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BRST SYMMETRY TOWARDS THE GAUSS CONSTRAINT FOR GENERAL RELATIVITY

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Quantization of systems with constraints can be carried on with several methods. In the Dirac's formulation the classical generators of gauge transformations are required to annihilate physical quantum states to ensure their gauge invariance. Carrying on BRST symmetry it is possible to get a condition on physical states which, differently from the Dirac's method, requires them to be invariant under the BRST transformation. Employing this method for the action of general relativity expressed in terms of the spin connection and tetrad fields with path integral methods, we construct the generator of BRST transformation associated with the underlying local Lorentz symmetry of the theory and write a physical state condition following from BRST invariance.

The condition we gain differs form the one obtained within Ashtekar's canonical formulation, showing how we recover the latter only by a suitable choice of the gauge fixing functionals. We finally discuss how it should be possible to obtain all the requested physical state conditions associated with all the underlying gauge symmetries of the classical theory using our approach.

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1. Introduction

The problem of quantization of constrained systems arises in many contexts of physical interest. The presence of constraints at a classical level avoids us to threat all the dynamical variables as independent ones, and entails several difficulties when we are to construct the quantum theory. In a program of canonical quantization which promotes all classical canonical variables to quantum operators one has to deal with the problem of imposing the constraints quantum mechanically. In the procedure *à la Dirac*[4] the constraint operators are imposed to annihilate physical states. This procedure stems from the observation that in the classical theory the constraint functions are generators of infinitesimal canonical transformations which don't alter the physical state of the system.

The Dirac's procedure is widely used in different contexts, including quantization of general relativity.[3,10] Nevertheless this procedure of quantization encounters several difficulties when we require the Dirac's conditions on physical states to be consistent with each other[4,6] and the physical states selected by constraint operators to posses a finite scalar product allowing a probabilistic interpretation:[6,12] moreover, in some cases this procedure can lead to a physical subspace of the entire Hilbert space that is curiously empty.[6] Other difficulties arise when one tries to implement Dirac's procedure, which are not properly to ascribe to Dirac's theory for constrained systems, but to the canonical quantization framework this procedure is developed in. As a matter of fact, our experience on quantum field theory in special relativity showed us how canonical quantization methods, when applied to systems with infinite degrees of freedom, lead to several inconsistencies:[7,11] for example, it is a remarkable fact that the Glashow - Weinberg - Salam theory for electroweak interactions cannot be consistently formulated by canonical quantization methods, while the only way it can be coherently written by is the Feynman's path integral. Even if Feynman's path integral can be derived after constructing the quantum theory by means of canonical quantization methods,[14] such inconsistencies make necessary to postulate the path integral approach as a founding element of the quantum theory when we deal with systems with infinite degrees of freedom.[11] For these reasons we developed all of our work avoiding to use the Dirac procedure for constrained systems and canonical quantization methods at all, employing a method to derive conditions on physical states based on BRST symmetry and path integral methods uniquely.

BRST symmetry[2,6,13,15] was conceived at first within non-abelian gauge theories and showed to apply to a really wide class of systems. There are different formulations for the BRST formalism, with substantial differences from each other. There exists a widely diffused formulation of BRST symmetry for constrained systems based on canonical quantization methods,[5,6] being employed in quantization of general relativity.[1] Another approach,[15] the one we followed in this work, to derive BRST symmetry is based entirely on path integral methods and is applicable to systems with infinite degrees of freedom avoiding those inconsistencies proper of canonical quantization methods we discussed above.

2. BRST symmetry for a non-abelian gauge theory

We start with an enlightening example, considering BRST symmetry for a non-abelian gauge theory. In order to compare path integral methods with canonical

quantization ones, one can[8] consider the Nöether charge following from BRST symmetry of the action and, taking an appropriate choice for the gauge fixing functionals in the DeWitt - Fadeev - Popov method, show it to be the generator of quantum BRST transformation within a canonical quantization framework.
Otherwise, using solely path integral methods, we show the BRST Nöether charge

$$Q \equiv \int d^3x \mathscr{J}^0(x)$$

related to the BRST current $\mathscr{J}^\mu$ to generate quantum BRST transformation by means of Ward's identities for the ensemble of gauge fields, ghost and antighost fields and Nakanishi - Lautrup fields,[15] designed by $\psi_i(x)$, i. e.

$$0 = \partial^x_\mu \left\langle \psi_{i_k}(x_k) \cdots \psi_{i_1}(x_1) \mathscr{J}^\mu(x) \right\rangle_{j=0} - i \sum_{l=1}^{k} \sigma^{i_1} \cdots \sigma^{i_l} \langle \psi_{i_k}(x_k) \cdots \\ \cdots \psi_{i_{l+1}}(x_{l+1}) \, s\psi_{i_l}(x) \psi_{i_{l-1}}(x_{l-1}) \cdots \psi_{i_1}(x_1) \rangle_{j=0} \delta^{(4)}(x - x_l) \,. \tag{1}$$

where $\sigma^i = \pm 1$ for ψ_i bosonic or fermionic respectively. The fact that in (1) the gauge fixing functionals are completely arbitrary allows us to infer a physical state condition on states of the gauge fields following from BRST invariance, given by the usual Gauss' constraint

$$\mathcal{D}_a F^{0a\alpha}(x) \, |\psi\rangle = 0. \tag{2}$$

The condition (2) differs from the one we'd obtain fixing the temporal gauge condition ${A_0}^\alpha = 0$ at a classical level and then quantizing by the Dirac method the classical Gauss' constraint $\mathcal{D}_a F^{0a\alpha}(x) = 0$ holding on the equations of motion. This is because in (2) the Yang-Mills tensor operator possesses some additional terms propotional to ${A_0}^\alpha$ with respect to the Yang-Mills tensor operator in the Dirac's procedure. Thus *according to BRST invariance the Dirac's procedure's Gauss' constraint does not annihilate physical states any more*: we will find a deeply analog result in Section 3 for general relativity.

3. BRST symmetry for general relativity

Afterward we turn our attention to general relativity expressed in first order formalism[9,10] in order to investigate the physicality condition for gravitational field's states arising from BRST invariance of the theory and following the same procedure employed for non-abelian gauge theories. We are to determine a physical state condition on quantum states without thinking of classical hamiltonian constraints in order to compare our physicality condition required by BRST symmetry and derived with path integral methods with the one obtained using the Dirac quantization method employed within Ashtekar's canonical formulation.[10,12] Employing the same method followed in Section (2) for general relativity, we arrive to the following physical state condition for the densitized triad[1,9,10,12] E^a_i

$$\mathcal{D}_a \left[E^a_j(x) + i e_{jb}(x) e_{0c}(x) \epsilon^{abc} \right] |\psi\rangle = 0. \tag{3}$$

Comparing our physicality condition with the one used in loop quantum gravity[10,12] and obtained fixing the temporal gauge condition already in the classical theory

$$\mathcal{D}_a E^a_j(x)\,|\psi\rangle = 0, \tag{4}$$

we find they differ by an additional term: thus *according to BRST invariance the Dirac's procedure's Gauss' constraint does not annihilate physical states any more*, just as we observed in Section 2 for a non-abelian gauge theory. We think the origin of this discrepancy to be in the choice of a particular gauge in the classical theory made within Ashtekar's approach and which was intentionally avoided in our work, in order to preserve a fundamental symmetry such as the local Lorentz's one. Finally we showed that we recover the Dirac's canonical condition in our BRST quantization only by a suitable choice of gauge fixing functionals within the DeWitt - Fadeev - Popov method, followed by an appropriate limiting procedure. Being such a limit possibly ill-defined, (3) and (4) cannot be trivially said to be equivalent.

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