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BUNCH LENGTHENING AND MICROWAVE INSTABILITY

(Part 2)

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This is a continuation of CERN/PS/BR 77-5 (Part 1)

CONTENTS

1. DERIVATION OF THE MODE-COUPLING MATRIX

The equations of motion for a single particle are

$$\frac{d\Delta E}{dt} = \frac{e\omega_0}{2\pi}\left[V_{rf}\sin\phi + \text{induced voltage}\right] \tag{14}$$

$$\frac{d\phi}{dt} = \eta h\omega_0 \frac{\Delta p}{p} = \frac{\eta h\omega_0}{\beta^2}\frac{\Delta E}{E} \tag{15}$$

where ΔE and ϕ are energy and phase (RF radians) deviations from the synchronous values, V_{rf} is the peak RF voltage per turn, and $\eta = 1/\gamma_T^2 - 1/\gamma^2$. The additional voltage due to the coupling impedance $Z(\omega)$ has a stationary part $V_0(\phi)$ induced by the stationary line density $\lambda_0(t)$ and an oscillating part $V_m(\phi,t)$ due to the perturbation $\lambda_m(\phi,t)$. In the following, we set $h = 1$ and $V_{rf}\sin\phi + V_0(\phi) = V_T\,\phi$ to simplify the derivation. Then (14) and (15) become

$$\ddot{\phi} + \omega_s^2\,\phi = -\,\omega_{so}^2\,\frac{V_m(\phi,t)}{V_{rf}} = -\,\omega_s^2\,\frac{V_m(\phi,t)}{V_T} \tag{16}$$

where ω_{so} is the synchrotron frequency (rad/sec) corresponding to the voltage V_{rf} and ω_s is the single-particle or incoherent frequency corresponding to the total voltage V_T.

For the stationary distribution, the particle orbits are circles in the normalized phase plane (Fig. 8), and the stationary distribution $\psi_0(r)$

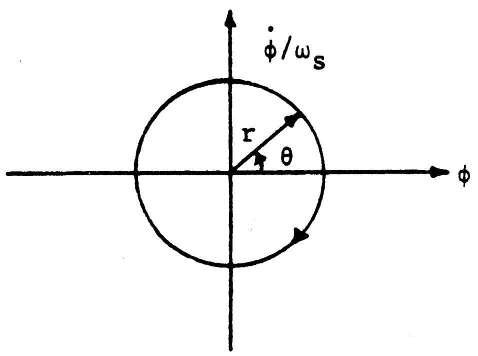

FIG. 8

depends only on the amplitude r and not on the synchrotron phase Θ. For small oscillations about the stationary distribution

$$\psi = \psi_0(r) + \psi_m(r,\Theta)e^{j\omega t} \tag{17}$$

where ψ_m satisfies the usual linearized Vlasov equation

$$j\omega\psi_m - \omega_s \frac{\partial\psi_m}{\partial\Theta} - \omega_s \frac{V_m(\phi)}{V_T}\frac{d\psi_0}{dr} \sin\Theta = 0. \tag{18}$$

Normalize ψ,

$$\int \psi d\phi d\dot{\phi} = \int \psi\omega_s r dr d\Theta = 1, \tag{19}$$

and introduce the line density

$$\lambda(\phi) = \int \psi d\dot{\phi} \ . \tag{20}$$

For N particles/bunch, $eN\lambda(\phi)$ = charge/radian, $eN\omega_0\lambda(\phi)$ = charge/sec, and

$$V_m(\phi) = - eN\omega_0 \sum_p Z(p)\tilde{\lambda}_m(p)e^{jp\phi} \tag{21}$$

where

$$\lambda(\phi) = \sum_p \tilde{\lambda}(p)e^{jp\phi}. \tag{22}$$

In the limit of zero intensity, $V_m = 0$, and the solutions of (18) are

$$\psi_m = R_m(r)e^{jm\Theta}$$

$$\omega = m\omega_s \tag{23}$$

where $R_m(r)$ is any function of r. For the general solution, write

$$\psi = \sum_m R_m(r) e^{jm\Theta} \tag{24}$$

and substitute into (18) to find the equations

$$j(\omega - m\omega_s) R_m(r)\ \delta_{mk} = \frac{eN\omega_0\omega_s}{V_T} \frac{d\psi_0}{dr} \sum_{p,k} j^m \frac{m}{pr} J_m(pr) \tilde{\lambda}_k(p) Z(p) \tag{25}$$

where δ_{mk} is the kronecker delta. The relation

$$\frac{1}{2\pi} \int_0^{2\pi} e^{jpr\cos\Theta - jm\Theta} \sin\Theta\ d\Theta = -j^m \frac{m}{pr} J_m(pr) \tag{26}$$

has been used where J_m is a Bessel function. Since

$$\tilde{\lambda}_k(p) = \frac{1}{2\pi} \int_0^{2\pi} e^{-jp\phi}\ \lambda_k(\phi) d\phi$$

$$= (-j)^k \int_0^\infty J_k(pr) R_k(r) \omega_s r dr, \tag{27}$$

equation (25) can be written as

$$j(\omega - m\omega_s) R_m(r) \delta_{mk} = \frac{m\omega_s}{V_T} eN\omega_0 \frac{1}{r} \frac{d\psi_0}{dr} j^m \sum_{p,k} (-j)^k \frac{Z(p)}{p} J_m(pr) \int_0^\infty J_k(pr') R_k(r') \omega_s r' dr'. \tag{28}$$

For low intensities, only the diagonal m=k term need be retained, and (28) reduces to an integral equation for the radial mode pattern $R_m(r)$ and eigenfrequency ω.

It is shown in the next section that the adjoint function $R_m^+(r)$ defined by

$$R_m(r) = \frac{1}{r}\frac{d\psi_0}{dr} R_m^+(r) \tag{29}$$

is orthogonal to $R_m(r)$. Multiply (28) by $R_m^+(r)$ and integrate over $\omega_s rdr$ to find the equations

$$j(\omega-m\omega_s)\int_0^\infty R_m^+ R_m \,\omega_s rdr\,\delta_{mk} = \frac{m\omega_s}{V_T} eN\omega_0 \sum_{p,k} \frac{Z(p)}{p} \tilde{\lambda}_m*(p)\tilde{\lambda}_k(p) . \tag{30}$$

The LHS of (30) can be expressed in terms of the line density rather than R(r). Consider the sum

$$\sum_{p=0}^{\infty} p\tilde{\lambda}_k^*(p)\tilde{\lambda}_k(p) = \sum_{p=0}^{\infty} p \iint_0^\infty J_k(pr)R_k(r)\omega_s rdr\; J_k(pr')R_k(r')\omega_s r'dr'$$

$$= \omega_s^2 \int_0^\infty R_k^2(r)rdr \tag{31}$$

since

$$\sum_{p=0}^{\infty} pJ_k(pr)J_k(pr') \simeq \int_0^\infty xJ_k(xr)J_k(xr')dx$$

$$= \frac{1}{r}\,\delta(r'-r). \tag{32}$$

For the parabolic distribution

$$\psi_0(r) = \frac{2}{\pi\omega_s\phi_0^2}(\phi_0^2-r^2), \tag{33}$$

$$\frac{1}{r}\frac{d\psi_0}{dr} = -\frac{4}{\pi\omega_s\phi_0^4} \tag{34}$$

and (31) becomes

$$\int_0^\infty R_k^+(r)R_k(r)\omega_s r dr = -\frac{1}{4}\pi\phi_0^4 \sum_{p=0}^{\infty} p\tilde{\lambda}_k^*(p)\tilde{\lambda}_k(p)$$

$$\simeq -\frac{1}{8}\pi\phi_0^4\, \bar{p} \sum_{p=-\infty}^{\infty} \tilde{\lambda}_k^*(p)\tilde{\lambda}_k(p), \tag{35}$$

where $\bar{p}$ is the central line in the mode spectrum $\tilde{\lambda}_k(p)$. For the sinusoidal modes of figures 2 and 3,

$$\bar{p}\, f_0 = \frac{k+1}{2\tau_L}, \tag{36}$$

so $\bar{p} = (k+1)/2B_0$. Thus (30) reduces to (3),

$$|\omega - m\omega_s - M_{mk}| = 0 \tag{37}$$

where the matrix M_{mk} is given by (5). The relations $\phi_0 = \pi B_0$ and $I_0 = eN\omega_0/2\pi$ have been used.

The main approximation in the derivation of (37) is the relation (36) for the central frequency $\bar{p}\, f_0$ of the mode spectrum. Somewhat smaller values have been measured experimentally[4] and are likely to occur for Gaussian bunches also. In any event, the correct value of $\bar{p}$ can be used in (37) if desired.

2. ADJOINT MODES

The diagonal m = k term of equation (28) has the form

$$(\omega - m\omega_s)R_m(r) = \frac{1}{r}\frac{d\psi_0}{dr} \int_0^\infty G_m(r,r')R_m(r')r'dr' \tag{38}$$

where

$$G_m(r,r') = -j\,\frac{m\omega_s^2}{V_T}\,eN\omega_0 \sum_p \frac{Z(p)}{p} J_m(pr)J_m(pr') = G_m(r',r) \qquad (39)$$

is a symmetric kernal. In general, the integral equation (38) has an infinite number of solutions for each azimuthal mode number m. These can be labelled by the index q which specifies the number of nodes in the radial mode pattern $R_{mq}(r)$ (see Fig. 9).

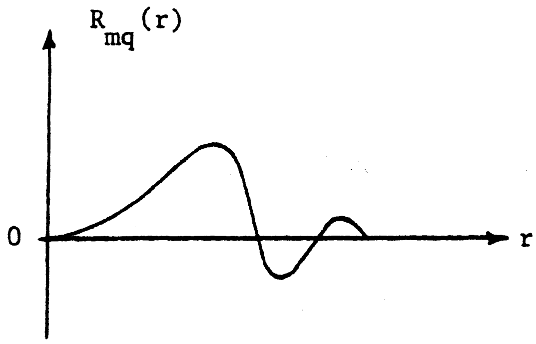

FIG. 9 : The radial mode pattern for q = 2.

For each mode, there is an adjoint mode, which satisfies the orthogonality relation

$$\int_0^\infty R^+_{mq}(r)\,R_{ml}(r)\,r\,dr = 0 \quad \text{unless } q = l. \qquad (40)$$

This can be shown as follows. By definition, the adjoint mode is the solution of the adjoint equation

$$\Delta\omega^+_q\,R^+_q(r) = \int_0^\infty G_m(r,r')\,\frac{d\psi_0}{dr'}\,R^+_q(r')\,dr' \qquad (41)$$

where the index m of R^+_{mq} is dropped, and $\Delta\omega^+_q \equiv \omega^+_q - m\omega_s$. Multiply (41) by $R_1(r)$, equation (40) by $R^+_q(r)$, integrate and subtract to find

$$(\Delta\omega^+_q - \Delta\omega_1) \int_0^\infty R^+_q(r)\, R_1(r) r dr = 0. \tag{42}$$

By comparing (38) and (41), it is clear that

$$R_q(r) = \frac{1}{r} \frac{d\psi_0}{dr} R^+_q(r) \tag{43}$$

$$\Delta\omega_q = \Delta\omega^+_q$$

and therefore (42) reduces to (40).

In part 1, only one radial mode (q=0) is kept for each azimuthal mode number m, and the second index on $R_{mq}(r)$ is dropped. The neglected higher-order radial modes are assumed to describe the single-particle incoherent motion. A different approach that includes some of the higher-order radial modes is given in reference 7.

3. RESONATOR

A parallel LCR circuit is assumed with impedance

$$Z(\omega) = -jR\sigma \frac{\omega}{\omega_r} \left[\frac{1}{\omega - \omega_r - j\sigma} - \frac{1}{\omega + \omega_r - j\sigma} \right] \tag{44}$$

where

$$\omega_r = \frac{1}{\sqrt{LC}} , \tag{45}$$

$$\sigma = \frac{1}{2RC} .$$

The form factor in equation 7 is

$$F_{mk} = \Delta_{mk} \frac{2}{\pi^2} \left| S_m \, S_k \right|$$

where

$$\Delta_{mk} = \begin{cases} 1 & m\text{-}k \text{ even} \\ -j(-1)^m & m\text{-}k \text{ odd,} \end{cases} \tag{46}$$

$$S_m = \frac{\sin\frac{\pi}{2}\left[y-(m+1)\right]}{y-(m+1)} + (-1)^m \frac{\sin\frac{\pi}{2}\left[y+(m+1)\right]}{y+(m+1)}, \tag{47}$$

and $y = \omega\tau_L/\pi = 2f\tau_L$. The second index on F_{mk} has been dropped in Part 1 for the m=k diagonal term. This is plotted in Fig. 5.

For the impedance (44), the matrix elements (5) become

$$M_{mk} = 0.27\ \varepsilon MB_0 \frac{m\omega_s}{m+1} \sum_{p=0}^{\infty} S_m(p)\ S_k(p) \frac{\sigma}{(\omega_p-\omega_r)^2+\sigma^2} \times$$

$$\times \begin{cases} (-1)^m\ \sigma & m\text{-}k \text{ odd} \\ (\omega_p-\omega_r) & m\text{-}k \text{ even} \end{cases} \tag{48}$$

where the summation is over the frequencies (1) or (2). The eigenvalues of the matrix $\left[m\omega_s+M_{mk}\right]$ were found by computer for different values of the intensity parameter ε and resonator bandwidth $\Delta f = 2\pi\sigma$. The threshold for instability is plotted in Fig. 7.

If only the three central diagonals are retained in the matrix, the results change by less than 5%, and the continued fraction

$$|A| = A_{11} - \cfrac{A_{12}\ A_{21}}{A_{22} - \cfrac{A_{23}\ A_{32}}{A_{33} - \cdots}} \tag{49}$$

can be used for evaluating the determinant.

For small bandwidths, coupled-bunch modes are unstable. The threshold is given by

$$|\Delta\omega_m| = \frac{1}{2}\frac{m}{m+1} S \tag{50}$$

where S is the synchrotron frequency spread between bunch center and bunch edge and $\Delta\omega_m$ is the coherent frequency shift

$$|\Delta\omega_m| = \frac{1}{3}\,\varepsilon\ MB_0 D\ \frac{m\omega_s}{m+1} \tag{51}$$

obtained from (7) and (8) with $F_{mm} = \frac{1}{2}$ and $D = \alpha/\sinh\alpha$ where $\alpha = 2\pi\Delta fT/M$ and T is the revolution period. The spread S is approximately

$$S = \frac{2}{3}\,\omega_s\,\frac{1+\Gamma^2}{1-\Gamma^2}(hB_0)^2 \tag{52}$$

where $\Gamma = \sin\phi_s$. Thus (50) becomes

$$\varepsilon\ MB_0 \leq \frac{1+\Gamma^2}{1-\Gamma^2}\ \frac{\sinh\alpha}{\alpha}(hB_0)^2\ , \tag{53}$$

which is shown in Fig. 7 for stationary buckets (Γ=0), with every third bucket filled (h=3M), and a bunch length $\frac{1}{10}$ of the bucket length (hB_0 = 0.1).

4. COASTING-BEAM STABILITY CRITERION

The usual Keil-Schnell criterion[3)] is

$$\left|\frac{Z}{n}\right| \leq \frac{0.7\pi}{2}\beta^2\ \frac{\gamma E_0|\eta|}{e\ \hat{I}}\left(\frac{\Delta p}{p}\right)^2_{FWHH} . \tag{54}$$

The bunch half-length ϕ_0 in RF radians and half-height Δp_0 are related by

$$\left(\frac{\Delta p}{p}\right)_0 = \frac{\omega_s}{h\omega_0|\eta|}\ \phi_0 \ . \tag{55}$$

For a parabolic distribution with

$$F(\Delta p) = \Delta p_0^2 - \Delta p^2 \ , \tag{56}$$

$$\left(\frac{\Delta p}{p}\right)^2_{FWHH} = 2\left(\frac{\Delta p}{p}\right)^2_0 \tag{57}$$

so

$$\left(\frac{\Delta p}{p}\right)_{FWHH} = \frac{eV_T|\cos\phi_s|}{\pi\gamma E_0\beta^2 h|\eta|}\ \phi_0^2 \tag{58}$$

and (54) becomes

$$\left|\frac{Z}{n}\right| \leq \frac{.7}{2}\ \frac{V_T|\cos\phi_s|}{h\hat{I}}\ \phi_0^2 \tag{59}$$

where ϕ_0 is related to the bunching factor B_0 by $\phi_0 = \pi h B_0$. The current is

$$\hat{I} = 1.5\ I_0/B_0 \qquad \text{peak current}$$

$$= hI_0 \qquad \text{average current}$$

and (59) reduces to (9) and (10).

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