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## Influence Surfaces of Plates Shaped as Sectors of a Circular Ring

by

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The aim of the authors of the present work \*) consists in determining an influence surface for deflection and other statical quantities like bending moments, twisting moments and the shear forces of a plate. This problem is identical with that of the determination of Green's function for the above-mentioned quantities.

As is well known, the statical quantities for a plate in polar coordinates can be expressed respectively by the following formulae:

a) bending moments

$$M_{r,j} = -N \left[ \frac{\partial^2 w_j}{\partial r^2} + \nu \left( \frac{1}{r} \frac{\partial w_j}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w_j}{\partial \varphi^2} \right) \right];$$

$$M_{\varphi,j} = -N \left[ \nu \frac{\partial^2 w_j}{\partial r^2} + \frac{1}{r} \frac{\partial w_j}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w_j}{\partial \varphi^2} \right];$$

b) twisting moments

$$M_{r\varphi,j} = -N(1 - \nu) \left[ \frac{1}{r} \frac{\partial^2 w_j}{\partial r \partial \varphi} - \frac{1}{r^2} \frac{\partial w_j}{\partial \varphi} \right];$$

c) shearing forces

$$T_{r,j} = -N \frac{\partial \nabla^2 w_j}{\partial r}; \quad T_{\varphi,j} = -N \frac{1}{r} \frac{\partial \nabla^2 w_j}{\partial \varphi}$$

where  $w_j$  is the deflection surface of the plate

$$N = \frac{E h^3}{12(1 - \nu^2)}$$

where  $E$  is the modulus of elasticity,  $\nu$  — Poisson's ratio and  $h$  the thickness of the plate.

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The main task was to determine Green's function for a wedge-shaped plate of infinite radius resting freely on its edges —  $\varphi = 0$ , and  $\varphi = \alpha$  (Fig. 1).

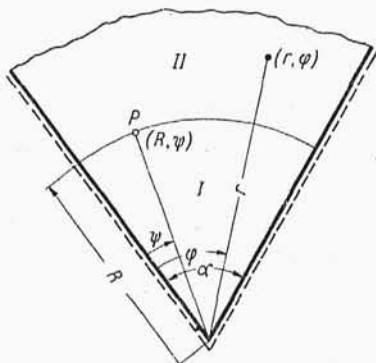


Fig. 1

The differential equation

$$\nabla^2 \nabla^2 w_j(r, \varphi) = 0 \quad j = I, II$$

of plate deflection was replaced by a system of differential equations

$$\nabla^2 w_j(r, \varphi) = \Phi_j(r, \varphi)$$

$$\nabla^2 \Phi_j(r, \varphi) = 0$$

where the index  $j = I$  is related to region I

( $\varrho = \frac{r}{R} \leq 1$ ) and the index  $j = II$  to region II

( $\varrho = \frac{r}{R} \geq 1$ ).

The function  $\Phi_j(r, \varphi)$  is proportional to the sum of the bending moments ( $M_{r,j} + M_{\varphi j}$ ) and can be presented in the following form:

$$(1a) \quad \Phi_I(r, \varphi) = \frac{P}{N\pi} \sum_{n=1,2,\dots}^{\infty} \frac{\varrho^{nk}}{n} \sin nk\psi \sin nk\varphi \quad \text{for } \varrho \leq 1,$$

$$\Phi_{II}(r, \varphi) = \frac{P}{N\pi} \sum_{n=1,2,\dots}^{\infty} \frac{\varrho^{-nk}}{n} \sin nk\psi \sin nk\varphi \quad \text{for } \varrho \geq 1,$$

where

$$k = \frac{\pi}{\alpha}, \quad \varrho = \frac{r}{R}$$

or in the closed form

$$(1b) \quad \Phi_j(r, \varphi) = -\frac{P}{4N\pi} \ln \frac{\cosh(k \ln \varrho) - \cos k(\varphi - \psi)}{\cosh(k \ln \varrho) - \cos k(\varphi + \psi)} \quad \text{for } \varrho \geq 1.$$

A simple calculation shows that the following relations exist between certain differential operations on function  $w_j$  and function  $\Phi$  and its first derivatives:

$$(2) \quad \begin{aligned} \frac{1}{k^2} \left( \frac{1}{\varrho} \frac{\partial w_j}{\partial \varrho} + \frac{1}{\varrho^2} \frac{\partial^2 w_j}{\partial \varphi^2} \right) &= \frac{1}{2} \Phi_j - \frac{1}{4} (\varrho - \varrho^{-1}) \frac{\partial \Phi_j}{\partial \varrho}, \\ \frac{1}{R^2} \frac{\partial^2 w_j}{\partial \varrho^2} &= \frac{1}{2} \Phi_j + \frac{1}{4} (\varrho - \varrho^{-1}) \frac{\partial \Phi_j}{\partial \varrho}, \\ \frac{1}{k^2} \left( \frac{1}{\varrho} \frac{\partial^2 w_j}{\partial \varphi \partial \varrho} - \frac{1}{\varrho^2} \frac{\partial w_j}{\partial \varphi} \right) &= (1 - \varrho^{-2}) \frac{\partial \Phi_j}{\partial \varphi}. \end{aligned}$$

Hence the statical quantities for a wedge-shaped plate of infinite radius charged with a concentrated force, and their influence surfaces can be represented by means of the function  $\Phi$  and its derivatives.

And so we get:

$$M_{r,j} = -N \left[ \frac{1+\nu}{2} \Phi + \frac{1-\nu}{4} (\rho - \rho^{-1}) \frac{\partial \Phi_j}{\partial \rho} \right],$$

$$M_{\varphi,j} = -N \left[ \frac{1+\nu}{2} \Phi_j - \frac{1-\nu}{2} (\rho - \rho^{-1}) \frac{\partial \Phi_j}{\partial \rho} \right],$$

$$M_{r\varphi,j} = -N \left[ \frac{1-\nu}{4} (1 - \rho^{-2}) \frac{\partial \Phi_j}{\partial \varphi} \right],$$

$$T_{r,j} = -\frac{N}{R} \frac{\partial \Phi_j}{\partial \rho},$$

$$T_{\varphi,j} = -\frac{N}{\rho R} \frac{\partial \Phi_j}{\partial \varphi}.$$

Fig. 2 a represents the influence surface  $8\pi M_r$ ; Fig. 2 b shows the influence surface  $8\pi M_\varphi$  at a point  $(R, \pi/8)$  for a wedge-shaped plate with central angle  $\pi/4$ .

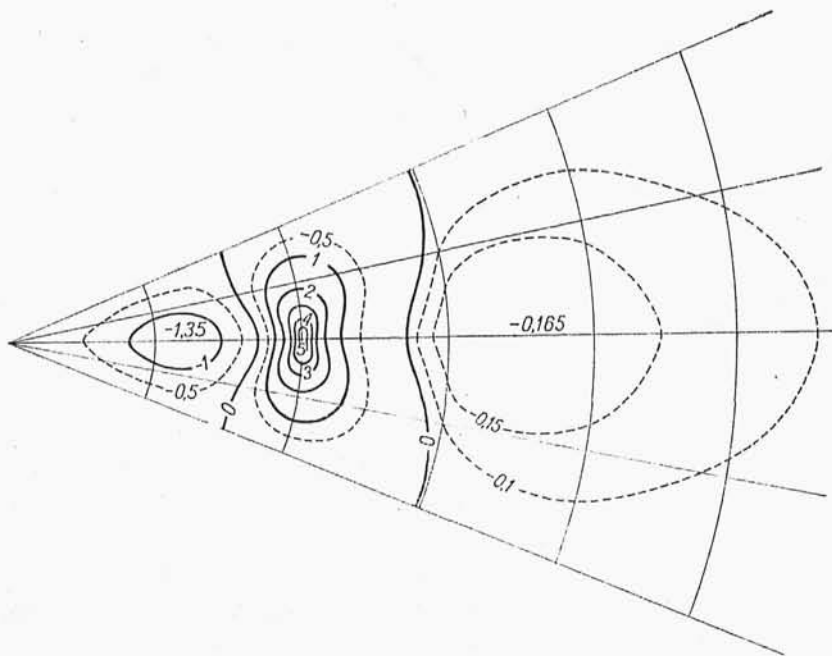


Fig. 2 a

The solution for a wedge of infinite radius contains, as a particular case, A. Nadai's [1] solution for a infinite plate band.

The solutions obtained for a wedge-shaped plate were used to determine Green's function for a plate shaped as a flat ring sector. The

solution for the deflection surface was represented as the sum of two functions:

$$(3) \quad w = w_j + w_1.$$

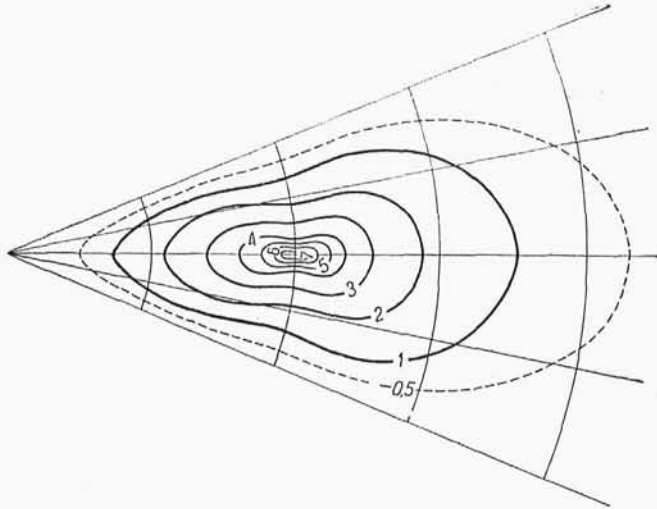


Fig. 2 b

Here  $w_j$  is Green's function for a wedge-shaped plate of infinite radius and function  $w_1$  takes into account the boundary conditions at the edges.

These particular cases where the section forces can be represented in closed form were elaborated in detail.

a. Plate in the form of a circular sector and resting freely on its edges  $\varphi = 0$  and  $\varphi = \alpha$  and clamped on an arc  $r = R_2$ .

A solution was obtained using equation (3). All the statical quantities determined for function  $w_j$  are given when discussing the case of a infinite wedge-shaped plate. We give here only the derivatives of the supplementary function  $w_1$  according to formula (3) by means of which we

are able to determine the statical quantities of the plate.

Thus:

$$\frac{1}{R^2} \frac{\partial^2 w_1}{\partial \varrho^2} = -\frac{1}{2} \Psi - \frac{1}{4} \frac{\partial \Psi}{\partial \varrho} (4\varrho - \varrho_2^2 \varrho^{-1} - 3\varrho \varrho_2^{-2}) - \frac{1}{4} \frac{\partial^2 \Psi}{\partial \varrho^2} (1 - \varrho_2^2 + \varrho^2 - \varrho^2 \varrho_2^{-2}),$$

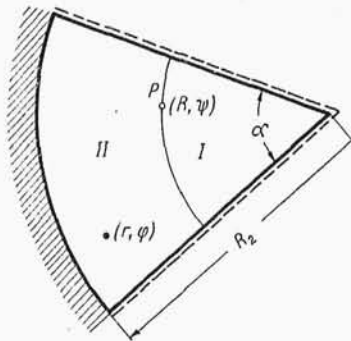


Fig. 3

$$\begin{aligned} \frac{1}{R^2} \left( \frac{1}{\varrho} \frac{\partial w_1}{\partial \varrho} + \frac{1}{\varrho^2} \frac{\partial^2 w_1}{\partial \varphi^2} \right) &= -\frac{1}{2} \Psi - \frac{1}{4} \frac{\partial \Psi}{\partial \varrho} (\varrho_2^2 \varrho^{-1} - \varrho \varrho_2^{-2}) + \\ &+ \frac{1}{4} \frac{\partial^2 \Psi}{\partial \varrho^2} (1 - \varrho_2^2 + \varrho^2 - \varrho^2 \varrho_2^{-2}), \\ (4) \quad \nabla^2 w_1 &= -\Psi - \frac{\partial \Psi}{\partial \varrho} (\varrho - \varrho \varrho_2^{-2}), \\ \frac{1}{R^2} \left( \frac{1}{\varrho} \frac{\partial^2 w_1}{\partial \varrho \partial \varphi} - \frac{1}{\varrho^2} \frac{\partial w_1}{\partial \varphi} \right) &= -\frac{1}{4} \frac{\partial \Psi}{\partial \varphi} (1 - \varrho_2^{-2}) - \frac{1}{4} \frac{\partial^2 \Psi}{\partial \varrho \partial \varphi} (1 - \varrho_2^{-2}) (\varrho - \varrho_2^2 \varrho^{-1}), \end{aligned}$$

where

$$\varrho_2 = \frac{R_2}{R},$$

but

$$(5) \quad \Psi(\varrho, \varphi) = -\frac{P}{4N\pi} \ln \frac{\cosh(k \ln \varrho \varrho_2^{-2}) - \cos k(\varphi - \psi)}{\cosh(k \ln \varrho \varrho_2^{-2}) - \cos k(\varphi + \psi)}.$$

b. For the Fig. 4 plate we introduce a new function

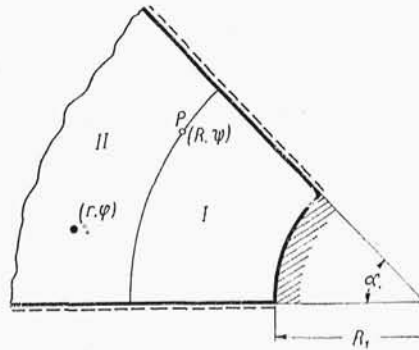


Fig. 4

$$(6) \quad \Theta(\varrho, \varphi) = -\frac{P}{4N\pi} \ln \frac{\cosh(k \ln \varrho \varrho_1^{-2}) - \cos k(\varphi - \psi)}{\cosh(k \ln \varrho \varrho_1^{-2}) - \cos k(\varphi + \psi)},$$

where

$$\varrho_1 = \frac{R_1}{R}.$$

In this case a solution was also obtained by using equation (3). The following relations exist between function  $\Theta(\varrho, \varphi)$  and the derivatives of function  $w_1$ :

$$\begin{aligned} \frac{1}{R^2} \frac{\partial^2 w_1}{\partial \varrho^2} &= -\frac{1}{2} \Theta - \frac{1}{4} \frac{\partial \Theta}{\partial \varrho} (4\varrho - \varrho_1^2 \varrho^{-1} - 3\varrho \varrho_1^{-2}) - \frac{1}{4} \frac{\partial^2 \Theta}{\partial \varrho^2} (1 - \varrho_1^2 + \varrho^2 - \varrho^2 \varrho_1^{-2}), \\ \frac{1}{R^2} \left( \frac{1}{\varrho} \frac{\partial w_1}{\partial \varrho} + \frac{1}{\varrho^2} \frac{\partial^2 w_1}{\partial \varphi^2} \right) &= -\frac{1}{2} \Theta - \frac{1}{4} \frac{\partial \Theta}{\partial \varrho} (\varrho_1^2 \varrho^{-1} - \varrho \varrho_1^{-2}) + \\ &+ \frac{1}{4} \frac{\partial^2 \Theta}{\partial \varrho^2} (1 - \varrho_1^2 + \varrho^2 - \varrho^2 \varrho_1^{-2}), \end{aligned}$$

$$(7) \quad \nabla^2 w_1 = -\Theta - \frac{\partial \Theta}{\partial \varrho} (\varrho - \varrho \varrho_1^{-2}),$$

$$\frac{1}{R^2} \left( \frac{1}{\varrho} \frac{\partial^2 w_1}{\partial \varrho \partial \varphi} - \frac{1}{\varrho^2} \frac{\partial w_1}{\partial \varphi} \right) = -\frac{1}{4} \frac{\partial \Theta}{\partial \varphi} (1 - \varrho_1^{-2}) - \frac{1}{4} \frac{\partial^2 \Theta}{\partial \varrho \partial \varphi} (1 - \varrho_1^{-2}) (\varrho - \varrho_1^2 \varrho^{-1}).$$

c. Finally, let us consider the case of a wedge-shaped plate (Fig. 5) charged with a force  $P$  and additionally supported on a curvilinear support

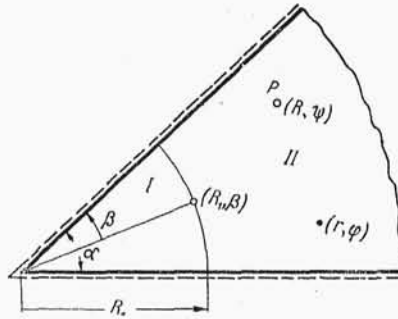


Fig. 5

$r = R_1$ . The solution was obtained in this case by using the method of integral equations [2] for plates with mixed boundary conditions. If  $R > R_1$ , i. e. if a force  $P$  is placed in region II, the plate deflection is expressed in the closed form by:

$$(8) \quad w(r, \varphi) = -\frac{PR_1^2}{32N\pi} \left( 1 - \frac{Tr^2}{R_1^2} \right) \left( \frac{R^2}{R_1^2} - 1 \right) \ln \frac{\cosh(k \ln \varrho) - \cos k(\varphi - \psi)}{\cosh(k \ln \varrho) - \cos k(\varphi + \psi)}.$$

The deflection in region II is expressed by different series for  $r > R$  and  $r < R$ , but their derivatives can be added up and presented in the closed form:

$$\frac{1}{R^2} \left( \frac{\partial^2 w}{\partial \varrho^2} \right) = \frac{1}{2} \Phi + \frac{1}{4} (\varrho - \varrho^{-1}) \frac{\partial \Phi}{\partial \varrho} - \frac{1}{4} \left[ (1 + \varrho_1^{-2}) \Theta + (2\varrho - \varrho_1^2 \varrho^{-1} - \varrho \varrho_1^{-2}) \frac{\partial \Theta}{\partial \varrho} \right] - \frac{1}{8} (1 + \varrho^2 - \varrho^2 \varrho_1^{-2} - \varrho_1^2) \frac{\partial^2 \Theta}{\partial \varrho^2}.$$

$$\frac{1}{R^2} \left( \frac{1}{\varrho} \frac{\partial w}{\partial \varrho} + \frac{1}{\varrho^2} \frac{\partial^2 w}{\partial \varrho^2} \right) = \frac{1}{2} \Phi - \frac{1}{4} (\varrho - \varrho^{-1}) \frac{\partial \Phi}{\partial \varrho} - \frac{1}{4} \left[ (1 + \varrho_1^{-2}) \Theta + (\varrho_1^2 \varrho^{-1} - \varrho \varrho_1^{-2}) \frac{\partial \Theta}{\partial \varrho} \right] + \frac{1}{8} (1 + \varrho^2 - \varrho^2 \varrho_1^{-2} - \varrho_1^2) \frac{\partial^2 \Theta}{\partial \varrho^2},$$

$$(9) \quad \nabla^2 w = \Phi - \frac{1}{2} \left[ (1 + \varrho_1^2) \Theta + (\varrho - \varrho \varrho_1^{-2}) \frac{\partial \Theta}{\partial \varrho} \right],$$

$$\frac{1}{R^2} \left( \frac{1}{\varrho} \frac{\partial^2 w}{\partial \varrho \partial \varphi} - \frac{1}{\varrho^2} \frac{\partial w}{\partial \varphi} \right) = \frac{1}{4} (1 - \varrho^{-2}) \frac{\partial \Phi}{\partial \varphi} - \frac{1}{8} \left[ (1 - \varrho^{-2} + \varrho_1^{-2} - \varrho_1^2 \varrho^{-2}) \frac{\partial \Theta}{\partial \varphi} + (\varrho + \varrho^{-1} - \varrho \varrho_1^{-2} - \varrho_1^2 \varrho^{-1}) \frac{\partial^2 \Theta}{\partial \varrho \partial \varphi} \right].$$

The function  $\Phi$  is expressed by formula (1) and  $\Theta$  by formula (6).

If in the given cases (a to c) we pass for  $\alpha$  to  $\rightarrow 0$  and for  $R_1$  to  $\rightarrow \infty$ , we shall obtain solutions for the semi-infinite strip or the well-known solution of S. Wojnowsky-Krieger [3] for a strip with additional rectilinear support.

A knowledge of influence surfaces in the plate systems we have been discussing makes possible to determine deflections and statical quantities for arbitrary plate charges.

#### REFERENCES

- [1] Nadai A., *Elastische Platten*, Berlin, 1925.
- [2] Nowacki W., *Płyty o mieszanych warunkach brzegowych*, *Archiwum Mechaniki Stosowanej*, **3** (1951), No. 3/4 (Plates With Mixed Edge Conditions).
- [3] Wojnowsky-Krieger S., *Beitrag zur Theorie der durchlaufenden Platten*, *Ing. Archiv*, 1938.

