BUILDING CARTESIAN PRODUCTS OF SURFACES WITH [0, 1]

BY
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1. **Introduction.** The following is a special case of Theorem 9.2, one of the chief results of this paper.

THEOREM. *Suppose that M is a 3-manifold with boundary, S is a compact 2-manifold with boundary in M such that $S \cap \mathrm{Bd}(M) = \mathrm{Bd}(S) \cap \mathrm{Bd}(M) = R$ either a 1-manifold with boundary or the empty set, and $\varepsilon > 0$.*

There is a $\delta > 0$ such that if f_0 and f_1 are homeomorphisms of S onto disjoint locally tame surfaces S_0 and S_1 in M where $f_e(S) \cap \mathrm{Bd}(M) = f_e(R)$ and f_e moves no point of S by as much as δ ($e = 0, 1$), then there is a homeomorphism g of $S \times [0, 1]$ onto a locally tame solid in M such that $g(S \times [0, 1]) \cap \mathrm{Bd}(M) = g(R \times [0, 1])$ and for each point y of S, $g(y, e) = f_e(y)$ ($e = 0, 1$) and the diameter of $g(y \times [0, 1])$ is less than ε.

This is the first of three papers (see also [13], [14]) where we investigate the global relation between two nice embeddings of a polyhedron in a 3-manifold with boundary where both embeddings approximate very closely a topological embedding of the polyhedron. In [13] we establish the following result.

THEOREM. *Suppose that M is a* pwl *3-manifold (with boundary), K is a finite polyhedron with no local cut points, K_a is a subpolyhedron of K with no degenerate components, f is a homeomorphism of K into M such that $f(K) \cap \mathrm{Bd}(M) = f(K_a)$, and $\varepsilon > 0$.*

There is a $\delta > 0$ such that if f_0 and f_1 are pwl *homeomorphisms of K into M so that $f_e(K) \cap \mathrm{Bd}(M) = f_e(K_a)$ and $d(f, f_e) < \delta$ ($e = 0, 1$) where d measures the distance between two maps of a space into a metric space, then there is a* pwl *ε-isotopy H_t ($0 \leqq t \leqq 1$) of M onto itself so that H_0 is the identity and $H_1 f_0 = f_1$.*

In [14] we develop an analogous result involving regular neighborhoods for the case of embeddings of polyhedra which have local cut points.

Both theorems which we stated are obtained by refining techniques used in [5]. We hope that the serious reader will acquaint himself with that paper especially with the proof of Theorem 3.2 there. The reader should also be familiar with [12].

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The lemmas of §§2–6 form a foundation on which we build the proofs of Theorem 7.1 here and Theorem 5.1 of [13]. In many cases the full strength of a lemma is needed only in [13].

In the remainder of this section we give several definitions. Many terms which we use are defined in [2]–[9], [12], and [25]. In particular our use of such terms as *general position*, *normally situated*, *universal curve*, $I(X, D)$, and *Property* Q is explained in [12, §§1 and 6].

We use pwl as an abbreviation for *piecewise linear(ly)*. A *Euclidean complex* is a rectilinear simplicial complex in some Euclidean space. A *Euclidean polyhedron* is a set which is the sum of the simplexes of a Euclidean complex. In general by *complex* we mean *geometric simplicial complex*. This is a partitioning of a separable metric space X into subsets with affine structures by means of a homeomorphism $f: |K| \to X$ where $|K|$ denotes the polyhedron underlying a Euclidean complex K. The simplexes of a geometric complex are the images $f(s)$ of the simplexes s of K. A *polyhedron* is the sum of the simplexes of a geometric complex. Suppose X is a polyhedron which receives its pwl structure by way of a homeomorphism $f: P \to X$ where P is a Euclidean polyhedron. We say a subset Y of X is a *polyhedron in* X or a *subpolyhedron of* X if $f^{-1}(Y)$ is a Euclidean polyhedron. If X and Y are polyhedra receiving their pwl structures by homeomorphisms $f: P \to X$ and $g: Q \to Y$ where P and Q are Euclidean polyhedra then a map $h: X \to Y$ is pwl if $g^{-1}hf$ is a pwl map of P into Q.

Suppose X is a separable metric space. If X is not regarded as a polyhedron then by a *triangulation* of X we mean a geometric complex whose underlying point set is X. If X is a polyhedron receiving its pwl structure by way of a homeomorphism $f: P \to X$ where P is a Euclidean polyhedron, then a *triangulation* of X is a geometric complex defined by means of a homeomorphism $g: |L| \to X$ such that $g^{-1}f$ is a pwl homeomorphism.

A pwl *n-cell* (*n-sphere*) is a polyhedron which is pwl homeomorphic to an n-simplex (the boundary of an $n+1$-simplex). A pwl *n-manifold* is a polyhedron such that each point has a closed polyhedral neighborhood which is a pwl n-cell. Notice that a pwl manifold can have a boundary.

Disjoint simplexes r and s in Euclidean space E^n (or of a complex L) are *joinable* if there is a simplex t of E^n (resp. of L) whose vertices consist of the vertices of r and s. We call t the *join* of r and s and denote this join by rs. The empty simplex is allowed here and every simplex is joinable to it. Complexes K_1 and K_2 in E^n (or subcomplexes K_1 and K_2 of a complex L) are *joinable* if each simplex of K_1 is joinable to each simplex of K_2 and the collection of joins of simplexes of K_1 with simplexes of K_2 together with their faces forms a complex (resp. a subcomplex of L). The complex just mentioned is called the *join* of K_1 and K_2 and is denoted by K_1K_2. Notice that local finiteness of complexes implies that only finite complexes can be joinable. Disjoint compact polyhedra K_1 and K_2 in E^n are *joinable* if K_1 and K_2 possess rectilinear triangulations T_{K_1} and T_{K_2} which are joinable. In this case K_1K_2 denotes the *join* of K_1 and K_2 by which is meant the polyhedron that underlies $T_{K_1}T_{K_2}$.

If K and L are joinable polyhedra in E^m and f and g are respectively maps of K and L into E^n, then by the *join* of the maps f and g we mean the map h of KL into E^n which is given by $h(\alpha x+(1-\alpha)y)=\alpha f(x)+(1-\alpha)g(y)$ for $x \in K$, $y \in L$, and $0 \leqq \alpha \leqq 1$. We do not insist that f and g be pwl. If K and L are joinable polyhedra in E^m and $f_i \to f$ and $g_i \to g$ are sequences of maps of K and L into E^n which converge to maps f and g, then the limit of the join of f_i and g_i is the join of f and g.

If s is a simplex in a complex K then by $st(s, K)$ we mean the polyhedron which is the sum of the simplexes of K that have s as a face. Since we take simplexes to be closed, $st(s, K)$ is compact. If X is a set in a complex K then by $N(X, K)$ we mean the polyhedron which is the sum of all simplexes of K that intersect X.

A subcomplex L of a complex K is *full* in K if for each simplex s of K, s belongs to L if every proper face of s belongs to L. A *first derived subdivision* K' of a complex K is a subdivision of K which is isomorphic to the first barycentric subdivision of K under an isomorphism that takes each simplex of K onto itself. An nth *derived subdivision* of a complex is defined by iteration. If L is a subcomplex of a complex K, and K' is a first derived subdivision of K with induced subdivision L' of L, then L' is full in K'.

If K is a finite polyhedron and L is a subpolyhedron such that $K=L \cup B^n$ and $B^n \cap L=B^{n-1} \subset \mathrm{Bd}\,(B^n)$ where B^n and B^{n-1} are pwl n- and $(n-1)$-cells, then we say that there is an *elementary collapse* from K to L or that K *collapses* to L ($K \searrow L$) by an elementary collapse. A polyhedron K *collapses* to a subpolyhedron L if there is a finite sequence of elementary collapses $K \searrow K_1 \searrow K_2 \cdots \searrow K_n=L$. We write this $K \searrow L$. If T_K is a triangulation of K in which each $\mathrm{Cl}\,(K_i-K_{i+1})$ and each $\mathrm{Cl}\,(\mathrm{Bd}\,(\mathrm{Cl}\,(K_i-K_{i+1}))-K_{i+1})$ is a simplex of T_K, then K *collapses simplicially* to L (in T_K). A finite polyhedron is *collapsible* if it collapses to a point.

A finite polyhedron N in a pwl n-manifold M is a *regular neighborhood* of a finite polyhedron K in M if

1. N is a pwl n-manifold,
2. N contains a neighborhood of K in M, and
3. N collapses to K.

If a pwl manifold M is a finite polyhedron which collapses to a subpolyhedron K, then M is a (intrinsic) *regular neighborhood* of K. If K is a finite polyhedron in a pwl manifold M, if T is a triangulation of a neighborhood of K in which K underlies a full subcomplex, and if T' is a first derived subdivision of T, then we call $N(K, T')$ a *derived neighborhood* of K in M.

In [24], [25] it is shown that if K is a finite polyhedron in a pwl manifold M then (1) a derived neighborhood of K in M is a regular neighborhood, (2) any two regular neighborhoods of K in M are pwl homeomorphic under a homeomorphism that is the identity on K, and (3) a regular neighborhood of K is a pwl cell if K is collapsible.

We use $[a, b]$ to denote the closed interval $a \leqq t \leqq b$ and (a, b) to denote the open interval $a < t < b$. Half open intervals are denoted by $[a, b)$ and $(a, b]$.

An *isotopy* H_t $(a \leqq t \leqq b)$ of a space X into a space Y is a continuous one parameter family of embeddings of X into Y. An *ambient isotopy* H_t $(a \leqq t \leqq b)$ of a space X or an *isotopy of a space X onto itself* is a continuous family of homeomorphisms of X onto itself. For such an isotopy we assume that H_a is the identity if $a=0$. Associated with an isotopy H_t $(a \leqq t \leqq b)$ of a space X into a space Y is a level preserving homeomorphism $H\colon X \times [a, b] \to Y \times [a, b]$ which is given by the rule $H(x, t) = (H_t(x), t)$.

Two definitions of the term pwl isotopy appear in the literature. Bing and Sanderson [5], [22] call an isotopy H_t $(a \leqq t \leqq b)$ of a polyhedron K into a polyhedron L piecewise linear if for each value of t, H_t is a pwl homeomorphism of K into L. We prefer to follow Gugenheim [16] and say that H_t is *piecewise linear* if the associated homeomorphism $H\colon K \times [a, b] \to L \times [a, b]$ is pwl. These two definitions are not equivalent, and this inequivalence causes us some difficulty since we employ isotopy results of [5], [22] involving isotopies that are not pwl in the Gugenheim sense. We get around this difficulty by showing how to approximate certain isotopies by pwl ones.

By the *track* of a set Z under an isotopy H_t $(a \leqq t \leqq b)$ is meant $\bigcup \{H_t(Z) \mid t \in [a, b]\}$. An isotopy H_t $(a \leqq t \leqq b)$ of a space X into a metric space Y is an *ε-isotopy* if the track under H_t of each point of X has diameter less than ε.

A *cylindrical sphere* in E^3 with axis L is the boundary of a 3-cell that is obtained by taking the closure of the part of a solid cylinder with axis L that lies between two planes perpendicular to L. The *bases* of the cylindrical sphere are the two 2-cells which compose the intersection of the sphere with the two planes just mentioned.

We use ρ to denote the metric on a metric space and I to denote the identity homeomorphism. If f and g are maps of a space X into a metric space Y we denote by $d(f, g)$ the least upper bound over points x of X of the distances $\rho(f(x), g(x))$. We will often be occupied with extending homeomorphisms defined on subpolyhedra of polyhedra. In order to conserve on symbols we will usually denote an extension of a homeomorphism with the same symbol used to denote the homeomorphism.

Consider a finite collection $D_1, \ldots, D_n$ of subdisks of a disk D. We say the collection has *Property Z* (in D) if (i) the interiors of the D_i's are mutually exclusive, (ii) no two of the D_i's intersect in a disconnected set, and (iii) $\bigcup D_i = D$. If the collection has Property Z and in addition each D_i is normally situated in D (that is, each D_i is either contained in Int (D) or intersects Bd (D) in an arc) then we say the collection has *Property Z′*. We say the collection has *Property* $Z(\varepsilon)$ or *Property* $Z'(\varepsilon)$ where $\varepsilon > 0$ if the collection has Property Z or Property Z' and in addition each D_i has diameter less than ε.

2. **Some simple lemmas on isotopies and homeomorphisms.**

Most of the isotopies and homeomorphisms which we construct in this paper and in [13] and which we claim are pwl can be shown to be pwl by the following proposition which is stated here without proof. It can be proved by using the lemmas of [25, Chapter 1] to imitate in higher dimensions proofs like those in [2, §1]. Recall that a polyhedron can be infinite, that is noncompact.

PROPOSITION 2.1. *Suppose that h is a homeomorphism of a polyhedron K onto a closed subset of a polyhedron L so that h is* pwl *on each element of a locally finite collection of compact subpolyhedra whose union is K.*

Then h is pwl.

LEMMA 2.1. *Suppose that B_1^n and B_2^n are* (pwl) *n-cells and h is a* (pwl) *homeomorphism of* $\text{Bd}(B_1^n)$ *onto* $\text{Bd}(B_2^n)$. *Then there is an extension of h to a* (pwl) *homeomorphism of B_1^n onto B_2^n.*

Proof. See [25, Lemma 10] for a proof of the pwl version. The topological version is proved similarly.

LEMMA 2.2. *Suppose that Δ^n is a n-simplex with barycenter b, and suppose that h is a* pwl *homeomorphism of Δ^n onto itself such that* $h|\text{Bd}(\Delta^n)=I$.

Then there is a pwl *isotopy H_t $(0\leqq t\leqq 1)$ of Δ^n onto itself such that* $H_t|\text{Bd}(\Delta^n)=I$ *and $H_1=h$.*

Proof. A proof of this lemma can be found in [25]; however, we repeat the proof in order to make several observations about the constructed isotopy.

There is no loss in assuming that $\Delta^n\subset E^n=E^n\times 0\subset E^n\times E^1=E^{n+1}$. Consider the prism $\Delta^n\times[0,1]$ in E^{n+1}. Define a level preserving pwl homeomorphism H of $\Delta^n\times[0,1]$ onto itself as follows. First set $H|\text{Bd}(\Delta^n)\times[0,1]=I$, $H|\Delta^n\times 0=I$, $H|\Delta^n\times 1=(h,1)$, and $H|b\times\frac{1}{2}=I$. Then define H on the rest of $\Delta^n\times[0,1]$ by the join of the two maps $H|\text{Bd}(\Delta^n\times[0,1])$ and $H|b\times\frac{1}{2}$.

The isotopy H_t is defined so that $H(y,t)=(H_t(y),t)$ $(y\in\Delta^n,\ t\in[0,1])$. □

COROLLARY 2.2$_a$. *If K is a polyhedron* (*possibly empty*) *in* $\text{Bd}(\Delta^n)$ *such that $h|bK=I$ then $H_t|bK=I$.*

Proof. If K is empty the problem reduces to showing that $H=I$ on $b\times[0,1]$. But $H|b\times\text{Bd}([0,1])=I$ and $H|b\times\frac{1}{2}=I$ so $H|b\times[0,1]=I$ by the join construction.

Suppose K is not empty. Let y be a point of K. Now $H=I$ on $(by)\times 1$ and on the part of $(by)\times[0,1]$ not in the 2-simplex $(b\times\frac{1}{2})((by)\times 1)$. But $H|(b\times\frac{1}{2})((by)\times 1)$ is the join of two identity maps and so is the identity. □

In a similar fashion one can show

COROLLARY 2.2$_b$. *If K is a polyhedron in* $\text{Bd}(\Delta^n)$ *such that h leaves each segment by invariant where y is a point of K, then H_t leaves each segment by invariant.*

COROLLARY 2.2$_c$. *If $\varepsilon>0$ there is a $\delta>0$ such that $d(h,I)<\delta$ implies that H_t is an ε-isotopy.*

Proof. Suppose this were not true. Then there would be a positive number η, there would be a sequence of homeomorphisms h_i of Δ^n onto itself such that $h_i|\mathrm{Bd}(\Delta^n)=I$ and $h_i \to I$, and there would be a sequence of level preserving homeomorphisms H^i of $\Delta^n \times [0, 1]$ onto itself constructed from the h_i's by the join construction such that each H^i moved some point by more than η. But since $\Delta^n \times [0, 1]$ is compact this would imply that $H^i \not\to I$ which contradicts the fact that the limit of the join of two maps is the join of the limits. □

LEMMA 2.3. *Suppose that B^n is a* (pwl) *n-cell and h is a* (pwl) *homeomorphism of B^n onto itself such that $h|\mathrm{Bd}(B^n)=I$.*

Then there is a (pwl) *isotopy H_t $(0 \leqq t \leqq 1)$ of B^n onto itself such that $H_t|\mathrm{Bd}(B^n)=I$ and $H_1=h$.*

Furthermore if $p \in \mathrm{Int}(B^n)$ and $h(p)=p$ then H_t can be constructed so that $H_t|p=I$.

Proof. We prove the pwl version of the lemma. See [1] for a proof of the topological version.

Define a pwl homeomorphism ϕ of B^n onto an n-simplex Δ^n which takes p onto the barycenter b of Δ^n. Now h induces $h'=\phi h \phi^{-1}: \Delta^n \to \Delta^n$. Apply Lemma 2.2 to find a pwl isotopy H'_t $(0 \leqq t \leqq 1)$ of Δ^n onto itself such that $H'_t|\mathrm{Bd}(\Delta^n)=I$ and $H'_1=h'$. Set $H_t=\phi^{-1}H'_t\phi$.

If $h(p)=p$ then $h'(b)=b$ so by Corollary 2.2_a, $H'_t|b=I$ and thus $H_t|p=I$. □

With the aid of Corollary 2.2_c we obtain the following lemma by imitating the previous construction. See also [17].

LEMMA 2.4. *Suppose that B^n is a* (pwl) *n-cell and $\varepsilon>0$.*

There is a $\delta>0$ such that if h is a (pwl) *δ-homeomorphism of B^n onto itself so that $h|\mathrm{Bd}(B^n)=I$, then there is a* (pwl) *ε-isotopy H_t $(0 \leqq t \leqq 1)$ of B^n onto itself so that $H_t|\mathrm{Bd}(B^n)=I$ and $H_1=h$.*

The next lemma shows that certain isotopies of pwl n-cells $(n \leqq 3)$ can be approximated by pwl isotopies.

LEMMA 2.5. *Suppose that B^n $(n \leqq 3)$ is a* pwl *n-cell, H_t $(0 \leqq t \leqq 1)$ is an isotopy of B^n onto itself such that $H_t|\mathrm{Bd}(B^n)=I$ and H_1 is a* pwl *homeomorphism, and $\varepsilon>0$.*

Then there is a pwl *isotopy H'_t $(0 \leqq t \leqq 1)$ of B^n onto itself such that $H'_t|\mathrm{Bd}(B^n)=I$, $H'_1=H_1$, and for each t, $d(H_t, H'_t) < \varepsilon$.*

Proof. Choose δ from Lemma 2.4 for the substitution $B^n \to B^n$ and $\varepsilon/3 \to \varepsilon$. Let $0=t_0<t_1<\cdots<t_m=1$ be numbers such that for each i, H_t $(t_i \leqq t \leqq t_{i+1})$ is a $\delta/3$-isotopy of B^n.

Use [2, Theorem 3] or a lower dimensional version of it to find for each i $(1 \leqq i \leqq m-1)$ a pwl homeomorphism h_i of B^n such that $d(h_i, H_{t_i})<\delta/3$ and $h_i|\mathrm{Bd}(B^n)=I$. Set $h_0=I$ and $h_m=H_1$.

From the inequality $d(h_i, h_{i+1}) \leqq d(h_i, H_{t_i})+d(H_{t_i}, H_{t_{i+1}})+d(H_{t_{i+1}}, h_{i+1})$ we find that $d(h_i, h_{i+1})=d(h_{i+1}h_i^{-1}, I)<\delta$ for each i. Thus from Lemma 2.4 there is for each i $(0 \leqq i \leqq m-1)$ a pwl $\varepsilon/3$-isotopy H_t^i $(0 \leqq t \leqq 1)$ of B^n onto itself such that $H_t^i|\text{Bd}(B^n)=I$ and $H_1^i=h_{i+1}h_i^{-1}$.

Define H_t' by the rule $H_0'=I$ and $H_t'=H^i_{(t-t_i)/(t_{i+1}-t_i)}H_{t_i}'=H^i_{(t-t_i)/(t_{i+1}-t_i)}h_i$ $(t_i \leqq t \leqq t_{i+1})$. From the triangle inequality we find that $d(H_t', H_t)<\varepsilon$ for each t. The restriction $H_t'|\text{Bd}(B^n)=I$, and $H_1'=(h_m h_{m-1}^{-1})h_{m-1}=h_m=H_1$. □

By employing Lemma 2.5 to appropriately alter the proofs of [22, Corollary 1 to Theorem 1], [5, Theorem 7.2] and making other small modifications we obtain the following two lemmas.

Lemma 2.6. *Suppose that D is a polyhedral disk in E^3, E is a polyhedral subdisk of D whose intersection with* Bd (D) *is an arc A, and O is an open polyhedron containing* $(D-E) \cup (\text{Bd}(E) \cap \text{Int}(D))$.

Then there is a pwl *isotopy H_t $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity except in O such that $H_1(D)=E$.*

Lemma 2.7. *Suppose that S is a polyhedral 2-sphere in E^3, D and E are polyhedral disks which span S from inside and have a common boundary on S, and h is a* pwl *homeomorphism of D onto E which is the identity on* $\text{Bd}(D)=\text{Bd}(E)$.

Then there is a pwl *isotopy H_t $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity except in* Int (S) *such that $H_1|D=h$.*

Lemma 2.8. *Suppose that S is a polyhedral 2-sphere in E^3 and D and E are polyhedral disks such that $D \cap S=E \cap S=A$ an arc on* $\text{Bd}(D) \cap \text{Bd}(E)$ *and* $(D-A) \cup (E-A) \subset \text{Int}(S)$.

Then if h is a pwl *homeomorphism of D onto E such that $h|A=I$, there is a* pwl *isotopy H_t $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity except in* Int (S) *so that $H_1|D=h$.*

Proof. Let F be a polyhedral disk on S such that $A \subset \text{Bd}(F)$. Both $D \cup F$ and $E \cup F$ are disks whose interiors contain Int (A). Use [12, Lemma 2.4] to find polyhedral 3-cells $C(D)$ and $C(E)$ in E^3 such that $D \cup F \subset \text{Bd}(C(D))$, $E \cup F \subset \text{Bd}(C(E))$, $C(D)-F \subset \text{Int}(S)$, and $C(E)-F \subset \text{Int}(S)$. Let D' and E' denote the respective polyhedral disks $\text{Bd}(C(D))-\text{Int}(F)$ and $\text{Bd}(C(E))-\text{Int}(F)$.

Extend h to a pwl homeomorphism of D' onto E' which is the identity on $\text{Bd}(D')=\text{Bd}(E')$. Lemma 2.7 now provides a pwl isotopy H_t $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity except on Int (S) so that $H_1|D'=h$. □

The next lemma is a mild extension of Theorems 7.3 and 7.5 of [5].

Lemma 2.9. *Suppose that $R_1, \ldots, R_i, \ldots, R_n$ are mutually exclusive polyhedral 2-spheres in E^3 whose diameters are less than ε.*

Suppose that $S_1, \ldots, S_j, \ldots, S_m$ are mutually exclusive compact polyhedral surfaces each of which is in general position with respect to every R_i.

Suppose that in each $S_j-(\bigcup R_i)$ there is a component U_j that lies in $\bigcap$ Ext (R_i) *so that every component of S_j-U_j is a disk of diameter less than ε which is normally situated in S_j.*

Suppose that X is a compact set in $(\bigcap \operatorname{Ext}(R_i)) \cup (\bigcup \operatorname{Cl}(U_j))$ *each component of which either intersects* $\bigcup \operatorname{Cl}(U_j)$ *or contains points whose distances from* $\bigcup R_i$ *exceed* 3ε.

Then there is a pwl 13ε*-isotopy* H_t $(0 \leqq t \leqq 1)$ *of* E^3 *onto itself which is the identity on* $X \cup (\bigcup \operatorname{Cl}(U_j))$ *and except in a* 3ε*-neighborhood of* $\bigcup R_i$ *such that for each* S_j *every component of* $H_1(S_j - \operatorname{Cl}(U_j))$ *is contained in some* $\operatorname{Int}(R_i)$.

Proof. We first consider a special case of the lemma.

No component of any $S_j - U_j$ *intersects* $\operatorname{Bd}(S_j)$. In this case a pwl 12ε-isotopy can be constructed in place of the promised 13ε-isotopy. This construction is made by simply imitating the steps in the proofs of [5, Theorems 7.3 and 7.5] and using Lemma 2.5 to convert the isotopies promised by [5, Theorems 7.1 and 7.2] to pwl isotopies at each place where one of these theorems is used in the proofs of [5, Theorems 7.3 and 7.5]. The special assumptions we have made concerning the nature of the U_j's make it unecessary to assume as in [5] that the S_j's are 2-spheres.

To finish the proof of the lemma we reduce the general case to the special case just mentioned. There is no loss in assuming that the R_i's have mutually exclusive interiors since any R_i can be thrown out if it is contained in the interior of another one. For each R_i let C_i denote the polyhedral cube whose boundary is R_i. For each S_j let $E_{j1}, \ldots, E_{jk}, \ldots$ denote the components of $S_j - U_j$ which intersect $\operatorname{Bd}(S_j)$.

By repeated applications of Lemma 2.6 we construct a pwl ε-isotopy H_t^1 $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity on $X \cup (\bigcup (S_j - \bigcup E_{jk}))$ and except in an ε-neighborhood of $\bigcup R_i$ so that for each E_{jk}, $H_1^1(E_{jk} - (E_{jk} \cap \operatorname{Cl}(U_j)))$ is contained in some $\operatorname{Int}(R_i)$.

Carve away thin mutually exclusive regular neighborhoods of the $H_1^1(E_{jk})$'s in the C_i's to obtain sets whose closures are polyhedral 3-cells $C_1', \ldots, C_i', \ldots$ where the boundary R_i' of each C_i' contains $(\bigcup (S_j - (\bigcup E_{jk}))) \cap R_i$ and is in general position with respect to every $H_1^1(S_j)$.

For each j let U_j' denote the subset $U_j \cup (\bigcup_k H_1^1(E_{jk}))$ of the surface $S_j' = H_1^1(S_j)$. Retain X, but replace each R_i, S_j, and U_j by R_i', S_j', and U_j' in the statement of the lemma. Now we have a situation which fits the special case so there is a pwl 12ε-isotopy H_t^2 $(0 \leqq t \leqq 1)$ of E^3 which is the identity on $X \cup (\bigcup \operatorname{Cl}(U_j'))$ and except in a 3ε-neighborhood of $\bigcup R_i'$ such that for each j every component of $H_1^2(S_j' - \operatorname{Cl}(U_j'))$ is contained in some $\operatorname{Int}(R_i')$.

The promised isotopy is given by $H_t = H_{2t}^1$ $(0 \leqq t \leqq \frac{1}{2})$ and $H_t = H_{2(t-1/2)}^2 H_{1/2}$ $(\frac{1}{2} \leqq t \leqq 1)$. It is a pwl 13ε-isotopy of E^3. Since $\bigcup U_j \subset \bigcup U_j'$ and a 3ε-neighborhood of $\bigcup R_i'$ is contained in a 3ε-neighborhood of $\bigcup R_i$, H_t is the identity on $X \cup (\bigcup \operatorname{Cl}(U_j))$ and except in a 3ε-neighborhood of $\bigcup R_i$. Each $\operatorname{Int}(R_i') \subset \operatorname{Int}(R_i)$ so every component of a $H_1(S_j - \operatorname{Cl}(U_j))$ is contained in some $\operatorname{Int}(R_i)$. □

3. Small isotopies of approximating disks into themselves.

Lemma 3.1. *Suppose that D is a polyhedral disk, R is a polyhedron in* $\operatorname{Bd}(D)$, *f is a homeomorphism of D into E^3, and* $\varepsilon > 0$.

There is a $\delta > 0$ *such that if* f' *is a* pwl *homeomorphism of* D *into* E^3 *where* $d(f, f') < \delta$, h *is a* pwl δ-*homeomorphism of* $f'(D)$ *onto itself which is the identity on* $f'(R)$, *and* L *is a polyhedron in* E^3 *whose intersection with* $f'(D)$ *is contained in* $f'(R)$, *then there is a* pwl ε-*isotopy* H_t $(0 \leqq t \leqq 1)$ *of* E^3 *onto itself which is the identity on* L *such that* $H_1 | f'(D) = h$.

Proof. First consider a special case.

The subpolyhedron $R = \mathrm{Bd}\,(D)$. In this case choose δ so that 3δ is subject to the restrictions on δ in Lemma 2.4 for the substitution $f(D) \to B^n$, $\varepsilon/6 \to \varepsilon$.

Let f' be a pwl homeomorphism of D into E^3 such that $d(f, f') < \delta$, let h be a pwl δ-homeomorphism of $f'(D)$ onto itself which is the identity on $f'(\mathrm{Bd}\,(D))$, and let L be a polyhedron in E^3 which misses $f'(\mathrm{Int}\,(D))$. Both $f'f^{-1}$ and $f(f')^{-1}$ are δ-homeomorphisms so $(f(f')^{-1})h(f'f^{-1})$ is a 3δ-homeomorphism of $f(D)$ onto itself which is the identity on $f(\mathrm{Bd}\,(D))$.

From Lemma 2.4 there is an $\varepsilon/6$-isotopy H_t^0 $(0 \leqq t \leqq 1)$ of $f(D)$ onto itself which is the identity on $f(\mathrm{Bd}\,(D))$ such that $H_1^0 = (f(f')^{-1})h(f'f^{-1})$. Define an isotopy H_t^1 $(0 \leqq t \leqq 1)$ of $f'(D)$ onto itself by $H_t^1 = (f'f^{-1})H_t^0(f(f')^{-1})$. We have $H_0^1 = I$, $H_1^1 = h$, and since the track of a point of $f(D)$ under $H_t^0(f(f')^{-1})$ has diameter less than $\varepsilon/6$ the track of a point of $f'(D)$ under H_t^1 has diameter less than $\varepsilon/6 + 2\delta < \varepsilon/3$.

From [12, Lemma 2.4] there are polyhedral 3-cells C_0 and C_1 in E^3 such that $C_0 \cap C_1 = \mathrm{Bd}\,(C_0) \cap \mathrm{Bd}\,(C_1) = f'(D)$ and $(C_0 \cup C_1) \cap L \subset f'(\mathrm{Bd}\,(D))$. From [10] we find embeddings $\lambda_e\colon \mathrm{Bd}\,(C_e) \times [0, 1] \to C_e$ $(e = 0, 1)$ so that for each point y of $\mathrm{Bd}\,(C_e)$, $\lambda_e(y, 0) = y$. We assume that each $\lambda_e(y \times [0, 1])$ has diameter less than $\varepsilon/6$.

Set $H_t^1 = I$ on $\mathrm{Bd}\,(C_0) \cup \mathrm{Bd}\,(C_1) - f'(\mathrm{Int}\,(D))$. Then define H_t^1 on $C_0 \cup C_1$ so that it is the identity except on $\bigcup \lambda_e(\mathrm{Bd}\,(C_e) \times [0, 1])$ and is given by $H_t^1(\lambda_e(y, s)) = (\lambda_e(H^1_{t(1-s)}(y), s))$ there. Now H_t^1 is an $\varepsilon/3 + 2\varepsilon/6$-isotopy of $C_0 \cup C_1$ which is the identity on $\mathrm{Bd}\,(C_0 \cup C_1)$. Unfortunately H_t^1 is not necessarily pwl.

From Lemma 2.4 and [2, Theorem 3] we find an $\varepsilon/3$-isotopy H_t^2 $(0 \leqq t \leqq 1)$ of $C_0 \cup C_1$ onto itself which is the identity on $\mathrm{Bd}\,(C_0) \cup \mathrm{Bd}\,(C_1)$ so that $H_1^2 H_1^1$ is a pwl homeomorphism of $C_0 \cup C_1$ onto itself. Let H_t^3 denote the ε-isotopy of $C_0 \cup C_1$ given by $H_t^3 = H_{2t}^1$ $(0 \leqq t \leqq \frac{1}{2})$ and $H_t^3 = H^2_{2(t-1/2)}$ $(\frac{1}{2} \leqq t \leqq 1)$. Lemma 2.5 together with [15] provides a pwl ε-isotopy H_t $(0 \leqq t \leqq 1)$ of $C_0 \cup C_1$ onto itself which is the identity on $\mathrm{Bd}\,(C_0 \cup C_1)$ so that $H_1 = H_1^3$. Extend H_t to the rest of E^3 by setting it equal to the identity outside $C_0 \cup C_1$. Since L fails to meet $\mathrm{Int}\,(C_0 \cup C_1)$ and since $H_1 | f'(D) = H_1^2 | f'(D) = h$, H_t satisfies the conditions in the conclusion of the special case of the lemma.

Now we prove the general case. Let δ' correspond to D, f, and $\varepsilon/2$ in the special case. Let δ be a positive number so small that a 3δ-homeomorphism of $f(\mathrm{Bd}\,(D))$ onto itself which is the identity on $f(R)$ is $\delta'/12$-isotopic to the identity in $f(\mathrm{Bd}\,(D))$ keeping $f(R)$ fixed.

Let f' be a pwl homeomorphism of D into E^3 such that $d(f, f') < \delta$, let h be a pwl δ-homeomorphism of $f'(D)$ onto itself which is the identity on $f'(R)$, and let L be a polyhedron in E^3 whose intersection with $f'(D)$ is contained in $f'(R)$.

The homeomorphism $(f(f')^{-1})h(f'f^{-1})$ is a 3δ-homeomorphism of $f(D)$ onto itself which is the identity on $f(R)$ so there is a $\delta'/12$-isotopy H_t^1 $(0 \leqq t \leqq 1)$ of $f(\mathrm{Bd}\,(D))$ onto itself which is the identity on $f(R)$ such that

$$H_1^1 = (f(f')^{-1})h(f'f^{-1})|f(\mathrm{Bd}\,(D)).$$

Define an isotopy H_t^2 $(0 \leqq t \leqq 1)$ of $f'(\mathrm{Bd}\,(D))$ onto itself by the rule

$$H_t^2 = (f^{-1}f')H_t^1(f(f')^{-1}).$$

It is a $\delta'/12+2\delta<\delta'/4$-isotopy of $f'(\mathrm{Bd}\,(D))$ onto itself which is the identity on $f'(R)$ such that $H_1^2=h|f'(\mathrm{Bd}\,(D))$.

By several applications of [12, Lemma 2.3] we enlarge $f'(D)$ to a polyhedral disk E in E^3 such that $f'(R)\subset \mathrm{Bd}\,(E)$, $E \cap L \subset f'(R)$, and $f'(\mathrm{Bd}\,(D)-R)\subset \mathrm{Int}\,(E)$. By using [10] to put a bicollar on $f'(\mathrm{Bd}\,(D)-R)$ in $\mathrm{Int}\,(E)$ which has fibers of diameter less than $\delta'/12$ and which tapers out near $f'(R)$, and then by imitating a step in the special case we extend H_t^2 $(0 \leqq t \leqq 1)$ to a $\delta'/4+2\delta'/12=5\delta'/12$-isotopy of E onto itself which is the identity on $\mathrm{Bd}\,(E)$.

Let C_0 and C_1 be polyhedral 3-cells such that $C_0 \cap C_1=\mathrm{Bd}\,(C_0) \cap \mathrm{Bd}\,(C_1)=E$ and $(C_0 \cup C_1)-E$ misses L. By collaring $\mathrm{Bd}\,(C_0)$ and $\mathrm{Bd}\,(C_1)$ in C_0 and C_1 with collars whose fibers have diameters less than $\delta'/12$ and then repeating a step in the special case we extend H_t^2 $(0 \leqq t \leqq 1)$ to a $7\delta'/12$-isotopy of $C_0 \cup C_1$ which is the identity on $\mathrm{Bd}\,(C_0 \cup C_1)$. As before we use Lemma 2.4 and [2, Theorem 3] to find a $\delta'/12$-isotopy H_t^3 $(0 \leqq t \leqq 1)$ of $C_0 \cup C_1$ onto itself which is the identity on $\mathrm{Bd}\,(C_0 \cup C_1) \cup f'(\mathrm{Bd}\,(D))$ such that $H_1^3H_1^2$ is a pwl homeomorphism of $C_0 \cup C_1$ onto itself and $H_1^3(f'(D))=f'(D)$.

Consider the $8\delta'/12$-isotopy H_t^4 $(0 \leqq t \leqq 1)$ of $C_0 \cup C_1$ onto itself which is given by the rule $H_t^4=H_{2t}^2$ $(0 \leqq t \leqq \frac{1}{2})$ and $H_t^4=H_{2(t-1/2)}^3H_{1/2}^4$ $(\frac{1}{2} \leqq t \leqq 1)$. It is the identity on $\mathrm{Bd}\,(C_0 \cup C_1)$ and H_1^4 is a pwl homeomorphism. From Lemma 2.5 and the fact [15] that C_0 and C_1 are pwl 3-cells there is a pwl $9\delta'/12$-isotopy H_t^5 $(0 \leqq t \leqq 1)$ of $C_0 \cup C_1$ onto itself which is the identity on $\mathrm{Bd}\,(C_0 \cup C_1)$ such that $H_1^5=H_1^4$. Extend H_t^5 to all of E^3 by setting it equal to the identity off $C_0 \cup C_1$.

Since $H_1^5|f'(\mathrm{Bd}\,(D))=h|f'(\mathrm{Bd}\,(D))$ and since $h(H_1^5)^{-1}$ is a $9\delta'/12+\delta<\delta'$-homeomorphism of $f'(D)$ onto itself which is the identity on $f'(\mathrm{Bd}\,(D))$ there is by the special case of the lemma a pwl $\varepsilon/2$-isotopy H_t^6 $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity on L so that $H_1^6|f'(D)=h(H_1^5)^{-1}$.

Set $H_t=H_{2t}^5$ $(0 \leqq t \leqq \frac{1}{2})$ and $H_t=H_{2(t-1/2)}^6H_{1/2}$ $(\frac{1}{2} \leqq t \leqq 1)$. It is a pwl $\delta'+\varepsilon/2<\varepsilon$-isotopy of E^3 onto itself which is the identity on L, and $H_1|f'(D)=H_1^6H_1^5|f'(D)=h$. □

4. **Stable graphs on disks.** Bing [5] calls a connected planar graph *stable* if each homeomorphism between two of its images in a 2-sphere can be extended to a homeomorphism of the 2-sphere onto itself. In [5, §9] he establishes results concerning stable graphs on spheres. With a few minor changes in the proofs of these

theorems one can obtain the following lemmas. Recall that Property Z is defined in the Introduction.

LEMMA 4.1. *Suppose that D is a disk and $\varepsilon > 0$.*

There is a $\delta > 0$ such that if G' is a finite graph in D which contains Bd (D) *and is such that each component of $D - G'$ has diameter less than δ, then there is a subgraph G of G' which also contains* Bd (D) *so that $G = \bigcup \text{Bd}(D_i)$ where $D_1, \ldots, D_n$ is a collection of subdisks of D with Property $Z(\varepsilon)$.*

LEMMA 4.2. *Suppose that D is a disk and $D_1, \ldots, D_n$ is a collection of subdisks which has Property Z.*

Then $G = \bigcup \text{Bd}(D_i)$ is connected, $G_O = \text{Cl}(G - \text{Bd}(D))$ is connected, and no Bd (D_i) *separates G.*

LEMMA 4.3. *Suppose that D is a (polyhedral) disk and $\varepsilon > 0$.*

There is a $\delta > 0$ such that if $D_1, \ldots, D_n$ is a collection of (polyhedral) subdisks of D with Property $Z(\delta)$, and if G denotes the graph $\bigcup \text{Bd}(D_i)$, then any (pwl) *ε-homeomorphism h of G into D which takes* Bd (D) *onto* Bd (D) *can be extended to a* (pwl) *3ε-homeomorphism of D onto itself.*

LEMMA 4.4. *A finite graph G on a disk D is stable if $G = \bigcup \text{Bd}(D_i)$ where $D_1, \ldots, D_n$ is a collection of disks with Property Z.*

Furthermore every homeomorphism of such a graph G into D which takes Bd (D) *onto itself can be extended to a homeomorphism of D onto itself.*

5. **Some special general position intersections.** Recall the definition of Property Q which we gave in [12]. One has a 3-manifold M with triangulation T whose i-skeleton is T_i, a disk D in M, a tame Sierpinski curve X normally situated in D (the closure of each component of $D - X$ is a disk which either lies in Int (D) or intersects Bd (D) in an arc), and a positive number η such that each component of $D - X$ has diameter less than η. A set $I(X, D)$ is defined to be the points of X which are not arcwise accessible from $D - X$. One supposes that D misses T_0 and $D \cap T_1$ is a finite collection of points in $I(X, D)$ where 1-simplexes of T pierce D. The quadruple (D, X, T_2, η) has *Property Q* if there is an η-homeomorphism g of D onto a polyhedral disk E in M such that

(i) g is the identity on X,

(ii) E is in general position with respect to T_2, and

(iii) $E \cap T_2 = X \cap T_2 = I(X, D) \cap T_2$.

Our interest in systems which have Property Q centers on a graph G_O which is defined by considering a particular η-homeomorphism g of D onto a polyhedral disk E which satisfies (i)–(iii). We define G_O to be the sum of the components of the general position intersection $E \cap T_2$ which contain points of T_1. In this section we show that if D' is a polyhedral disk which is homeomorphically very close to D, if D' is in general position with respect to T_2, if the cardinality of $D' \cap T_1$ is the same as the cardinality of $D \cap T_1$, and if G'_O denotes the graph which is the sum of the components of $D' \cap T_2$ which contain points of T_1, then there is a homeomorphism h of G_O onto G'_O such that for each simplex s of T, $h(s \cap G_O) = s \cap G'_O$.

LEMMA 5.1. *Suppose that D is a disk in E^3, L is a straight line whose intersection with D is a point p where L pierces D, and $\varepsilon > 0$.*

There is a $\delta > 0$ such that if g is a δ-homeomorphism of D onto a polyhedral disk D' in E^3, then there is a pwl *isotopy H_t $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity except in a ε-neighborhood of p so that $H_1(D') \cap L$ is a point p' where a neighborhood of p' in $H_1(D')$ lies in the plane through p' perpendicular to L. (There is a $\delta > 0$ such that if g_0 and g_1 are δ-homeomorphisms of D onto disjoint polyhedral disks D_0 and D_1 in E^3, then there is a* pwl *isotopy H_t $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity except in an ε-neighborhood of p so that $H_1(D_e) \cap L$ $(e = 0, 1)$ is a point p_e where a neighborhood of p_e in $H_1(D_e)$ lies in the plane through p_e perpendicular to L.)*

Proof. We prove only the case where there are to be two homeomorphisms g_0 and g_1.

Let O be an open set of diameter less than ε which contains p and whose closure misses Bd (D). Choose subdisks D_1 and D_2 of D in O and a polyhedral cylindrical sphere C in O with axis L whose bases miss D so that $p \in \text{Int}(D_2)$, $D_2 \subset \text{Int}(C)$, $D_2 \subset \text{Int}(D_1)$, and $C \cap D \subset \text{Int}(D_1)$.

Let δ be a positive number less than the distance from D to the bases of C and less than each of the numbers $\rho(D - D_1, C)$, $\rho(D - D_2, L)$, and $\rho(D_2, C)$. From [5, §§5 and 6] we may also assume that δ is so small that if g' is a δ-homeomorphism of D onto a polyhedral disk D' in general position with respect to C and if U' denotes the component of $D' - C$ which contains Bd (D'), then exactly one component of Cl $(U') \cap C$ separates the bases of C.

Let g_0 and g_1 be δ-homeomorphisms of D onto disjoint polyhedral disks D_0 and D_1 in E^3. Since D is compact there is a positive number $\delta' < \delta$ such that g_0 and g_1 are δ'-homeomorphisms of D.

Let H_t^1 $(0 \leqq t \leqq 1)$ be a pwl $(\delta - \delta')$-isotopy of E^3 onto itself which is the identity except in O so that $H_1^1(D_0 \cup D_1)$ is in general position with respect to C. Let U_e $(e = 0, 1)$ denote the component of $H_1^1(D_e) - C$ which contains $H_1^1(\text{Bd}(D_e))$. Let $E_{e1}, \ldots, E_{ei}, \ldots$ $(e = 0, 1)$ denote the disks which are the components of $H_1^1(D_e) - U_e$. The boundary of exactly one E_{ei} $(e = 0, 1)$, say E_{e1}, separates the bases of C.

From Lemma 2.9 there is a pwl isotopy H_t^2 $(0 \leqq t \leqq 1)$ of E^3 which is the identity on Cl $(U_0 \cup U_1)$ and except in O such that each $H_1^2(\text{Int}(E_{ei}))$ $(e = 0, 1)$ is contained in Int (C). Each Bd $(E_{ei}) = H_1^2(\text{Bd}(E_{ei}))$ $(e = 0, 1; i > 1)$ bounds a disk on C which misses the bases of C. By repeated applications of [5, Theorem 7.1] (we use Lemma 2.5 to convert the isotopies promised by the theorem to pwl isotopies) we push the successive $H_1^2(E_{ei})$'s over to $C - C \cap L$ and then push them slightly outside C and so construct a pwl isotopy H_t^3 $(0 \leqq t \leqq 1)$ of E^3 which is the identity on

$U_0 \cup U_1 \cup H_1^2(E_{01} \cup E_{11})$ and except in O such that $H_1^3H_1^2H_1^1(D_e) - H_1^2(E_{e1}) \subset \text{Ext}(C) \cap (E^3 - L)$ $(e=0, 1)$.

Let E'_{01} and E'_{11} denote the disjoint disks on C whose boundaries are respectively Bd (E_{01}) and Bd (E_{11}). Each of these disks contains a base of C. Two further applications of [5, Theorem 7.1] like those in the last paragraph give us a pwl isotopy H_t^4 $(0 \leqq t \leqq 1)$ of E^3 which is the identity on

$$H_1^3H_1^2H_1^1(D_0 \cup D_1) - H_1^2(\text{Int}(E_{01} \cup E_{11}))$$

and except in O so that $H_1^4H_1^2(E_{e1}) = E'_{e1}$ $(e=0, 1)$.

Define H_t by the rule $H_0 = I$ and $H_t = H^i_{4(t-(i-1)/4)}H_{(i-1)/4}$ $(1 \leqq i \leqq 4; (i-1)/4 \leqq t \leqq i/4)$. This pwl isotopy is the identity except in the ε-set O and $H_1(D_e) \cap L$ $(e=0, 1)$ is the point $p_e = E'_{e1} \cap L$. Since each E'_{e1} contains a base of C a neighborhood of p_e $(e=0, 1)$ is contained in the plane through p_e perpendicular to L. □

Lemma 5.2 is a simple extension of Lemma 5.1 and we omit the proof.

Lemma 5.2. *Suppose that T is a rectilinear triangulation of E^3 with i-skeleton T_i, D is a disk in E^3 such that $D \cap T_1$ is a finite collection of points $p_1, \ldots, p_k, \ldots$ where 1-simplexes of T pierce D, and $\varepsilon > 0$.*

There is a $\delta > 0$ such that if g is a δ-homeomorphism of D onto a polyhedral disk D' in general position with respect to T_2, then there is a pwl *isotopy H_t $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity except in mutually exclusive ε-neighborhoods $O_1, \ldots, O_k, \ldots$ of the p_k's so that $H_1(D')$ is in general position with respect to T_2 and $H_1(D') \cap T_1$ is a finite collection of points $p'_1, \ldots, p'_k, \ldots$ where each $p'_k \in O_k$. (There is a $\delta > 0$ such that if g_0 and g_1 are δ-homeomorphisms of D onto disjoint polyhedral disks D_0 and D_1 in general position with respect to T_2, then there is a* pwl *isotopy H_t $(0 \leqq t \leqq 1)$ of E^3 onto itself which is the identity except in mutually exclusive ε-neighborhoods $O_1, \ldots, O_k, \ldots$ of the p_k's so that $H_1(D_0 \cup D_1)$ is in general position with respect to T_2 and $H_1(D_e) \cap T_1$ $(e=0, 1)$ is a finite collection of points $p_{e1}, \ldots, p_{ek}, \ldots$ where each $p_{ek} \in O_k$.)*

Lemma 5.3. *Suppose that T is a rectilinear triangulation of E^3 with i-skeleton T_i, D is a disk in E^3, X is a tame Sierpinski curve normally situated in D, and η is a positive number such that (D, X, T_2, η) has Property Q.*

Let G_O denote the finite graph in $I(X, D)$ which consists of the components of $D \cap T_2$ which intersect T_1, and let G denote the graph $G_O \cup$ Bd (D).

Suppose $\varepsilon > 0$, and suppose $G_O \cap$ Bd (D) contains at least two points.

There is a $\delta > 0$ such that if g is a δ-homeomorphism of D onto a polyhedral disk D' in general position with respect to T_2 so that the cardinality of $D' \cap T_1$ is the same as the cardinality of $D \cap T_1$, then there is a homeomorphism π of G onto a finite graph $G' = G'_O \cup$ Bd (D') where G'_O is the sum of the components of $D' \cap T_2$ which intersect T_1 so that

1. *for each point p of $D \cap T_1$, $\pi(p)$ is a point of T_1 which is contained in an ε-neighborhood of p, and π takes no other points of G into T_1,*

2. *for each 2-simplex* Δ *of* T *and each component* t *of* $G_O \cap \Delta$, $\pi(t)$ *is an arc component of* $G'_O \cap \Delta$ *that lies in an* ε*-neighborhood of* t, *and*

3. *for each arc* s *in* Bd (D) *which is the closure of a component of* Bd $(D)-G_O$, $\pi(s)$ *is the closure of a component of* Bd$(D')-G'_O$ *and it lies in an* ε*-neighborhood of* s.

Proof. Choose $\varepsilon_1<\varepsilon$ so small that for each 2-simplex Δ of T, ε_1-neighborhoods of the components of $G_O \cap \Delta$ are mutually exclusive sets which miss T_0, for each 2-simplex Δ of T and each component t of $G_O \cap \Delta$ an ε_1-neighborhood of t intersects $T_2-\text{Int}(\Delta)$ in at least as many components as there are endpoints of t on Bd (Δ), and for each point q of G_O on Bd (D) there is an arc $s(q)$ on Bd (D) of diameter less than $\varepsilon/2$ which contains the intersection of Bd (D) with an ε_1-neighborhood of q. By requiring ε_1 to be very small we may assume that the $s(q)$'s are mutually exclusive.

Since 1-simplexes of T pierce D at the points of $D \cap T_1$ it follows that if δ is sufficiently small and g is a δ-homeomorphism of D onto a disk D' of the type described in the hypothesis of the lemma there will have to be a point p' of Int $(D') \cap T_1$ in an ε_1-neighborhood of each point p of $D \cap T_1$. Hence we will be able to define π on $D \cap T_1$ so that Condition 1 is satisfied.

Let Δ be a 2-simplex of T and let $t_1, \dots, t_i, \dots$ denote the arcs which are the components of $G_O \cap \Delta$. We show what restrictions to place on δ so that we will be able to define π on $G_O \cap \Delta$.

Let τ be a 3-simplex of T which has Δ as a face, and let U be a connected open set in E^3 containing Δ so that $U-\text{Bd}(\tau)$ has exactly two components, U_{-1} and U_1. For each arc t_i let t_{ai} be an arc in $U \cap G_O \cap \text{Bd}(\tau)$ which contains a neighborhood of t_i in $G_O \cap \text{Bd}(\tau)$. We assume that the t_{ai}'s are mutually exclusive and are contained in ε_1-neighborhoods of the t_i's. Each $t_{ai}-t_i$ misses Δ. Let G_a denote the graph $\bigcup t_{ai}$.

Use [12, Lemma 3.1] and the fact that $G_a \subset I(X, D)$ to construct an embedding ϕ of $G_a \times [-1, 1]$ into $U \cap D$ so that

1. for each point y of G_a, $\phi(y, 0)=y$, $\phi(y \times [-1, 1]) \subset \text{Bd}(D) \cap s(y)$ if $y \in \text{Bd}(D)$, and $\phi(y \times [-1, 1])$ misses Bd (D) if $y \in \text{Int}(D)$,

2. for each t_{ai} and each endpoint y of t_{ai}, $\phi(y \times [-1, 1])$ misses Δ if y misses Δ and $\phi(y \times [-1, 1])$ misses $T_2-\text{Int}(\Delta)$ if $y \in \text{Int}(\Delta)$,

3. each $\phi(t_{ai} \times [-1, 1])$ is contained in an ε_1-neighborhood of t_i,

4. $\phi(G_a \times [-1, 1]) \cap (G_O \cap \text{Bd}(\tau))=G_a$,

5. $\phi(G_a \times e) \subset I(X, D) \cap U_e$ $(e=-1, 1)$.

Here are the restrictions on δ which will enable us to define π on $G_a \cap \Delta$: (1) $\delta<\rho(\phi(G_a \times [-1, 1]), E^3-U)$, (2) $\delta<\rho(\phi(G_a \times e), E^3-U_e)$ $(e=-1, 1)$, (3) for each t_{ai} and each endpoint y of t_{ai}, $\delta<\rho(\phi(y \times [-1, 1]), T_2-\text{Int}(\Delta))$ if $y \in \text{Int}(\Delta)$ and $\delta<\rho(\phi(y \times [-1, 1]), \Delta)$ if $y \notin \Delta$, and (4) for each t_{ai} a δ-neighborhood of $\phi(t_{ai} \times [-1, 1])$ is contained in an ε_1-neighborhood of t_{ai}.

We choose δ subject to restrictions like those in the preceding paragraph for each 2-simplex of T and subject to the restriction mentioned in the second paragraph of the proof. Further we assume that δ is sufficiently small so that for each arc s with endpoints q_1 and q_2 where s is the closure of a component of $\text{Bd}(D)-G_O$, a δ-neighborhood of $s \cup s(q_1) \cup s(q_2)$ is contained in an ε-neighborhood of s.

Let g be a δ-homeomorphism of D onto a polyhedral disk D' in general position with respect to T_2 so that the cardinality of $D' \cap T_1$ is the same as the cardinality of $D \cap T_1$, and let G'_O and G' be defined as indicated in the hypothesis of the lemma.

From the second paragraph of the proof π can be defined to take $D \cap T_1$ onto $D' \cap T_1$ so that for each point p of $D \cap T_1$, $\pi(p)$ is contained in an ε_1-neighborhood of p. Thus Condition 1 is satisfied in the conclusion of the lemma.

We show how to define π on $G_O \cap \Delta$.

Consider an arc t_i and suppose that it has endpoints $p_1(i)$ and $p_2(i)$ on $\text{Bd}(\Delta)$. In this case t_{ai} has endpoints $p_{a1}(i)$ and $p_{a2}(i)$ in $\text{Bd}(\tau)-\Delta$. From Conditions 2, 3 and 4 on the choice of δ, $\text{Bd}(\tau)$ separates $g\phi(t_{ai}\times -1)$ from $g\phi(t_{ai}\times 1)$ and there is an arc t'_{ai} in $g\phi(t_{ai}\times[-1,1]) \cap \text{Bd}(\tau)$ which runs from a point $p'_{a1}(i)$ of

$$g\phi(p_{a1}(i)\times[-1,1]) \cap (\text{Bd}(\tau)-\Delta)$$

to a point $p'_{a2}(i)$ of $g\phi(p_{a2}(i)\times[-1,1]) \cap (\text{Bd}(\tau)-\Delta)$. The arc t'_{ai} is contained in an ε_1-neighborhood of t_i, and since an ε_1-neighborhood of t_i intersects $\text{Bd}(\tau)-\text{Int}(\Delta)$ in at least two components t'_{ai} must intersect Δ in a subarc t'_i whose endpoints can only be $p'_1(i)=\pi(p_1(i))$ and $p'_2(i)=\pi(p_2(i))$. Define π on t_i so that $\pi(t_i)=t'_i$.

Suppose an arc t_i has just one endpoint $p(i)$ on $\text{Bd}(\Delta)$. In this case t_{ai} has one endpoint $p_a(i)$ in $\text{Bd}(\tau)-\Delta$ and one endpoint $q(i)$ in $\text{Int}(\Delta)$. As before $\text{Bd}(\tau)$ separates $g\phi(t_{ai}\times -1)$ from $g\phi(t_{ai}\times 1)$ and there is an arc t'_{ai} which runs from a point $p'_a(i)$ in $g\phi(p_a(i)\times[-1,1]) \cap (\text{Bd}(\tau)-\Delta)$ to a point $q'(i)$ in $g\phi(q(i)\times[-1,1]) \cap \text{Int}(\Delta)$. This arc is contained in an ε_1-neighborhood of t_i and it intersects $\text{Bd}(\Delta)$ only in $p'(i)=\pi(p(i))$ so it contains a subarc t'_i in Δ which runs from $p'(i)$ to $q'(i)$. Define π on t_i so that $\pi(t_i)=t'_i$.

Since the arcs t'_i account for all the components of $G'_O \cap \Delta$ we have defined π to take $G_O \cap \Delta$ onto $G'_O \cap \Delta$. For each remaining 2-simplex of T define π in the manner just described. Now π takes G_O onto G'_O so that Condition 2 is satisfied in the conclusion of the lemma.

From Condition 1 on the construction of ϕ each $\pi(q)\in g(s(q))$ where $q \in G_O \cap \text{Bd}(D)$. Since the $s(q)$'s are mutually exclusive we can define π on $\text{Bd}(D)$ by slightly adjusting $g|\text{Bd}(D)$ so that it takes each q onto $\pi(q)$. If s is the closure of a component of $\text{Bd}(D)-G_O$ with endpoints q_1 and q_2 then $\pi(s)\subset g(s \cup s(q_1) \cup s(q_2))$ which is contained in an ε-neighborhood of s by the choice of δ. Thus Condition 3 is satisfied in the conclusion of the lemma. □

6. **A construction involving Property Q.** We describe a construction which is employed in the proofs of Theorem 7.1 here and Theorem 5.1 of [13]. Hypotheses

are introduced before Lemmas 6.1, 6.4, 6.5, and 6.6. Once introduced a hypothesis is to be kept for the remainder of §6. The same is to hold for the numbers η, δ_1, δ, ε_1, and ε_2 which appear at various places in §6.

Consider a polyhedral disk D, a homeomorphism f of D into E^3, a positive number η, and a rectilinear triangulation T of E^3 with i-skeleton T_i and with mesh less than η. Let $\Delta_1, \ldots, \Delta_j, \ldots$ denote the 2-simplexes of T. From [12, Theorem 6.1] there is a tame Sierpinski curve X normally situated in D, and there is an η-homeomorphism h of E^3 onto itself which is the identity outside an η-neighborhood of $f(D)$ so that $(hf(D), h(X), T_2, \eta)$ has Property Q. Let G_{IV} denote the finite graph which consists of $hf(\mathrm{Bd}\,(D))$ and the components of $h(X) \cap T_2$ that meet T_1. Set $G_{\mathrm{III}} = h^{-1}(G_{\mathrm{IV}})$.

From the definition of Property Q there is a homeomorphism ϕ of $hf(D)$ onto a polyhedral disk E in general position with respect to T_2 so that $\phi | h(X) = I$ and $E \cap T_2 = h(X) \cap T_2$.

LEMMA 6.1. *Let δ_1 be a positive number.*

There is a positive number δ so that if h' is a homeomorphism of E^3 for which $d(h, h') < \delta_1/2$ and f' is a homeomorphism of D into E^3 for which $d(f, f') < \delta$ then $d(hf, h'f') < \delta_1$.

Proof. Suppose the lemma were not true. Then there would be a homeomorphism h' such that $d(h, h') < \delta_1/2$, and there would be a sequence of homeomorphisms $f_1, \ldots, f_n, \ldots$ of D into E^3 such that $f_n \to f$ and for each n, $d(hf, h'f_n) \geqq \delta_1$. But by the continuity of h', $h'f_n \to h'f$ so there is an N such that $d(h'f_n, h'f) < \delta_1/2$ $(n \geqq N)$. But then $d(h'f_n, hf) \leqq d(h'f_n, h'f) + d(h'f, hf) < \delta_1/2 + \delta_1/2 = \delta_1$. □

LEMMA 6.2. *Let ε_1 be a positive number.*

If η is sufficiently small then each component of $hf(D) - G_{\mathrm{IV}}$ has diameter less than ε_1.

Proof. Suppose η is so small that each 3η-subset of $f(D)$ lies in an $\varepsilon_1/6$-disk which is normally situated in $f(D)$.

Let x and y be a pair of points in $hf(D) - G_{\mathrm{IV}}$ such that $\rho(x, y) \geqq \varepsilon_1$. We show that G_{IV} separates x from y. Denote by Z the sum of all 2-simplexes of T whose distances from $x \cup y$ exceed $\varepsilon_1/3$. Since $\eta < \varepsilon_1/3$, Z separates x from y in E^3 so $Z \cap hf(D)$ separates x from y in $hf(D)$. Let Z_1 be a subset of $Z \cap hf(D)$ which is irreducible with respect to separating x from y. The unicoherence of $hf(D)$ shows that Z_1 is connected.

Either Z_1 fails to intersect T_1 or it intersects T_1. In the first case it lies in some 2-simplex of T and so has diameter less than η; thus $h^{-1}(Z_1)$ has diameter less than 3η so it is contained in an $\varepsilon_1/6$-disk that is normally situated in $f(D)$. But then Z_1 is contained in an $\varepsilon_1/6+2\eta$- or $\varepsilon_1/3$-disk that is normally situated in $hf(D)$. Such a disk neither separates $hf(D)$ nor intersects $x \cup y$ since $\rho(Z, x \cup y) > \varepsilon_1/3$. We conclude that Z_1 intersects T_1. But then Z_1 intersects $h(I(X, f(D)))$. Since $h(A(X, f(D)))$

misses T_2 and separates $h(f(D)-X)$ from $h(I(X,f(D)))$, $Z_1 \subset G_{\text{IV}}$ so G_{IV} separates x from y. □

LEMMA 6.3. *Let ε_2 be a positive number.*

If ε_1 is sufficiently small then G_{III} contains a stable subgraph G_{I} containing $f(\text{Bd}(D))$ such that $G_{\text{I}} = \bigcup \text{Bd}(D_m^{\text{I}})$ where $D_1^{\text{I}}, \ldots, D_m^{\text{I}}, \ldots$ is a collection of subdisks of $f(D)$ which has Property $Z'(\varepsilon_2)$.

Proof. Each component of $f(D)-G_{\text{III}}$ is the preimage of a component of $hf(D)-G_{\text{IV}}$ so it has diameter less than $\varepsilon_1+2\eta<2\varepsilon_1$.

Lemma 4.1 together with the fact that G_{IV} arises from the general position intersection of the polyhedral disk E with T_2 shows that the (D_m^{I})'s can be found provided $2\varepsilon_1$ is subject to the restrictions on δ in Lemma 4.1 for the substitution $f(D)\to D$ and $\varepsilon_2\to\varepsilon$. □

Set $G_{\text{II}}=h(G_{\text{I}})$ and for each m, $D_m^{\text{II}}=h(D_m^{\text{I}})$. Each D_m^{II} has diameter less than $\varepsilon_2+2\eta<2\varepsilon_2$. For each D_m^{II} let r_m denote $\text{Cl}(\text{Bd}(D_m^{\text{II}}) \cap hf(\text{Int}(D)))$. If $D_m^{\text{II}} \subset hf(\text{Int}(D))$ then $r_m=\text{Bd}(D_m^{\text{II}})$, and if D_m^{II} intersects $hf(\text{Bd}(D))$ then r_m is the arc in $\text{Bd}(D_m^{\text{II}})$ which spans $hf(\text{Bd}(D))$. Let G_{IO} and G_{IIO} denote the respective graphs $\text{Cl}(G_{\text{I}} \cap f(\text{Int}(D)))$ and $\text{Cl}(G_{\text{II}} \cap hf(\text{Int}(D)))$. Lemma 4.2 shows that G_{IO} and G_{IIO} are connected.

Let T' be a subdivision of T in which $h(X) \cap T_2$ and $E=\phi hf(D)$ underlie full subcomplexes, and let T'' be a first derived subdivision of T'. For each r_m define a polyhedron $L(r_m)$ to be the sum of the components of the $(N(r_m, T'') \cap \Delta_j)$'s that contain nondegenerate subsets of r_m.

LEMMA 6.4. *Each $L(r_m)$ is a 2-manifold with boundary such that $r_m \subset \text{Int}(L(r_m))$. If r_m is an arc then $L(r_m)$ is a 2-cell, and if r_m is a simple closed curve then $L(r_m)$ is an annulus.*

Proof. The reader may verify that each $L(r_m)$ is a 2-manifold with boundary which collapses to r_m and which is such that $r_m \subset \text{Int}(L(r_m))$.

If r_m is an arc then $L(r_m)$ is a 2-cell since a 2-cell is the only surface which collapses to an arc. Suppose that r_m is a simple closed curve. The disk $\phi(D_m^{\text{II}})$ intersects $L(r_m)$ in exactly r_m, and it can be fattened up into a polyhedral 3-cell whose boundary contains $L(r_m)$. But then $L(r_m)$ must be an annulus since an annulus is the only planar surface which collapses to a simple closed curve. □

Let T''' be a first derived subdivision of T''. Consider the regular neighborhoods $N(G_{\text{IIO}}, T''')$ and for each m, $N(r_m, T''')$. Since G_{IIO} is connected $N(G_{\text{IIO}}, T''')$ is a cube-with-handles. Each $N(G_{\text{IIO}}, T''') \cap L(r_m)$ is a regular neighborhood of r_m in $\text{Int}(L(r_m))$.

LEMMA 6.5. *Each $L(r_m)$ separates $N(G_{\text{IIO}}, T''')$ and every $N(r_m, T''')$ into two components. For each r_m one of the components $N_O(r_m, T''')$ of $N(r_m, T''')-L(r_m)$ contains no point of G_{IIO} and is also a component of $N(G_{\text{IIO}}, T''')-L(r_m)$. Furthermore the*

$N_O(r_m, T''')$'s are mutually exclusive and are in fact exactly the components of $N(G_{\text{II}O}, T''')-(\bigcup L(r_m))$.

Proof. Since E underlies a full subcomplex of T', $N(E, T''')$ is a regular neighborhood of E in E^3. Similarly each $N(\phi(D_m^{\text{II}}), T''')$ is a regular neighborhood of $\phi(D_m^{\text{II}})$ in E^3.

Each $L(r_m)$ is two sided so it separates the corresponding $N(r_m, T''')$ into two components. Since r_m separates E into the two components $\phi(D_m^{\text{II}}-r_m)$ and $E-\phi(D_m^{\text{II}})$, each of the two nonintersecting connected sets $N(\phi(D_m^{\text{II}}-r_m), T''')-L(r_m)$ and $N(E-\phi(D_m^{\text{II}}), T''')-L(r_m)$ intersects just one of the components of $N(r_m, T''')-L(r_m)$. Thus $N(E, T''')-L(r_m)$ has two components—$N_{Om}(E, T''')$ which contains $\phi(D_m^{\text{II}}-r_m)$ and $N_{Um}(E, T''')$ which contains $E-\phi(D_m^{\text{II}})$. This shows that $N(r_m, T''')-L(r_m)$ has exactly two components since it is a subset of $N(E, T''')-L(r_m)$ and can have at most two components. Let these two components be denoted by $N_O(r_m, T''')\subset N_{Om}(E, T''')$ and $N_U(r_m, T''')\subset N_{Um}(E, T''')$. Since $G_{\text{II}O}-r_m\subset(E-\phi(D_m^{\text{II}}))$ we see that $N_O(r_m, T''')$ is also a component of $N(G_{\text{II}O}, T''')-L(r_m)$ and contains no point of $G_{\text{II}O}-r_m$.

No two of the $N_{Om}(E, T''')$'s intersect for if they did some r_m would fail to separate E. Thus the $N_O(r_m, T''')$'s are mutually exclusive, and since

$$N(G_{\text{II}O}, T''') = \left(\bigcup N_O(r_m, T''')\right) \cup \left(\bigcup L(r_m)\right)$$

the $N_O(r_m, T''')$'s are the components of $N(G_{\text{II}O}, T''')-(\bigcup L(r_m))$. □

Let ε_3 be a positive number. Let f' be a pwl homeomorphism of D into E^3 such that $f'(D)$ is in general position with respect to T_2, the cardinality of $f'(D)\cap T_1$ is the same as the cardinality of $hf(D)\cap T_1$, and $d(hf, f')<\delta_1$. Lemma 5.3 shows that if δ_1 is sufficiently small there is a homeomorphism π' of G_{IV} onto the graph G'_{IV} that consists of $f'(\text{Bd}(D))$ together with the components of $f'(D)\cap T_2$ that intersect T_1. The homeomorphism π' takes $hf(D)\cap T_1$ onto $f'(D)\cap T_1$ and takes each point p of $hf(D)\cap T_1$ into an ε_3-neighborhood of itself. Further for each 2-simplex Δ of T and each arc component t of $G_{\text{IV}}\cap\Delta$ that intersects T_1, π' takes t onto a component t' of $G'_{\text{IV}}\cap\Delta$ that lies in an ε_3-neighborhood of t. Finally for each arc s which is the closure of a component of $hf(\text{Bd}(D))-G_{\text{IV}}$, π' takes s onto an arc s' that lies in an ε_3-neighborhood of s and is the closure of a component of $f'(\text{Bd}(D))-\text{Cl}(G'_{\text{IV}}\cap f'(\text{Int}(D)))$.

Lemma 6.6. *Let ε_4 be a positive number.*

If δ_1, ε_2, and ε_3 are sufficiently small then $\pi'|G_{\text{II}}$ can be extended to an ε_4-homeomorphism π of $hf(D)$ onto $f'(D)$ so that for each D_m^{II}, $\text{Int}(N_O(r_m, T'''))\cup\pi(r_m)$ contains a neighborhood of $\pi(r_m)$ in $\pi(D_m^{\text{II}})$ and so that each $\pi(D_m^{\text{II}})$ has diameter less than ε_4.

Proof. Each D_m^{II} has diameter less than $2\varepsilon_2$ so by requiring that $\varepsilon_2<\varepsilon_4/2$ we insure that each D_m^{II} has diameter less than ε_4.

For each D_m^{II} let O_m be an open set in E^3 of diameter less than ε_4 which contains D_m^{II}. Require ε_3 to be so small that an ε_3-neighborhood of each r_m is contained in O_m and does not intersect Bd $(L(r_m))$.

For each r_m use [12, Lemma 3.1] and the fact that $I(h(X), hf(D)) \subset E$ to construct a homeomorphism λ_m of $r_m \times [-1, 1]$ onto a set A_m in $hf(D) \cap \text{Int}\,(N(r_m, T'''))$ so that $r_{m,-1} = \lambda_m(r_m \times -1) \subset \text{Int}\,(N_O(r_m, T'''))$, $r_{m,1} = \lambda_m(r_m \times 1) \subset \text{Int}\,(N_U(r_m, T'''))$, and for each point y of r_m, $\lambda_m(y, 0) = y$, $\lambda_m(y \times [-1, 1]) \subset hf(\text{Bd}\,(D))$ if $y \in hf(\text{Bd}\,(D))$, and $\lambda_m(y \times [-1, 1]) \subset hf(\text{Int}\,(D))$ if $y \in hf(\text{Int}\,(D))$. We assume that the λ_m's are such that each $D_m^{\text{II}} \cup A_m \subset O_m$, and for each Δ_j there is a point y of r_m such that $\lambda_m(y \times [-1, 1])$ misses Δ_j.

Require δ_1 to be so small that for each D_m^{II} a δ_1-neighborhood of $D_m^{\text{II}} \cup A_m$ is contained in O_m, a δ_1-neighborhood of A_m is contained in $N(r_m, T''')$, a δ_1-neighborhood of $r_{m,-1}$ is contained in $N_O(r_m, T''')$, and a δ_1-neighborhood of $r_{m,1}$ is contained in $N_U(r_m, T''')$. Also require δ_1 to be so small that for each 2-simplex Δ_j of T and each A_m, a δ_1-neighborhood of Δ_j does not contain a subset of A_m which separates $r_{m,-1}$ from $r_{m,1}$.

From the conditions on ε_3 and δ_1 and the fact that G'_{IV} contains all the components of $f'(D) \cap T_2$ that intersect T_1, each $r'_m = \pi'(r_m) \subset f'((hf)^{-1}(A_m)) \cap \text{Int}\,(L(r_m))$ and it separates $f'((hf)^{-1}(r_{m,-1})) \subset \text{Int}\,(N_O(r_m, T'''))$ from

$$f'((hf)^{-1}(r_{m,1})) \subset \text{Int}\,(N_U(r_m, T'''))$$

in $f'((hf)^{-1}(A_m))$.

From the last condition on δ_1 no component of a $f'((hf)^{-1}(A_m)) \cap L(r_m)$ other than r'_m separates $f'((hf)^{-1}(r_{m,-1}))$ from $f'((hf)^{-1}(r_{m,1}))$ in $f'((hf)^{-1}(A_m))$. Consider the disk F_m in $f'((hf)^{-1}(D_m^{\text{II}} \cup A_m))$ whose boundary intersects $f'(\text{Int}\,(D))$ in exactly Int (r'_m). From the preceding remark $F_m \cap (r'_m \cup \text{Int}\,(N_O(r_m, T''')))$ contains a neighborhood of r'_m in F_m. Define π on D_m^{II} so that it extends $\pi'|\text{Bd}\,(D_m^{\text{II}})$ and sends D_m^{II} onto F_m.

The proof that π is a homeomorphism is the same as the proof of Lemma 4.4 (see [5, Theorem 9.5]). Since each $D_m^{\text{II}} \cup F_m \subset O_m$, π is an ε_4-homeomorphism and each $\pi(D_m^{\text{II}})$ has diameter less than ε_4. □

7. **Building cartesian products of disks with** [0, 1].

The theorem which follows is a modification of a theorem of Bing's [5, Theorem 3.2]. His proof of that theorem can be used as a rough model for the proof of Theorem 7.1 here. We prefer to use pwl homeomorphisms where Bing used homeomorphisms and rectilinear triangulations of E^3 where Bing used fences.

THEOREM 7.1. *Suppose that D is a polyhedral 2-cell, f is a homeomorphism of D into E^3, and $\varepsilon > 0$.*

There is a $\delta > 0$ such that if f_0 and f_1 are pwl *homeomorphisms of D onto disjoint polyhedral disks D_0 and D_1 in E^3 so that $d(f, f_e) < \delta$ $(e = 0, 1)$, then there is a* *polyhedral stable graph G in D which contains* Bd (D) *and there is a* pwl *homeomorphism g of $G\times[0, 1]$ into E^3 so that*

1. *each component of $D-G$ has diameter less than ε,*

2. *for each point y of G, $g(y, e)=f_e(y)$ $(e=0, 1)$ and the diameter of $g(y\times[0, 1])$ is less than ε, and*

3. $g(G\times[0, 1])\cap D_e=f_e(G)$ $(e=0, 1)$.

Proof. The proof is broken into five steps. We introduce at various places a δ_1, η, γ, and several ε's where the size of each of these numbers is provisional upon the numbers introduced later. We employ the sequence of lemmas in §6, but we do not attempt to make the indices of our ε's conform to the indices of the ε's in §6.

Step 1. *Locating the graph G.* Let ε_1 be a positive number. From §6 there is a positive number $\eta<\varepsilon_1/2$, a tame Sierpinski curve X normally situated in $f(D)$, a rectilinear triangulation T of E^3 with mesh less than η and i-skeleton T_i, and an η-homeomorphism h of E^3 onto itself which is the identity outside an η-neighborhood of $f(D)$ so that $(hf(D), h(X), T_2, \eta)$ has Property Q. Furthermore if G_{IV} denotes the graph which consists of $hf(\mathrm{Bd}\,(D))$ together with the components of $hf(D)\cap T_2$ that intersect T_1, and if G_{III} denotes $h^{-1}(G_{\mathrm{IV}})$, then there is a finite collection $D^{\mathrm{I}}_1,\ldots, D^{\mathrm{I}}_m,\ldots$ of subdisks of $f(D)$ which have Property $Z'(\varepsilon_1)$ so that $G_{\mathrm{I}}=\bigcup \mathrm{Bd}\,(D^{\mathrm{I}}_m)$ is a stable subgraph of G_{III}. For each D^{I}_m let D^{II}_m denote the $\varepsilon_1+2\eta<2\varepsilon_1$-disk $h(D^{\mathrm{I}}_m)$, let G_{II} denote the graph $h(G_{\mathrm{I}})$, and let $G_{\mathrm{II}O}$ denote the connected (Lemma 4.2) subgraph $\mathrm{Cl}\,(G_{\mathrm{II}}\cap hf(\mathrm{Int}\,(D)))$.

Let γ be a positive number. Consider the graph $f^{-1}(G_{\mathrm{I}})$. Let G be a polyhedral graph in D which is so close to $f^{-1}(G_{\mathrm{I}})$ that there is a γ-homeomorphism θ of D onto itself which is the identity on Bd (D) and which takes G onto $f^{-1}(G_{\mathrm{I}})$. Let G_O denote the connected graph $\mathrm{Cl}\,(G\cap \mathrm{Int}\,(D))$, and for each D^{I}_m let D^0_m denote the polyhedral disk $\theta^{-1}f^{-1}(D^{\mathrm{I}}_m)$. We assume that ε_1 and γ are sufficiently small so that each D^0_m has diameter less than ε. Now Condition 1 is satisfied in the conclusion of the theorem.

Step 2. *Converting* pwl *approximations to f into* pwl *approximations to hf.* Here we take pwl approximations f_0 and f_1 to f, convert them to pwl approximations to hf, and adjust these new approximations slightly so that they carry D onto polyhedral disks which intersect T_2 in a nice way.

Let T', T'', and T''' be subdivisions of T as indicated in §6. For each m let r_m, $L(r_m)$, and $N_O(r_m, T''')$ be defined as in §6. Let $\Delta_1,\ldots,\Delta_j,\ldots$ denote the 2-simplexes of T, and let $t_1,\ldots, t_i,\ldots$ denote the arc components of the $(G_{\mathrm{II}O}\cap\Delta_j)$'s. Let $p_1,\ldots,p_k,\ldots$ denote the points of $hf(D)\cap T_1$.

Let δ_1 be a positive number. Use [4], [19] to find a pwl homeomorphism h_a of E^3 onto itself such that $d(h_a, h)<\delta_1/2$ and $d(h_a, I)<\eta$.

Now let f_0 and f_1 be pwl homeomorphisms of D onto disjoint polyhedral disks D_0 and D_1 in E^3 such that $d(f, f_e)<\delta$ $(e=0, 1)$. Lemma 6.1 shows that if δ is sufficiently small then $d(h_a f_e, hf) < \delta_1$ $(e=0, 1)$. Let h_b be a pwl δ_1-homeomorphism of E^3 onto itself so that $h_b h_a(D_0 \cup D_1)$ is in general position with respect to T_2.

Let ε_2 be a positive number such that ε_2-neighborhoods of the p_k's are mutually exclusive and miss T_0. From Lemma 5.2 we can require δ_1 to be sufficiently small so that there is a pwl ε_2-homeomorphism h_c of E^3 onto itself so that $h_c h_b h_a(D_0 \cup D_1)$ is in general position with respect to T_2 and $h_c h_b h_a(D_e) \cap T_1$ $(e=0, 1)$ consists of points p_{ke} where each p_{ke} is contained in an ε_2-neighborhood of p_k. Let G^e_{IV} $(e=0, 1)$ denote the graph which consists of $h_c h_b h_a(\mathrm{Bd}\,(D_e))$ together with the components of $h_c h_b h_a(D_e) \cap T_2$ that intersect T_1.

Let ε_3 be a positive number such that ε_3-neighborhoods of the components of the $(h(X) \cap \Delta_j)$'s are mutually exclusive. Now $d(h_c h_b h_a f_e, hf) < \delta_1 + \delta_1 + \varepsilon_2$ $(e=0, 1)$. From Lemma 5.3 we find that by requiring both δ_1 and ε_2 to be sufficiently small there is a homeomorphism π'_e $(e=0, 1)$ of G_{IV} onto G^e_{IV} which takes each p_k onto p_{ke}, which takes each component t of a $h(X) \cap \Delta_j$ onto an arc in $h_c h_b h_a(D_e) \cap \Delta_j$ that lies in an ε_3-neighborhood of t, and which takes each component s of $hf(\mathrm{Bd}\,(D)) - \mathrm{Cl}\,(G_{\mathrm{IV}} \cap hf(\mathrm{Int}\,(D)))$ onto a component of

$$h_c h_b h_a(\mathrm{Bd}\,(D_e)) - \mathrm{Cl}\,(G^e_{\mathrm{IV}} \cap h_c h_b h_a(\mathrm{Int}\,(D_e)))$$

that lies in an ε_3-neighborhood of s. For each i set $t^e_i = \pi'_e(t_i)$ and for each m set $r^e_m = \pi'_e(r_m)$.

Let ε_4 be a positive number. From Lemma 6.6 we find that δ_1, ε_1, ε_2, and ε_3 can be required to be sufficiently small so that $\pi'_e|G_{\mathrm{II}}$ $(e=0, 1)$ can be extended to an ε_4-homeomorphism π_e of $hf(D)$ onto $h_c h_b h_a(D_e)$ where each $\pi_e(D^{\mathrm{II}}_m)$ has diameter less than ε_4, where each $r^e_m \subset L(r_m) \cap \mathrm{Int}\,(N(r_m, T'''))$, and where each

$$r^e_m \cup \mathrm{Int}\,(N_O(r_m, T'''))$$

contains a neighborhood of r^e_m in $\pi_e(D^{\mathrm{II}}_m)$.

Step 3. *A trial embedding of* $G_O \times [0, 1]$. In this step we adjust $h_c h_b h_a(D_0 \cup D_1)$ slightly by a homeomorphism h_d of E^3 which keeps $\pi_0(G_O) \cup \pi_1(G_O)$ fixed, and then we construct a trial embedding $g' \colon G_O \times [0, 1] \to E^3$ so that $g'(G_O \times (0, 1))$ misses $h_d h_c h_b h_a(D_0 \cup D_1)$.

For each Δ_j and each t_i in $hf(\mathrm{Int}\,(D)) \cap \Delta_j$ that has both endpoints on $\mathrm{Bd}\,(\Delta_j)$ let F_i denote the polyhedral disk in $N(t_i, T''') \cap \Delta_j$ whose boundary consists of $t^0_i \cup t^1_i$ together with two arcs in $N(t_i, T''') \cap \mathrm{Bd}\,(\Delta_j)$. For each Δ_j and each t_i in Δ_j that has only one endpoint on $\mathrm{Bd}\,(\Delta_j)$ let F_i be a polyhedral disk in $N(t_i, T''') \cap \Delta_j$ whose boundary contains $t^0_i \cup t^1_i$, whose intersection with $\mathrm{Bd}\,(\Delta_j)$ is the arc in $N(t_i, T''') \cap \mathrm{Bd}\,(\Delta_j)$ between $t^0_i \cap \mathrm{Bd}\,(\Delta_j)$ and $t^1_i \cap \mathrm{Bd}\,(\Delta_j)$, and whose intersection with $h_c h_b h_a(D_0 \cup D_1)$ is $t^0_i \cup t^1_i$. See Figure 7.1. Since each t_i underlies a subcomplex of T' the F_i's associated with a given Δ_j are mutually exclusive.

We wish to have each disk F_i as the image of $t_i \times [0, 1]$ under the trial embedding g'. Thus for g' to have the properties indicated at the beginning of this step we need to clear each $F_i - (\pi_0(t_i) \cup \pi_1(t_i))$ of points of $h_c h_b h_a(D_0 \cup D_1)$. We do this in the next two paragraphs.

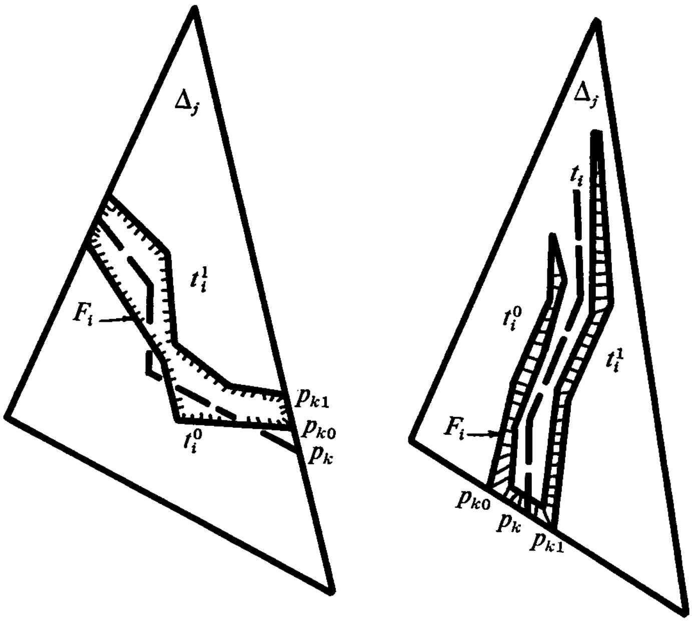

FIGURE 7.1

For each t_i which has both endpoints on T_1 let $F_{i,-1}$ and $F_{i,1}$ be a pair of polyhedral disks missing T_2 which lie to either side of F_i so that $F_{i,-1} \cup F_{i,1}$ separates in $h_c h_b h_a(D_e)$ $(e=0, 1)$, $\pi_e(G_{\text{IIO}})$ from every point of $h_c h_b h_a(D_e) \cap \text{Int}(F_i)$. We assume that the F_{ij}'s are mutually exclusive. Fatten up the F_{ij}'s into mutually exclusive polyhedral cubes missing T_2 whose boundaries R_{ij} are 2-spheres in general position with respect to $h_c h_b h_a(D_0 \cup D_1)$. Let U_e $(e=0, 1)$ denote the component of $h_c h_b h_a(D_e) - \bigcup_{i,j} R_{ij}$ which contains the connected graph $\pi_e(G_{\text{IIO}})$. Since the F_{ij}'s separate in $h_c h_b h_a(D_e)$ $(e=0, 1)$, $\pi_e(G_{\text{IIO}})$ from the Int (F_i)'s no point of U_e can lie in an Int (F_i). See Figure 7.2.

Each R_{ij} is contained in a 3-simplex of T and so has diameter less than $\eta < \varepsilon_4$. Each component of $h_c h_b h_a(D_e) - U_e$ $(e=0, 1)$ lies in some $\pi_e(D_m^{\text{II}})$ and so has diameter less than ε_4. We apply Lemma 2.9 to find a pwl $13\varepsilon_4$-homeomorphism h_d of E^3 onto itself which is the identity on $\text{Cl}(U_0 \cup U_1)$ and on all the F_i's that intersect $h_c h_b h_a(\text{Bd}(D_0) \cup \text{Bd}(D_1))$ so that each component of $h_d h_c h_b h_a(D_e) - \text{Cl}(U_e)$ $(e=0, 1)$ lies in some Int (R_{ij}). Each Int (F_i) is free of points of

$$h_d h_c h_b h_a(D_0 \cup D_1).$$

Construct a pwl homeomorphism g' of $G_O \times [0, 1]$ into E^3 as follows. Define g' on $G_O \times \{0, 1\}$ so that for each point $y = (hf\theta)^{-1}(p_k)$, $g'(y, e) = p_{ke} = \pi_e(p_k)$, and so that g' takes each arc $((hf\theta)^{-1}(t_i)) \times e$ $(e=0, 1)$ pwl onto t_i^e. Define g' on each $(hf\theta)^{-1}(p_k) \times [0, 1]$ so that it takes this arc pwl onto the arc in $N(p_k, T''') \cap T_1$ between p_{k0} and p_{k1}. For each endpoint $q(i)$ of a t_i that lies in Bd (D) define g' on

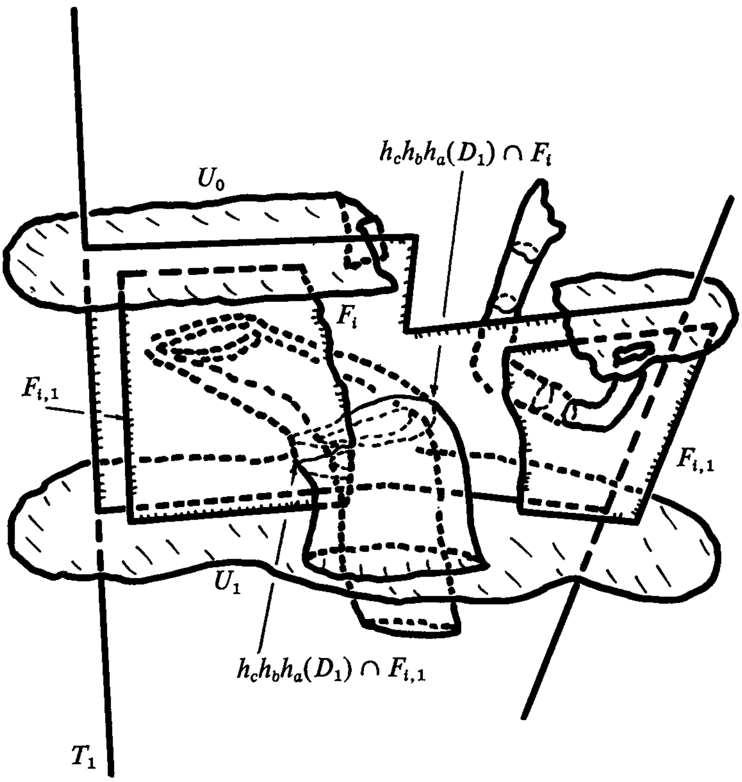

FIGURE 7.2

$((hf\theta)^{-1}(q(i))) \times [0, 1]$ so that it takes this arc pwl onto the part of Bd $(F_i) - T_1$ between $\pi_0(q(i))$ and $\pi_1(q(i))$. Now g' has been defined so that it takes each Bd $(((hf\theta)^{-1}(t_i)) \times [0, 1])$ pwl onto Bd (F_i). Use Lemma 2.1 to define g' so that it takes $((hf\theta)^{-1}(t_i)) \times [0, 1]$ pwl onto F_i. So defined g' takes $G_O \times [0, 1]$ pwl onto $\bigcup F_i$ and $g'(G_O \times (0, 1))$ misses $h_d h_c h_b h_a(D_0 \cup D_1)$.

Step 4. *Extending g' to a trial embedding of* $G \times [0, 1]$. For each point y of G_O, $g'(y \times [0, 1])$ has diameter less than η since $g'(y \times [0, 1])$ is contained in some 2-simplex of T. Furthermore each $h_d(\pi_e(D_m^{\mathrm{II}}))$ $(e=0, 1)$ has diameter less than $\varepsilon_4 + 2(13\varepsilon_4) = 27\varepsilon_4$. Thus each set

$$E_m = h_d(\pi_0(D_m^{\mathrm{II}}) \cup \pi_1(D_m^{\mathrm{II}})) \cup g'(((hf\theta)^{-1}(r_m)) \times [0, 1])$$

has diameter less than $2(27\varepsilon_4) + \varepsilon_4 = 55\varepsilon_4$. Notice that E_m is a disk if D_m^{II} intersects $hf(\mathrm{Bd}\,(D))$.

From the fact that the $N_O(r_m, T''')$'s are mutually exclusive, the fact that each $\pi_e(r_m) \cup \mathrm{Int}\,(N_O(r_m, T'''))$ $(e=0, 1)$ contains a neighborhood of $\pi_e(r_m)$ in $\pi_e(D_m^{\mathrm{II}})$ (Lemmas 6.5 and 6.6), and the fact that h_d is the identity on Cl $(U_0 \cup U_1)$ we can use [12, Lemma 2.4] to construct polyhedral cubes C_m of diameter less than $55\varepsilon_4$

for the (D_m^{II})'s that intersect $hf(\text{Bd}(D))$ so that each $\text{Bd}(C_m)$ contains E_m and so that the $(C_m - E_m)$'s are mutually exclusive sets which miss

$$h_d h_c h_b h_a(D_0 \cup D_1) \cup g'(G_0 \times [0, 1]).$$

Define g' on $\text{Bd}(D) \times \{0, 1\}$ so that it extends $g'|(G_O \cap \text{Bd}(D)) \times \{0, 1\}$ and so that it sends each $(D_m^0 \cap \text{Bd}(D)) \times e$ $(e=0, 1)$ pwl onto $h_d\pi_e(D_m^{\text{II}} \cap hf(\text{Bd}(D)))$. For each D_m^0 that meets $\text{Bd}(D)$, g' takes $\text{Bd}((D_m^0 \cap \text{Bd}(D)) \times [0, 1])$ pwl onto $\text{Bd}(E_m)$. Use Lemma 2.1 to define g' so that it takes each $(D_m^0 \cap \text{Bd}(D)) \times [0, 1]$ pwl onto $\text{Bd}(C_m) - \text{Int}(E_m)$. From the construction of the C_m's we see that g' is now a pwl homeomorphism of $G \times [0, 1]$ into E^3 such that $g'(G \times (0, 1))$ misses $h_d h_c h_b h_a(D_0 \cup D_1)$ and for each point y of G the diameter of $g'(y \times [0, 1])$ is less than $55\varepsilon_4$.

Step 5. *The embedding g.* Let H_1 denote the homeomorphism $h_d h_c h_b h_a$. It is an $\eta + \delta_1 + \varepsilon_2 + 13\varepsilon_4 < 16\varepsilon_4$-homeomorphism of E^3 onto itself. Construct a pwl homeomorphism f_e' $(e=0, 1)$ of D onto $H_1(D_e)$ by defining f_e' first on G so that for each point y of G, $f_e'(y) = g'(y, e)$, and then using Lemma 2.1 to extend f_e' to all of D so that each $f_e'(D_m^0) = h_d\pi_e(D_m^{\text{II}})$.

We suppose now that the γ in Step 1 is sufficiently small so that each $hf(D_m^0) \cup D_m^{\text{II}}$ has diameter less than ε_4. Since $h_d\pi_e$ $(e=0, 1)$ is a $14\varepsilon_4$-homeomorphism each $hf(D_m^0) \cup h_d\pi_e(D_m^{\text{II}})$ has diameter less than $29\varepsilon_4$ so $d(hf, f_e') < 29\varepsilon_4$. Then since h is an ε_4-homeomorphism of E^3 we see that $d(f, f_e') < 30\varepsilon_4$ $(e=0, 1)$ and thus

$$d(f, H_1^{-1}f_e') < 30\varepsilon_4 + 16\varepsilon_4 = 46\varepsilon_4 \qquad (e = 0, 1).$$

This shows that $d(I, f_e(H_1^{-1}f_e')^{-1}) < 46\varepsilon_4 + \delta$ $(e=0, 1)$.

From Lemma 3.1 we may require ε_4 and δ to be so small that there is a pwl $\varepsilon/2$-homeomorphism H_2^e $(e=0, 1)$ of E^3 onto itself such that $H_2^e|D_e = f_e(H_1^{-1}f_e')^{-1}$ and H_2^e is the identity except on a finite polyhedron K_e containing D_e. There is no loss in assuming that K_0 does not intersect K_1. Define a pwl $\varepsilon/2$-homeomorphism H_2 of E^3 by setting $H_2 = H_2^e$ on K_e $(e=0, 1)$ and the identity elsewhere.

The promised embedding g is defined to be $H_2H_1^{-1}g'$. If y is a point of G then $g(y, e) = H_2H_1^{-1}g'(y, e) = H_2H_1^{-1}f_e'(y) = f_e(y)$ $(e=0, 1)$. The diameter of each $g(y \times [0, 1])$ is less than $55\varepsilon_4 + 2(16\varepsilon_4) + \varepsilon/2$ so by requiring ε_4 to be less than $\varepsilon/174$ we cause Condition 2 to be satisfied in the conclusion of the theorem. That Condition 3 is satisfied follows from the fact that $g'(G \times (0, 1))$ misses $H_1(D_0 \cup D_1)$.

This completes the proof of the theorem. □

THEOREM 7.2. *Suppose that D is a polyhedral* 2-*cell, M is a* pwl 3-*manifold, f is a homeomorphism of D into* Int (M), *and* $\varepsilon > 0$.

There is a $\delta > 0$ *such that if* f_0 *and* f_1 *are* pwl *homeomorphisms of D onto disjoint polyhedral disks* D_0 *and* D_1 *in M where* $d(f, f_e) < \delta$ $(e=0, 1)$, *then there is a* pwl *homeomorphism g of* $D \times [0, 1]$ *into* Int (M) *so that for each point y of D*, $g(y, e) = f_e(y)$ $(e=0, 1)$ *and the diameter of* $g(y \times [0, 1])$ *is less than* ε.

Proof. From [18], $f(D)$ is contained in the interior of a polyhedral cube-with-handles K. Since K can be pwl embedded in E^3 under a uniformly continuous homeomorphism we might as well restrict our proof to the case where $M=E^3$.

Let $\varepsilon_1<\varepsilon/4$ be a positive number less than one tenth the diameter of $f(D)$. Let $\varepsilon_2<\varepsilon_1$ be a positive number such that the image under f of each ε_2-set in D has diameter less than ε_1. Let δ be subject to the restrictions on δ in Theorem 7.1 for the substitution $D\to D, f\to f$, and $\varepsilon_2\to\varepsilon$.

Let f_0 and f_1 be pwl homeomorphisms of D onto disjoint polyhedral disks D_0 and D_1 and E^3 such that $d(f, f_e)<\delta$ $(e=0, 1)$. From Theorem 7.1 there is a finite collection $D_1^0, \dots, D_m^0, \dots$ of polyhedral subdisks of D which has Property $Z(\varepsilon_2)$ so that $G=\bigcup \text{Bd}(D_m^0)$ is a stable graph. Further there is a pwl homeomorphism g' of $G\times[0, 1]$ into E^3 such that $g'(G\times[0, 1])\cap D_e=g'(G\times e)$ $(e=0, 1)$ and for each point y in G, $g'(y, e)=f_e(y)$ $(e=0, 1)$ and the diameter of $g'(y\times[0, 1])$ is less than ε_2. Define g on $D\times\{0, 1\}$ so that for each point y of D, $g(y, e)=f_e(y)$ $(e=0, 1)$. Set $g=g'$ on $G\times[0, 1]$.

For each disk D_m^0 consider the polyhedral 2-sphere $g(\text{Bd}(D_m^0\times[0, 1]))$. It bounds [15] a pwl 3-cell C_m. Use Lemma 2.1 to extend g so that it takes $D_m^0\times[0, 1]$ pwl onto C_m.

No two of the C_m's share interior points for if some $\text{Int}(C_i)\cap\text{Int}(C_j)$ were nonempty $(i\neq j)$ there would be a point say of $D_0-f_0(D_i^0)$ in $\text{Int}(C_i)$. But since $D_0-f_0(D_i^0)$ is connected $D_0-f_0(D_i^0)$ would be contained in $\text{Int}(C_i)$ so D_0 would be contained in C_i. This is impossible since the diameter of D_0 exceeds $10\varepsilon_1-2\delta>8\varepsilon_1$ and the diameter of C_i is less than $\varepsilon_1+2\delta+\varepsilon_1<4\varepsilon_1$. Thus g is a pwl homeomorphism of $D\times[0, 1]$ into E^3 such that for each point y of D, $g(y, e)=f_e(y)$ $(e=0, 1)$. Since each C_m has diameter less than $4\varepsilon_1<\varepsilon$ each $g(y\times[0, 1])$ has diameter less than ε. $\square$

By using two dimensional techniques analogous to those used in the proofs of Theorem 7.1 and Theorem 7.2 we obtain the following two dimensional version of Theorem 7.2.

THEOREM 7.3. *Suppose that A is a polygonal arc, M is a* pwl *2-manifold, f is a homeomorphism of A into* $\text{Int}(M)$, *and* $\varepsilon>0$.

There is a $\delta>0$ such that if f_0 and f_1 are pwl *homeomorphisms of A onto disjoint polygonal arcs A_0 and A_1 in* $\text{Int}(M)$ *where* $d(f, f_e)<\delta$ $(e=0, 1)$, *then there is a* pwl *homeomorphism g of $A\times[0, 1]$ into* $\text{Int}(M)$ *so that for each point y of D,* $g(y, e)=f_e(y)$ *and the diameter of* $g(y\times[0, 1])$ *is less than ε.*

8. **Piecing together cartesian products of disks with** [0, 1].

In this section we introduce a pair of constructions which enable us to prove our general cartesian product theorem by piecing together cartesian products of disks with [0, 1].

For the first construction consider a 2-simplex $\Delta=v\sigma$ where σ is a 1-simplex with vertices v_a and v_b. Let α_a and α_b be a pair of positive numbers less than 1/3, and let

$\sigma(\alpha)$ denote the line segment in Δ whose endpoints are $\alpha_a v+(1-\alpha_a)v_a$ and $\alpha_b v+(1-\alpha_b)v_b$. Let $\Delta(\alpha)$ denote the 2-simplex $v\sigma(\alpha)$, and let $E(\alpha)$ denote the disk $\text{Cl}(\Delta-\Delta(\alpha))$. For each point p of $\text{Int}(\sigma)$ let $r_p(\alpha)$ denote the line segment in Δ with one endpoint at p and the other on $\text{Int}(\sigma(\alpha))$ which is perpendicular to σ (if such a segment exists). Let $p(\alpha)$ denote the point $r_p(\alpha)\cap\sigma(\alpha)$. For each pair of points q_1 and q_2 in $\text{Int}(\sigma)$ let $\sigma_{q_1q_2}$ denote the part of σ between q_1 and q_2, and if the $r_{q_i}(\alpha)$'s are defined let $E_{q_1q_2}(\alpha)$ denote the closure of the component of $E(\alpha)-(r_{q_1}(\alpha)\cup r_{q_2}(\alpha))$ between $r_{q_1}(\alpha)$ and $r_{q_2}(\alpha)$. Let $\sigma_{q_1q_2}(\alpha)$ denote the arc $E_{q_1q_2}(\alpha)\cap\sigma(\alpha)$ (if $E_{q_1q_2}(\alpha)$ is defined).

Let f be a homeomorphism of Δ into a pwl 3-manifold M such that $f(\Delta)\cap\text{Bd}(M)=f(\sigma)$. Let ε_1 be a positive number. Suppose that f_0 and f_1 are pwl homeomorphisms of Δ onto disjoint polyhedral disks D_0 and D_1 in M such that $f_e(\Delta)\cap\text{Bd}(M)=f_e(\sigma)$ and $d(f,f_e)<\varepsilon_1$ $(e=0, 1)$. Suppose further that g is a pwl homeomorphism of $(\Delta(\alpha)\cup\sigma)\times[0,1]$ into M such that $g(\Delta(\alpha)\times[0,1])\subset\text{Int}(M)$, $g(\Delta(\alpha)\times(0,1))$ misses $D_0\cup D_1$, $g(\sigma\times[0,1])\subset\text{Bd}(M)$, and for each point y of $\Delta(\alpha)\cup\sigma$, $g(y,e)=f_e(y)$ $(e=0, 1)$ and the diameter of $g(y\times[0,1])$ is less than ε_1.

For each point p of σ where $r_p(\alpha)$ is defined let $J_p(\alpha)$ denote the simple closed curve $(\bigcup f_e(r_p(\alpha)))\cup g(\text{Bd}(r_p(\alpha))\times[0,1])$. Let $B(\alpha)$ denote the disk $(\bigcup f_e(E(\alpha)))\cup g(\sigma(\alpha)\times[0,1])$ and $A(\alpha)$ the annulus $B(\alpha)\cup g(\sigma\times[0,1])$. For each pair of points q_1 and q_2 in σ for which $E_{q_1q_2}(\alpha)$ is defined let $B_{q_1q_2}(\alpha)$ denote the disk $(\bigcup f_e(E_{q_1q_2}(\alpha)))\cup g(\sigma_{q_1q_2}(\alpha)\times[0,1])$ and $A_{q_1q_2}(\alpha)$ the annulus $B_{q_1q_2}(\alpha)\cup g(\sigma_{q_1q_2}\times[0,1])$.

Lemma 8.1. *Let p be a point of* $\text{Int}(\sigma)$ *and* $\varepsilon_2>0$.

There is an $\eta>0$ such that if α_a, α_b, and ε_1 are less than η, then there is a polyhedral disk $F_p(\alpha)$ in M whose diameter is less than ε_2 so that $\text{Bd}(F_p(\alpha))=J_p(\alpha)$ *and* $\text{Int}(F_p(\alpha))$ *is contained in* $\text{Int}(M)$ *and misses* $B(\alpha)\cup g(\Delta(\alpha)\times[0,1])$.

Proof. Let y_1 and y_2 be points of $\text{Int}(\sigma)$ such that $p\in\text{Int}(\sigma_{y_1y_2})$, and let y_3 and y_4 be points of $\text{Int}(\sigma_{y_1y_2})$ such that $p\in\text{Int}(\sigma_{y_3y_4})$. Let G denote the disk $v\sigma_{y_3y_4}$. Let C be a pwl 3-cell which misses $f(v)$ and contains a neighborhood of $f(\sigma)$ in M. Let $\varepsilon_3<\varepsilon_2$ be a positive number such that (1) $2\varepsilon_3$-neighborhoods of $f(y_1)$, $f(y_2)$ and $f(G)$ are mutually exclusive, (2) a $2\varepsilon_3$-neighborhood of $f(p)$ misses a $2\varepsilon_3$-neighborhood of $f(\text{Cl}(\Delta-G))$, (3) a $2\varepsilon_3$-neighborhood of $f(\sigma)$ is contained in C, and (4) a $2\varepsilon_3$-neighborhood of $f(v)$ misses C.

First we show how to choose η so that an ε_3-disk $F_p(\alpha)$ can be found whose interior misses $A(\alpha)\cup\text{Bd}(M)$. Let C_1 be a pwl 3-cell of diameter less than ε_3 such that $C_1\cap\text{Bd}(M)$ is a polyhedral disk K_1 whose interior contains $f(p)$. Let q_1 and q_2 be points of σ such that $p\in\text{Int}(\sigma_{q_1q_2})$ and $f(\sigma_{q_1q_2})\subset\text{Int}(K_1)$. Let C_2 be a pwl 3-cell in $\text{Int}(C_1)\cup\text{Int}(K_1)$ such that $C_2\cap\text{Bd}(M)$ is a disk K_2 where $f(p)\in\text{Int}(K_2)$ and $K_2\cap f(\sigma)\subset f(\text{Int}(\sigma_{q_1q_2}))$. Choose C_2 so that there is a polyhedral disk L_2 in $\text{Bd}(C_2)-\text{Int}(K_2)$ such that L_2 fails to intersect $f(\sigma_{pq_2})$ and $L_2\cap\text{Bd}(K_2)$ is an arc whose interior contains $f(\sigma_{q_2p})\cap\text{Bd}(K_2)$.

Choose $\eta < \varepsilon_3$ so that if α_a and α_b are less than η then (1) $r_p(\alpha)$, $r_{q_1}(\alpha)$, and $r_{q_2}(\alpha)$ exist, (2) a 3η-neighborhood of $f(r_p(\alpha))$ is contained in C_2, (3) a 3η-neighborhood of $f(E_{q_1q_2}(\alpha))$ is contained in C_1, (4) a 3η-neighborhood of $f(E(\alpha))$ is contained in C, (5) a 3η-neighborhood of $f(\text{Cl}\,(E(\alpha)-E_{q_1q_2}(\alpha)))$ fails to intersect C_2, (6) a 3η-neighborhood of $f(E_{pq_1}(\alpha)) \cap (\text{Bd}\,(C_2)-\text{Int}\,(K_2))$ is contained in L_2, and (7) a 3η-neighborhood of $f(E_{q_2p}(\alpha))$ fails to intersect L_2.

Suppose now that α_a, α_b, and ε_1 are less than η. From the conditions on η we find that $J_p(\alpha) \subset \text{Int}\,(C_2) \cup \text{Int}\,(K_2)$, $A_{q_1q_2}(\alpha) \subset \text{Int}\,(C_1) \cup \text{Int}\,(K_1)$, $A(\alpha) \cap C_2 \subset \text{Int}\,(A_{q_1q_2}(\alpha))$, $A_{pq_1}(\alpha) \cap (\text{Bd}\,(C_2)-\text{Int}\,(K_2)) \subset \text{Int}\,(L_2) \cup \text{Int}\,(L_2 \cap K_2)$, and $A_{q_2p}(\alpha) \cap L_2$ is empty. By shifting C_2 a small amount if necessary we can suppose that $\text{Bd}\,(C_2)$ is in general position with respect to $A(\alpha)$.

From the preceding remarks the components of $A(\alpha) \cap L_2$ are simple closed curves in $\text{Int}\,(A_{pq_1}(\alpha))$, and one of these curves, call it J, is nontrivial in $A_{pq_1}(\alpha)$ since L_2 separates the boundary components of $A_{pq_1}(\alpha)$. Now J bounds a polyhedral

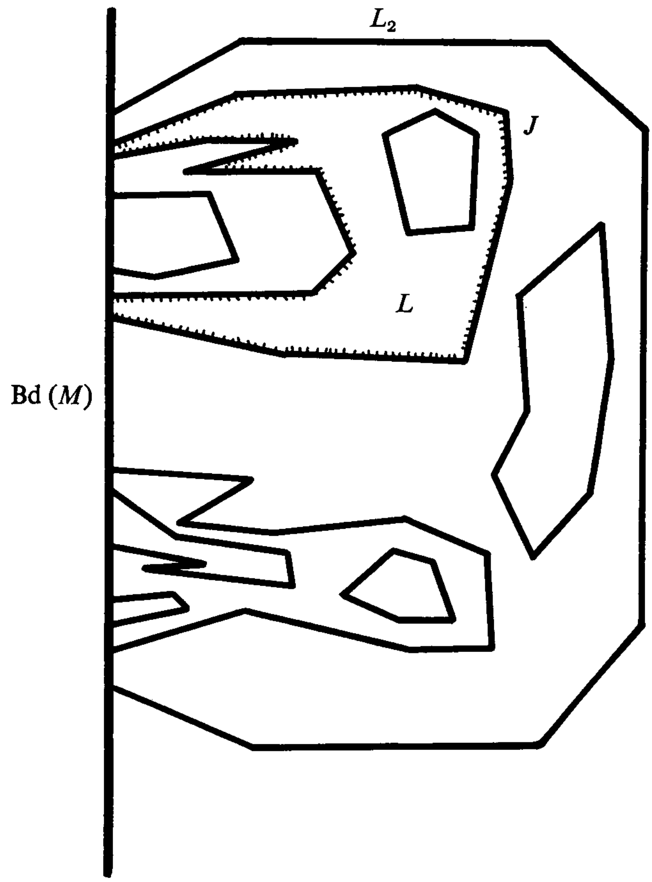

FIGURE 8.1

disk L in L_2 and Int $(L) \cap A(\alpha)$ consists of finitely many mutually exclusive simple closed curves in Int $(B_{q_1q_2}(\alpha))$. Thus by repeated applications of [5, Theorem 7.1] as in the proof of [5, Theorem 7.3] we find a pwl homeomorphism h_1 of C_1 onto itself which is the identity on Bd (C_1) and on J so that $h_1(\text{Int}(L))$ misses $A(\alpha)$. See Figure 8.1.

To obtain the promised disk we define a pwl homeomorphism h_2 of $h_1(L)$ into C_1 which slides J onto $J_p(\alpha)$ and takes $h_1(\text{Int}(L))$ into Int $(C_1)-A(\alpha)$. For $F_p(\alpha)$ we take $h_2h_1(L)$. Since it is contained in C_1 it has diameter less than ε_3.

To insure that Int $(F_p(\alpha))$ misses $g(\Delta(\alpha)\times[0, 1])$ we require η to be so small that polyhedral ε_3-disks $F_{y_1}(\alpha)$ and $F_{y_2}(\alpha)$ like $F_p(\alpha)$ can also be found. From the conditions on η and ε_3 the three disks $F_p(\alpha)$, $F_{y_1}(\alpha)$, and $F_{y_2}(\alpha)$ are mutually exclusive, and $A_{y_1y_2}(\alpha) \cup F_{y_1}(\alpha) \cup F_{y_2}(\alpha)$ is a polyhedral 2-sphere which bounds a polyhedral cube C_3 in C. Furthermore $F_p(\alpha)$ must span Bd (C_3) from Int (C_3) since otherwise it could not attach onto $J_p(\alpha)$ along $g(p\times[0, 1])$.

Suppose that there were points of $g(\Delta(\alpha)\times[0, 1])$ in Int $(F_p(\alpha))$. From Condition 2 on ε_3 these points would have to belong to the connected set $g((G-\sigma(\alpha))\times[0, 1])$. But by Condition 4 on ε_3, $g(v\times[0, 1])$ misses C_3 so there would have to be points of $g((G-\sigma(\alpha))\times[0, 1])$ in Int $(F_{y_1}(\alpha)) \cup$ Int $(F_{y_2}(\alpha))$ which is ruled out by Condition 1 on ε_3. □

The second construction which we employ is similar to the first one. Consider a pair of 2-simplexes $\Delta_0=v_0\sigma$ and $\Delta_1=v_1\sigma$ where $\Delta_0\cap\Delta_1=\sigma$ and σ is a 1-simplex with vertices v_a and v_b. Let α_a and α_b be a pair of positive numbers less than 1/3. Define $\sigma_{q_1q_2}$ as before and define $\sigma_j(\alpha)$, $\Delta_j(\alpha)$, $r_{pj}(\alpha)$, $p_j(\alpha)$, $E_j(\alpha)$, $E_{q_1,q_2,j}(\alpha)$, and $\sigma_{q_1,q_2,j}(\alpha)$ $(j=0, 1)$ to correspond to the objects without the subscripts j in the preceding construction.

Let f be a homeomorphism of $\Delta_0\cup\Delta_1$ into the interior of a pwl 3-manifold M. Let ε_1 be a positive number. Suppose that f_0 and f_1 are pwl homeomorphisms of $\Delta_0\cup\Delta_1$ onto disjoint polyhedral disks D_0 and D_1 in Int (M) such that $d(f, f_e)<\varepsilon_1$ $(e=0, 1)$. Suppose further that g is a pwl homeomorphism of $(\Delta_0(\alpha)\cup\Delta_1(\alpha))\times[0, 1]$ into Int (M) such that $g((\Delta_0(\alpha)\cup\Delta_1(\alpha))\times(0, 1))$ misses $D_0\cup D_1$ and for each point y of $\Delta_0(\alpha)\cup\Delta_1(\alpha)$, $g(y, e)=f_e(y)$ $(e=0, 1)$ and the diameter of $g(y\times[0, 1])$ is less than ε_1.

For each point p of Int (σ) for which $r_{p0}(\alpha)$ and $r_{p1}(\alpha)$ are defined let $J_p(\alpha)$ denote the simple closed curve $(\bigcup_{e,j} f_e(r_{pj}(\alpha))) \cup g((p_0(\alpha)\cup p_1(\alpha))\times[0, 1])$. Let $A(\alpha)$ denote the annulus $(\bigcup_{e,j} f_e(E_j(\alpha))) \cup g((\sigma_0(\alpha)\cup\sigma_1(\alpha))\times[0, 1])$. For each appropriate pair of points q_1 and q_2 of σ, let $A_{q_1q_2}(\alpha)$ denote the closed annulus between $J_{q_1}(\alpha)$ and $J_{q_2}(\alpha)$.

LEMMA 8.2. *Let p be a point of* Int (σ) *and* $\varepsilon_2>0$.

There is a positive number $\eta<\varepsilon_2$ such that if α_a, α_b, and ε_1 are less than η, then there is a polyhedral disk $F_p(\alpha)$ in Int (M) *of diameter less than ε_2 so that* Bd $(F_p(\alpha))$ $=J_p(\alpha)$ *and* Int $(F_p(\alpha))$ *misses* $A(\alpha)\cup g((\Delta_0(\alpha)\cup\Delta_1(\alpha))\times[0, 1])$.

Proof. We only sketch the proof. Details are similar to details of the preceding proof.

Find a small polyhedral 3-cell C whose interior contains $f(\sigma_{qp})$ where q is a second point of Int (σ) near p. Let $x \in$ Int (σ_{qp}), and let S be a very small polyhedral 2-sphere in Int (C) such that $f(x)$ belongs to the interior of S in C and $f(\sigma) \cap S$ is contained in $f(\text{Int}\,(\sigma_{qp}))$. By requiring η to be very small we find that S separates $J_x(\alpha)$ from $J_p(\alpha)$ and that $S \cap A(\alpha) \subset \text{Int}\,(A_{qp}(\alpha))$. We can suppose that S has been slightly adjusted so that it is in general position with respect to $A(\alpha)$. Now some component of $S \cap A(\alpha)$ is a simple closed curve J which separates $J_q(\alpha)$ from $J_p(\alpha)$ in $A(\alpha)$ and bounds a disk K in S such that each component of Int $(K) \cap A(\alpha)$ is a trivial loop in $A_{qp}(\alpha)$. As in the proof of Lemma 8.1 we find a pwl homeomorphism h of K into C so that $h(J)=J_p(\alpha)$ and $h(\text{Int}\,(K))$ fails to intersect $A(\alpha)$. For $F_p(\alpha)$ we take $h(K)$. □

9. Building cartesian products of arbitrary surfaces with [0, 1].

THEOREM 9.1. *Suppose that M is a* pwl *3-manifold, S is a* pwl *2-manifold, $R \subset \text{Bd}\,(S)$ is either a 1-manifold with boundary or the empty set, and f is a homeomorphism of S onto a closed subset of M such that $f(S) \cap \text{Bd}\,(M) = f(R)$.*

Suppose that μ is a positive continuous function on S.

There is a positive continuous function ν on S such that if f_0 and f_1 are pwl *homeomorphisms of S onto disjoint polyhedral surfaces S_0 and S_1 in M where $f_e(S) \cap \text{Bd}\,(M) = f_e(R)$ $(e=0, 1)$ and for each point y of S, $\rho(f(y), f_e(y)) < \nu(y)$ $(e=0, 1)$, then there is a* pwl *homeomorphism g of $S \times [0, 1]$ into M so that $g(S \times [0, 1]) \cap \text{Bd}\,(M) = g(R \times [0, 1])$ and for each point y of S, $g(y, e) = f_e(y)$ $(e=0, 1)$ and the diameter of $g(y \times [0, 1])$ is less than $\mu(y)$.*

Proof. It is sufficient to consider the case where S is connected. Further we may assume that for each positive number t, $\mu^{-1}([t, \infty))$ is compact. If μ does not have this property it can be cut down in size to a continuous function which does have the property.

Consider the two pwl 3-manifolds M and Bd $(M) \times [0, 1]$ where the pwl structure of Bd (M) is inherited from M. By identifying each point y of Bd (M) with the point $(y, 0)$ of Bd $(M) \times [0, 1]$ we obtain a new pwl 3-manifold M' whose interior contains M. Give M' a metric which extends the metric on M. At certain points in this proof we find it convenient to regard f as a homeomorphism of S into M'.

Let T be a triangulation of S of sufficiently fine mesh so that for each simplex s of T there is a pwl 3-cell $C(s)$ in M whose diameter is less than one third the minimum value of μ on $N(s, T)$ and which contains a neighborhood of $f(N(s, T))$. We assume that T is such that (1) every component of R contains more than one 1-simplex of T, (2) for no vertex v of T is $N(v, T) = S$, and (3) each 2-simplex of T which intersects Bd (M) intersects it in a 1-simplex or a vertex. From Condition 3 we may assume that $C(s)$ misses Bd (M) if s misses R and that $C(s)$ intersects

Bd (M) in exactly a 2-cell if s intersects R. Let $\Delta_1, \ldots, \Delta_i, \ldots$ denote the 2-simplexes, $\sigma_1, \ldots, \sigma_j, \ldots$ the 1-simplexes, and $v_1, \ldots, v_k, \ldots$ the vertices of T. Let T' denote the first barycentric subdivision of T, and for each simplex s of T let $b(s)$ denote the barycenter of s.

For each Δ_i let $H(\Delta_i)$ be a polyhedral 3-manifold with connected boundary such that $H(\Delta_i)$ either misses Bd (M) or intersects it in a disk, a neighborhood of $H(\Delta_i)$ is contained in $C(\Delta_i)$, and a neighborhood of $f(\Delta_i)$ is contained in $H(\Delta_i)$. Choose the $H(\Delta_i)$'s so that each $f(\mathrm{Cl}\,(S-N(\Delta_i, T')))$ misses $H(\Delta_i)$ and $H(\Delta_j)$ misses $H(\Delta_i)$ if Δ_j misses Δ_i.

For each v_k let $H(v_k)$ be a polyhedral 3-manifold with connected boundary such that $H(v_k)$ either misses Bd (M) or intersects it in a disk, a neighborhood of $H(v_k)$ is contained in $C(v_k)$, and a neighborhood of $f(N(v_k, T'))$ is contained in $H(v_k)$. Choose the $H(v_k)$'s so that each $f(\mathrm{Cl}\,(S-N(v_k, T)))$ misses $H(v_k)$ and so that $H(v_k)$ misses $H(v_j)$ if v_k and v_j are not faces of a common simplex of T.

For each σ_j let $H(b(\sigma_j))$ be a polyhedral cube such that $H(b(\sigma_j))$ either misses Bd (M) or intersects it in a disk, a neighborhood of $f(b(\sigma_j))$ is contained in $H(b(\sigma_j))$, and for each $v_k \in \sigma_j$ a neighborhood of $H(b(\sigma_j))$ is contained in $H(v_k)$. Choose the $H(b(\sigma_j))$'s so that they are mutually exclusive and so that $H(b(\sigma_j))$ does not intersect $H(\Delta_i)$ or $H(v_k)$ unless σ_j is a face of Δ_i or v_k is a face of σ_j.

From our assumption that $H(s)$ either misses Bd (M) or intersects it in a disk, that Bd $(H(s))$ is connected, and that $H(s) \subset C(s)$ it follows that any polyhedral 2-sphere in Int $(H(s)) \cup$ Int $(H(s) \cap \mathrm{Bd}\,(M))$ bounds a polyhedral 3-cell in $H(s)$ and hence [15] a pwl 3-cell. Thus any pwl homeomorphism of the boundary of a pwl 3-cell into Int $(H(s)) \cup$ Int $(H(s) \cap \mathrm{Bd}\,(M))$ can be extended to a pwl homeomorphism of the 3-cell into $H(s)$ by Lemma 2.1.

For each σ_j let $\varepsilon(\sigma_j)$ be a positive number such that a $3\varepsilon(\sigma_j)$-neighborhood of $f(b(\sigma_j))$ is contained in $H(b(\sigma_j))$, for every Δ_i containing σ_j a $3\varepsilon(\sigma_j)$-neighborhood of $f(\Delta_i)$ is contained in $H(\Delta_i)$, and for every $v_k \in \sigma_j$ a $3\varepsilon(\sigma_j)$-neighborhood of $f(N(v_k, T'))$ is contained in $H(v_k)$. Suppose that $\sigma_j \subset R$ and is a face of $\Delta_{i(j)} = v_{k(j)}\sigma_j$. In this case let $\eta(\sigma_j) < \frac{1}{2}$ be subject to the restrictions on η in Lemma 8.1 when $\Delta_{i(j)}$, $f|\Delta_{i(j)}$, M, $b(\sigma_j)$, and $\varepsilon(\sigma_j)$ are substituted for the appropriate items. If σ_j is a face of two 2-simplexes, $\Delta_{i_0(j)} = v_{k_0(j)}\sigma_j$ and $\Delta_{i_1(j)} = v_{k_1(j)}\sigma_j$, let $\eta(\sigma_j) < \frac{1}{2}$ be subject to the restrictions on η in Lemma 8.2 when $f|\Delta_{i_0(j)} \cup \Delta_{i_1(j)}$, M', $b(\sigma_j)$, and $\varepsilon(\sigma_j)$ are substituted for the appropriate items. If $\sigma_j \subset \mathrm{Bd}\,(S)$ but $\sigma_j \not\subset R$ set $\eta(\sigma_j) = 1$.

Let $\alpha_1, \ldots, \alpha_k, \ldots$ be positive numbers such that each $\alpha_k < \eta(\sigma_j)$ if $v_k \in \sigma_j$. For each σ_j let $v_{k_a(j)}$ and $v_{k_b(j)}$ denote the endpoints of σ_j. For a 2-simplex $\Delta_i = v_{k(j)}\sigma_j$ which has σ_j as a face define a line segment σ_{ij} in Δ_i as follows. If $\sigma_j \subset \mathrm{Bd}\,(S)$ but $\sigma_j \not\subset R$ set $\sigma_{ij} = \sigma_j$. Otherwise let σ_{ij} be the line segment in Δ_i from $\alpha_{k_a(j)}v_{k(j)} + (1-\alpha_{k_a(j)})v_{k_a(j)}$ to $\alpha_{k_b(j)}v_{k(j)} + (1-\alpha_{k_b(j)})v_{k_b(j)}$. For each Δ_i let $\Delta_i(\alpha)$ denote the closure of the component of $\Delta_i - \bigcup_j \sigma_{ij}$ which contains $b(\Delta_i)$.

For each σ_j that either lies in R or is not contained in Bd (S) let r_j denote the polyhedral arc such that for every Δ_i that has σ_j as a face $r_j \cap \Delta_i$ is the line segment in Δ_i from $b(\sigma_j)$ to σ_{ij} which is perpendicular to σ_j. For each v_k let D_k denote the polyhedral 2-cell which is the closure of the component of

$$N(v_k, T)-((\bigcup \Delta_i(\alpha)) \cup (\bigcup r_j))$$

that contains v_k. See Figure 9.1. We assume that the α_k's are so small that for each σ_j and each $v_k \in \sigma_j$ a $2\varepsilon(\sigma_j)$-neighborhood of $f(D_k)$ is contained in $H(v_k)$, and for each r_j a $2\varepsilon(\sigma_j)$-neighborhood of $f(r_j)$ is contained in $H(b(\sigma_j))$.

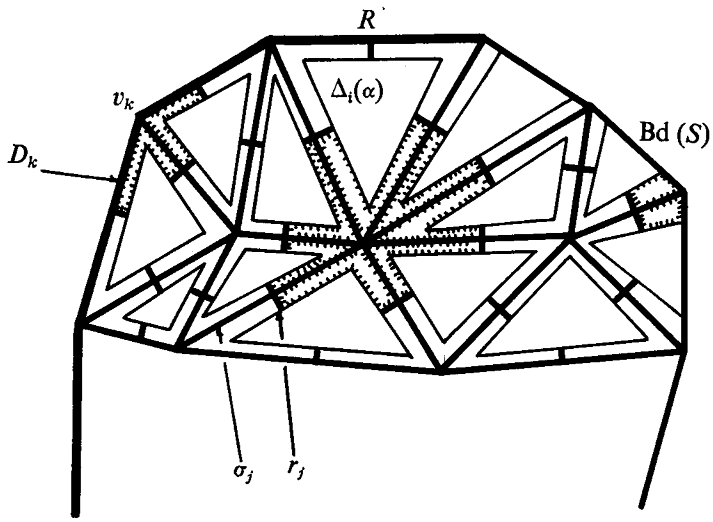

FIGURE 9.1

For each Δ_i let $\varepsilon(\Delta_i)$ be a positive number such that (1) $\varepsilon(\Delta_i)<\eta(\sigma_j)$ for every σ_j that is a face of Δ_i, (2) a $2\varepsilon(\Delta_i)$-neighborhood of $f(\Delta_i(\alpha))$ is contained in Int (M) and fails to intersect a $2\varepsilon(\Delta_j)$-neighborhood of $f(\Delta_j)$ $(j\neq i)$, (3) for every σ_j that is not a face of Δ_i a $2\varepsilon(\Delta_i)$-neighborhood of $f(\Delta_i)$ fails to intersect $H(b(\sigma_j))$, (4) if $v_j \in \Delta_i$ and $v_k \neq v_j$ a $2\varepsilon(\Delta_i)$-neighborhood of $f(v_j)$ misses $H(v_k)$, (5) if $\Delta_i = v_{k(j)}\sigma_j$ where $\sigma_j \subset R$ a $2\varepsilon(\Delta_i)$-neighborhood of $f(v_{k(j)}\sigma_{ij})$ is contained in Int (M), and (6) if a σ_j is the face of 2-simplexes $\Delta_{i_0(j)}=v_{k_0(j)}\sigma_j$ and $\Delta_{i_1(j)}=v_{k_1(j)}\sigma_j$ then a $2\varepsilon(\Delta_{i_0(j)})$-neighborhood of $f(v_{k_0(j)}\sigma_{i_0(j),j})$ misses a $2\varepsilon(\Delta_{i_1(j)})$-neighborhood of $f(v_{k_1(j)}\sigma_{i_1(j),j})$.

For each Δ_i that has a face σ_j in R let $\delta(\Delta_i)>0$ be subject to the restrictions on δ in Theorem 7.2 when Δ_i, M', $f|\Delta_i$, and $\varepsilon(\Delta_i)$ are substituted for the appropriate items, and let $\delta(\Delta_i)$ also be subject to the limitations on δ in Theorem 7.3 when σ_j, Bd (M), $f|\sigma_j$, and $\varepsilon(\Delta_i)$ are substituted for the appropriate items. For each remaining Δ_i use Theorem 7.2 to find $\delta(\Delta_i)$ by substituting Δ_i, M', $f|\Delta_i$, and $\varepsilon(\Delta_i)$.

Let ν be a positive continuous function on S whose maximum value on each Δ_i is less than $\delta(\Delta_i)$.

Suppose now that f_0 and f_1 are pwl homeomorphisms of S into M such that $f_e(S) \cap$ Bd $(M)=f_e(R)$ $(e=0, 1)$ and for each point y of S, $\rho(f(y), f_e(y))<\nu(y)$ $(e=0, 1)$.

For each Δ_i there is from Theorem 7.2 a pwl homeomorphism g_i of $\Delta_i \times [0, 1]$ into M' such that for every point y of Δ_i, $g_i(y, e) = f_e(y)$ $(e=0, 1)$ and the diameter of $g_i(y \times [0, 1])$ is less than $\varepsilon(\Delta_i)$. For each σ_j in R there is from Theorem 7.3 a pwl homeomorphism g'_j of $\sigma_j \times [0, 1]$ into Bd (M) such that for every point y of σ_j, $g'_j(y, e) = f_e(y)$ $(e=0, 1)$ and the diameter of $g'_j(y \times [0, 1])$ is less than $\varepsilon(\Delta_i)$ if σ_j is a face of Δ_i. Condition 2 on the $\varepsilon(\Delta_i)$'s shows that the $g_i(\Delta_i(\alpha) \times [0, 1])$'s are mutually exclusive and are contained in Int (M) and that the $g_i(\Delta_i(\alpha) \times (0, 1))$'s miss $S_0 \cup S_1$. Define g on $S \times \{0, 1\}$ so that for each point y of S, $g(y, e) = f_e(y)$ $(e=0, 1)$. Define g on each $\Delta_i(\alpha) \times [0, 1]$ to be g_i. For each σ_j in R define g on $b(\sigma_j) \times [0, 1]$ to be g'_j.

Conditions 1, 5, and 6 on the $\varepsilon(\Delta_i)$'s enable us to apply Lemmas 8.1 and 8.2 and so find for each σ_j where an r_j is defined a polyhedral disk F_j of diameter less than $\varepsilon(\sigma_j)$ such that Bd $(F_j) = g(\text{Bd}\,(r_j \times [0, 1]))$ and Int $(F_j) \subset$ Int (M) and misses each $g(\Delta_i \times \{0, 1\} \cup \Delta_i(\alpha) \times [0, 1])$ where Δ_i has σ_j as a face. From our assumptions about the sizes of the α_k's a neighborhood of each F_j is contained in $H(b(\sigma_j))$ so the F_j's are mutually exclusive. From Condition 3 on the $\varepsilon(\Delta_i)$'s the Int (F_j)'s miss all $g(\Delta_i(\alpha) \times [0, 1])$'s. Use Lemma 2.1 to define g on each $r_j \times [0, 1]$ so that g takes it pwl onto F_j.

From our assumptions about the smallness of the α_k's and from Condition 1 on the $\varepsilon(\Delta_i)$'s we find that for each $v_k \in$ Int (R),

$$g(\text{Bd}\,((D_k \cap R) \times [0, 1])) \subset \text{Int}\,(H(v_k) \cap \text{Bd}\,(M)).$$

Use Lemma 2.1 to define g on $(D_k \cap R) \times [0, 1]$ so that g takes it pwl into Int $(H(v_k) \cap \text{Bd}\,(M))$. Similarly for each $v_k \in$ Bd (R),

$$g((D_k \cap R) \times \{0, 1\} \cup (\text{Bd}\,(D_k \cap R) \cap \text{Int}\,(R)) \times [0, 1])$$

is contained in Int $(H(v_k) \cap \text{Bd}\,(M))$ and we can define g to take the rest of $(D_k \cap R) \times [0, 1]$ pwl into Int $(H(v_k) \cap \text{Bd}\,(M))$ as in Step 3 of the proof of Theorem 7.1 so that $g(v_k \times (0, 1))$ misses $g(R \times \{0, 1\})$. Condition 1 among the assumptions about the triangulation T of S insures that $g(v_k \times [0, 1]) \cap g(v_j \times [0, 1])$ is empty if v_k and v_j are distinct points of Bd (R). Since the $H(b(\sigma_j))$'s are mutually exclusive $g | R \times [0, 1]$ fails to be a homeomorphism only if some

$$g(\text{Int}\,((D_{k_1} \cap R) \times [0, 1])) \cap g(\text{Int}\,((D_{k_2} \cap R) \times [0, 1])) \qquad (k_1 \neq k_2)$$

is nonempty. But that implies that $g(v_{k_1} \times [0, 1])$ intersects $g(\text{Int}\,((D_{k_2} \cap R) \times [0, 1]))$ which is impossible by Condition 4 on the $\varepsilon(\Delta_i)$'s.

For each v_k that does not lie in Cl (Bd $(S) - R$), g takes Bd $(D_k \times [0, 1])$ pwl into Int $(H(v_k)) \cup$ Int $(H(v_k) \cap \text{Bd}\,(M))$ so by our previous remarks we can extend g to take $D_k \times [0, 1]$ pwl into $H(v_k)$. Let K denote the polyhedron

$$R \cup \text{Cl}\,(S - \bigcup \{D_k \mid v_k \in \text{Cl}\,(\text{Bd}\,(S) - R)\}).$$

For the same reason that $g | R \times [0, 1]$ is a homeomorphism, $g | K \times [0, 1]$ is a homeomorphism, and by construction $g(K \times [0, 1]) \cap \text{Bd}\,(M) = g(R \times [0, 1])$.

By using [12, Lemma 2.4] as in the proof of Lemma 2.8 and as in Step 4 of the proof of Theorem 7.1 we find a collection of mutually exclusive polyhedral 3-cells B_k for the v_k's in Cl (Bd $(S)-R$) so that each $B_k \subset H(v_k)$, each

$$B_k \cap (S_0 \cup S_1 \cup g(K \times [0, 1]))$$

is contained in Bd (B_k) and is in fact the disk $g((D_k \times \{0, 1\}) \cup ((D_k \cap K) \times [0, 1]))$, and each $B_k - B_k \cap g(R \times [0, 1]) \subset \text{Int}(M)$. For those v_k's we use Lemma 2.1 and [15] to extend g so that it takes each $D_k \times [0, 1]$ pwl onto B_k.

Now g is a homeomorphism of $S \times [0, 1]$ into M such that $g(S \times [0, 1]) \cap \text{Bd}(M) = g(R \times [0, 1])$ and for each point y of S, $g(y, e) = f_e(y)$ $(e = 0, 1)$. Since for each Δ_i, $g(\Delta_i \times [0, 1]) \subset C(\Delta_i) \cup (\bigcup \{C(v_k) \mid v_k \in \Delta_i\})$, and since these C's have diameters less than one third of the minimum value of μ on Δ_i, each $g(y \times [0, 1])$ has diameter less than $\mu(y)$. From the assumption on μ that each $\mu^{-1}([t, \infty))$ $(t > 0)$ is compact $g(S \times [0, 1])$ is a closed subset of M so g is a pwl homeomorphism by Proposition 2.1. This completes the proof of the theorem. □

Here is a topological version of Theorem 9.1.

THEOREM 9.2. *Suppose that M is a 3-manifold with boundary, S is a surface in M such that $S \cap \text{Bd}(M) = \text{Bd}(S) \cap \text{Bd}(M) = R$ either a 1-manifold with boundary or the empty set, and μ is a positive continuous function on S.*

There is a positive continuous function ν on S such that if f_0 and f_1 are homeomorphisms of S onto disjoint locally tame surfaces in M such that $f_e(S) \cap \text{Bd}(M) = f_e(R)$ $(e = 0, 1)$ and for each point y of S, $\rho(y, f_e(y)) < \nu(y)$ $(e = 0, 1)$, then there is a homeomorphism g of $S \times [0, 1]$ onto a locally tame solid in M so that $g(S \times [0, 1]) \cap \text{Bd}(M) = g(R \times [0, 1])$ and for each point y of S, $g(y, e) = f_e(y)$ $(e = 0, 1)$ and the diameter of $g(y \times [0, 1])$ is less than $\mu(y)$.

Proof. Since an open subset of M can always be found which contains S as a closed subset it is sufficient to consider the case where S is a closed connected subset of M. From [2], [4], M can be triangulated so we may assume that it is a pwl 3-manifold. Similarly since surfaces can be triangulated there is a pwl 2-manifold Σ and a homeomorphism f of Σ onto S. Set $R' = f^{-1}(R)$.

Let ν' be a positive continuous function on Σ which is subject to the restrictions on ν in Theorem 9.1 when M, Σ, R', f and $(\mu f)/3$ are substituted for the appropriate items. From the proof of Theorem 9.1 we know that for each positive number t, $(\nu')^{-1}([t, \infty))$ is compact. Define ν to be $\frac{1}{2}\nu' f^{-1}$.

Let λ be a nonnegative continuous function on M which is positive on S and which is so small that for each point y of S, $\mu(y)/6$ is greater than the maximum value of λ over all points of M whose distances from y do not exceed $2/3\mu(y)$.

Suppose now that f_0 and f_1 are homeomorphisms of S into M such that $f_e(S) \cap \text{Bd}(M) = f_e(R)$ $(e = 0, 1)$ and for each point y of S, $\rho(y, f_e(y)) < \nu(y)$ $(e = 0, 1)$. Let f'_e $(e = 0, 1)$ denote the homeomorphism $f_e f$. For each point y of Σ we have $\rho(f(y), f'_e(y)) < \frac{1}{2}\nu'(y)$.

Use [2], [20] to find a homeomorphism H of M onto itself such that $H(S_0 \cup S_1)$ is locally polyhedral and for each point y of M, $\rho(y, H^{-1}(y)) < \lambda(y)$. We assume that H moves points so little that for each point y of Σ, $\rho(f(y), Hf'_e(y)) < \nu'(y)$ $(e=0, 1)$. From [15, §9] we may assume that Hf'_e $(e=0, 1)$ takes Σ pwl onto $H(S_e)$. From the fact that each $(\nu')^{-1}([t, \infty))$ $(t>0)$ is compact $H(S_e)$ $(e=0, 1)$ is a closed subset of M and thus Hf'_e is a pwl homeomorphism of Σ into M.

Theorem 9.1 provides a pwl homeomorphism g' of $\Sigma \times [0, 1]$ into M such that $g'(\Sigma \times [0, 1]) \cap \text{Bd}\,(M) = g'(R' \times [0, 1])$ and for each point y of Σ, $g'(y, e) = Hf'_e(y)$ $(e=0, 1)$ and the diameter of $g'(y \times [0, 1])$ is less than $(\mu f(y))/3$.

Define g by the rule $g(y, t) = H^{-1}g'(f^{-1}(y), t)$. It is a homeomorphism of $S \times [0, 1]$ into M such that $g(S \times [0, 1]) \cap \text{Bd}\,(M) = g(R \times [0, 1])$. For each point y of S we have $g(y, e) = H^{-1}g'(f^{-1}(y), e) = H^{-1}Hf'_e(f^{-1}(y)) = f_e(y)$ $(e=0, 1)$. The diameter of each $g'(f^{-1}(y) \times [0, 1])$ is less than $\mu(y)/3$ so since $\nu'(f^{-1}(y))$ is certainly less than $\mu(y)/3$, $g'(f^{-1}(y) \times [0, 1])$ is contained in a $2/3\mu(y)$-neighborhood of $f(y)$. From the conditions on λ, H^{-1} moves no point of $g'(f^{-1}(y) \times [0, 1])$ by as much as $\mu(y)/6$ so $g(y \times [0, 1]) = H^{-1}(g'(f^{-1}(y) \times [0, 1]))$ has diameter less than

$$2/3\mu(y) + 2/6\mu(y) = \mu(y). \quad \square$$

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