Online Appendix for "Building Routines: Learning, Cooperation and the Dynamics of Incomplete Relational Contracts"

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Appendix B (continued)

Proof of Proposition 1: By Assumption 1 we know there exists an equilibrium (s_1^0, s_2^0) in which player 1 stays in the first period. Denote (V_1^0, V_2^0) the associated initial values. Since player 1 has the option to exit and player 2 can choose not to reciprocate, we necessarily have $V_1^0 \geq 0$ and $V_2^0 \geq \pi$. Now consider an equilibrium (s_1, s_2) on the Pareto frontier of Γ_{FI}. Assume that there is a history h_t^1 attainable[1] on the equilibrium path, at which player 1 decides to exit. We now show that (s_1, s_2) cannot be efficient.

Let us first consider the case where in the subgame starting from h_t^1, s_1 prescribes that player 1 should never stay again in equilibrium. Consider the alternative strategies $\tilde{s}_1$ and $\tilde{s}_2$ defined by:

$$\forall i \in \{1,2\},\ \tilde{s}_i(h^i) = \left\{ \begin{array}{ll} s_i(h^i) & \text{if } \nexists h' \ s.t.\ h^i = h_t^1 \sqcup h' \\ s_i^0(\hat{h}^i) & \text{if } h^i = h_t^1 \sqcup \hat{h}^i \end{array} \right. .^2$$

By construction $(\tilde{s}_1, \tilde{s}_2)$ is also an equilibrium and dominates (s_1, s_2).

Let us now consider the case in which following h_t^1 there is an attainable equilibrium history at which player 1 stays under s_1. This implies that at h_t^1 the continuation values associated with s_1 and s_2 are positive and player 2 gets strictly positive value. Consider the alternative strategies in which history h_t^1 is skipped:

$$\forall i \in \{1,2\},\ \tilde{s}_i(h^i) = \left\{ \begin{array}{ll} s_i(h^i) & \text{if } \nexists h' \ s.t.\ h^i = h_t^1 \sqcup h' \\ s_i(h_t^1 \sqcup (E, \emptyset, \emptyset, \emptyset) \sqcup h') & \text{if } h^i = h_t^1 \sqcup h' \end{array} \right. .$$

By construction $(\tilde{s}_1, \tilde{s}_2)$ is also an equilibrium and it strictly dominates (s_1, s_2). This concludes the proof.

[1] i.e. a history that can be reached with positive probability.

[2] For conciseness, the description of strategy $\tilde{s}_2$ omits the initial element $\{\mathcal{N}\}$ from history h^2.

Lemma B.1 establishes that there exist parameter values such that Assumptions 1 and 2 hold together.

Lemma B.1: *Pick parameter values* $k > 0$, $\pi > 0$, $\delta > 1/2$, a pair (p, q) such that $p > q > \delta$ and $\frac{1-q}{\delta} < 1 - \frac{1-pq}{pq}\frac{1-\delta}{\delta}$, and b^0 such that $\frac{1-q}{\delta} < \frac{k}{pqb^0} < 1 - \frac{1-pq}{pq}\frac{1-\delta}{\delta}$. The following hold,

(i) $\forall c > 0, \quad \delta q \overline{V}_2^0 > \delta(\overline{V}_2^0 - \underline{V}_2^D)$.

(ii) Let $c_{max} = \max\{c | \delta q \overline{V}_2^0 \geq c\}$ and $c_{min} = \min\{c | \delta(\overline{V}_2^0 - \underline{V}_2^D) \leq c\}$. We have that $c_{max} > \max\{\frac{\delta\pi}{p}, c_{min}\}$.

(iii) For any $c \in (\max\{\frac{\delta\pi}{p}, c_{min}\}, c_{max})$, $\frac{1}{1-\delta}q\pi > \overline{V}_2^0$ and both Assumptions 1 and 2 hold together.

Proof of Lemma B.1: Let us begin with point (i). We have that

$$\begin{aligned}
\delta q \overline{V}_2^0 &= \frac{\delta}{1-\delta}q\pi - q\frac{\delta}{1-\delta}\frac{k}{qb^0}c \\
\delta(\overline{V}_2^0 - \underline{V}_2^D) &= \frac{\delta}{1-\delta}\left(1 - \frac{1-\delta}{1-\delta(1-p)}\right)\pi - \frac{\delta}{1-\delta}\frac{k}{qb^0}c.
\end{aligned}$$

Note that $1 - \frac{1-\delta}{1-\delta(1-p)}$ is increasing in p and that for $p = 1$, it is equal to δ, which is strictly less than q. This implies that $q > 1 - \frac{1-\delta}{1-\delta(1-p)}$ and hence $\delta q \overline{V}_2^0 > \delta(\overline{V}_2^0 - \underline{V}_2^D)$ for all $c > 0$. This shows point (i).

Regarding point (ii), the fact that $c_{max} > c_{min}$ simply follows from point (i). Let us now show that $c_{max} > \delta\pi/p$. We have that $c_{max} = \frac{\delta}{1-\delta}q\pi\left(\frac{\delta}{1-\delta}\frac{k}{b^0} + 1\right)^{-1}$. Hence,

$$c_{max} > \delta\pi/p \iff \pi\left(\frac{\delta}{1-\delta}\frac{k}{pqb^0}pq + 1\right) < \frac{1}{1-\delta}pq\pi.$$

The fact that $\frac{k}{pqb^0} < 1 - \frac{1-pq}{pq}\frac{1-\delta}{\delta}$ implies this last inequality holds. This proves point (ii).

We now turn to point (iii). Let us first show that $q\frac{1}{1-\delta}\pi > \overline{V}_2^0$. We have,

$$q\frac{1}{1-\delta}\pi > \overline{V}_2^0 \iff q > 1 - \frac{k}{pqb^0}\frac{pc}{\pi} \iff \frac{k}{pqb^0} > \frac{\pi}{pc}(1-q).$$

Since $c > \frac{\delta\pi}{p}$, we have that $\frac{\pi}{pc}(1-q) < (1-q)/\delta$. Since b^0 is picked such that $\frac{k}{pqb^0} > (1-q)/\delta$, we have that indeed $q\frac{1}{1-\delta}\pi > \overline{V}_2^0$. This, along with points (i) and (ii), implies that Assumptions 1 and 2 hold together.

Proof of Lemma 1: Since $\delta(\overline{V}_2^0 - \underline{V}_2^D) < c$, there exists $\mu > 0$ such that for all $b^1 \in [b^0, b^0+\mu]$, $\delta(\overline{V}_2^1 - \underline{V}_2^D) < c - \mu$. Let us now pick a value of K independent of $b^1 \in [b^0, b^0+\mu]$. For any K, define

$$\underline{V}_2^{D,K} \equiv \frac{1-\delta^{K+1}(1-p)^{K+1}}{1-\delta(1-p)}\pi.$$

As K goes to infinity, $\underline{V}_2^{D,K}$ converges to $\underline{V}_2^D$. Furthermore, since $\delta(\overline{V}_2^1 - \underline{V}_2^D) < c - \mu$, there exists K large enough, such that for all $b^1 \in [b^0, b^0+\mu]$, $\delta(\overline{V}_2^1 - \underline{V}_2^{D,K}) < c - \mu/2$.

Consider an equilibrium (s_1, s_2) and a revelation stage $h_t^{2|1}$ for action a^1. Denote by $\hat{\eta}$ the probability that player 1 exits in the next K periods. Let us consider subsequent histories h_s^2, with $t < s \leq t+K$, such that a^1 is still unconfirmed and the confirmed action a^0 has not been available. On the equilibrium path such histories have probability at least $(1-q)^{s-t+1}(1-p)^{s-t}$ and hence following such histories, exit can only occur with probability less than

$$\frac{\hat{\eta}}{(1-q)^{s-t+1}(1-p)^{s-t}} \leq \frac{\hat{\eta}}{(1-q)^{K+1}(1-p)^{K}}.$$

Out of equilibrium, if player 2 deviates by taking only costless actions, the likelihood that a^1 is still unconfirmed and the confirmed action a^0 has not been available is $(1-p)^{s-t}$. Hence using such a strategy, player 2 obtains at least payoff

$$\begin{aligned}\underline{V}_2^{D,K,\hat{\eta}} &\geq \sum_{s=t}^{t+K} \delta^{s-t}(1-p)^{s-t}\left(1 - \frac{\hat{\eta}}{(1-q)^{K+1}(1-p)^{K}}\right)\pi \\ &\geq \left(1 - \frac{\hat{\eta}}{(1-q)^{K+1}(1-p)^{K}}\right)\frac{1-\delta^{K+1}(1-p)^{K+1}}{1-\delta(1-p)}\pi.\end{aligned}$$

For revelation to be incentive compatible, we must have $\delta(\overline{V}_2^1 - \underline{V}_2^{D,K,\hat{\eta}}) < c$, which implies that

$$\hat{\eta} \geq \frac{(1-q)^{K+1}(1-p)^{K}}{\delta \underline{V}_2^{D,K}}\left[c - \delta(\overline{V}_2 - \underline{V}_2^{D,K})\right] \equiv \eta > 0.$$

Hence there exist $\mu > 0$, $K \in \mathbb{N}$ and $\eta > 0$ such that for all $b^1 \in [b^0, b^0+\mu]$, at any revelation stage for action a^1, there is probability greater than η that player 1 exits in the next K periods.

Proof of Proposition 4: We begin with point (i). It is intuitively clear that when b^1 becomes large, revealing action a_1 creates value. However, providing incentives for revelation sometimes requires inefficient punishment, and value must be destroyed on some equilibrium paths. Hence, the delicate part of the proof is to show that after any history, including histories where inefficient punishment is required on the equilibrium path, a^1 will be confirmed with positive probability after any history where player 1 stays.

Consider a Pareto efficient equilibrium (s_1, s_2) and a history h_t^1 at which player 1 stays. We first consider the case in which action a^0 has been confirmed before h_t^1. By Assumption

2, $\frac{1}{1-\delta}q\pi > \overline{V}_2^0$. Hence, there exists $\overline{\Delta}$ high enough such that for all $b^1 > b^0 + \overline{\Delta}$, we have

$$\begin{aligned} V_2^* &\equiv q\frac{1}{1-\delta}\left(\pi - \frac{1}{q\sqrt{b^1}}c\right) > \overline{V}_2^0 \\ V_1^* &\equiv q\frac{1}{1-\delta}(-k+\sqrt{b^1}) > \frac{1}{1-\delta}(-k+b^0). \end{aligned}$$

The proof of point (i) uses the two following facts. First, by construction, there exist $\tau_1 \in \mathbb{N}$, and $\nu_1 > 0$ such that at any history h_s^2 where player 1's continuation value is greater than V_1^*, or player 2's continuation value is greater than V_2^*, there must be probability at least ν_1 that action a^1 is confirmed in the next τ_1 periods. Second, by point (ii) of Assumption 2, player 1 cannot be induced to stay if player 2 never takes action a^1, and only takes action a^0 when action a^1 is unavailable. This implies that there exist $\tau_2 \in \mathbb{N}$ and $\nu_2 > 0$ such that if player 1 stays at some history h_t^1, then player 2 must take action a^1, or take action a^0 at a history where a^1 is available, with probability at least ν_2 in the next τ_2 periods.

Consider h_s^2 with $s > t$, an equilibrium history at which player 2 takes action a^0, and action a^1 is available. Denote $V_1(h_s^2)$ and $V_2(h_s^2)$ the players' continuation values at such a history. If $V_1(h_s^2) \geq V_1^*$ or $V_2(h_s^2) \geq V_2^*$, we know that action a^1 must be taken with probability at least ν_1 in the next τ_1 periods.

Assume temporarily that at h_s^2, players' have continuation values such that $V_1(h_s^2) < V_1^*$ and $V_2(h_s^2) < V_2^*$. Let us show that if this is the case, then (s_1, s_2) cannot be efficient. Indeed, consider the modified strategies $(\hat{s}_1, \hat{s}_2)$ that coincide with (s_1, s_2) except following equilibrium history h_s^2. At history h_s^2, strategies $(\hat{s}_1, \hat{s}_2)$ prescribe that player 2 take action a^1. If a^1 is immediately confirmed, then in the continuation game, on the equilibrium path, player 1 stays every period and player 2 takes action a^1 with probability $\frac{1}{pq\sqrt{b^1}}$ whenever it is available (using public randomizations). If action a^1 fails when player 2 takes it at history h_s^2, then $(\hat{s}_1, \hat{s}_2)$ prescribe that player 1 always exits and player 2 only takes unproductive actions. Under strategies $(\hat{s}_1, \hat{s}_2)$, players obtain values V_2^* and V_1^* at history h_s^2. This increases both players' continuation values and implies that starting from h_s^2, $(\hat{s}_1, \hat{s}_2)$ is indeed an equilibrium. In particular, since taking a costly action was incentive compatible for player 2 under (s_1, s_2), it is also incentive compatible under $(\hat{s}_1, \hat{s}_2)$. Note that players obtain these higher continuation values only if actions a^0 and a^1 are both confirmed. We also know that if player 2 deviates before h_s^2, then histories at which a^0 and a^1 are both confirmed are not reachable. Hence, improving players' utility at equilibrium histories where actions a^0 and a^1 are confirmed increases continuation values on the equilibrium path but does not change player 2's payoffs upon deviation. This implies that $(\hat{s}_1, \hat{s}_2)$ is an equilibrium of the overall game. Since it dominates (s_1, s_2), which is by assumption Pareto efficient, we obtain a contradiction. This yields that $V_1(h_s^2) \geq V_1^*$ or $V_2(h_s^2) \geq V_2^*$. Altogether, this implies that whenever player 1 stays, there is probability at least $q\nu_1\nu_2$ that action a^1 will be confirmed in the next $\tau_1 + \tau_2$ periods.

We now turn to the case where a^0 is not confirmed at history h_t^1. Since player 1 stays at history h_t^1, there must be probability $\nu_2 > 0$ that player 2 takes action a^0 or a^1 in the next τ_2 periods. Consider a history h_s^2 at which player 2 takes action a^0. Since by Assumption 2,

$\delta\pi/p < c$ and $q > \delta$, it follows that $(1-q)\frac{\delta}{1-\delta}\pi < c$. In words, this means that obtaining profit π forever if action a^0 fails does not cover player 2's cost of taking a productive action. Hence, there exist $\tau_3 \in \mathbb{N}$ and $\nu_3 > 0$ such that whenever player 2 takes action a^0 and a^0 is confirmed, there is probability greater than ν_3 that player 1 stays at least once in the next τ_3 periods. This puts us in the configuration discussed above. Altogether we can conclude that at h_t^1 there is probability at least $q^2\nu_1\nu_2\nu_3$ that action a^1 will be confirmed in the next $\tau_1+\tau_2+\tau_3$ periods. This proves point (i).

We now turn to point (ii). To begin, we consider the case where a^0 is confirmed at some history $h_{t_0}^1$ and no other action has been revealed.[3] Let us define the sets of values $\mathcal{U}_0$, $\mathcal{U}_{0,1}$ and $\mathcal{U}_{0,1}^{K,\eta}$ as follows:

(i) $\mathcal{U}_0$ is the set of Pareto efficient equilibrium values in the complete information game where only a^0 is productive, at a history $h_t^2 \in \mathcal{H}^2$ where a^0 is available.

(ii) $\mathcal{U}_{0,1}$ is the set of Pareto efficient equilibrium values in the complete information game where a^0 and a^1 are productive, at a history $h_t^2 \in \mathcal{H}^2$ where a^1 is available.

(iii) $\mathcal{U}_{0,1}^{K,\eta}$ is the set of values sustainable in the complete information game where a^0 and a^1 are productive, in equilibria such that player 1 exits with probability greater than η in the next K periods, at a history $h_t^2 \in \mathcal{H}^2$ where a^1 is available.

Consider a pair of values $(V_1, V_2) \in \mathcal{U}_{0,1}$. Since player 1 never exits in equilibrium there exists a positive number r such that $V_2 = \frac{1}{1-\delta}(\pi - prc)$. Since player 2 always has the option to take unproductive actions, we have that $V_2 \geq \pi$. This implies that $r < \frac{\delta\pi}{pc}$, which, by point (i) of Assumption 2, implies that $r < 1$. Hence (V_1, V_2) can be achieved under complete information by having player 1 never exit in equilibrium, and player 2 take action a^1 with probability r when it is available. By considering the strategy in which player 1 never exits in equilibrium and player 2 takes action a^0 with probability r whenever it is available, it follows that as b^1 goes to b^0, the set of values $\mathcal{U}_{0,1}$ converges to $\mathcal{U}_0$. More formally, for all $\epsilon > 0$, there exists b^1 close enough to b^0 such that for all $(V_1, V_2) \in \mathcal{U}_{0,1}$, there exists $(\hat{V}_1, \hat{V}_2) \in \mathcal{U}_0$ such that $\hat{V}_1 \geq V_1 - \epsilon$ and $\hat{V}_2 \geq V_2 - \epsilon$.

Furthermore, for any $K \in \mathbb{N}$ and $\eta > 0$, there exists $\alpha > 0$ such that for all $(V_1', V_2') \in \mathcal{U}_{0,1}^{K,\eta}$, there exists $(V_1, V_2) \in \mathcal{U}_{0,1}$ such that $V_1 \geq V_1' + \alpha$ and $V_2 \geq V_2' + \alpha$. This implies that we can pick $\underline{\Delta} > 0$ small enough so that for all $b^1 \in (b^0, b^0 + \underline{\Delta})$, first, Lemma **??** holds, and second, whenever $(V_1', V_2') \in \mathcal{U}_{0,1}^{K,\eta}$, there exists $(\hat{V}_1, \hat{V}_2) \in \mathcal{U}_0$ such that $\hat{V}_1 \geq V_1' + \alpha/2$ and $\hat{V}_2 \geq V_2' + \alpha/2$.

Let us consider a revelation stage $h_t^{2|1}$ for action a^1, such that no other revelation stages have occurred between $h_{t_0}^1$ and $h_t^{2|1}$. By Lemma 1, there exist K and $\eta > 0$ such that values (V_1^{Rev}, V_2^{Rev}) at $h_t^{2|1}$ are dominated by values in $\mathcal{U}_{0,1}^{K,\eta}$. This implies that for all $b^1 < b^0 + \underline{\Delta}$, there exists $(\hat{V}_1, \hat{V}_2) \in \mathcal{U}_0$ such that $\hat{V}_1 > V_1^{Rev}$ and $\hat{V}_2 > V_2^{Rev}$.

[3] Note that unproductive actions may have been taken at histories that are not revelation stages.

Let us denote by $(V_1^{Conf,0}, V_2^{Conf,0})$ continuation values at the history $h^1_{t_0}$ where action a^0 was confirmed. We must have $V_1^{Conf,0} \geq 0$ and $\delta V_2^{Conf,0} \geq c$. Define the pair of real numbers

$$\begin{aligned} \hat{V}_1^{Conf,0} &\equiv V_1^{Conf,0} + prob(h_t^{2|1}) \left(\hat{V}_1 - V_1^{Rev} \right) \\ \hat{V}_2^{Conf,0} &\equiv V_2^{Conf,0} + prob(h_t^{2|1}) \left(\hat{V}_2 - V_2^{Rev} \right), \end{aligned}$$

obtained by replacing revelation values at $h_t^{2|1}$ with continuation values not involving revelation of action a^1. We have that $\hat{V}_1^{Conf,0} > V_1^{Conf,0}$ and $\hat{V}_2^{Conf,0} > V_2^{Conf,0}$. Repeat the same replacement operation at all first revelation stages occurring after action a^0 is confirmed. We obtain values $\widetilde{V}_1^{Conf,0} > V_1^{Conf,0}$ and $\widetilde{V}_2^{Conf,0} > V_2^{Conf,0}$. By construction these values are such that player 1 only ever gets benefit b^0, and they dominate the original values involving further revelation. The first question is whether such values correspond to a continuation equilibrium. Let us show that this is indeed the case.

Between $h^1_{t_0}$ and a consecutive revelation stage $h_t^{2|1}$ for action a^1, all revealed actions are confirmed. Proposition 3 implies that (s_1, s_2) prescribes no exit on the equilibrium path between $h^1_{t_0}$ and $h_t^{2|1}$. Hence, there exists $r > 0$ such that $\widetilde{V}_1^{Conf,0}$ and $\widetilde{V}_2^{Conf,0}$ can be written

$$\widetilde{V}_1^{Conf,0} = -\frac{1}{1-\delta} k + \frac{1}{1-\delta} prqb^0 \quad \text{and} \quad \widetilde{V}_2^{Conf,0} = \frac{1}{1-\delta}\pi - \frac{1}{1-\delta} prc.$$

We have that $\widetilde{V}_1^{Conf,0} > 0$ and $\delta \widetilde{V}_2^{Conf,0} > c$. By point (i) of Assumption **??**, we have that $c/\delta > \pi$. The fact that $\widetilde{V}_2^{Conf,0} > \pi$ implies that $r < \overline{r} < 1$. Hence values $\widetilde{V}_1^{Conf,0}$ and $\widetilde{V}_2^{Conf,0}$ are supported by the continuation equilibrium in which player 1 always stays on the equilibrium path and player 2 cooperates at rate r whenever action a^0 is available.

To finish the proof, we must show that incentive constraints at histories preceding $h^1_{t_0}$ still hold after changing continuation strategies at $h^1_{t_0}$. By assumption, history $h^1_{t_0}$ is such that no action is revealed and unconfirmed. This implies that $h^1_{t_0}$ is not attainable by earlier deviations from player 2. Therefore, increasing continuation values at $h^1_{t_0}$ does not increase player 2's payoffs upon deviation and increases equilibrium continuation values. As a result, increasing continuation values at $h^1_{t_0}$ can only improve earlier incentive compatibility constraints.

This concludes the proof of point (ii): for all $b^1 \in (b^0, b^0 + \underline{\Delta})$, efficient equilibria should involve no further revelation upon confirmation of a^0. An identical proof holds in the case where a^1 is confirmed and no other action has been revealed.