qquad r_c(t;x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ist} g_{is}(y) h_{is}(x)\,ds,$$

where $i = \sqrt{-1}$, and the Laplace transform (2.20) is inverted, using the imaginary axis as integration road (in fact we can take any road, parallel to the imaginary axis, of the type $c + i\mathbb{R}$, with $c > a_1 \approx -2.3381$, a_1 being the largest zero of the Airy function Ai on the negative real axis, see [1], p.478).

Using properties of Airy functions, it is easily seen from (2.24) that $r_c(t;x,y)$ has the following properties:

$$(2.25)\qquad \lim_{y\downarrow 0} r_c(t;x,y) = 0,$$

(this also follows directly from (2.23)),

$$(2.26)\qquad \lim_{y\downarrow 0} \frac{\partial}{\partial y} r_c(t;x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ist} \mathrm{Ai}(i\xi + (4c)^{1/3}x)/\mathrm{Ai}(i\xi)\,ds,$$

where $\xi = (2c^2)^{-1/3}s$,

$$(2.27)\qquad \lim_{y\downarrow 0} \frac{\partial^2}{\partial y^2} r_c(t;x,y) = 0\,.$$

It is also clear from the representation (2.24) that for each positive integer k and each $x > 0$ the limits

$$\lim_{y\downarrow 0} \frac{\partial^k}{\partial y^k} r_c(t;x,y)$$

exist and are finite.

By l' Hôpital's rule we now obtain from (2.23) and (2.25) to (2.27)

$$(2.28)\qquad \lim_{y\downarrow 0} \frac{\partial}{\partial y} E^x\{\exp(-2c\int_0^t B(u)\,du \mid B(t) = y\}$$

$$= \lim_{y\downarrow 0} \frac{\partial^2}{\partial y^2}\{p_t^\partial(x,y)\frac{\partial}{\partial y} r_c(t;x,y) - r_c(t;x,y)\frac{\partial}{\partial y}p_t^\partial(x,y)\}/\frac{\partial^2}{\partial y^2}p_t^\partial(x,y)^2$$

$$= 0,$$

(in fact, we only need that the limit at the left-hand side of (2.28) is finite). Thus we get from Corollary 2.1.(i), Lemma 2.4.(iii) and (2.28),

$$\lim_{y\uparrow a} \frac{\partial}{\partial y} q^{\partial}(s,x;t,y) = \exp\{-\tfrac{2}{3}c^2(t^3-s^3) - 2cs(a-x)\}\cdot$$

$$\cdot\ E^0\{\exp(-2c\int_0^{t-s} B(u)du) \mid B(t-s)=x\}\cdot$$

$$\cdot\ \lim_{y\uparrow a}\frac{\partial}{\partial y}\, p^{\partial}_{t-s}(a-x,a-y),$$

for $x < a$. Since $\lim_{y\uparrow a}\frac{\partial}{\partial y} p^{\partial}_{t-s}(a-x,a-y) = \frac{\partial}{\partial y} p^{\partial}_{t-s}(a-x,a-y)\big|_{y=a} =$ $= -2\,\psi_{a-x}(t-s)$ (with the notation of the statement of Theorem 2.1), the result now follows from Lemma 2.2.

Ad (ii). As in the proof of part (i), we have that the density of τ_a at t is given by

$$-\tfrac{1}{2}\exp\{-\tfrac{2}{3}c^2(t^3-s^3) - 2cs(a-x)\}\cdot$$

$$\cdot\ \lim_{y\uparrow a}\frac{\partial}{\partial y}\, r_c(t-s;a-x,a-y)$$

But by (2.26), the Laplace transform of $-\frac{1}{2}\lim_{y\uparrow a}\frac{\partial}{\partial y} r_c(\cdot;a-x,a-y)$ is given by the function $\hat{h}_{c,a-x}$. □

3. The maximum and the location of the maximum of $\{W(t) - ct^2 : t \geq s\}$.

Throughout this section, we will use the same notation as in section 2; in particular, the process $\{X(t): t \geq s\}$, with corresponding probability measure $Q_c^{(s,x)}$ on the Borel σ-field of $C([s,\infty);\mathbb{R})$, will denote the process $\{W(t) - ct^2: t \geq s\}$, where $\{W(t): t \geq s\}$ is Brownian motion, starting at $x + cs^2$ at time s.

Consider the probability

$$(3.1) \qquad Q_c^{(s,x)}\{X(t) < a, \quad \text{for all} \quad t \geq s\}.$$

It follows from the space-homogeneity of Brownian motion that

$$Q_c^{(s,x)}\{X(t) < a, \quad \text{for all} \quad t \geq s\}$$

$$= Q_c^{(s,x-a)}\{X(t) < 0, \quad \text{for all} \quad t \geq s\}.$$

Hence, defining

$$(3.2) \qquad K_c(s,x) = Q_c^{(s,-x)}\{X(t) < 0, \quad \text{for all} \quad t \geq s\},$$

we can denote (3.1) by $K_c(s,a-x)$.

The following theorem determines analytically the function $s \to K_c(s,x)$, for each $x > 0$ (clearly $K_c(s,x) = 0$, for each $x \leq 0$).

Theorem 3.1. *Let* $K_c(s,x)$ *be defined by* (3.2). *Then, for each* $x > 0$ *and* $s \in \mathbb{R}$,

$$(3.3) \qquad K_c(s,x) = \exp\{\tfrac{2}{3} c^2 s^3 - 2csx\}\psi_x(s),$$

where the function $\psi_x : \mathbb{R} \to \mathbb{R}_+$ *has Fourier transform*

$$(3.4) \qquad \hat{\psi}_x(\lambda) = \int_{\mathbb{R}} e^{i\lambda s}\psi_x(s)ds =$$

$$= \pi(2c^2)^{-1/3}\{Ai(i\xi)Bi(i\xi+z) - Bi(i\xi)Ai(i\xi+z)\}/Ai(i\xi),$$

where $\xi = (2c^2)^{-1/3}\lambda$, $z = (4c)^{1/3}x$, *and* Ai *and* Bi *are the Airy functions defined on* p.446 *of* [1].

The somewhat technical proof of Theorem 3.1 is given in the Appendix. Functions of the type $K_c(s,x)$ were studied in Chernoff's (1964) paper on estimators of the mode of a distribution, and apparently Theorem 3.1 solves a long-standing question concerning the analytical characterization of these functions. As an immediate corollary to Theorem 3.1 we obtain the joint distribution of the maximum and the location of the maximum of the process X, starting at x at time s.

COROLLARY 3.1. *Let* $Q_c^{(s,x)}$ *be the probability measure, corresponding to the process* $\{X(t): t \geq s\}$, *starting at* x *at time* s, *where* $X(t) = W(t) - ct^2$, *and* $\{W(t): t \geq s\}$ *is Brownian motion, starting at* $x + cs^2$ *at time* s. *Let* M *and* τ_M *denote the maximum and the location of the maximum, respectively, of the process* $\{X(t): t \geq s\}$ *(note that* M *is a.s. finite and* τ_M *is a.s. finite and unique under* $Q_c^{(s,x)}$*). Then we have, for* $a > x$ *and* $t > s$,

$$(3.5) \qquad Q_c^{(s,x)}\{\tau_M \in dt, \ M \in da\}$$

$$= Q_c^{(s,x)}\{\tau_a \in dt\} k_c(t) da$$

$$= \exp\{-\tfrac{2}{3}c^2(t^3 - s^3) - 2cs(a-x)\} \cdot h_{c,a-x}(t-s) k_c(t) dt da,$$

where $k_c(t) = \lim_{x \downarrow 0} \frac{\partial}{\partial x} K_c(t,x)$ *(see (3.2)), and where the function* $u \to h_{c,a-x}(u)$, $u \geq 0$, *has Laplace transform*

$$(3.6) \qquad \hat{h}_{c,a-x}(\lambda) = \mathrm{Ai}((4c)^{1/3}(a-x) + \xi)/\mathrm{Ai}(\xi), \qquad \xi = (2c^2)^{-1/3}\lambda > 0.$$

The function $t \to k_c(t)$, $t \in \mathbb{R}$ *can be written*

$$(3.7) \qquad k_c(t) = \exp(\tfrac{2}{3}c^2t^3) g_c(t), \qquad t \in \mathbb{R},$$

where the function $g_c: \mathbb{R} \to \mathbb{R}_+$ *has the Fourier transform*

$$(3.8) \qquad \hat{g}_c(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda s} g_c(s) ds = 2^{1/3}c^{-1/3}/\mathrm{Ai}(i(2c^2)^{-1/3}\lambda), \qquad \lambda \in \mathbb{R}.$$

PROOF. Let the transition density $q^\partial(s,x;t,y)$ be defined as in Lemma 2.1 and 2.2; i.e. q^∂ is the transition density of the process X, killed when reaching a. Then, by a similar argument as used in Lemma 2.2 we can write if $a > x,y$,

$$Q_c^{(s,x)}\{\tau_M > t, \quad M \in da\}$$
$$= \{\int_{-\infty}^{a} q^{\partial}(s,x;t,y)k_c(t,a-y)dy\}da,$$

where

$$k_c(t,z) = \frac{\partial}{\partial z} K_c(t,z).$$

Hence the joint density of (τ_M,M) at (t,a) is given by

$$(3.9) \qquad -\frac{\partial}{\partial t}\int_{-\infty}^{a} q^{\partial}(s,x;t,y)k_c(t,a-y)dy ,$$

if $a > x$ and $t > s$.

Since $\{X(t): t \geq s\} = \{W(t) - ct^2: t \geq s\}$, the function $k_c(t,z)$ satisfies the (backward) equation

$$\frac{\partial}{\partial t} k_c(t,z) = -\frac{1}{2}\frac{\partial^2}{\partial z^2} k_c(t,z) + 2ct \frac{\partial}{\partial z} k_c(t,z)$$

and q^{∂} satisfies the (forward) equation

$$\frac{\partial}{\partial t} q^{\partial}(s,x;t,y) = \frac{1}{2}\frac{\partial}{\partial y^2} q^{\partial}(s,x;t,y) + 2ct \frac{\partial}{\partial y} q^{\partial}(s,x;t,y),$$

if $t > s$ and $a > x, y$. Hence we get, after a straightforward computation, using integration by parts,

$$-\frac{\partial}{\partial t}\int_{-\infty}^{a} q^{\partial}(s,x;t,y)k_c(t,a-y)dy$$
$$= \int_{-\infty}^{a} k_c(t,a-y)\frac{\partial}{\partial t}q^{\partial}(s,x;t,y)dy - \int_{-\infty}^{a} q^{\partial}(s,x;t,y)\frac{\partial}{\partial t}k_c(t,a-y)dy$$
$$= -\frac{1}{2}k_c(t)\partial_4 q^{\partial}(s,x;t,a),$$

where $\partial_4 q^{\partial}(s,x;t,a) = \lim_{y \uparrow a} \frac{\partial}{\partial y} q^{\partial}(s,x;t,y)$. Since, by Lemma 2.2,

$$Q_c^{(s,x)}\{\tau_a \in dt\} = -\frac{1}{2}\partial_4 q^{\partial}(s,x;t,a)dt,$$

(3.5) now follows from (3.9). The Laplace transform $\hat{h}_{c,a-x}(\lambda)$ in (3.6) is given by part (ii) of Theorem 2.1.

Finally, the Fourier transform of the function $g_c\colon \mathbb{R} \to \mathbb{R}_+$ can be computed using Theorem 3.1. By Theorem 3.1, we can write

$$k_c(t,z) = \exp\{\tfrac{2}{3}c^2t^3 - 2ctz\}\cdot\{-2ct\psi_z(t) + \tfrac{\partial}{\partial z}\psi_z(t)\}.$$

Furthermore, we have by (3.4),

$$\text{(3.10)} \qquad \lim_{z\downarrow 0} \frac{\partial}{\partial z}\hat{\psi}_z(\lambda) = (2/c)^{1/3}/\mathrm{Ai}(i\xi),$$

where $\xi = (2c^2)^{-1/3}\lambda$, using the relation $\mathrm{Ai}(z)\mathrm{Bi}'(z) - \mathrm{Ai}'(z)\mathrm{Bi}(z) = \pi^{-1}$. Since

$$\lim_{z\downarrow 0} k_c(t,z) = \exp(\tfrac{2}{3}c^2t^3) \lim_{z\downarrow 0} \frac{\partial}{\partial z}\psi_z(t),$$

(3.8) now follows from (3.10). □

The following corollary gives the corresponding result for two-sided Brownian motion

<u>COROLLARY 3.2</u>. *Let* $\{W(t): t \in \mathbb{R}\}$ *be two-sided Brownian motion, originating from zero. Define*

$$M = \sup\{W(t) - ct^2: t \in \mathbb{R}\}$$

and

$$\tau_M = \sup\{t \in \mathbb{R}: W(t) - ct^2 \text{ is maximal}\},$$

i.e. τ_M *is the a.s. unique location of the (a.s. finite) maximum* M. *The joint density of* (τ_M, M) *at* (t,a), $t \in \mathbb{R}$, $a > 0$, *is given by*

$$\text{(3.11)} \qquad f_c(t,a) = g_c(|t|)h_{c,a}(|t|)\psi_a(0),$$

where the functions g_c *and* $h_{c,a}$ *are as in Corollary* 3.1, *and the function* $\psi_a\colon \mathbb{R} \to \mathbb{R}_+$ *is defined as in Theorem* 3.1.

<u>PROOF</u>. Let $t > 0$, $a > 0$, let $M_+ = \max\{W(t) - ct^2: t \geq 0\}$ be the maximum of the process $\{W(t) - ct^2: t \in \mathbb{R}\}$, *restricted to* $[0,\infty)$, and let τ_{M_+} be

the location of this maximum. By Corollary 3.1, we have

$$Q_c^{(0,0)}\{M_+ \in da,\ \ \tau_M \in dt\} = h_{c,a}(t)g_c(t)dadt$$

We also have

$$\Pr\{W(s) - cs^2 < a,\ \ \text{for all}\ \ s < 0\}$$

$$= \Pr\{W(s) - cs^2 < a,\ \ \text{for all}\ \ s > 0\}$$

$$= Q_c^{(0,0)}\{X(s) < a,\ \ \text{for all}\ \ s > 0\}$$

$$= K_c(0,a) = \psi_a(0),$$

by (3.3) in Theorem 3.1. Relation (3.11) now follows, for the case $t > 0$. The case $t < 0$ is treated in a completely similar way. □

The particularly simple form of the marginal density of the location of the maximum of the process $\{W(t) - ct^2 : t \in \mathbb{R}\}$ is given in the following corollary.

COROLLARY 3.3. *The density of the random variable*

$$Z = \sup\{t \in \mathbb{R} : W(t) - ct^2 \text{ is maximal}\}$$

is given by

$$f_Z(t) = \tfrac{1}{2} g_c(t) g_c(-t), \tag{3.12}$$

where the function g_c *has the Fourier transform given by* (3.8).

PROOF. By Corollary 3.2 we have

$$f_Z(t) = g_c(|t|) \int_0^\infty h_{c,a}(|t|)\psi_a(0)da. \tag{3.13}$$

Suppose $t > 0$. By part (ii) of Theorem 2.1 the density of τ_a at t, under the probability measure $Q_c^{(0,0)}$ is given by $h_{c,a}(t)$. Hence, by Lemma 2.2,

$$\exp(-\tfrac{2}{3}c^2t^3)h_{c,a}(t) = Q_c^{(0,0)}\{\tau_a \in dt\}/dt = -\tfrac{1}{2}\partial_4 q^\partial(0,0;t,a).$$

Since $\psi_a(0) = Q_c^{(0,0)}\{X(s) < a, \text{ for all } s \ge 0\} = \Pr\{W(s) - cs^2 < a, \text{ for all } s \le 0\}$, we get

$$\exp\{-\tfrac{2}{3}c^2t^3\} \int_0^\infty h_{c,a}(t)\psi_a(0)\,da = -\tfrac{1}{2}\int_0^\infty \Pr\{W(s) - cs^2 < a, \text{ all } s \le 0\} \cdot \partial_4 q^\partial(0,0;t,a)\,da$$

$$= \tfrac{1}{2}\int_0^\infty \partial_2 q^\partial(-t,a;0,0) \cdot \Pr\{W(s) - cs^2 < a, \text{ all } s \ge 0\}da,$$

where the last equality follows from a simple time reversal argument. We also have

$$\tfrac{1}{2}\int_0^\infty \partial_2 q^\partial(-t,a;0,0) \cdot \Pr\{W(s) - cs^2 < a, \text{ all } s \ge 0\}da$$

$$= \tfrac{1}{2}\lim_{y\uparrow 0} \frac{\partial}{\partial y}\int_0^\infty q^\partial(-t,y;0,-a) \cdot Q_c^{(0,-a)}\{X(s) < 0, \text{ all } s \ge 0\}da$$

$$= \tfrac{1}{2}\lim_{y\uparrow 0} \frac{\partial}{\partial y} Q_c^{(-t,y)}\{X(s) < 0, \text{ all } s \ge -t\}$$

$$= \tfrac{1}{2}k_c(-t).$$

Hence, by (3.7) and (3.13)

$$f_Z(t) = g_c(t)\int_0^\infty h_{c,a}(t)\psi_a(0)\,da$$

$$= \tfrac{1}{2}g_c(t)k_c(-t)\exp\{-\tfrac{2}{3}c^2t^3\}$$

$$= \tfrac{1}{2}g_c(t)g_c(-t).$$

The case $t < 0$ follows by symmetry. □

The tail behavior of the random variable $Z = \sup\{t \in \mathbb{R} : W(t) - ct^2 \text{ is maximal}\}$ is given in Corollary 3.4.

<u>COROLLARY 3.4</u>. *The function* g_c, *defining the density* f_Z *in Corollary* 3.3, *has the following properties*

(i) $$g_c(t) = (4c)^{1/3}\sum_{n=1}^\infty \exp((2c^2)^{1/3}a_nt)/Ai'(a_n),$$

if $t < 0$, *where the* a_n *are the zeros of the Airy function* Ai *on the negative real axis.*

(ii) $$g_c(t) \sim 4ct \exp(-\tfrac{2}{3}c^2t^3), \qquad \text{as} \quad t \to \infty .$$

Hence we have

(iii) $$f_Z(t) \sim \tfrac{1}{2}(4c)^{4/3}|t|\exp\{-\tfrac{2}{3}c^2|t|^3 + (2c^2)^{1/3}a_n|t|\}/\mathrm{Ai}'(a_1),$$

as $|t| \to \infty$, *where* $a_1 \approx -2.3381$ *is the largest zero of the Airy function* Ai *and where* $\mathrm{Ai}'(a_1) \approx 0.7022$ *(see* [1], p.478*)*.

PROOF. By (3.8) we can write

(3.14) $$g_c(t) = (2/c)^{1/3}\frac{1}{2\pi i}\int_{c_1 - i\infty}^{c_1 + i\infty} e^{-tu}/\mathrm{Ai}((2c^2)^{-1/3}u)\,du$$

where $c_1 > a_1$, a_1 being the largest zero of the Airy function Ai. Here we use the fact that Ai is an analytic function and that

$$|1/\mathrm{Ai}(c_1 + is)| \sim 2\sqrt{\pi}\;|s|^{1/4}\exp\{-\tfrac{2}{3}^{\frac{1}{2}}|s|^{3/2} + c_1\sqrt{\tfrac{1}{2}|s|}\},$$

as $|s| \to \infty$, $s \in \mathbb{R}$.

If $t < 0$, we can shift the integration road to the left, obtaining the series of residues given in (i).

Ad (ii). Again using the representation (3.14) of $g_c(t)$, it is seen that the integrand has a saddlepoint at (approximately) $u = 2c^2t^2$, as $t \to \infty$, using the relation

$$\mathrm{Ai}(z) \sim \exp(-\tfrac{2}{3}z^{3/2})/2\pi^{\frac{1}{2}}z^{1/4}, \qquad \mathrm{Re}\; z \to \infty.$$

Hence, taking $c_1 = 2c^2t^2$, we obtain by Laplace's method (see e.g. Olver (1974), section 3.7)

$$\exp(\tfrac{2}{3}c^2t^3)g_c(t) \sim \frac{2}{2\pi i}^{3/2} c^2\int_{-i\infty}^{i\infty} 2\sqrt{\pi}\exp(\tfrac{1}{2}c^2t^3y^2)\,dy$$

$$= 4ct\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy = 4ct$$

Finally, part (iii) of the corollary follows immediately from (i) and (ii).
□

REMARK. It is clear from part (iii) of Corollary 3.4 that the density of Z has a very thin tail. Using the expansion 10.4.59 in [1], it is possible to give a complete asymptotic expansion of the density $f_Z(t)$, as $t \to \infty$, just by plugging in this representation of Ai in the proof of (ii) in Corollary 3.4 and using Watson's lemma (see e.g. Olver (1974), p.71). We shall, however, not go into this.

The representation of $g_c(t)$, for $t < 0$, given in Corollary 3.4.(i), has been derived from Theorem 4.3 in Groeneboom (1984), using different methods, by N.M. Temme (personal communication).

4. Excursion integrals and the Grenander estimator.

Let $\{W(t): t \in \mathbb{R}\}$ be two-sided standard Brownian motion on $\mathbb{R}$, originating from zero, and let the process $\{V(a): a \in \mathbb{R}\}$ be defined by

$$V(a) = \sup\{t \in \mathbb{R}: \ W(t) - (t-a)^2 \ \text{is maximal}\}. \tag{4.1}$$

It is easy to see that V is an increasing pure jump process, generated by Brownian motion sample paths. A picture of the situation is shown in figure 4.1.

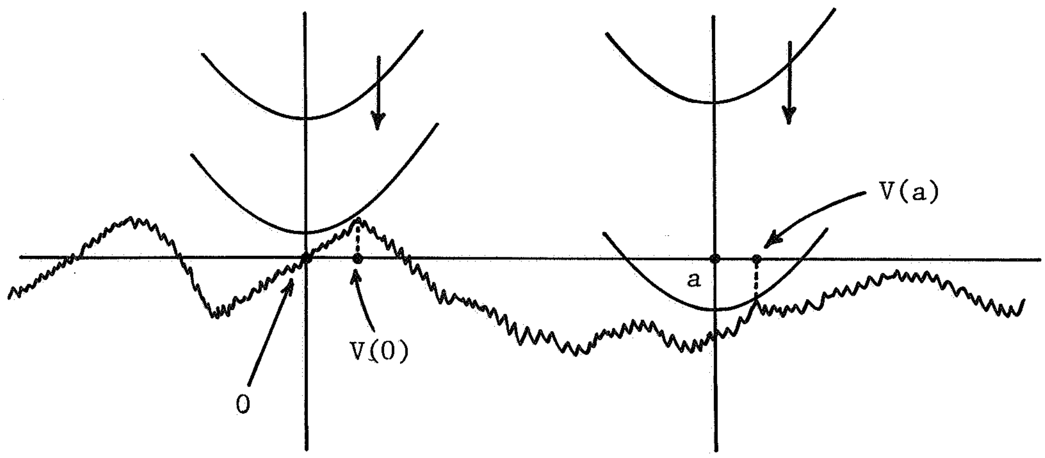

Figure 4.1.

V(a) is the location of the point where the parabola $f(t) = (t-a)^2 + c$, sliding down along the line $t = a$, hits two-sided Brownian motion, originating from zero.

The process $\{V(a): a \in \mathbb{R}\}$ plays a fundamental role in describing the *global* behavior of the Grenander maximum likelihood estimator (MLE) of a monotone density. In particular, if $\mathcal{F}$ is the class of nonincreasing left continuous densities on the interval $[0,\infty)$, and $X_1,\ldots,X_n$ is a sample generated by a density $f \in \mathcal{F}$, then the Grenander MLE f_n of f, under the restriction that f_n should belong to $\mathcal{F}$, is given by a left continuous version of the slope of the concave majorant $\hat{F}_n$ of the empirical distribution function F_n, based on $X_1,\ldots,X_n$, and the asymptotic variance of the L_1-distance $\|f_n - f\|_1$ can be expressed in terms of covariance structure of the process $\{V(a): a \in \mathbb{R}\}$ (under some regularity conditions on the density f). For details (and pictures) we refer to Groeneboom (1984).

It is clear that the process $\{V(a): a \in \mathbb{R}\}$ generates the endpoints of "excursions below parabolas", so it is perhaps no surprise that the structure of the process $\{V(a): a \in \mathbb{R}\}$ can be described in terms of functionals of ordinary Brownian excursions (with the help of the Cameron-Martin-Girsanov formula).

We recall the definition of a Brownian excursion. A Brownian excursion on $[0,1]$ is a nonhomogeneous Markov process $\{e(t): t \in [0,1]\}$ with marginal densities

$$(4.2)\qquad f_{e(t)}(x) = 2x^2\exp\{-x^2/(2t(1-t))\}/\{2\pi t^3(1-t)^3\}^{\frac{1}{2}},$$

and transition densities

$$(4.3)\qquad f_{e(t)|e(s)}(y|x) =$$

$$= \{n_{t-s}(y-x) - n_{t-s}(y+x)\} \cdot (1-s)^{3/2}\, y\exp\{-y^2/(2(1-t))\} \cdot$$

$$\cdot \{(1-t)^{3/2}\, x\exp\{-x^2/(2(1-s))\}\}^{-1}, \qquad x,\ y > 0,$$

where $n_u(x) = u^{-\frac{1}{2}}\phi(x/\sqrt{u})$ and ϕ is the standard normal density (see e.g. Itô & McKean (1976), p.76). Intuitively speaking, a Brownian excursion is a Brownian bridge, "conditioned to be positive" (see e.g. Durrett et al (1977) and Blumenthal (1983)). More generally, we can consider excursions $\bar{e}$ on an interval $[a,b]$, which are obtained from the excursions defined by (4.2) and (4.3) by putting

$$(4.4)\qquad \bar{e}(t) = (b-a)^{\frac{1}{2}}e((t-a)/(b-a)), \qquad t \in [a,b].$$

Now let $v(t,x,w)$ be defined by

$$(4.5)\qquad v(t,x,w) = Q_1^{(t,x)}\{\tau_0 \in dw\}/dw,$$

where τ_0 and $Q_1^{(t,x)}$ are defined as in Theorem 2.1. We will show that the infinitesimal generator of the time-space process $\{(a,V(a)): a \in \mathbb{R}\}$ can be expressed in terms of the function

(4.6) $$v_2(t,w) = -\lim_{x \uparrow 0} \frac{\partial}{\partial x} v(t,x,w)$$

and the function

(4.7) $$k_1(t) = \lim_{x \downarrow 0} \frac{\partial}{\partial x} K_1(t,x),$$

where $K_1(t,x)$ is defined by (3.2), with $c = 1$. We will first show that $v_2(t,w)$ can be expressed in terms of an expectation of a function of a Brownian excursion integral (Lemma 4.1) and we will compute the Laplace transform of the density of this Brownian excursion integral (Lemma 4.2).

LEMMA 4.1. *Let* $v_2(t,w)$ *be defined by* (4.5) *and* (4.6). *Then we have*

(4.8) $$v_2(t,w) = \{2\pi(w-t)^3\}^{-\frac{1}{2}} \exp\{-\tfrac{2}{3}(w^3 - t^3)\} \cdot E \exp\{-2\int_t^w e(u)du\},$$

where $\{e(u): u \in [t,w]\}$ *is a Brownian excursion on* $[t,w]$ *(see* (4.2) *to* (4.4)*; we write* $e(u)$ *also for excursions defined on intervals different from* $[0,1]$*).*

PROOF. By part (i) of Theorem 2.1 we have, for $x > 0$,

$$Q_1^{(t,-x)}\{\tau_0 \in dw\} =$$

$$= \exp\{-\tfrac{2}{3}(w^3 - t^3) - 2tx\}\psi_x(w-t) \cdot$$
$$\cdot E^0\{\exp(-2\int_0^{w-t} B(u)du) \mid B(w-t) = x\}dw.$$

where $\psi_z(u) = (2\pi u^3)^{-\frac{1}{2}} z \exp(-z^2/2u)$, $u, z > 0$, and $\{B(u): u \geq 0\}$ is a Bes(3) process, starting at zero at time 0. Moreover, by part (ii) of Theorem 2.1, we have, for $u > 0$,

$$\psi_x(u)E^0\{\exp(-2\int_0^u B(z)dz) \mid B(u) = x\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda u} Ai(4^{1/3}x + i\xi)/Ai(i\xi)d\lambda,$$

where $\xi = 2^{-1/3}\lambda$. Define, for $x, u > 0$,

(4.9) $$F_u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda u} Ai(2^{2/3}x + i\xi)/Ai(i\xi)d\lambda.$$

The function $F_u(x)$ can be represented as the series of residues

$$F_u(x) = 2^{1/3} \sum_{n=1}^{\infty} \frac{Ai(2^{2/3}x + a_n)}{Ai'(a_n)} \exp(2^{1/3}a_n u)$$

where the a_n are the zeros of the function Ai on the negative halfline. Hence we have

$$\text{(4.10)} \qquad \lim_{x \downarrow 0} F_u'(x) = 2 \sum_{n=1}^{\infty} \exp(2^{1/3}a_n u) ,$$

$$\lim_{x \downarrow 0} F_u''(x) = 0,$$

and

$$\lim_{x \downarrow 0} F'''(x) = 2^{7/3} \sum_{n=1}^{\infty} a_n \exp(2^{1/3}a_n u).$$

Thus we get, applying l'Hôpital's rule,

$$\lim_{x \downarrow 0} \frac{\partial}{\partial x} E^0\{\exp(-2\int_0^u B(z)dz) \mid B(u) = x\}$$

$$= \lim_{x \downarrow 0} \frac{\partial}{\partial x} \{F_u(x)/\psi_x(u)\}$$

$$= \frac{1}{2} \lim_{x \downarrow 0} \frac{\frac{\partial}{\partial x}\{F_u''(x)\psi_x(u) - F_u(x) \frac{\partial^2}{\partial x^2} \psi_x(u)\}}{(\frac{\partial}{\partial x} \psi_x(u))^2}$$

$$= 0.$$

We therefore obtain, by Lemma 2.4,

$$v_2(t,w) = \exp\{-\tfrac{2}{3}(w^3 - t^3)\} \cdot$$

$$\cdot \lim_{x \downarrow 0} E^0\{\exp(-2\int_0^{w-t} B(z)dz) \mid B(w-t) = x\} \cdot \frac{\partial}{\partial x} \psi_x(w-t)$$

$$= \{2\pi(w-t)^3\}^{-\frac{1}{2}} \exp\{-\tfrac{2}{3}(w^3 - t^3)\} \cdot E \exp\{-2\int_t^w e(z)dz\},$$

noting that $E \exp\{-2\int_t^w e(z)dz\} = E \exp\{-2(w-t)^{3/2}\int_0^1 e(z)dz\}$. □

The following lemma gives an analytic characterization of the function $v_2(t,w)$ and also gives the Laplace transform of the excursion integral ${}_0\int^1 e(u)du$.

Lemma 4.2. *Let* $v_2(t,w)$ *be defined as in Lemma* 2.1. *Then we have*

(i) $$v_2(t,w) = \exp\{-\tfrac{2}{3}(w^3 - t^3)\}p(w-t),$$

where the function $u \to p(u)$, $u \ge 0$, *satisfies the relation*

(4.11) $$\int_0^\infty e^{-\lambda u}\{p(u) - (2\pi u^3)^{-\frac{1}{2}}\}du = 2^{2/3}Ai'(\xi)/Ai(\xi) + \sqrt{2\lambda}, \qquad \xi = 2^{-1/3}\lambda.$$

(ii) *The function* $p\colon \mathbb{R}_+ \to \mathbb{R}$, *defined in* (i) *has the following representation*

(4.12) $$p(u) = 2\sum_{n=1}^\infty \exp(2^{1/3}a_n u), \qquad u > 0,$$

where the a_n *are the zeros of the function* Ai *on the negative halfline.*

(iii) *The density of the random variable* ${}_0\int^1 e(u)du$ *has the Laplace transform*

(4.13) $$E\exp(-\lambda\int_0^1 e(u)du) = \lambda\sqrt{2\pi}\sum_{n=1}^\infty \exp(2^{-1/3}a_n\lambda^{2/3}), \qquad \lambda > 0,$$

where the a_n *are defined as in* (ii).

Proof. Ad (i). By part (ii) of Theorem 2.1 we have (arguing as in the proof of Lemma 4.1),

$$v_2(t,w) = \exp\{-\tfrac{2}{3}(w^3 - t^3)\}\cdot\lim_{x\downarrow 0} F'_{w-t}(x),$$

where $F_u(x)$ is defined by (4.9), for $x, u > 0$. Hence

$$p(u) = \lim_{x\downarrow 0} F'_u(x), \qquad u > 0.$$

For $x > 0$, the function $u \to F'_u(x)$ has the Laplace transform

$$\int_0^\infty e^{-\lambda u} F'_u(x)du = 2^{2/3}Ai'(\xi + 2^{2/3}x)/Ai(\xi), \qquad \xi = 2^{-1/3}\lambda.$$

Thus we get

$$\lim_{x \downarrow 0} \int_0^\infty e^{-\lambda u}\{F_u'(x) - \frac{\partial}{\partial x}(2\pi u^3)^{-\frac{1}{2}}\, x\, \exp(-x^2/2u)\}du$$

$$= \lim_{x \downarrow 0} \{2^{2/3}Ai'(\xi + 2^{2/3}x)/Ai(\xi) + \sqrt{2\lambda}\, \exp(-x\sqrt{2\lambda})\}$$

$$= 2^{2/3}Ai'(\xi)/Ai(\xi) + \sqrt{2\lambda}.$$

Since we also have

$$\lim_{x \downarrow 0} \int_0^\infty e^{-\lambda u}\{F_u'(x) - \frac{\partial}{\partial x}(2\pi u^3)^{-\frac{1}{2}}\, x\, \exp(-x^2/2u)\}du$$

$$= \int_0^\infty e^{-\lambda u}\{p(u) - (2\pi u^3)^{-\frac{1}{2}}\}du,$$

(4.11) follows (noting that $p(u) - (2\pi u^3)^{-\frac{1}{2}} = O(1), \quad u \downarrow 0)$.

Ad (ii). This is just relation (4.10).

Ad (iii). By Brownian scaling, we have

$$E\, \exp\{-2\int_0^t e(u)du\} = E\, \exp\{-2t^{3/2}\int_0^1 e(u)du\}.$$

Thus (4.13) follows from part (ii) by taking $\lambda = 2t^{3/2}$. □

The structure of the jump process $\{V(a): a \in \mathbb{R}\}$, defined by (4.1), is determined in the following theorem.

Theorem 4.1. *Let* B *denote the set of bounded Borel measurable functions* $f: \mathbb{R}^2 \to \mathbb{R}$, *and let* $\{P_t: t \geq 0\}$ *be the semigroup of linear operators on* B *defined by*

(4.14) $\quad [P_t f](a,x) = E\{f(a+t, V(a+t)) \mid V(a) = x\}.$

Let $C_c^\infty(\mathbb{R}^2)$ *be the space of functions* $f: \mathbb{R}^2 \to \mathbb{R}$, *which have compact support and continuous derivatives of all orders, and let* $C(\mathbb{R}^2)$ *be the space of continuous functions* $f: \mathbb{R}^2 \to \mathbb{R}$. *Then the semigroup* $\{P_t: t \geq 0\}$ *has the infinitesimal generator* $G: C_c^\infty(\mathbb{R}^2) \to C(\mathbb{R}^2)$, *defined by*

$$(4.15)\qquad [Gf](a,x) = \frac{\partial}{\partial a} f(a,x) + $$

$$+ 2\int_x^\infty (y-x)\{k_1(y-a)/k_1(x-a)\}\{f(a,y)-f(a,x)\}v_2(x-a,y-a)\,dy$$

$$= \frac{\partial}{\partial a} f(a,x) +$$

$$+ 2\int_x^\infty (y-x)\{g_1(y-a)/g_1(x-a)\}\{f(a,y)-f(a,x)\}p(y-x)\,dy$$

where the functions k_1 *and* g_1 *are defined as in Corollary* 3.1 *and the function* $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is defined as in Lemma* 4.2. *In particular the function* g_1 *has Fourier transform*

$$(4.16)\qquad \hat{g}_1(\lambda) = 2^{1/3}/\mathrm{Ai}(i2^{-1/3}\lambda), \qquad \lambda \in \mathbb{R},$$

(see (3.8)*), and the Laplace transform of the function* $p_0\colon \mathbb{R}_+ \to \mathbb{R}$, *defined by*

$$(4.17)\qquad p_0(u) = p(u) - (2\pi u^3)^{-\frac{1}{2}}, \qquad u > 0$$

(the regularization removes the singularity of the function p *of order* $(2\pi u^3)^{-\frac{1}{2}}$ *at zero) is given by*

$$(4.18)\qquad \hat{p}_0(\lambda) = 2^{2/3}\{\mathrm{Ai}'(\xi)/\mathrm{Ai}(\xi)\} + \sqrt{2\lambda}, \qquad \xi = 2^{-1/3}\lambda,$$

(see (4.11)*).*

Proof. First we note that for the process $\{V(a)\colon a \in \mathbb{R}\}$ of *locations* of maxima the "pinning down" of two-sided Brownian motion at zero is immaterial; we could just as well pin down Brownian motion at another place, without changing the structure of the process $\{V(a)\colon a \in \mathbb{R}\}$. Now consider the process $\{X(t)\colon t \ge t_0\}$, starting at x_0 at time t_0, where $X(t) = W(t) - t^2$, and $\{W(t)\colon t \ge t_0\}$ is (one-sided) Brownian motion, starting at $x_0 + t_0^2$ at time t_0. Let M denote the maximum of the process $\{X(t)\colon t \ge t_0\}$, and let τ_M denote the (a.s. unique) location of this maximum. Then τ_M is a last-exit time for the process

$$\{(X(u),M(u))\ :\ u \ge t_0\},$$

where $M(u) = \max\{X(z): t_0 \le z \le u\}$, since τ_M is the time of the last visit to the set $\{(x,x): x \ge x_0\}$. From the results on the decomposition of Markov processes at last-exit times in Meyer, Smythe and Walsh (1972) it then follows that, conditionally, given $\tau_M = t_1$ and $M = a$, the process $\{X(t): t \ge t_1\}$ is a (nonhomogeneous) diffusion, which we will denote by $\{Y(t): t \ge t_1\}$, with transition probabilities

$$(4.19)\qquad \Pr\{Y(w) \in dy | Y(t) = x\}$$

$$= K_1(t,a-x)^{-1}\, q^{\partial}(t,x;w,y) K_1(w,a-y), \qquad t_1 < t < w;\quad x,\, y < a,$$

where $K_1(u,z)$ is defined by (3.2), and where $q^{\partial}(t,x;w,y)$ is the transition density of the process X, killed when reaching a (see (2.3)). The marginal densities of the process Y are given by

$$(4.20)\qquad \Pr\{Y(w) \in dy\} = 2k_1(t_1)^{-1} v(-w,y-a,-t_1) K_1(w,a-y),$$

where the functions v and k_1 are defined by (4.5) and (4.7), respectively. This follows from (4.19), by taking the limit as $t \downarrow t_1$ and $x \uparrow a$, noticing that $q^{\partial}(t,x;w,y) = q^{\partial}(-w,y;-t,x)$ and that, by Lemma 2.2,

$$v(-w,y-a,-t_1) = -\tfrac{1}{2}\partial_4\, q^{\partial}(-w,y;-t,a).$$

Define, for $b > 0$,

$$\tau(b) = \sup\{t \ge t_1:\ Y(t) + 2b(t-t_1) \text{ is maximal}\}.$$

It is easily seen that we have

$$(4.21)\qquad \Pr\{V(b) \in dt_2 | V(0) = t_1\} = \Pr\{\tau(b) \in dt_2\}.$$

A sample path of the process $\{Y(t) + 2b(t-t_1) : t \ge t_1\}$ can only have a maximum at $t_2 > t_1$, if the sample path is of the form shown in Figure 4.2.

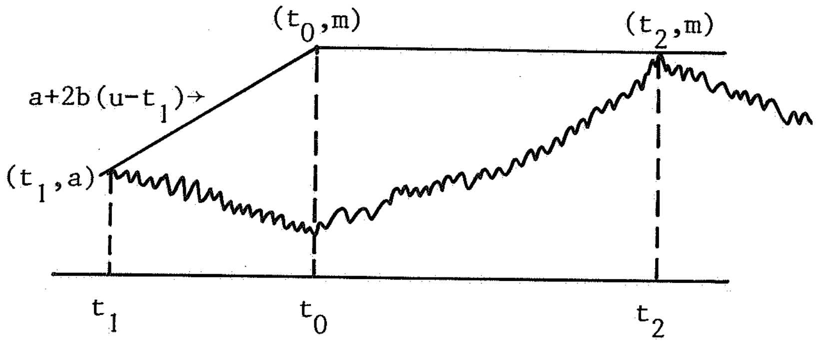

Figure 4.2.

The sample path in Figure 4.2 attains its maximum value m at time t_2; the point (t_0,m) is the intersection of the lines $f(u) = a + 2b(u - t_1)$ and $f(u) = m$, hence $t_0 = t_1 + (m-a)/(2b)$.

Let $\{Z(t): t \geq t_1\}$ be the process defined by $Z(t) = Y(t) + 2b(t - t_1)$. By (4.20) we have

$$(4.22) \qquad \Pr\{Z(t_0) \in dy\}/dy = \Pr\{Y(t_0) \in -2b(t-t_1) + dy\}/dy$$

$$= 2k_1(t_1)^{-1} \, v(-t_0,y-m,-t_1)K_1(t_0,m-y).$$

Let M_b denote the maximum of the process $\{Z(t): t \geq t_1\}$ and let $\tau(b)$ be the (a.s. unique) location of this maximum (see (4.21)). We will show

$$(4.23) \qquad \Pr\{M_b \in dm, \tau(b) \in dt_2\}$$

$$= 2k_1(t_1)^{-1} \int_{-\infty}^{m} v(-t_0,y-m,-t_1)v(t_0-b,y-m,t_2-b)dy \; k_1(t_2-b)dmdt_2$$

For the proof of (4.23), we consider the process $\{U(t): t \geq t_0\}$, starting at y at time t_0, with corresponding probability measure $R^{(t_0,y)}$ on the Borel field of $C([t_0,\infty); \mathbb{R})$, where

$$(4.24) \qquad U(t) = W(t) - t^2 + 2b(t - t_1),$$

and $\{W(t): t \geq t_0\}$ is Brownian motion, starting at $y + t_0^2 - 2b(t_0 - t_1)$ at time t_0.

Let $r^\partial(s,x;t,y)$ be the transition density of the process U, killed when reaching m, let $L(t,x)$ be defined by

$$L(t,x) = R^{(t,-x)}\{U(z) < 0, \quad \text{for all} \quad z \geq t\}, \qquad x > 0,$$

and let

$$\ell(t,x) = \frac{\partial}{\partial x} L(t,x).$$

Furthermore, let M denote the maximum of the process $\{U(t): t \geq t_0\}$, and let τ_M be the (a.s. unique) location of this maximum. Then we have, arguing as in the proof of Corollary 3.1,

$$R^{(t_0,y)}\{\tau_M > t_2, \quad M \in dm\} = \{\int_{-\infty}^{m} r^\partial(t_0,y;t_2,z)\ell(t_2,m-z)dz\}dm,$$

and the joint density of (τ_M,M) at (t_2,m) is given by

$$-\frac{1}{2}\ell(t_2)\,\partial_4\, r^\partial(t_0,y;t_2,m), \tag{4.25}$$

where

$$\ell(t) = \lim_{x \downarrow 0} \ell(t,x).$$

But by (4.24), we can write

$$U(t) = W(z) - z^2, \qquad z = t - b, \tag{4.26}$$

where $\{W(z): z \geq t_0 - b\}$ is Brownian motion, starting at $y + (t_0 - b)^2$ at time $t_0 - b$. Hence we get from (4.25) and (4.26)

$$R^{(t_0,y)}\{\tau_M \in dt_2, \quad M \in dm\}/dt_2dm$$

$$= -\frac{1}{2}k_1(t_2 - b)\,\partial_4\, q^\partial(t_0 - b,y;t_2 - b,m)$$

$$= k_1(t_2 - b)v(t_0 - b,y - m,t_2 - b).$$

Relation (4.23) now follows, since by (4.22)

$$\Pr\{M_b \in dm, \quad \tau(b) \in dt_2\}$$
$$= \int_{-\infty}^{m} (2k_1(t_1))^{-1} v(-t_0,y-m,-t_1) K_1(t_0,m-y)) \cdot$$
$$\cdot (K_1(t_0,m-y))^{-1} R^{(t_0,y)}\{\tau_M \in dt_2, \quad M \in dm\}) dy$$

Furthermore, since $a \le m \le a + 2b(t_2-t_1)$, if $\tau(b) = t_2$, (see Figure 4.2), we obtain from (4.23) by integration with respect to m

$$\Pr\{\tau(b) \in dt_2\}/dt_2$$
$$= 2k_1(t_1)^{-1}\{\int_a^{a+2b(t_2-t_1)} \int_{-\infty}^{m} v(-t_0,y-m,-t_1) \cdot$$
$$\cdot v(t_0 - b, y - m, t_2 - b) k_1(t_2 - b) dy\} dm$$

Letting b tend to zero, we obtain

$$\Pr\{\tau(b) \in dt_2\}/dt_2 = 4bk_1(t_1)^{-1} k_1(t_2) \cdot \tag{4.27}$$
$$\cdot \int_{t_1}^{t_2} \{\int_{-\infty}^{0} v(-t_0,y,-t_1) v(t_0,y,t_2) dy\} dt_0$$
$$+ o(b), \qquad \text{as} \quad b \downarrow 0,$$

making the change of variables $m = a + 2b(t_0 - t_1)$.

By Theorem 2.1.(i) and (4.5) we have

$$v(-t_0,y,-t_1) v(t_0,y,t_2)$$
$$= \tfrac{1}{2} E\{\exp(-2\int_{t_1}^{t_2} e(u)du \mid e(t_0) = -y\} \cdot f_{e(t_0)}(-y) \cdot$$
$$\cdot \{2\pi(t_2 - t_1)^3\}^{-\frac{1}{2}} \exp\{-\tfrac{2}{3}(t_2^3 - t_1^3)\},$$

where $\{e(u): \ t_1 \le u \le t_2\}$ is a Brownian excursion on $[t_1,t_2]$, and $f_{e(t_0)}(-y)$ is the density of $e(t_0)$ at $-y$ (> 0). This is easily seen by gluing the two (conditioned) Bessel processes of Theorem 2.1, on

$[-t_0,-t_1]$ and $[t_0,t_2]$ respectively, together at t_0 (not unlike the construction in section 2.10 of Itô & McKean (1974)), applying time reversal and translation on the Bessel process on $[-t_0,-t_1]$. Hence we get

$$(4.28)\qquad \int_{-\infty}^{0} v(-t_0,y,-t_1)v(t_0,y,t_2)dy$$
$$= \tfrac{1}{2}\{2\pi(t_2-t_1)^3\}^{-\frac{1}{2}} \exp\{-\tfrac{2}{3}(t_2^3-t_1^3)\}\, E\exp\{-2\int_{t_1}^{t_2} e(u)du\}.$$

Thus we obtain, from (4.27), (4.28) and (4.8)

$$(4.29)\qquad \Pr\{\tau(b)\in dt_2\}/dt_2$$
$$= 2bk_1(t_1)^{-1}k_2(t_2)\{2\pi(t_2-t_1)\}^{-\frac{1}{2}} \exp\{-\tfrac{2}{3}(t_2^3-t_1^3)\}\cdot$$
$$\cdot\, E\exp\{-2\int_{t_1}^{t_2} e(u)du\} + o(b)$$
$$= 2bk_1(t_1)^{-1}k_2(t_2)(t_2-t_1)v_2(t_1,t_2) + o(b).$$

The first equality in (4.15) now follows from (4.21) and (4.29), noting that the distribution of $V(a)-a$ is independent of a (and hence equal to that of $V(0)$). The second equality follows from Lemma 4.2.(i) and (3.7). Finally, (4.18) also follows from Lemma 4.2.(i). □

A different version of Theorem 4.1 is given in section 4 of Groeneboom (1984) (see Theorem 4.1 of that section), where also a different approach, based on integral equations is given. The integral equations are further analyzed in Temme (1984).

We finally want to note that the process $\{V(a): a\in\mathbb{R}\}$ not only describes the limiting global behavior of the Grenander maximum likelihood estimator of a (smooth and strictly decreasing) density (see Groeneboom (1984)), but also describes the limiting behavior of certain "isotonic" estimators of distribution functions and hazard functions. In particular, by using the properties of this process, a simple proof of results in Kiefer & Wolfowitz (1976) can be given, which at the same time clarifies the connection between these results (on the estimation of concave distribution functions) and results on the estimation of a monotone density. These statistical applications will be discussed elsewhere.

5. Appendix

Proof of Theorem 3.1. We have, for $x > 0$,

$$K_c(s,x) = Q_c^{(s,-x)}\{X(t) < 0, \quad t \geq s\} =$$

$$= \lim_{t\to\infty} Q_c^{(s,-x)}\{X(u) < 0, \quad s \leq u \leq t\}.$$

Furthermore, by Corollary 2.1,

$$Q_c^{(s,-x)}\{X(u) < 0, \quad s \leq u \leq t\} = \int_{-\infty}^{0} q^{\partial}(s,-x;t,y)dy$$

$$= \exp\{-\tfrac{2}{3}c^2(t^3 - s^3) - 2csx\}\cdot \int_0^{\infty} \exp(2cty)r_c(t-s;x,y)dy,$$

where

$$r_c(u;x,y) = p_u^{\partial}(x,y)\, E^x\{\exp(-2c\int_0^u B(z)dz)\,|\,B(u) = y\}, \tag{5.1}$$

see (2.19). We will show that

$$\lim_{t\to\infty} \exp\{-\tfrac{2}{3}c^2t^3\}\int_0^{\infty}\exp(2cty)r_c(t-s;x,y)dy \tag{5.2}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is\lambda}\, \Phi_x(\lambda)d\lambda.$$

from which (3.3) and (3.4) immediately follow.

First of all, since, by (5.1), $r_c(t-s;x,y) \leq 1$, we have for each $M > 0$,

$$\lim_{t\to\infty} \exp\{-\tfrac{2}{3}c^2t^3\}\int_0^{M}\exp(2cty)r_c(t-s;x,y)dy$$

$$\leq \lim_{t\to\infty} \exp\{-\tfrac{2}{3}c^2t^3 + 2ctM\} = 0.$$

Taking $M > x$, we obtain from (2.20)

$$\int_M^{\infty} e^{2cty}r_c(t-s;x,y)dy = \tag{5.3}$$

$$= \int_M^{\infty} e^{2cty}\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t-s)\lambda}2g_{i\lambda}(x)h_{i\lambda}(y)d\lambda\}dy$$

where $g_{i\lambda}(x)$ and $h_{i\lambda}(y)$ are defined by (2.14) and (2.15) (with λ replaced by $i\lambda$). We note that, if $y > x > 0$, the Laplace transform

$$\lambda \to 2g_\lambda(x)h_\lambda(y)$$

can be inverted along any line of the form $c_1 + i\mathbb{R}$, parallel to the imaginary axis, with $c_1 > a_1$, and a_1 the largest zero of the Airy function Ai on the negative halfline (this will become clear from the computations below).

We now show that we can interchange the order of integration in the expression at the right-hand side of (5.3). We have, by (2.14) and (2.15),

(5.4)
$$g_{i\lambda}(x)h_{i\lambda}(y) =$$
$$= \pi(4c)^{-1/3}\mathrm{Ai}(i\xi + y_1)\mathrm{Ai}(i\xi)^{-1}\cdot$$
$$\cdot\{\mathrm{Ai}(i\xi)\mathrm{Bi}(i\xi + x_1) - \mathrm{Bi}(i\xi)\mathrm{Ai}(i\xi + x_1)\},$$

where $\xi = (2c^2)^{-1/3}$, $x_1 = (4c)^{1/3}x$ and $y_1 = (4c)^{1/3}y$. First suppose that $\xi > 0$. By 10.4.9 in [1], we can write

(5.5)
$$\mathrm{Bi}(z) = i\mathrm{Ai}(z) - 2ie^{\pi i/3}\mathrm{Ai}(z\,e^{-2\pi i/3})$$

Hence we have

$$\mathrm{Ai}(i\xi)\mathrm{Bi}(i\xi + x_1) - \mathrm{Bi}(i\xi)\mathrm{Ai}(i\xi + x_1) =$$
$$= 2\,e^{-\pi i/6}\{\mathrm{Ai}(i\xi)\mathrm{Ai}(e^{-\pi i/6}(\xi - ix_1)) - \mathrm{Ai}(e^{-\pi i/6}\xi)\mathrm{Ai}(i\xi + x_1)\},$$

and therefore

(5.6)
$$|\mathrm{Ai}(i\xi)\mathrm{Bi}(i\xi + x_1) - \mathrm{Bi}(i\xi)\mathrm{Ai}(i\xi + x_1)| \sim$$
$$\sim \pi^{-\frac{1}{2}}\xi^{-1/4}\exp(\tfrac{1}{2}x_1\sqrt{2\xi}), \qquad \text{as} \quad \xi \to \infty$$

using the asymptotic equivalence

(5.7)
$$\mathrm{Ai}(z) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}z^{-1/4}\exp(-\tfrac{2}{3}z^{3/2}), \qquad |z| \to \infty, \quad |\arg(z)| < \pi,$$

(see 10.4.59 in [1]), and the expansion

$$|\exp\{-\frac{2}{3}(e^{-\pi i/6}(\xi - ix_1))^{3/2}\}| =$$

$$= \exp\{-\frac{1}{3}\xi^{3/2}\sqrt{2} + \frac{1}{2}x_1\sqrt{2\xi}\}(1 + O(\xi^{-\frac{1}{2}})),$$

as $\xi \to \infty$. A similar analysis shows

(5.8) $$|Ai(i\xi + y_1)/Ai(i\xi)| =$$

$$= O(\exp\{-\frac{2}{3}r^{3/2}(\cos\frac{3}{2}\theta + 2^{-\frac{1}{2}}\sin^{3/2}\theta)\}),$$

as $r \to \infty$, where $r = |y_1 + i\xi|$, $y_1 = r\cos\theta$ and $\xi = r\sin\theta$, $0 \le \theta \le \frac{1}{2}\pi$.

Since $\cos\frac{3}{2}\theta + 2^{-\frac{1}{2}}\sin^{3/2}\theta = r^{-3/2}\mathrm{Re}\{(i\xi + y)^{3/2} - (i\xi)^{3/2}\}$, the function $f: \theta \to \cos\frac{3}{2}\theta + 2^{-\frac{1}{2}}\sin^{3/2}\theta$ is strictly positive on the interval $[0,\frac{1}{2}\pi)$, and is zero at $\theta = \frac{1}{2}\pi$. A Taylor expansion of the function f in a neighborhood of $\theta = \frac{1}{2}\pi$ shows

(5.9) $$f(\theta) \sim \frac{3}{2}\cdot 2^{-\frac{1}{2}}\cos\theta, \qquad \text{as } \theta \uparrow \frac{1}{2}\pi,$$

and hence

(5.10) $$|Ai(i\xi + y_1)/Ai(i\xi)| = O(\exp(-\frac{1}{2}y_1\sqrt{2\xi})),$$

as $\xi \to \infty$ and $y_1/\xi \to 0$.

Thus, by (5.4) to (5.10) and the choice of $M > x$, we have, if $y \ge M$,

$$|e^{2cty}g_{i\lambda}(x)h_{i\lambda}(y)| = O(\exp\{2cty - c_1(y_1\sqrt{2\xi} + y_1^{3/2})\})$$

for a fixed constant $c_1 > 0$, as $\lambda \to \infty$ and/or $y \to \infty$, implying that the function

(5.11) $$(y,\lambda) \to e^{2cty}g_{i\lambda}(x)h_{i\lambda}(y)$$

is absolutely integrable on $[M,\infty) \times (0,\infty)$.

Similarly, using the representation

$$\mathrm{Bi}(z) = -i\mathrm{Ai}(z) + 2ie^{-\pi i/3}\mathrm{Ai}(ze^{2\pi i/3}),$$

instead of (5.5) (see 10.4.9 in [1]), it is seen that the function (5.11) is absolutely integrable on $[M,\infty) \times (-\infty,0)$. Hence we can apply Fubini's theorem, yielding

$$\int_M^\infty e^{2cty}\{\int_{-\infty}^\infty e^{i(t-s)\lambda} g_{i\lambda}(x)h_{i\lambda}(y)d\lambda\}dy$$
$$= \int_{-\infty}^\infty e^{i(t-s)\lambda} g_{i\lambda}(x)\{\int_M^\infty e^{2cty}h_{i\lambda}(y)dy\}d\lambda.$$

Fix $\lambda \in \mathbb{R}$. Then, as $t \to \infty$,

$$(5.12) \qquad \int_M^\infty \exp(2cty)h_{i\lambda}(y)dy = \int_M^\infty \exp(2cty)\mathrm{Ai}(i\xi + y_1)dy$$
$$\sim (4c)^{-1/3}\exp\{\tfrac{2}{3}c^2t^3 - it\lambda\}.$$

This asymptotic relation can be derived by first writing (using the change of variables $y = ct^2u$)

$$\int_M^\infty \exp(2cty)\mathrm{Ai}(i\xi + y_1)dy$$
$$= ct^2\int_{m/ct^2}^\infty \exp(2c^2t^3u)\mathrm{Ai}(i\xi + (2c^2)^{2/3}t^2u)du,$$

and next, using (5.7), by expanding the integrand at $u = 1$, which is the approximate location of its saddle point for large t. This yields

$$\int_M^\infty \exp(2cty)\mathrm{Ai}(i\xi + y_1)dy$$
$$\sim 2^{-7/6}\pi^{-\frac{1}{2}}c^{2/3}t^{3/2}\exp(\tfrac{2}{3}c^2t^3 - it\lambda) \cdot \int_0^\infty \exp\{-\tfrac{1}{2}c^2t^3(u-1)^2\}du$$
$$\sim (4c)^{-1/3}\exp(\tfrac{2}{3}c^2t^3 - it\lambda), \qquad t \to \infty$$

Thus we obtain, for fixed $a > 0$,

$$(5.13) \qquad \frac{1}{\pi}\exp(-\tfrac{2}{3}c^2t^3)\int_{-a}^{a}\exp(i(t-s)\lambda)g_{i\lambda}(x)\{\int_M^\infty \exp(2cty)\mathrm{Ai}(i\xi + y_1)dy\}d\lambda$$

$$\to \frac{1}{2\pi} \int_{-a}^{a} \exp(-is\lambda)\Phi_x(\lambda)\,d\lambda, \qquad \text{as} \quad t \to \infty$$

Hence we are through, if we can show that we can take $a = \infty$ in (5.13), or, stated differently, that

$$(5.14) \qquad \lim_{t,a\to\infty} \exp(-\tfrac{2}{3}c^2t^3) \int_{\lambda > a} \exp(i(t-s)\lambda)g_{i\lambda}(x)\,\cdot \\ \cdot \int_M^\infty (2cty)Ai(i\xi + y_1)\,dyd\lambda = 0,$$

and similarly that the integral over the region $(-\infty,-a) \times [M,\infty)$ tends to zero as $t \to \infty$ and $a \to \infty$. We will only show (5.14), since the other case can be treated in a completely similar way.

By the change of variables $y = ct^2u$ and $\lambda = 2c^2t^2v$, we get

$$\int_{\lambda > a} \left|\exp(i(t-s)\lambda)g_{i\lambda}(x)\int_M^\infty \exp(2cty)Ai(i\xi + y_1)\right|dyd\lambda =$$

$$\sim 2^{-3/2}\pi^{-1}c^2t^3 \int_{v > a/2c^2t^2} \exp(\tfrac{1}{2}x_1t(2c^2)^{1/3}\sqrt{2v})\,dv\,\cdot \\ \cdot \int_{u > M/ct^2} \exp\{2c^2t^3(u - \tfrac{2}{3}\mathrm{Re}\{(u+iv)^{3/2} - (iv)^{3/2}\})\}du,$$

as $a \to \infty$. If $0 \le v < 2$, we have

$$(5.15) \qquad \int_{M/ct^2}^\infty \exp\{2c^2t^3(u - \tfrac{2}{3}\mathrm{Re}\{(u+iv)^{3/2} - (iv)^{3/2}\})\}du \\ = O(\int_{M/ct^2}^\infty \exp\{2c^2t^3(f(v) - g(u,v))\}du),$$

uniformly in $v \in [0,2)$, where

$$f(v) = u_1 - \tfrac{2}{3}\mathrm{Re}\{(u_1 + iv)^{3/2} - (iv)^{3/2}\},$$

$$g(u,v) = \tfrac{1}{4}\{\mathrm{Re}(u_1 + iv)^{-\frac{1}{2}}\}(u - u_1)^2,$$

and u_1 is the unique solution of the equation (in u)

$$(5.16) \qquad \mathrm{Re}\{(u+iv)^{\frac{1}{2}}\} = 1.$$

It is clear that the root u_1 of the equation (5.16) is a strictly decreasing function of $v \in [0,2]$, with $u_1 = 1$, if $v = 0$, and $u_1 = 0$, if $v = 2$. The function f is also strictly decreasing in v, with $f(0) = \frac{1}{3}$ and $f(2) = 0$. For the proof, we write $u_1 = r\cos\theta$, $v = r\sin\theta$, which yields

$$(5.17) \qquad u_1 - \tfrac{2}{3}\mathrm{Re}\{(u_1+iv)^{3/2} - (iv)^{3/2}\}$$

$$= r\cos\theta - \tfrac{2}{3}r^{3/2}\{\cos\tfrac{3}{2}\theta - 2^{-\frac{1}{2}}\sin^{3/2}\theta\}$$

$$= \tfrac{1}{3} + \mathrm{tg}^2\tfrac{1}{2}\theta - \tfrac{4}{3}\mathrm{tg}^{3/2}\tfrac{1}{2}\theta,$$

if $r^{\frac{1}{2}}\cos\frac{1}{2}\theta = \mathrm{Re}(u_1+iv)^{\frac{1}{2}} = 1$, as can be verified by writing $\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta$, $\cos\frac{3}{2}\theta = \cos^3\frac{1}{2}\theta - 3\cos\frac{1}{2}\theta\sin^2\frac{1}{2}\theta$, and $2^{-\frac{1}{2}}\sin^{3/2}\theta = 2\sin^{3/2}\frac{1}{2}\theta\cos^{3/2}\frac{1}{2}\theta$. Since $\mathrm{tg}\,\theta = v/u_1$, and u_1 is a decreasing function of v, it is seen from (5.17) that $f(v)$ is a strictly decreasing function of v (using that $\mathrm{tg}^2\frac{1}{2}\theta - \frac{4}{3}\mathrm{tg}^{3/2}\frac{1}{2}\theta$ is strictly decreasing in θ, $0 \le \theta \le \frac{1}{2}\pi$).

If $v \ge 2$, we have

$$(5.18) \qquad \int_{M/ct^2}^{\infty} \exp\{2c^2t^3(u - \tfrac{2}{3}\mathrm{Re}\{(u+iv)^{3/2} - (iv)^{3/2}\})\}du$$

$$= O\{\int_{M/ct^2}^{\infty} \exp\{-2^{\frac{1}{2}}c^2t^3v^{\frac{1}{2}}u\}du\}$$

$$= O\{\frac{1}{c^2t^3\sqrt{2v}}\exp(-ctM\sqrt{2v})\},$$

as $t \to \infty$ uniformly in $v \in [2,\infty)$, since in this case the function $u \to u - \frac{2}{3}\mathrm{Re}(u+iv)^{3/2}$, $u \ge 0$, is decreasing on $[0,\infty)$.

Thus we obtain, from (5.15), (5.17) and (5.18)

$$\int_{M/ct^2}^{\infty} \exp\{2c^2t^3(u - \tfrac{2}{3}\mathrm{Re}\{(u+iv)^{3/2} - (iv)^{3/2}\})\}du$$

$$= O(\exp(c_1t^3 - ctM\sqrt{2v})),$$

if $v \ge \delta > 0$, uniformly in $v \in [\delta,\infty)$, where c_1 is a positive constant (depending on δ) such that $c_1 < \frac{2}{3}$. This shows

(5.19) $$\exp(-\tfrac{2}{3}c^2t^3)\int_{v\geq\delta}\exp(\tfrac{1}{2}x_1t(2c^2)^{1/3}\sqrt{2v})dv\;\cdot$$

$$\int_{u>M/ct^2}\exp\{2c^2t^3(u-\tfrac{2}{3}\mathrm{Re}\{(u+iv)^{3/2}-(iv)^{3/2}\})\}du$$

$$\to 0, \quad \text{as} \quad t\to\infty,$$

since $x_1 = (4c)^{1/3}x$ and hence $ctM > \frac{1}{2}x_1\, t(2c^2)^{1/3}\sqrt{2v}$ (using $M > x$). Finally, if $v\to 0$, but $\lambda = 2c^2t^2v\to\infty$, we get

(5.20) $$t^3\int_{M/ct^2}\exp\{2c^2t^3(u-\tfrac{2}{3}\mathrm{Re}\{(u+iv)^{3/2}-(iv)^{3/2}\})\}du$$

$$= O\{\exp(\tfrac{2}{3}t^3-\tfrac{2}{3}c^2t^3(2v)^{3/2})\},$$

using the same techniques as in the derivation of (5.12).

Relation (5.14) now follows from (5.19) and (5.20). □

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