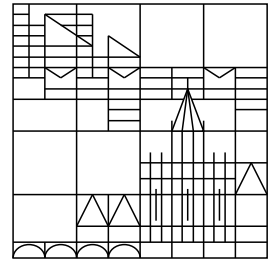Universität Konstanz

Bunches of Cones in the Divisor Class Group — A new Combinatorial Language for Toric Varieties

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BUNCHES OF CONES IN THE DIVISOR CLASS GROUP — A NEW COMBINATORIAL LANGUAGE FOR TORIC VARIETIES

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ABSTRACT. As an alternative to the description of a toric variety by a fan in the lattice of one parameter subgroups, we present a new language in terms of what we call bunches — these are certain collections of cones in the vector space of rational divisor classes. The correspondence between these bunches and fans is based on classical Gale duality. The new combinatorial language allows a much more natural description of geometric phenomena around divisors of toric varieties than the usual method by fans does. For example, the numerically effective cone and the ample cone of a toric variety can be read off immediately from its bunch. Moreover, the language of bunches appears to be useful for classification problems.

INTRODUCTION

The most important feature of a toric variety X is that it is completely described by a fan Δ in the lattice N of one parameter subgroups of the big torus $T_X \subset X$. Applying a linear Gale transformation to the set of primitive generators v_ϱ of the rays $\varrho \in \Delta^{(1)}$ of Δ, gives a new vector configuration in a rational vector space $K_\mathbb{Q}$. This opens an alternative combinatorial approach to the toric variety X: The vector space $K_\mathbb{Q}$ is isomorphic to the rational divisor class group of X, and one can shift combinatorial information between the spaces $N_\mathbb{Q}$ and $K_\mathbb{Q}$.

In toric geometry, this principle has been used to study the projective case, compare [19]: Roughly speaking, if we consider all fans Σ in N having their rays among $\Delta^{(1)}$, then the (quasi-)projective Σ correspond to so called Gelfand-Kapranov-Zelevinski cones in $K_\mathbb{Q}$. These cones subdivide the cone generated by the Gale transform of the vector configuration $\{v_\varrho; \ \varrho \in \Delta^{(1)}\}$, and the birational geometry of the associated toric varieties is reflected by the position of their Gelfand-Kapranov-Zelevinski cones.

If one leaves the (quasi-)projective setting, then there are generalizations of Gelfand-Kapranov-Zelevinski cones, compare e.g. [14]; but so far there seems to be no concept which is simple enough to serve for practical purposes in toric geometry. Our aim is to fill this gap and to propose a natural combinatorial language which also works in the non quasiprojective case. The combinatorial data are certain collections — which we call *bunches* — of overlapping cones in the vector space of rational divisor classes. As we shall see, this approach gives very natural descriptions of geometric phenomena connected with divisors.

In order to give a first impression of the language of bunches, let us present it here for in a special case, namely for toric varieties with free class group. Consider a sequence $(w_1, \ldots, w_n)$ of not necessarily pairwise distinct points in some lattice $K \cong \mathbb{Z}^k$ and the following polyhedral convex cones in the associated rational vector space $K_{\mathbb{Q}}$:

$$\vartheta := \mathrm{cone}(w_1, \ldots, w_n), \qquad \vartheta_i := \mathrm{cone}(w_j; \ j \neq i).$$

We impose the following regularity conditions: Every set $W_i := \{w_j; \ j \neq i\}$ should generate the lattice K, and any intersection $\vartheta_i^{\circ} \cap \vartheta_j^{\circ}$ of relative interiors should be nonempty. By an *admissible weight cone* in K we mean a cone $\tau \subset K_{\mathbb{Q}}$ generated by some of the w_i such that $\tau^{\circ} \cap \vartheta^{\circ} \neq \emptyset$ holds.

Definition. A *(free standard) bunch* in $\mathbb{Z}^k$ is a collection Θ of admissible weight cones in K such that

(i) every cone ϑ_i, where $i = 1, \ldots, n$, contains a cone of Θ,
(ii) an admissible weight cone σ belongs to Θ if and only if it satisfies

$$\emptyset \neq \sigma^{\circ} \cap \tau^{\circ} \neq \tau^{\circ} \quad \text{for all } \tau \in \Theta \text{ with } \tau \neq \sigma.$$

How to construct a toric variety from such a bunch Θ? The first step is to *unpack* the combinatorial information encoded in Θ. For this, let $E := \mathbb{Z}^n$, and let $Q\colon E \to K$ denote the linear surjection sending the canonical base vector e_i to w_i. Denote by $\gamma \subset E_{\mathbb{Q}}$ the positive orthant. We define the *covering collection* of Θ as

$$\mathrm{cov}(\Theta) := \{\gamma_0 \preceq \gamma; \ \gamma_0 \text{ minimal with } Q(\gamma_0) \supset \tau \text{ for some } \tau \in \Theta\}.$$

The next step is to *dualize* the information contained in $\mathrm{cov}(\Theta)$. This is done by a procedure close to a linear Gale transformation, for the classical setup see e.g. [11] and [17]: Consider the exact sequence arising from $Q\colon E \to K$, and the corresponding dualized exact sequence:

$$0 \longrightarrow M \longrightarrow E \stackrel{Q}{\longrightarrow} K \longrightarrow 0,$$

$$0 \longrightarrow L \longrightarrow F \stackrel{P}{\longrightarrow} N \longrightarrow 0.$$

Note that $P\colon F \to N$ is not the dual homomorphism of $Q\colon E \to K$. Let $\delta := \gamma^{\vee} \subset F_{\mathbb{Q}}$ be the dual cone. The crucial observation then is that we obtain a fan $\Delta(\Theta)$ in the lattice N having as its maximal cones the images

$$P(\gamma_0^{\perp} \cap \delta), \quad \gamma_0 \in \mathrm{cov}(\Theta).$$

Definition. The *toric variety associated to the bunch* Θ is $X_{\Theta} := X_{\Delta(\Theta)}$.

The toric variety X_{Θ} is *nondegenerate* in the sense that it has no torus factors. Moreover, X_{Θ} is *2-complete*, that means if $X_{\Theta} \subset X$ is an open toric embedding such that the complement $X \setminus X_{\Theta}$ is of codimension at least 2, then $X = X_{\Theta}$. Here comes a first collection of examples, compare Section 10:

The weighted
projective space
$\mathbb{P}_{1,2,3}$

A Hirzebruch
surface

Oda's example
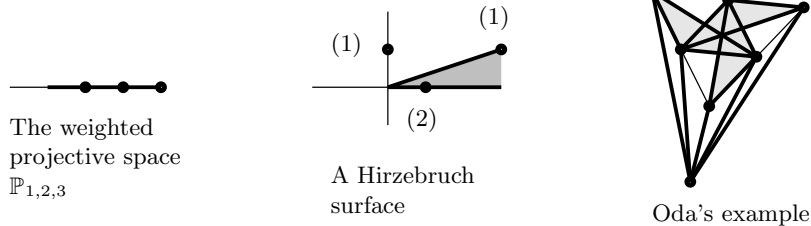
Introducing a suitable notion of a morphism, we can extend the assignment $\Theta \mapsto X_\Theta$ from bunches to 2-complete nondegenerate toric varieties to a contravariant functor. In fact, we even obtain a weak antiequivalence, see Theorem 6.3:

Theorem. *The functor $\Theta \mapsto X_\Theta$ induces a bijection on the level of isomorphism classes of bunches and nondegenerate 2-complete toric varieties. In particular, every complete toric variety arises from a bunch.*

In order to read off geometric properties of X_Θ directly from the bunch Θ, one has to translate the respective fan-theoretical formulations via the above Gale transformation into the language of bunches. This gives for example:

- X_Θ is $\mathbb{Q}$-factorial if and only if every cone $\tau \in \Theta$ is of full dimension, see Proposition 7.2.
- X_Θ is smooth if and only if every cone $Q(\gamma_0)$, where γ_0 in $\mathrm{cov}(\Theta)$, is regular and of full dimension, see Proposition 7.3.
- The orbits X_Θ have fixed points in their closures if and only if all cones $Q(\gamma_0)$, where $\gamma_0 \in \mathrm{cov}(\Theta)$ are simplicial, see Proposition 7.5.

As mentioned, the power of the language of bunches lies in the description of geometric phenomena around divisors, because $K_\mathbb{Q}$ turns out to be the rational divisor class group of X_Θ. For example, we obtain very simple descriptions for the classes of rational Cartier divisors, the cone $\mathrm{C}^{\mathrm{sa}}(X_\Theta)$ of semiample classes and the cone $\mathrm{C}^{\mathrm{a}}(X_\Theta)$ of ample classes, see Theorem 9.2:

Theorem. *For the toric variety X_Θ arising from a bunch Θ, there are canonical isomorphisms:*

$$\mathrm{Pic}_\mathbb{Q}(X_\Theta) \cong \bigcap_{\tau \in \Theta} \mathrm{lin}(\tau), \qquad \mathrm{C}^{\mathrm{sa}}(X_\Theta) \cong \bigcap_{\tau \in \Theta} \tau, \qquad \mathrm{C}^{\mathrm{a}}(X_\Theta) \cong \bigcap_{\tau \in \Theta} \tau^\circ.$$

Note that the last isomorphism gives a quasiprojectivity criterion in the spirit of [8] and [20], see Corollary 9.3. Moreover, we can derive from the above Theorem a simple Fano criterion, see Corollary 9.8. Finally, we get back Reid's Toric Cone Theorem, see [21], even with a new description of the Mori Cone, see Corollary 9.10:

Corollary. *Suppose that X_Θ is complete and simplicial. Then the cone of numerically effective curve classes in $H_2(X, \mathbb{Q})$ is given by*

$$\mathrm{NE}(X_\Theta) \cong \sum_{\tau \in \Theta} \tau^\vee.$$

In particular, this cone is convex and polyhedral. Moreover, X_Θ is projective if and only if $\mathrm{NE}(X_\Theta)$ is strictly convex.

Bunches can also be used for classification problems. For example, once the machinery is established, Kleinschmidt's classification [15] becomes very simple and can even be slightly improved, see Proposition 10.1; below we use instead of a sequence a set of weight vectors w and attach to each w a multplicity $\mu(w)$:

Theorem. *The smooth 2-complete toric varieties X with $\mathrm{Cl}(X) \cong \mathbb{Z}^2$ correspond to bunches $\Theta = \{\mathrm{cone}(w_1, w_n)\}$ given by*

- *weight vectors $w_1 := (1,0)$, and $w_i := (b_i, 1)$ with $0 = b_n < b_{n-1} < \ldots < b_2$,*
- *multiplicities $\mu_i := \mu(w_i)$ with $\mu_1 > 1$, $\mu_n > 0$ and $\mu_2 + \ldots + \mu_n > 1$.*
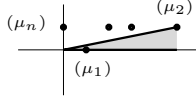
Moreover, the toric variety X defined by such a bunch Θ is always projective, and it is Fano if and only if we have

$$b_2(\mu_3 + \ldots + \mu_n) < \mu_1 + b_2\mu_3 + \ldots + b_{n-1}\mu_{n-1}.$$

In general the functor from bunches to toric varieties is neither injective nor surjective on morphisms, see Examples 8.8 and 8.9. But if we restrict to $\mathbb{Q}$-factorial toric varieties, then the language of bunches provides also a tool for the study of toric morphisms, see Theorem 8.3:

Theorem. *There is an equivalence from the category of simple bunches to the category of full $\mathbb{Q}$-factorial toric varieties.*

CONTENTS

1. FANS AND TORIC VARIETIES

In this section, we recall some basic facts on the correspondence between fans and toric varieties, and thereby fix our notation used later. For details, we refer to the books of Oda [18] and Fulton [10]. We begin with introducing the necessary terminology from convex geometry.

By a lattice we mean a free finitely generated $\mathbb{Z}$-module. The associated rational vector space of a lattice N is $N_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} N$. If $P \colon F \to N$ is a lattice homomorphism, then we denote the induced linear map $F_{\mathbb{Q}} \to N_{\mathbb{Q}}$ of rational vector spaces again by P.

By a cone in a lattice N we always mean a polyhedral (not necessarily strictly) convex cone in the associated rational vector space $N_{\mathbb{Q}}$. Let N be a lattice, and let $M := \mathrm{Hom}(N, \mathbb{Z})$ denote the dual lattice of N. The orthogonal space and the dual cone of a cone σ in N are

$$\sigma^{\perp} := \{u \in M_{\mathbb{Q}};\ u|_{\sigma} = 0\}, \qquad \sigma^{\vee} := \{u \in M_{\mathbb{Q}};\ u|_{\sigma} \geq 0\}.$$

If σ_0 is a face of σ, then we write $\sigma_0 \preceq \sigma$. The dimension of a cone σ is the dimension of the linear space $\mathrm{lin}(\sigma)$ generated by σ. The relative interior of a cone σ is denoted by σ°, and the set of its k-dimensional faces is denoted by $\sigma^{(k)}$.

The primitive generators of a cone σ in a lattice N are the primitive lattice vectors of the extremal rays of σ. A cone is called simplicial if its primitive generators are linearly independent, and it is called regular if its primitive generators can be complemented to a lattice basis of N.

Definition 1.1. (i) A *fan* in a lattice N is a finite collection Δ of strictly convex cones in N such that for each $\sigma \in \Delta$ also all $\sigma_0 \preceq \sigma$ belong to Δ and for any two $\sigma_i \in \Delta$ we have $\sigma_1 \cap \sigma_2 \preceq \sigma_i$.

(ii) A *map of fans* Δ_i in lattices N_i is a lattice homomorphism $F \colon N_1 \to N_2$ such that for every $\sigma_1 \in \Delta_1$ there is a $\sigma_2 \in \Delta_2$ containing $F(\sigma_1)$.

Recall that the compatibility condition $\sigma_1 \cap \sigma_2 \preceq \sigma_i$ in the above definition is equivalent to the existence of a *separating linear form* for the cones σ_1 and σ_1, i.e., a linear form u on N such that

$$u_{|\sigma_1} \geq 0, \qquad u_{|\sigma_2} \leq 0, \qquad u^\perp \cap \sigma_i = \sigma_1 \cap \sigma_2.$$

If we replace in Definition 1.1 "strictly convex" with "convex", we obtain the category of *quasifans*. For a fan Δ in N, we denote by $|\Delta|$ its support, that is the union of all its cones. Moreover, $\Delta^{\max}$ is the set of maximal cones of Δ, and $\Delta^{(k)}$ is the set of all k-dimensional cones of Δ.

In the sequel, we shall often make use of a well known universal lifting construction, which makes the set of primitive generators of the rays of a given fan into a lattice basis, compare for example [6]:

Construction 1.2. Let Δ be a fan in a lattice N, and let $\mathrm{R} := \Delta^{(1)}$. Let $C \colon \mathbb{Z}^{\mathrm{R}} \to N$ be the map sending the canonical base vector e_ϱ to the primitive generator $v_\varrho \in \varrho$. For $\sigma \in \Delta^{\max}$, set

$$\widetilde{\sigma} := \mathrm{cone}(e_\varrho; \; \varrho \in \sigma^{(1)}).$$

Then the cones $\widetilde{\sigma}$, where $\sigma \in \Delta^{\max}$, are the maximal cones of a fan $\widetilde{\Delta}$ consisting of faces of the positive orthant in $\mathbb{Q}^{\mathrm{R}}$. Moreover, $C \colon \mathbb{Z}^{\mathrm{R}} \to N$ is a map of the fans $\widetilde{\Delta}$ and Δ.

Now we turn to toric varieties. Throughout the entire paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero, and the word point refers to a closed point.

Definition 1.3. (i) A *toric variety* is a normal variety X containing an algebraic torus T_X as an open subset such that the group structure of T_X extends to a regular action on X.

(ii) A *toric morphism* is a regular map $X \to Y$ of toric varieties that restricts to a group homomorphism $T_X \to T_Y$.

The correspondence between fans and toric varieties is obtained as follows: Let Δ be a fan in a lattice N, and let $M := \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice of N. For every cone $\sigma \in \Delta$ one defines an affine toric variety:

$$X_\sigma := \mathrm{Spec}(\sigma^\vee \cap M).$$

For any two such X_{σ_i}, one has canonical open embeddings of $X_{\sigma_1 \cap \sigma_2}$ into X_{σ_i}. Patching together all X_σ along these open embeddings gives a toric variety X_Δ. The assignment $\Delta \mapsto X_\Delta$ is functorial; it is even a (covariant) equivalence of categories.

In the sequel, we shall frequently restrict our investigations to toric varieties that behave reasonably. For that purpose, we consider the following geometric properties:

Definition 1.4.   (i) A toric variety X is called *nondegenerate* if it admits no
   toric decomposition $X \cong X' \times \mathbb{K}^*$.
  (ii) We call a toric variety X *2-complete* if it does not admit a toric open embeding
   $X \subset X'$ with $X' \setminus X$ nonempty of codimension at least two.
 (iii) We call a toric variety X *full* if, it is 2-complete and every T_X-orbit has a
   fixed point in its closure.

   The notion of 2-completeness already occurs in [2]; it generalizes completeness in
the sense that a toric variety is complete if and only if it is "1-complete". In terms
of fans, the properties introduced in Definition 1.4 are characterized as follows:

Remark 1.5. Let X be the toric variety arising from a fan Δ in a lattice N.
  (i) X is nondegenerate if and only if the support $|\Delta|$ generates $N_{\mathbb{Q}}$ as a vector
   space.
 (ii) X is 2-complete if and only if the fan Δ cannot be enlarged without adding
   new rays.
 (iii) X is full if and only if Δ is as in (ii) and every maximal cone of Δ is of full
   dimension.

2. THE CATEGORY OF BUNCHES

   In this section, we introduce the language of bunches. Intuitively, one should
think of a bunch as a collection of pairwise overlapping lattice cones, which satisfies
certain irredundancy and maximality properties.
   The precise definition of the category of bunches is performed in three steps.
The first one is to introduce the category of projected cones:

Definition 2.1.   (i) A *projected cone* is a pair $(E \xrightarrow{Q} K, \gamma)$, where $Q \colon E \to K$
   is an epimorphism of lattices and $\gamma \subset E_{\mathbb{Q}}$ is a strictly convex simplicial cone
   of full dimension.
  (ii) A *morphism of projected cones* $(E_i \xrightarrow{Q_i} K_i, \gamma_i)$ is a homomorphism $\Phi \colon E_1 \to E_2$ such that $\Phi(\gamma_1) \subset \gamma_2$ holds and there is a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\;\Phi\;} & E_2 \\ {\scriptstyle Q_1}\big\downarrow & & \big\downarrow{\scriptstyle Q_2} \\ K_1 & \xrightarrow[\;\overline{\Phi}\;]{} & K_2 \end{array}$$

   In the second step, we give the definition of bunches. Such a bunch will live in
a projected cone $(E \xrightarrow{Q} K, \gamma)$. By a *projected face* in K we mean the image $Q(\gamma_0)$
of a face $\gamma_0 \preceq \gamma$.

Definition 2.2. A *bunch* in $(E \xrightarrow{Q} K, \gamma)$ is a collection Θ of projected faces in K
with the following property: A projected face $\tau_0 \subset K_{\mathbb{Q}}$ belongs to Θ if and only if

$$(2.2.1) \qquad \emptyset \neq \tau_0^{\circ} \cap \tau^{\circ} \neq \tau^{\circ} \qquad \text{holds for all } \tau \in \Theta \text{ with } \tau \neq \tau_0.$$

   Note that condition 2.2.1 yields irredundancy in the sense that a bunch Θ con-
tains no pair τ_0, τ with $\tau_0^{\circ} \subsetneq \tau^{\circ}$. On the other hand, 2.2.1 ensures maximality in
the following sense: If a projected face $\tau_0 = Q(\gamma_0)$ overlaps every cone of Θ, then
at least some $Q(\gamma_1)$ with $\gamma_1 \preceq \gamma_0$ belongs to Θ.

Example 2.3. Let $E := \mathbb{Z}^n$, and let $K = \mathbb{Z}$. Moreover, fix a sequence $w_1, \ldots, w_n$ of positive integers having greatest common divisor one. This gives an epimorphism

$$Q \colon E \to K, \qquad e_i \mapsto w_i,$$

where e_i denotes the i-th canonical base vector. Setting $\gamma := \mathrm{cone}(e_1, \ldots, e_n)$, we obtain a projected cone $(E \xrightarrow{Q} K, \gamma)$, and $\Theta := \{Q(\gamma)\}$ is a bunch.

Finally, as the third step, we have to fix the notion of a morphism of bunches. For this, we first have to "unpack" the combinatorial information contained in a bunch. This is done by constructing a further collection of cones:

Definition 2.4. Let Θ be a bunch in a projected cone $(E \xrightarrow{Q} K, \gamma)$. The *covering collection* of Θ is

$$\mathrm{cov}(\Theta) := \{\gamma_0 \preceq \gamma;\ \gamma_0 \text{ minimal with } Q(\gamma_0) \supset \tau \text{ for some } \tau \in \Theta\}.$$

As Example 2.3 shows, $\mathrm{cov}(\Theta)$ will in general comprise much more cones than Θ itself. We can reconstruct the bunch from its covering collection:

$$\Theta = \{\tau;\ \tau \text{ minimal with } \tau = Q(\gamma_0) \text{ for some } \gamma_0 \in \Theta\}.$$

In general, for an element $\gamma_0 \in \mathrm{cov}(\Theta)$, the image $Q(\gamma_0)$ need not be an element of Θ. For later purposes, the following observation will be crucial:

Lemma 2.5 (Overlapping Property)**.** *Let Θ be a bunch in $(E \xrightarrow{Q} K, \gamma)$. For any two $\gamma_1, \gamma_2 \in \mathrm{cov}(\Theta)$, we have $Q(\gamma_1)^\circ \cap Q(\gamma_2)^\circ \neq \emptyset$.*

Proof. Let $\sigma_i := Q(\gamma_i)$. By the definition of $\mathrm{cov}(\Theta)$, there exist cones $\tau_1, \tau_2 \in \Theta$ with $\tau_i \subset \sigma_i$. Now assume that the relative interiors of the cones σ_i are disjoint. Then there is a proper face $\sigma_0 \prec \sigma_1$ such that $\sigma_1 \cap \sigma_2$ is contained in σ_0.

Clearly, $\tau_1 \cap \tau_2$ is contained in σ_0. Moreover, by the condition 2.2.1, the intersection $\tau_1^\circ \cap \tau_2^\circ$ is not empty. In particular, τ_1° meets σ_0. Since σ_0 is a face of σ_1, we conclude $\tau_1 \subset \sigma_0$. Thus $\gamma_0 := Q^{-1}(\sigma_0) \cap \gamma_1$ is a proper face of γ_1 such that $Q(\gamma_0) = \sigma_0$ contains an element of Θ. This contradicts minimality of γ_1. $\qquad\square$

We come back to the definition of a morphism of bunches. It is formulated in terms of the respective covering collections:

Definition 2.6. Let Θ_i be bunches in projected cones $(E_i \xrightarrow{Q_i} K_i, \gamma_i)$. A *morphism* from Θ_1 to Θ_2 is a morphism $\Phi \colon E_1 \to E_2$ of the projected cones such that for every $\alpha_2 \in \mathrm{cov}(\Theta_2)$ there is an $\alpha_1 \in \mathrm{cov}(\Theta_1)$ with $\Phi(\alpha_1) \subset \alpha_2$.

This concludes the definition of the category of bunches. The notion of an isomorphism is characterized as follows:

Proposition 2.7. *A morphism Φ of bunches Θ_1 and Θ_2 is an isomorphism if and only if Φ is an isomorphism of the ambient projected cones and the induced map $\overline{\Phi}$ defines a bijection $\Theta_1 \to \Theta_2$.*

Proof. Let the bunch Θ_i live in the projected cone $(E_i \xrightarrow{Q_i} K_i, \gamma_i)$. Suppose first that $\Phi \colon E_1 \to E_2$ is an isomorphism of the bunches. Then there is a morphism of bunches $\Psi \colon E_2 \to E_1$ from Θ_2 to Θ_1 such that Φ and Ψ are inverse to each other as lattice homomorphisms. Note that Φ and Ψ are as well inverse to each other as morphisms of projected cones.

In order to see that $\overline{\Phi} \colon K_1 \to K_2$ defines a bijection $\Theta_1 \to \Theta_2$, it suffices to show that Φ defines a bijection $\mathrm{cov}(\Theta_1) \to \mathrm{cov}(\Theta_2)$. By bijectivity of Φ and Ψ, we only

have to show that for every $\alpha_1 \in \mathrm{cov}(\Theta_1)$ the image $\Phi(\alpha_1)$ belongs to $\mathrm{cov}(\Theta_2)$. This is done as follows:

Given $\alpha_1 \in \mathrm{cov}(\Theta_1)$, we apply Definition 2.6 to Ψ, and obtain an $\alpha_2 \in \mathrm{cov}(\Theta_2)$ with $\Psi(\alpha_2) \subset \alpha_1$. Applying Φ gives $\alpha_2 \subset \Phi(\alpha_1)$. Again by Definition 2.6, we find an $\widetilde{\alpha}_1 \in \mathrm{cov}(\Theta_1)$ with $\Phi(\widetilde{\alpha}_1) \subset \alpha_2$. Thus we have $\Phi(\widetilde{\alpha}_1) \subset \Phi(\alpha_1)$. By the definition of a bunch, we obtain $\widetilde{\alpha}_1 = \alpha_1$. Consequently, $\Phi(\widetilde{\alpha}_1) = \alpha_2$ belongs to $\mathrm{cov}(\Theta_2)$.

Now suppose that Φ is an isomorphism of projected cones and the $\overline{\Phi}$ defines a bijection $\Theta_1 \to \Theta_2$. Let $\Psi \colon E_2 \to E_1$ denote the inverse of Φ as a morphism of projected cones. The only thing we have to show is that Ψ is a morphism from Θ_2 to Θ_1. This is done below:

Let $\alpha_1 \in \mathrm{cov}(\Theta_1)$. Then $\alpha_2 := \Phi(\alpha_1)$ is a face of γ_2. We check $\alpha_2 \in \mathrm{cov}(\Theta_2)$. Choose $\tau_1 \in \Theta_1$ with $\tau_1 \subset Q_1(\alpha_1)$. By assumption, $\tau_2 := \overline{\Phi}(\alpha_1)$ belongs to Θ_2. Moreover, we have $\tau_2 \subset Q_2(\alpha_2)$. Since Φ is an isomorphism of projected cones and $\alpha_1 \in \mathrm{cov}(\Theta_1)$ holds, α_2 is even a minimal face of γ_2 with $\tau_2 \subset Q_2(\alpha_2)$. But this means $\alpha_2 \in \mathrm{cov}(\Theta_2)$. $\qquad\square$

The reminder of this section is devoted to the visualization of bunches. The idea is that one should be able to recover many basic properties of a bunch Θ without knowing the ambient projected cone $(E \xrightarrow{Q} K, \gamma)$. This will work for the following important class of bunches:

Definition 2.8. By a *free* bunch we mean a bunch Θ in a projected cone $(E \xrightarrow{Q} K, \gamma)$, where γ is a regular cone in E.

The following construction shows that every free bunch arises from a certain collections of data in some lattice K:

Construction 2.9. Let K be a lattice, and let $(w_1, \dots, w_n)$ be a sequence in K. We speak of the *weight vectors* w_i, and call any cone generated by some of the w_i a *weight cone*. Let Θ be a collection of weight cones in K satisfying Condition 2.2.1, i.e., a weight cone τ_0 belongs to Θ if and only if

$$(2.9.1) \qquad \emptyset \neq \tau_0^{\circ} \cap \tau^{\circ} \neq \tau^{\circ} \qquad \text{holds for all } \tau \in \Theta \text{ with } \tau \neq \tau_0.$$

Then there is an associated projected cone $(E \xrightarrow{Q} K, \gamma)$ with the lattice $E := \mathbb{Z}^n$, the cone $\gamma := \mathrm{cone}(e_1, \dots, e_n)$ spanned by the canonical base vectors, and the map $Q \colon E \to K$ sending e_i to w_i. By construction, the collection Θ is a bunch in the projected cone $(E \xrightarrow{Q} K, \gamma)$.

This construction allows us to visualize bunches: We draw the set $\{w_1, \dots, w_n\}$, attach to each w_i its multiplicity μ_i, and indicate the cones of the bunch. For example, in the setting of 2.3, the bunch Θ defined by the sequence $1, 3, 5, 5$ arises from a picture like this:
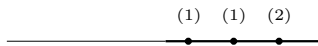
As one might expect, this picture describes the threedimensional weighted projective space $\mathbb{P}_{1,3,5,5}$. Moreover, the smooth complete toric varieties with Picard group $\mathbb{Z}^2$ will arise from sequences in $\mathbb{Z}^2$ and a collection $\Theta = \{\tau\}$ according to the following picture:
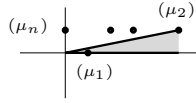
In order to compare two bunches arising from Construction 2.9, there is no need to determine the covering collection. Namely, using Proposition 2.7, we obtain:

Remark 2.10. Two sets of data $(K; w_1, \ldots, w_n; \Theta)$ and $(K'; w_1', \ldots, w_n'; \Theta')$ as in 2.9 have isomorphic associated free bunches if and only if there is a lattice isomorphism $\overline{\Phi} \colon K \to K'$ such that

(i) $(w_1', \ldots, w_n')$ and $(\overline{\Phi}(w_1), \ldots, \overline{\Phi}(w_n))$ differ only by enumeration,
(ii) the collections Θ' and $\{\overline{\Phi}(\tau); \ \tau \in \Theta\}$ coincide.

3. The basic duality lemmas

In this section, we provide basic duality statements for translating from the language of bunches into the language of fans. First we need a concept of a dual of a given projected cone $(E \xrightarrow{Q} K, \gamma)$. For this, note that $(E \xrightarrow{Q} K, \gamma)$ determines two exact sequences of lattice homomorphisms

$$0 \longrightarrow M \longrightarrow E \xrightarrow{Q} K \longrightarrow 0,$$

$$0 \longrightarrow L \longrightarrow F \xrightarrow{P} N \longrightarrow 0,$$

where M is the kernel of $Q \colon E \to K$, and the second sequence arises from the first one by applying $\mathrm{Hom}(?, \mathbb{Z})$; note that P is not the dual homomorphism of Q. Let $\delta := \gamma^\vee$ denote the dual cone. Then δ is again strictly convex, simplicial and of full dimension.

Definition 3.1. We call $(F \xrightarrow{P} N, \delta)$ the *dual projected cone* of $(E \xrightarrow{Q} K, \gamma)$.

In the sequel, fix a projected cone $(E \xrightarrow{Q} K, \gamma)$, and denote the associated dual projected cone by $(F \xrightarrow{P} N, \delta)$. Recall that we have the face correspondence, see for example [18, Appendix A]:

Remark 3.2. The sets of faces of the cones $\gamma \subset E_\mathbb{Q}$ and $\delta \subset F_\mathbb{Q}$ are in order reversing correspondence via

$$\mathrm{faces}(\gamma) \to \mathrm{faces}(\delta), \qquad \gamma_0 \mapsto \gamma_0^* := \gamma_0^\perp \cap \delta.$$

Our task is to understand the relations between the projected faces $Q(\gamma_0)$, where $\gamma_0 \preceq \gamma$, and the images $P(\gamma_0^*)$ of the corresponding faces. The following observation is central:

Lemma 3.3 (Invariant Separation Lemma). *Let* $\gamma_1, \gamma_2 \preceq \gamma$, *let* $\delta_i := \gamma_i^*$, *and let* $L := \ker(P)$. *Then the following statements are equivalent:*

(i) *There is an L-invariant separating linear form for δ_1 and δ_2.*
(ii) *For the relative interiors $Q(\gamma_i)^\circ$ we have $Q(\gamma_1)^\circ \cap Q(\gamma_2)^\circ \neq \emptyset$.*

Proof. Let $M := \ker(Q)$. Recall that there is an L-invariant separating linear form for δ_1 and δ_2 if and only if

$$(3.3.1) \qquad M \cap \left(\delta_1^\vee \cap (\delta_1 \cap \delta_2)^\perp\right)^\circ \cap - \left(\delta_2^\vee \cap (\delta_1 \cap \delta_2)^\perp\right)^\circ \quad \neq \quad \emptyset.$$

Note that $\delta_i^\vee$ equals $\mathrm{lin}(\gamma_i) + \gamma$. Moreover, $(\delta_1 \cap \delta_2)^\perp$ equals $\mathrm{lin}(\gamma_1) + \mathrm{lin}(\gamma_2)$, because the cone δ is simplicial. Hence we obtain

$$\left(\delta_1^\vee \cap (\delta_1 \cap \delta_2)^\perp\right)^\circ \;\; = \;\; ((\mathrm{lin}(\gamma_1) + \gamma) \cap (\mathrm{lin}(\gamma_1) + \mathrm{lin}(\gamma_2)))^\circ$$
$$= \;\; \mathrm{lin}(\gamma_1) + \gamma_2^\circ.$$

For the second equality we used simpliciality of γ. The analogous simplification works for $\delta_2^\vee \cap (\delta_1 \cap \delta_2)^\perp$. Thus 3.3.1 holds if and only if

$$M \cap (\mathrm{lin}(\gamma_1) + \gamma_2^\circ) \cap (\mathrm{lin}(\gamma_2) - \gamma_1^\circ) \;\; \neq \;\; \emptyset.$$

But the left hand side simplifies to $M \cap (\gamma_2 - \gamma_1)^\circ$. Consequently 3.3.1 is equivalent to condition (ii). $\qquad\square$

Let us mention here that simpliciality of the cones γ and δ is essential for the Invariant Separation Lemma:

Example 3.4. Consider the projected cone $(E \xrightarrow{Q} K, \gamma)$, where the lattices are $E := \mathbb{Z}^3$ and $K := \mathbb{Z}$, the map Q is the projection onto the third coordinate, and the cone γ is given in terms of canonical base vectors by

$$\gamma = \mathrm{cone}(e_1 + e_3, e_2 + 2e_3, e_1 - 2e_3, e_2 - e_3).$$

Denote the dual projected cone by $(F \xrightarrow{P} N, \delta)$. Then $L := \ker(P)$ is the sublattice generated by the dual base vector e_3^*, and the cone δ is given by

$$\delta = \mathrm{cone}(e_1^*, e_2^*, e_1^* + 2e_2^* - e_3^*, 2e_1^* + e_2^* + e_3^*).$$

The faces $\gamma_1 := \mathrm{cone}(e_2 - e_3)$ and $\gamma_2 := \mathrm{cone}(e_1 + e_3)$ don not satisfy Condition 3.3 (ii). Nevertheless, the corresponding faces

$$\gamma_1^* = \mathrm{cone}(e_1^*, 2e_1^* + e_2^* + e_3^*), \qquad \gamma_2^* = \mathrm{cone}(e_2^*, e_1^* + 2e_2^* - e_3^*),$$

admit L-invariant separating linear forms. For example we can take the linear form $e_1 - e_2 \in E$.

Next we compare injectivity of Q with surjectivity of P along corresponding faces (of course, the roles of Q, δ_0 etc. and P, γ_0, etc. can be interchanged in the statement):

Lemma 3.5. *For a face $\gamma_0 \preceq \gamma$ and $\delta_0 := \gamma_0^*$, the following statements are equivalent:*

  (i) *P maps $\mathrm{lin}(\delta_0)$ onto $N_\mathbb{Q}$,*
  (ii) *Q is injective on $\mathrm{lin}(\gamma_0)$.*

Proof. Let $M := \ker(Q)$ and $L := \ker(P)$. Using the fact that $\mathrm{lin}(\delta_0)$ and $\mathrm{lin}(\gamma_0)$ are the orthogonal spaces of each other, we obtain the assertion by dualizing:

$$\mathrm{lin}(\delta_0) + L_\mathbb{Q} = F_\mathbb{Q} \quad \Longleftrightarrow \quad \mathrm{lin}(\gamma_0) \cap M_\mathbb{Q} = \{0\}. \qquad \square$$

If we take the lattice structure into consideration, then the situation becomes a slightly more involved. The essential observation is:

Lemma 3.6. *For a face $\gamma_0 \preceq \gamma$ and $\delta_0 := \gamma_0^*$, the following statements are equivalent:*

  (i) *P maps $\mathrm{lin}(\delta_0) \cap F$ onto N,*
  (ii) *Q maps $\mathrm{lin}(\gamma_0) \cap E$ isomorphically onto a primitive sublattice of K.*

Proof. Set $L := \ker(P)$. Assume that (i) holds. Then the snake lemma provides an exact sequence

$$0 \longrightarrow L \cap \mathrm{lin}(\delta_0) \longrightarrow L \longrightarrow F/(\mathrm{lin}(\delta_0) \cap F) \longrightarrow 0.$$

The dual lattice of $F/(\mathrm{lin}(\delta_0) \cap F)$ is canonically isomorphic to $E \cap \mathrm{lin}(\gamma_0)$. Hence, applying $\mathrm{Hom}(?, \mathbb{Z})$ gives an exact sequence

$$0 \longrightarrow E \cap \mathrm{lin}(\gamma_0) \overset{Q}{\longrightarrow} K \longrightarrow \mathrm{Hom}(L \cap \mathrm{lin}(\delta_0), \mathbb{Z}) \longrightarrow 0.$$

This implies condition (ii). The reverse direction can be settled by similar arguments. $\square$

Finally, we consider morphisms $\Phi \colon E_1 \to E_2$ of projected cones $(E_i \overset{Q_i}{\longrightarrow} K_i, \gamma_i)$. These data define a commutative diagram of lattices with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & E_1 & \overset{Q_1}{\longrightarrow} & K_1 & \longrightarrow & 0 \\
& & \downarrow & & \Phi \downarrow & & \downarrow \bar{\Phi} & & \\
0 & \longrightarrow & M_2 & \longrightarrow & E_2 & \underset{Q_2}{\longrightarrow} & K_2 & \longrightarrow & 0
\end{array}
$$

Applying $\mathrm{Hom}(?, \mathbb{Z})$ to this diagram, we obtain the following commutative diagram, again with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L_1 & \longrightarrow & F_1 & \overset{P_1}{\longrightarrow} & N_1 & \longrightarrow & 0 \\
& & \uparrow & & \Psi \uparrow & & \uparrow \bar{\Psi} & & \\
0 & \longrightarrow & L_2 & \longrightarrow & F_2 & \underset{P_2}{\longrightarrow} & N_2 & \longrightarrow & 0
\end{array}
$$

Remark 3.7. The dual map $\Psi \colon F_2 \to F_1$ is a morphism of the dual projected cones $(F_i \overset{P_i}{\longrightarrow} N_i, \delta_i)$ of $(E_i \overset{Q_i}{\longrightarrow} K_i, \gamma_i)$.

Now, consider faces $\alpha_i \preceq \gamma_i$, and let $\beta_i := \alpha_i^*$ denote the corresponding faces of the cones δ_i.

Lemma 3.8. *We have* $\Phi(\alpha_1) \subset \alpha_2$ *if and only if* $\Psi(\beta_2) \subset \beta_1$ *holds.*

Proof. We only have to verify one implication. The other then is a simple consequence of $\alpha_i = \beta_i^*$. So, suppose $\Phi(\alpha_1) \subset \alpha_2$. Then we obtain

$$\Psi(\beta_2) = \Psi(\alpha_2^{\perp} \cap \delta_2) \subset \Psi(\alpha_2^{\perp}) \cap \Psi(\delta_2) \subset \alpha_1^{\perp} \cap \delta_1 = \beta_1. \qquad \square$$

4. Bunches and fans

In this section, we compare bunches with fans. We shall show that the category of bunches is equivalent to the category of "maximal projectable fans", see Theorem 4.6. The latter category is defined as follows:

Definition 4.1. (i) Let $(F \overset{P}{\longrightarrow} N, \delta)$ be a projected cone, and let $L := \ker(P)$.
(a) A *projectable fan* in $(F \overset{P}{\longrightarrow} N, \delta)$ is a fan Σ consisting of faces of δ such that any two maximal cones of Σ can be separated by an L-invariant linear form.

(b) We call a projectable fan Σ in $(F \xrightarrow{P} N, \delta)$ *maximal* if any $\delta_0 \prec \delta$, which can be separated by L-invariant linear forms from the maximal cones of Σ, belongs to Σ.

(ii) A *morphism* of projectable fans Σ_i (maximal or not) in projected cones $(F_i \xrightarrow{P_i} N_i, \delta_i)$ is a morphism $\Psi \colon F_1 \to F_2$ of projected cones which is in addition a map of the fans Σ_i.

Note that in the terminology of [14], a projectable fan is nothing but a "locally coherent costring". We shall demonstrate later by means of an example the importance of the maximality condition (b), see 4.7.

We define now a functor $\mathfrak{F}$ from bunches to maximal projectable fans. Let Θ be a bunch in the projected cone $(E \xrightarrow{Q} K, \gamma)$. Consider the associated dual projected cone $(F \xrightarrow{P} N, \delta)$ and the following subfan of the fan of faces of δ:

$$\Sigma := \{\sigma \preceq \delta; \; \sigma \preceq \gamma_0^* \text{ for some } \gamma_0 \in \text{cov}(\Theta)\}.$$

Lemma 4.2. Σ *is a maximal projectable fan in* $(F \xrightarrow{P} N, \delta)$.

Proof. Let $L := \ker(F)$. By the Overlapping Property 2.5 of the Covering Collection $\text{cov}(\Theta)$ and the Invariant Separation Lemma 3.3, any two maximal cones of Σ can be separated by L-invariant linear forms. So we only have to verify the maximality condition 4.1 (i) (b) for Σ.

Suppose that the face $\sigma \preceq \delta$ can be separated by L-invariant linear forms from the maximal cones of Σ but does not belong to Σ. The projected face $\tau_0 := Q(\sigma^*)$ does not belong to Θ, because otherwise any minimal face $\gamma_0 \preceq \sigma^*$ projecting onto τ_0 would belong to $\text{cov}(\Theta)$, which contradicts the choice of σ.

The Invariant Separation Lemma 3.3 yields $\tau_0^\circ \cap \tau^\circ \neq \emptyset$ for every $\tau \in \Theta$. Since τ_0 is not an element of Θ, it has to contain some element of Θ. But then some face of σ^* belongs to the Covering Collection $\text{cov}(\Theta)$. Again this contradicts the choice of the face $\sigma \preceq \delta$. $\qquad\square$

The assignment $\Theta \mapsto \Sigma$ extends canonically to morphisms. Namely, let Θ_i be bunches in projected cones $(E_i \xrightarrow{Q_i} K_i, \gamma_i)$. Let Σ_i denote the associated maximal projectable fans in the respective dual projected cones $(F_i \xrightarrow{P_i} N_i, \delta_i)$.

Lemma 3.8 tells us that for every morphism $\Phi \colon E_1 \to E_2$ of the bunches Θ_1 and Θ_2, the dual map $\Psi \colon F_2 \to F_1$ is a morphism of the maximal projectable fans Σ_2 and Σ_1. Thus, we obtain:

Proposition 4.3. *The assignments* $\Theta \mapsto \Sigma$ *and* $\Phi \mapsto \Psi$ *define a contravariant functor* $\mathfrak{F}$ *from the category of bunches to the category of maximal projectable fans.* $\qquad\square$

Now we go the other way round. Consider a maximal projectable fan Σ in a projected cone $(F \xrightarrow{P} N, \delta)$. Let $(E \xrightarrow{Q} K, \gamma)$ denote the associated dual projected cone. Define Θ to be the set of the minimal cones among all projected faces in K arising from Σ:

$$\Theta := \{\tau_0; \; \tau_0 \text{ minimal with } \tau_0 = Q(\delta_0^*) \text{ for some } \delta_0 \in \Sigma^{\max}\}.$$

Lemma 4.4. Θ *is a bunch in* $(E \xrightarrow{Q} K, \gamma)$.

Proof. We verify Property 2.2.1 for a given $\tau_0 \in \Theta$. According to the Invariant Separation Lemma 3.3, we have $\tau_0^\circ \cap \tau^\circ \neq \emptyset$ for any further $\tau \in \Theta$. Moreover,

since Θ consists of minimal cones, τ° is not contained in τ_0° provided that $\tau \in \Theta$ is different from τ_0.

Conversely, let the projected face τ_0 satisfy 2.2.1. Choose $\gamma_0 \preceq \tau_0$ with $Q(\gamma_0) = \tau_0$. The Invariant Separation Lemma 3.3 tells us that $\delta_0 := \gamma_0^*$ belongs to Σ. Let $\delta_1 \in \Sigma$ be a maximal cone with $\delta_0 \preceq \delta_1$, and consider the image $\tau_1 := Q(\delta_1^*)$. Then we have $\tau_1 \subset \tau_0$, because $\delta_1^* \preceq \delta_0^*$ holds.

By the definition of the collection Θ, there is a cone $\tau_2 \in \Theta$ with $\tau_2 \subset \tau_1$. In particular, we have $\tau_2 \subset \tau_0$. Applying once more the Invariant Separation Lemma 3.3, gives even $\tau_2^\circ \subset \tau_0^\circ$. Thus Property 2.2.1 yields $\tau_0 = \tau_2$. This shows $\tau_0 \in \Theta$. $\qquad\square$

According to Lemma 3.8, associating to a map Ψ of maximal projectable fans its dual map Φ makes this construction functorial. Thus we have:

Proposition 4.5. *The assignments $\Sigma \mapsto \Theta$ and $\Psi \mapsto \Phi$ define a contravariant functor $\mathfrak{B}$ from the category of maximal projectable fans to the category of bunches.* $\qquad\square$

Summing up, we arrive at the main result of this section, namely the following duality statement:

Theorem 4.6. *The functors $\mathfrak{F}$ and $\mathfrak{B}$ are inverse to each other. In particular, the categories of bunches and maximal projectable fans are dual to each other.* $\qquad\square$

Let us outline here the role of the maximality condition 4.1 (i) (b) in this result. The following example shows that there is no hope for a similar statement on (non maximal) projectable fans:

Example 4.7. Consider the projected cone $(F \xrightarrow{P} N, \delta)$, where the lattices are $F := \mathbb{Z}^3$ and $N := \mathbb{Z}^2$, the cone δ is the positive orthant in $F_\mathbb{Q}$, and the projection map is given by

$$P\colon F \to N, \qquad (v_1, v_2, v_3) \mapsto (v_1 - v_3, v_2 - v_3).$$

Let Σ be the fan in F having $\sigma_1 := \mathrm{cone}(e_1, e_2)$ and $\sigma_2 := \mathrm{cone}(e_1, e_3)$ as its maximal cones. Then Σ is a projectable fan but it is not maximal in the sense of Definition 4.1 (i) (b).

Since the only projected faces in $K \cong \mathbb{Z}$ are the trivial cone and the positive ray, there is no way to reconstruct Σ via face duality from the dual projected cone $(E \xrightarrow{Q} K, \gamma)$ and a collection of projected faces in K.

In the rest of this section, we associate to any maximal projectable fan its "quotient fan". So, let Σ be a maximal projectable fan in a weighted lattice $(F \xrightarrow{P} N, \delta)$. Then the images $P(\sigma)$, where σ runs through the maximal cones of Σ are the maximal cones of a quasifan Σ' in N.

We reduce Σ' to a fan as follows: Let $L' \subset N$ be the primitive sublattice generating the minimal cone of Σ', let $N' := N/L'$, and let $P'\colon N \to N'$ denote the projection.

Definition 4.8. The *quotient fan* of Σ is $\Delta := \{P'(\sigma'); \; \sigma' \in \Sigma'\}$.

Note that $R := P' \circ P\colon F \to N'$ is a map of the fans Σ and Δ. In fact, this is a special case of a more general construction, see [1, Theorem 2.3]. In our setting, it is easy to see that everything is compatible with morphisms. Thus we obtain:

Proposition 4.9. *The assignment $\Sigma \mapsto \Delta$ defines a covariant functor $\mathfrak{Q}$ from the category of maximal projectable fans to the category of fans.* $\qquad\square$

The following simple example shows that dividing by L' in the construction of quotient fan is indeed necessary:

Example 4.10. Consider the lattices $E := \mathbb{Z}^2$ and $K := \mathbb{Z}$, the map $Q \colon E \to K$, $(u_1, u_2) \mapsto u_1 + u_2$, and the positive orthant $\gamma \subset \mathbb{Q}^2$. Let Θ be the bunch consisting just of the trivial cone $\{0\}$. Then the quasifan Σ' determined by Θ consists of the single cone $\sigma := \mathbb{Q}$.

We note an observation on the composition $\mathfrak{Q} \circ \mathfrak{F}$. Consider a bunch Θ in $(E \xrightarrow{Q} K, \gamma)$ and its associated maximal projectable fan Σ in $(F \xrightarrow{P} N, \delta)$. Let Δ be the quotient fan of Σ, and, as before, let $R \colon F \to N'$ be the projection.

Proposition 4.11. *There is a canonical order reversing bijection*

$$\{\gamma_0 \preceq \gamma; \ \tau^\circ \subset Q(\gamma_0)^\circ \ for \ some \ \tau \in \Theta\} \to \Delta, \qquad \gamma_0 \mapsto R(\gamma_0^*).$$

Proof. The inverse map is given by $\sigma \mapsto (R^{-1}(\sigma) \cap |\Sigma|)^*$. $\qquad\square$

5. Combinatorics of quotients

Here we present the first application of the language of bunches. We consider the action of a subtorus on a $\mathbb{Q}$-factorial nondegenerate affine toric variety and give a combinatorial description of the maximal open subsets admitting a good quotient by this action. This complements results of [4] on torus actions on $X = \mathbb{C}^n$.

Let us first recall the basic concepts concerning good quotients. Let the reductive group G act on a variety X by means of a morphism $G \times X \to X$. A *good quotient* for this action is a G-invariant affine morphism $p \colon X \to Y$ such that the canonical map $\mathcal{O}_Y \to p_*(\mathcal{O}_X)^G$ is an isomorphism. If it exists, then the good quotient space is usually denoted by $X /\!\!/ G$.

In general, a G-variety X need not admit a good quotient $X \to X /\!\!/ G$, but there frequently exist many invariant open subsets $U \subset X$ with good quotient $U \to U /\!\!/ G$. It is one of the central tasks of Geometric Invariant Theory to describe all these open subsets, see [3, Section 7.2]. In the course of this problem, one reasonably looks for maximal $U \subset X$ in the following sense, see [3, Section 7.2] and [4]:

Definition 5.1. An open subset $U \subset X$ is called *G-maximal*, if there is a good quotient $p \colon U \to U /\!\!/ G$ and there is no open $U' \subset X$ admitting a good quotient $p' \colon U' \to U' /\!\!/ G$ such that U is a proper p'-saturated subset of U'.

In the setting of subtorus actions, the maximal open subsets with good quotient can be characterized in terms of fans. This relies on the following observation due to Święcicka, see [22, Proposition 2.5]:

Proposition 5.2. *Let X be a toric variety, and let $T \subset T_X$ be a subtorus of the big torus. If $U \subset X$ is a T-maximal subset, then U is invariant under T_X.*

Now, let X be the toric variety arising from a fan Δ in a lattice N, let $T \subset T_X$ be the subtorus corresponding to a primitive sublattice $L \subset N$. By the above proposition, the T-maximal subsets $U \subset X$ correspond to certain subfans of Δ. The characterization of these fans is standard, see e.g. [12, Proposition 1.3]:

Proposition 5.3. *Let $U \subset X$ be the open T_X-invariant subset defined by a subfan Σ of Δ.*

(i) *There is a good quotient $U \to U /\!/ T$ if and only if any two maximal cones of Σ can be separated by an L-invariant linear form.*

(ii) *U is T-maximal if and only if (i) holds and every $\sigma \in \Delta$ that can be separated by L-invariant linear forms from the maximal cones of Σ belongs to Σ.* $\qquad\square$

Though this is a complete combinatorial description of all T-maximal subsets, it has two drawbacks in practice: On the one hand, the ambient space of the combinatorial data might be of quite big dimension, and, on the other hand, for the explicit checking of the conditions there may be large numbers of cones to go through. The language of bunches makes the situation more clear.

Let X be an affine nondegenerate $\mathbb{Q}$-factorial toric variety arising from a cone δ in a lattice F, and let $T \subset T_X$ be the subtorus corresponding to a sublattice $L \subset F$. Setting $N := F/L$, we obtain a projected cone $(F \xrightarrow{P} N, \delta)$. Moreover, we have the dual projected cone $(E \xrightarrow{Q} K, \gamma)$, where K is canonically isomorphic to the lattice of characters of the small torus $T \subset T_X$.

In order to describe the T-maximal subsets of X, we use the functor $\mathfrak{F}$ associating to a bunch Θ in $(E \xrightarrow{Q} K, \gamma)$ a maximal projectable fan $\mathfrak{F}(\Theta)$ in $(F \xrightarrow{P} N, \delta)$. The resulting statement generalizes and complements the results of [4]:

Theorem 5.4. *The assignment $\Theta \mapsto X_{\mathfrak{F}(\Theta)}$ defines a one-to-one correspondence between the bunches in $(E \xrightarrow{Q} K, \gamma)$ and the T-maximal open subsets of X.*

Proof. By the definition of a maximal projectable fan and Proposition 5.3, the toric open subvariety $X_{\mathfrak{F}(\Theta)}$ is indeed T-maximal. Hence the assignment is well defined. Moreover, it is of course injective. Surjectivity follows from Proposition 5.2. $\qquad\square$

Remark 5.5. In the setting of Theorem 5.4, the good quotient of the T-action on $X_{\mathfrak{F}(\Theta)}$ is the toric morphism $X_{\mathfrak{F}(\Theta)} \to X_{\mathfrak{Q}(\mathfrak{F}(\Theta))}$ arising from the projection $\mathfrak{F}(\Theta) \to \mathfrak{Q}(\mathfrak{F}(\Theta))$ onto the quotient fan.

Some of the good quotients are of special interest: A *geometric quotient* for an action of a reductive group G on a variety X is a good quotient that separates orbits. Geometric quotients are denoted by $X \to X/G$.

Again, for subtorus actions on toric varieties, there is a description in terms of fans. Let X be the toric variety arising from a fan Δ in a lattice N, let $T \subset T_X$ be the subtorus corresponding to a primitive sublattice $L \subset N$. Existence of a geometric quotient is characterized as follows, see [13, Theorem 5.1]:

Proposition 5.6. *The action of T on X admits a geometric quotient if and only if the projection $P \colon N \to N/L$ is injective on the support Σ.*

Let us translate this into the language of bunches. As before, consider a projected cone $(F \xrightarrow{P} N, \delta)$ and its associated dual projected cone $(E \xrightarrow{Q} K, \gamma)$.

Definition 5.7. A bunch Θ in $(E \xrightarrow{Q} K, \gamma)$ is called *geometric* if $\dim(\tau) = \dim(K)$ holds for every $\tau \in \Theta$.

We consider the affine toric variety $X := X_\delta$ and the subtorus $T \subset T_X$ corresponding to the sublattice $L \subset N$. The above notion yields what we are looking for:

Proposition 5.8. *Let Θ be a bunch in $(E \xrightarrow{Q} K, \gamma)$. The open toric subvariety $X_{\mathfrak{F}(\Theta)} \subset X$ admits a geometric quotient by the action of T if and only if Θ is geometric.*

Proof. $X_{\mathfrak{F}(\Theta)}$ admits a geometric quotient by T if and only if $P \colon F \to N$ is injective on $|\Sigma|$. In our situation, the latter is equivalent to saying that $P \colon F \to N$ is injective on the maximal cones of Σ, that means on the cones γ_0^* with γ_0 an element of $\mathrm{cov}(\Theta)$.

Thus Lemma 3.5 tells us that $X_{\mathfrak{F}(\Theta)}$ admits a geometric quotient if and only if every cone $Q(\gamma_0)$, $\gamma_0 \in \mathrm{cov}(\Theta)$, is of full dimension in K. Since the elements of Θ occur among these cones and for any two cones of $\mathrm{cov}(\Theta)$ their relative interiors intersect, we obtain the desired characterization. $\qquad\square$

6. Standard bunches and toric varieties

We introduce the class of standard bunches. The main result of this section, Theorem 6.3, says that every nondegenerate 2-complete toric variety can be described by a standard bunch, and, moreover, the isomorphism classes of free standard bunches correspond to the isomorphism classes of nondegenerate 2-complete toric varieties having free class group.

Throughout the entire section, we fix a projected cone $(E \xrightarrow{Q} K, \gamma)$. Let $\gamma_1, \ldots, \gamma_n$ denote the facets of γ, that means the one codimensional faces. In this setting, we have certain data in K, namely

$$\vartheta := Q(\gamma), \qquad \vartheta_i := Q(\gamma_i), \qquad K_i := Q(\mathrm{lin}(\gamma_i) \cap E).$$

Definition 6.1. We say that a bunch Θ in $(E \xrightarrow{Q} K, \gamma)$ is a *standard bunch* if the following conditions are fulfilled:

(i) For all i we have $K_i = K$.
(ii) For all $\tau \in \Theta$ we have $\tau^\circ \cap \vartheta^\circ \neq \emptyset$.
(iii) Every ϑ_i contains an element $\tau \in \Theta$.
(iv) For all i, j we have $\vartheta_i^\circ \cap \vartheta_j^\circ \neq \emptyset$.

If Θ is a standard bunch in $(E \xrightarrow{Q} K, \gamma)$, and the cone $\gamma \subset E_{\mathbb{Q}}$ is regular, then we speak of the *free standard bunch* Θ.

Note that a geometric bunch Θ in $(E \xrightarrow{Q} K, \gamma)$ is a standard bunch if and only if it satisfies the conditions 6.1 (i) and (iii). The constructions of Section 4 provide a functor from standard bunches to toric varieties:

Definition 6.2. Let Θ be a standard bunch in $(E \xrightarrow{Q} K, \gamma)$, and let $\Delta := \mathfrak{Q}(\mathfrak{F}(\Theta))$ be the quotient fan of the maximal projectable fan corresponding to Θ. The *toric variety associated to* Θ is $X_\Theta := X_\Delta$.

We come to the main result of this section. Recall from Section 1 that X is nondegenerate, if it does not admit a toric decomposition $X \cong X' \times \mathbb{K}^*$. Moreover, X is 2-complete, if any toric open embedding $X \subset X'$ with $\mathrm{codim}(X' \setminus X) \geq 2$ is an isomorphism.

Theorem 6.3. *The assignment* $\mathfrak{T} \colon \Theta \mapsto X_\Theta$ *defines a contravariant functor from the category of standard bunches to the category of nondegenerate 2-complete toric varieties. Moreover,*

(i) *Every nondegenerate 2-complete toric variety is isomorphic to a toric variety X_Θ with a standard bunch Θ.*
(ii) *$\mathfrak{T}$ induces a bijection from the isomorphism classes of free standard bunches to the isomorphism classes of nondegenerate 2-complete toric varieties with free class group.*

For the proof of Theorem 6.3, we have to do some preparations. We need a torsion free version of Cox's construction 1.2 for nondegenerate fans, i.e. fans Δ in a lattice N such that the support $|\Delta|$ generates the vector space $N_{\mathbb{Q}}$:

Definition 6.4. Let $(F \xrightarrow{P} N, \delta)$ be a projected cone, Σ a fan in F, and Δ a fan in N. We say that these data form a *reduced Cox construction* for Δ if

(i) $\Sigma^{(1)}$ equals $\delta^{(1)}$ and P induces bijections $\Sigma^{(1)} \to \Delta^{(1)}$ and $\Sigma^{\max} \to \Delta^{\max}$,

(ii) P maps the primitive generators of δ to primitive lattice vectors,

We show now that every nondegenerate fan admits reduced Cox contructions. In fact, these reduced Cox constructions will be even compatible with a certain type of maps of fans.

Let Δ_i be nondegenerate fans in lattices N_i. Moreover, let $\overline{\Psi} \colon N_1 \to N_2$ be any isomorphism of lattices that is a map of the fans Δ_1 and Δ_2. Suppose that $\overline{\Psi}$ induces a bijection on the sets of rays $\Delta_1^{(1)}$ and $\Delta_2^{(1)}$.

Lemma 6.5. *There exist projected cones* $(F_i \xrightarrow{P_i} N_i, \delta_i)$, *fans* Σ_i *in* F_i *defining reduced Cox constructions for* Δ_i, *a lattice isomorphism* $\Psi \colon F_1 \to F_2$, *and a commutative diagram of maps of fans*

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\Psi} & F_2 \\
{\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\
N_1 & \xrightarrow[\overline{\Psi}]{} & N_2
\end{array}
$$

The map Ψ *induces a bijection of the sets of rays* $\Sigma_1^{(1)}$ *and* $\Sigma_2^{(1)}$. *If* $\overline{\Psi}$ *maps* Δ_1 *isomorphically onto a subfan of* Δ_2, *then* Ψ *maps* Σ_1 *isomorphically onto a subfan of* Σ_2.

Proof. First we perform Cox's original construction 1.2 for the fans Δ_i. Denote by R_i the set of rays of Δ_i. For every maximal cone $\sigma \in \Delta_i$ let

$$\widetilde{\sigma} := \mathrm{cone}(e_{\varrho};\ \varrho \in \mathrm{R}_i,\ \varrho \subset \sigma).$$

Then these cones $\widetilde{\sigma}$ are the maximal cones of a fan $\widetilde{\Delta}_i$ in $\mathbb{Z}^{\mathrm{R}_i}$. Moreover, we have canonical projections sending the canonical base vectors to the primitive lattice vectors of the corresponding rays:

$$C_i \colon \mathbb{Z}^{\mathrm{R}_i} \to N_i, \qquad e_{\varrho} \mapsto v_{\varrho}.$$

Surely these C_i are maps of fans. Moreover, we can lift $\overline{\Psi} \colon N_1 \to N_2$ to a lattice isomorphism $\Psi \colon \mathbb{Z}^{\mathrm{R}_1} \to \mathbb{Z}^{\mathrm{R}_2}$ by setting

$$\Psi(e_{\varrho}) := e_{\overline{\Psi}(\varrho)} \quad \text{for every } \varrho \in \mathrm{R}_1.$$

Since the maps C_i need not be surjective, they cannot serve as projections of projected cones. Thus we have to perform a reduction step: Let $L_i := \ker(C_i)$. Choose any section $s_1 \colon N_{1\mathbb{Q}} \to \mathbb{Q}^{\mathrm{R}_1}$ of the projection C_1 and set

$$F_1 := s_1(N_1) \oplus L_1 \subset \mathbb{Q}^{\mathrm{R}_1}, \qquad F_2 := \Psi(F_1).$$

Then we can view δ_i and Σ_i as well as objects in the lattices F_i. Note that the δ_i need no longer be regular, but remain simplicial. Together with the surjections $P_i \colon F_i \to N_i$ and the lifting $\Psi \colon F_1 \to F_2$ we have constructed the desired data. $\square$

The above proof shows in particular that we cannot expect uniqueness of reduced Cox constructions for a given fan.

Lemma 6.6. *Let Δ be a nondegenerate fan in a lattice N. Then the associated toric variety X has free class group $\mathrm{Cl}(X)$ if and only if Δ admits a reduced Cox construction Σ in the projected cone $(F \xrightarrow{P} N, \delta)$ with a regular cone δ.*

Proof. Again, we consider Cox's construction 1.2 for Δ. Let R denote the set of rays of Δ, and let $C \colon \mathbb{Z}^{\mathrm{R}} \to N$ be the map sending the canonical base vector e_ϱ to the primitive lattice vector $v_\varrho \in \varrho$.

If the toric variety X has free class group then the lattice homomorphism C is surjective, see [6]. Hence it defines the desired reduced Cox construction with $F = \mathbb{Z}^{\mathrm{R}}$ and δ the positive orthant in $\mathbb{Q}^{\mathrm{R}}$.

Conversely, let $(F \xrightarrow{P} N, \delta)$ and Σ be a reduced Cox construction of Δ with δ regular. Then C factors as $C = P \circ S$, where $S \colon \mathbb{Z}^{\mathrm{R}} \to F$ maps the canonical base vectors to the primitive generators of the cone δ. It follows that C is surjective. Hence $\mathrm{Cl}(X)$ is free. $\qquad\square$

Lemma 6.7. *Let $(F_i \xrightarrow{P_i} N_i, \delta_i)$ and Σ_i be reduced Cox constructions for Δ_i such that the δ_i are regular cones. Then every isomorphism $\overline{\Psi} \colon N_1 \to N_2$ of the fans Δ_i admits a unique lifting $\Psi \colon F_1 \to F_2$ to an isomorphism of the fans Σ_i.*

Proof. For $\varrho \in \Delta_i^{(1)}$, let $e_\varrho^i \in F_i$ denote the primitive generator of δ_1 above the primitive vector of ϱ. Then define $\Psi \colon F_1 \to F_2$ by setting $\Psi(e_\varrho^1) := e_{\overline{\Psi}(\varrho)}^2$. $\qquad\square$

Let Θ be a bunch in $(E \xrightarrow{Q} K, \gamma)$, and let Σ denote the associated maximal projectable fan in the dual projected cone $(F \xrightarrow{P} N, \delta)$. Let Δ denote the quasifan in N obtained by projecting the maximal cones of Σ.

Lemma 6.8. *The following statements are equivalent:*
 (i) *Δ is a fan and $P \colon F \to N$ is a reduced Cox construction.*
 (ii) *Θ is a standard bunch.*

Proof. Having in mind that Σ is a (maximal) projectable fan and thus satisfies Property 4.1 (i) (a), we may reformulate (i) as follows:
 (a) *P maps every primitive generator of δ to a primitive lattice vector in N.*
 (b) *for every $\widehat{\sigma} \in \Sigma$ the image $P(\widehat{\sigma})$ is a strictly convex cone,*
 (c) *every extremal ray of δ is a ray of Σ,*
 (d) *the map P induces a bijection $\Sigma^{(1)} \to \Delta^{(1)}$ of the sets of rays.*
The task then is to show that the conditions 6.1 (i) to (iv) hold if and only if (a) to (d) hold. This amounts to dualizing these conditions.

Lemma 3.6 tells us that (a) is equivalent to 6.1 (i). The equivalence of (b) and 6.1 (ii) as well as the equivalence of (c) and 6.1 (iii) are simple consequences of the Invariant Separation Lemma 3.3 and the definition of the maximal projectable fan Σ associated to Θ.

Now suppose that (a) to (d) hold. By the preceding consideration, we only have to check that 6.1 (iv) is valid. But by the Invariant Separation Lemma 3.3 this follows from the fact that P separates the rays of Σ.

Conversely, if 6.1 (i) to (iv) are valid, we only have to show that P induces an injection on the rays of Σ. Again by the Invariant Separation Lemma 3.3, this follows from 6.1 (iv). $\qquad\square$

Proof of Theorem 6.3. We show that for a standard bunch Θ, the associated toric variety X_Θ is 2-complete. Let $X_\Theta \subset X$ be an open toric embedding with nonempty complement of codimension at least two. Comparing suitable reduced Cox constructions of X_Θ and X with the aid of Lemma 6.5 and Proposition 5.3, we see that the maximal projectable fan associated to Θ does not satisfy 4.1 (ii). A contradiction.

So the functor $\mathfrak{T} \colon \Theta \mapsto X_\Theta$ is well defined. The fact that it is surjective on isomorphism classes follows from existence of reduced Cox constructions and Lemma 6.8. The correspondence of isomorphism classes of free bunches with isomorphism classes of nondegenerate 2-complete toric varieties with free class group is a direct application of Lemmas 6.6 and 6.7. $\qquad\square$

We conclude this section with some examples showing that the last three conditions of Definition 6.1 are independent from each other:

Example 6.9. Let $E := \mathbb{Z}^2$ and $K := \mathbb{Z}$, and let $\gamma := \mathrm{cone}(e_1, e_2)$. Define a map $Q_{a_1,a_2} \colon E \to \mathbb{Z}$ by sending e_i to a_i. We consider the following bunches Θ in the resulting projected cone:

(i) For $a_1 = 1$ and $a_2 = -1$, let Θ consist of the trivial cone $\{0\}$. Then 6.1 (ii) and (iii) hold, but (iv) does not.

(ii) For $a_1 = a_2 = 1$, let Θ consist of the trivial cone $\{0\}$. Then 6.1 (iii) and (iv) hold, but (ii) does not.

Example 6.10. Let $E := \mathbb{Z}^3$ and $K := \mathbb{Z}$, and let $\gamma := \mathrm{cone}(e_1, e_2, e_3)$. Define a map $Q \colon E \to \mathbb{Z}$ by sending e_1, e_2 to 1 and e_3 to -1. The bunch $\Theta := \{\mathbb{Q}_{\leq 0}\}$ in resulting projected cone satisfies 6.1 (ii) and (iv), but not (iii).

7. A VERY FIRST DICTIONARY

Fix a standard bunch Θ in a projected cone $(E \xrightarrow{Q} K, \gamma)$, and let $X := X_\Theta$ denote the associated toric variety. In this section, we characterize basic geometric properties of X in terms of the bunch Θ.

Let $(F \xrightarrow{P} N, \delta)$ be the dual projected cone. Denote by Σ the maximal projectable fan associated to Θ, and let Δ be the quotient fan of Σ. Recall from Lemma 6.8 that these data form a reduced Cox construction of Δ. In particular, Δ lives in N, and we have

$$X = X_\Delta, \qquad \dim(X) = \mathrm{rank}(E) - \mathrm{rank}(K).$$

We study now $\mathbb{Q}$-factoriality, smoothness, existence of fixed points and completeness. For this we need the following observation:

Lemma 7.1. *Consider a face $\gamma_0 \in \mathrm{cov}(\Theta)$ and the corresponding maximal cone $P(\gamma_0^*)$ of Δ. Then we have:*

(i) *$Q(\gamma_0)$ is of full dimension if and only if $P(\gamma_0^*)$ is simplicial.*

(ii) *$Q(\gamma_0)$ is simplicial if and only if $P(\gamma_0^*)$ is of full dimension.*

Proof. We prove (i). Let $Q(\gamma_0)$ be of full dimension. By Lemma 3.5, the map P is injective along γ_0^*. In particular, $P(\gamma_0^*)$ is simplicial. Conversely, let $P(\gamma_0^*)$ be simplicial. Since P induces a bijection from the rays of γ_0^* to the rays of $P(\gamma_0^*)$, it is injective along γ_0^*. Thus Lemma 3.5 yields that $Q(\gamma_0)$ is of full dimension.

We turn to (ii). If $P(\gamma_0^*)$ is of full dimension, we see as before that $Q(\gamma_0)$ is simplicial. For the converse we show that Q is injective along γ_0: For every ray ϱ of $Q(\gamma_0)$, choose a ray τ of γ_0 with $Q(\tau) = \varrho$. Then the cone $\gamma_1 \preceq \gamma_0$ generated by

these rays τ is mapped bijectively onto $Q(\gamma_0)$. By minimality of γ_0 as an element of $\mathrm{cov}(\Theta)$, we conclude $\gamma_1 = \gamma_0$. $\square$

The first statements of this section are the following characterizations of $\mathbb{Q}$-factoriality and smoothness:

Proposition 7.2. *The toric variety X is $\mathbb{Q}$-factorial if and only if Θ consists of cones of full dimension in K.*

Proof. The toric variety X is $\mathbb{Q}$-factorial if and only if all cones of Δ are simplicial. By Lemma 7.1 this is equivalent to saying that all cones $Q(\gamma_0)$, $\gamma_0 \in \mathrm{cov}(\Theta)$, are of full dimension. The latter holds if and only if Θ consists of cones of full dimension, because every $Q(\gamma_0)$ contains some $\tau \in \Theta$. $\square$

Characterizing smoothness of toric varieties in terms of bunches involves the lattice structure:

Proposition 7.3. *The toric variety X_Θ is smooth if and only if for every $\gamma_0 \in \mathrm{cov}(\Theta)$ we have*

  (i) *γ_0^* is a regular cone in F.*
  (ii) *Q maps $\mathrm{lin}(\gamma_0) \cap E$ onto K.*

Proof. Suppose that (i) and (ii) hold. To verify smoothness of X, we have to show that all cones $P(\gamma_0^*)$, $\gamma_0 \in \mathrm{cov}(\Theta)$, are regular. By 6.4 (ii) and Condition (i), the primitive generators of $P(\gamma_0^*)$ span the sublattice $P(\mathrm{lin}(\gamma_0^*) \cap F)$ of N. By Lemma 3.6 and Condition (ii), this sublattice is primitive. This proves regularity of the cone $P(\gamma_0^*)$.

Conversely, let X be smooth. For $\gamma_0 \in \mathrm{cov}(\Theta)$, consider the sublattice $F_0 \subset F$ spanned by the primitive generators of γ_0^*. Since $P(\gamma_0^*)$ is regular, Property 6.4 (ii) yields that $P(F_0)$ is a primitive sublattice of N. Since P is injective along γ_0^*, we see that also F_0 is primitive. This gives Condition (i). Condition (ii) then follows from Lemma 3.6. $\square$

Existence of global regular functions on a toric variety is characterized as follows:

Proposition 7.4. *We have $\mathcal{O}(X) = \mathbb{K}$ if and only if Q contracts no ray of γ to a point and the image $Q(\gamma)$ is strictly convex.*

Proof. $\mathcal{O}(X) = \mathbb{K}$ holds if and only if the rays of Δ generate $N_\mathbb{Q}$ as a convex cone. This is valid if and only if $\delta + \ker(P)_\mathbb{Q}$ equals $F_\mathbb{Q}$. By dualizing this condition is equivalent to $\gamma \cap \ker(Q) = \{0\}$. This in turn holds if and only if Q contracts no ray of γ and $Q(\gamma)$ is strictly convex. $\square$

Recall that we speak of a full toric variety X if X is 2-complete and every T_X-orbit contains a fixed point in its closure.

Proposition 7.5. *The toric variety X is full if and only if all cones $Q(\gamma_0)$, where $\gamma_0 \in \mathrm{cov}(\Theta)$, are simplicial.*

Proof. Existence of fixed points in the T_X-orbit closures means that all maximal cones of Δ are of full dimension. Thus Lemma 7.1 gives the assertion. $\square$

Finally, we also can characterize completeness of a toric variety in terms of its bunch:

Proposition 7.6. *The toric variety X_Θ is complete if and only if Θ consists of simplicial cones and any face $\gamma_0 \preceq \gamma$ satisfying $\tau^\circ \subsetneq Q(\gamma_0)^\circ$ for some $\tau \in \Theta$ and $\gamma_1 \preceq \gamma_0$ for precisely one $\gamma_1 \in \mathrm{cov}(\Theta)$ belongs to $\mathrm{cov}(\Theta)$.*

Proof. This is a direct translation of the following characterization of completeness in terms of fans: X is complete if and only if all maximal cones of the corresponding fan Δ are of full dimension and any cone of Δ contained in only one maximal cone is itself maximal. $\qquad\square$

One may ask if a full toric variety X with $\mathcal{O}(X) = \mathbb{K}$ is already complete. If $\dim(X) \leq 3$ holds, then the answer is positive. For $\dim(X) \geq 4$, there are counterexamples, compare [9, Remark 2]. However, we shall see in Section 10 that every smooth 2-complete toric variety with class group $\mathbb{Z}^2$ is complete.

8. Full $\mathbb{Q}$-factorial toric varieties

In this section we introduce the class of simple bunches, and we show that $\Theta \mapsto X_\Theta$ defines an equivalence of categories between the simple bunches and the full $\mathbb{Q}$-factorial toric varieties.

Definition 8.1. By a *simple bunch* we mean a standard bunch Θ in a projected cone $(E \xrightarrow{Q} K, \gamma)$ such that Q maps $E \cap \mathrm{lin}(\gamma_0)$ isomorphically onto K for every $\gamma_0 \in \mathrm{cov}(\Theta)$.

Note that the cones of a simple bunch are of full dimension and simplicial, but they need not be regular. Simple bunches could as well be defined directly, i.e. without requiring them to be standard bunches:

Remark 8.2. A bunch Θ in a projected cone $(E \xrightarrow{Q} K, \gamma)$ is simple if and only if it has the following properties:

  (i) Q maps $\mathrm{lin}(\gamma_0) \cap E$ isomorphically onto K for every $\gamma_0 \in \mathrm{cov}(\Theta)$,
  (ii) For every facet $\gamma_0 \preceq \gamma$ we have $Q(\mathrm{lin}(\gamma_0) \cap E) = K$.
  (iii) For every facet $\gamma_0 \preceq \gamma$ there is a $\tau \in \Theta$ with $\tau \subset Q(\gamma_0)$.

For the formulation of the main result of this section, recall from Definition 1.4 that a toric variety X is full, if it is 2-complete and every T_X-orbit has a fixed point in its closure.

Theorem 8.3. *The assignment $\Theta \mapsto X_\Theta$ defines an equivalence of the category of simple bunches with the category of full $\mathbb{Q}$-factorial toric varieties.*

The crucial observation for the proof is that full $\mathbb{Q}$-factorial toric varieties admit a universal reduced Cox construction. First we show existence:

Lemma 8.4. *Let Δ be a simplicial fan in a lattice N such that any maximal cone is of full dimension. Then there are $(F \xrightarrow{P} N, \delta)$ and Σ defining a reduced Cox construction of Δ such that P induces for every $\widehat{\sigma} \in \Sigma^{\max}$ an isomorphism*

$$P_{\widehat{\sigma}} \colon F \cap \mathrm{lin}(\widehat{\sigma}) \to N.$$

Proof. As usual, let R be the set of rays of Δ, let $P \colon \mathbb{Z}^{\mathrm{R}} \to N$ be the homomorphism sending the canonical base vector e_ϱ to the primitive vector $v_\varrho \in \varrho$, and let $\delta \subset \mathbb{Q}^{\mathrm{R}}$ be the cone generated by the e_ϱ. For any cone $\sigma \in \Delta$ set

$$\widehat{\sigma} := \mathrm{cone}(e_\varrho;\ \varrho \in \sigma^{(1)}).$$

Then these cones form a fan Σ in the lattice $\mathbb{Z}^{\mathrm{R}}$. In order to achieve the desired properties, we refine the lattice $\mathbb{Z}^{\mathrm{R}}$ as follows: Think of P for the moment as a map of the vector spaces $\mathbb{Q}^{\mathrm{R}}$ and $N_{\mathbb{Q}}$ and consider the set

$$P^{-1}(N) \cap |\Sigma|.$$

Since Δ is simplicial, this set generates a lattice $F \subset \mathbb{Q}^{\mathrm{R}}$. Moreover, the restriction $P_{\widehat{\sigma}} \colon F \cap \mathrm{lin}(\widehat{\sigma}) \to N$ of P is an isomorphism for every $\widehat{\sigma} \in \Sigma^{\mathrm{max}}$. Here we use that the maximal cones of Δ are of full dimension.

We may view δ and Σ as well as data in the lattice F. Since $P \colon F \to N$ is surjective, this gives in particular a projected cone $(F \xrightarrow{P} N, \delta)$. Now it is straightforward to verify the defining properties of a reduced Cox construction for these data. $\square$

We shall call a reduced Cox construction with the property of Lemma 8.4 a *free Cox construction*. Uniqueness of free Cox constructions is a consequence of the following lifting property:

Lemma 8.5. *Let Δ_i be fans in lattices N_i, and let $(F_i \xrightarrow{P_i} N_i, \delta_i)$ and Σ_i be free Cox constructions. Then every map $\overline{\Psi} \colon N_1 \to N_2$ of the fans Δ_i admits a unique lifting to a map $\Psi \colon F_1 \to F_2$ of the fans Σ_i:*

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\ \Psi\ } & F_2 \\
{\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\
N_1 & \xrightarrow[\overline{\Psi}]{} & N_2
\end{array}
$$

Proof. The lifting can be constructed locally. For every maximal cone σ_1 of Δ_1, fix a maximal cone σ_2 of Δ_2 such that $\overline{\Psi}(\sigma_1) \subset \sigma_2$. Let $\widehat{\sigma}_i \in \Sigma_i$ denote the cones lying over σ_i. By the property of a free Cox construction, we obtain a unique commutative diagram

$$
\begin{array}{ccc}
F_1 \cap \mathrm{lin}(\widehat{\sigma}_1) & \xrightarrow{\ \Psi_{\widehat{\sigma}_1}\ } & F_2 \cap \mathrm{lin}(\widehat{\sigma}_2) \\
{\scriptstyle P_1}\Big\updownarrow{\scriptstyle s_{\widehat{\sigma}_1}} & & {\scriptstyle s_{\widehat{\sigma}_2}}\Big\updownarrow{\scriptstyle P_2} \\
N_1 & \xrightarrow[\overline{\Psi}]{} & N_2
\end{array}
$$

for every maximal cone σ_1 of Δ, where $s_{\widehat{\sigma}_i}$ are the sections mapping the primitive generators of σ_i to those of $\widehat{\sigma}_i$. Note that these are unique. Therefore, the liftings $\Psi_{\widehat{\sigma}_1}$ patch together to the desired linear map. $\square$

Proof of Theorem 8.3. First observe that X_{Θ} is indeed full and $\mathbb{Q}$-factorial: Since for every $\gamma_0 \in \mathrm{cov}(\Theta)$ the cone $Q(\gamma_0)$ is simplicial and of full dimension, Lemma 7.1 ensures that also the maximal cones of the fan Δ defining X are of full dimension and simplicial.

Next we show that, up to isomorphism, every full $\mathbb{Q}$-factorial toric variety X is of the form X_{Θ} with a simple bunch. We may assume that X arises from a fan Δ in a lattice N. Lemma 8.4 provides a free Cox construction $(F \xrightarrow{P} N, \delta)$ and Σ of Δ. By Lemma 3.6, the bunch Θ corresponding to the maximal projectable fan Σ is as wanted.

Finally, we have to show bijectivity on the level of morphisms. Due to Lemma 8.5 the morphisms between full $\mathbb{Q}$-factorial simplicial toric varieties are in one-to-one correspondence with the maps of maximal projectable fans defined by their free Cox constructions. By Theorem 4.6, the latter morphisms correspond to the morphisms of the associated bunches. $\square$

Since the free simple bunches are precisely those describing full smooth toric varieties, we obtain:

Corollary 8.6. $\Theta \mapsto X_\Theta$ *defines an equivalence of categories from free simple bunches to full smooth toric varieties.* $\square$

We conclude this section with some observations and examples. The first remark gives the geometric interpretation of the free Cox construction:

Remark 8.7. In terms of toric varieties, the free Cox construction $p \colon \widehat{X} \to X$ arises from the usual Cox construction $c \colon \widetilde{X} \to X$ by dividing $\widetilde{X}$ by the (finite) group generated by all isotropy groups of the action of $\ker(Q)$ on $\widetilde{X}$.

The following example shows that for nonsimplicial toric varieties the functor $\mathfrak{T}$ is not surjective on the level of morphisms:

Example 8.8. We present a toric morphism that cannot be lifted to the Cox constructions. In $\mathbb{Z}^3$ consider the vectors

$$v_1 := (1,0,0), \quad v_2 := (1,0,1), \quad v_3 := (0,1,0), \quad v_4 := (-3,2,2), \quad v_5 := (0,1,1).$$

Let Δ_2 be the fan of faces of the cone generated by $v_1, \dots, v_4$, and let Δ_1 be the subdivision of Δ_2 at v_5. Mapping the i-th canonical base vector to v_i, we obtain projections

$$P_1 \colon \mathbb{Z}^5 \to \mathbb{Z}^3, \qquad P_2 \colon \mathbb{Z}^4 \to \mathbb{Z}^3.$$

Let Σ_1 and Σ_2 be the fans above Δ_1 and Δ_2, respectively. Note that these data are in fact reduced Cox constructions.

We claim that the identity $\varphi \colon \mathbb{Z}^3 \to \mathbb{Z}^3$ does not admit a lifting. In fact, since v_5 equals $3/2 v_1 + 1/2 v_4$, a possible lifting $\Phi \colon \mathbb{Z}^5 \to \mathbb{Z}^4$ must satisfy

$$\Phi(0,0,0,0,1) \in (3/2,0,0,1/2) + \mathbb{Q}(5,-2,-2,1),$$

where $(5,-2,-2,1)$ generates $\ker(P_2)$. An explicit calculation shows that the right hand sice does not contain integral points with nonnegative coefficients.

This excludes existence of a map $\Phi \colon \mathbb{Z}^5 \to \mathbb{Z}^4$ of the fans Σ_1 and Σ_2 lifting $\varphi \colon \mathbb{Z}^3 \to \mathbb{Z}^3$.

The next example shows that in the nonsimplicial case the functor $\mathfrak{T}$ is not injective on the level of morphisms.

Example 8.9. We give a toric morphism that admits two different liftings. Let Δ_2 be the fan of faces of the cone generated by the vectors

$$(1,0,0), \quad (0,1,0), \quad (-1,0,1), \quad (0,-1,1).$$

Let Δ_1 be the subdivision of Δ_2 at $(0,0,1)$. Then there are two different liftings of the identity map $\mathbb{Z}^3 \to \mathbb{Z}^3$ to the respective Cox-constructions, namely the maps

$\mathbb{Z}^5 \to \mathbb{Z}^4$ defined by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

9. Invariant divisors and divisor classes

In this section, we come to one of the most powerful parts of the language of bunches: We study geometric properties of divisors and divisor classes.

Let us fix the notation. As usual, Θ is a standard bunch in a projected cone $(E \xrightarrow{Q} K, \gamma)$, and $X := X_\Theta$ is the associated toric variety with its big torus $T := T_X$. Let $(F \xrightarrow{P} N, \delta)$ be the dual projected cone, Σ the maximal projectable fan corresponding to Θ, and Δ the quotient fan of Σ. Recall that these data define a reduced Cox construction of Δ, and we have $X = X_\Delta$.

Our first task is to relate the lattice K to the divisor class group $\mathrm{ClDiv}(X)$. Let $v_1, \ldots, v_r$ be the primitive generators of the one dimensional cones of Δ, and let $\widehat{v}_1, \ldots, \widehat{v}_r$ be the primitive generators of the rays of δ, numbered in such a way that we always have $P(\widehat{v}_i) = v_i$.

Note that every ray $\widehat{\varrho}_i := \mathbb{Q}_{\geq 0} \widehat{v}_i$ of δ has a unique "opposite" ray $\widehat{\eta}_i \in \gamma^{(1)}$, that means that $\widehat{\eta}_i$ is the unique ray not contained in $\widehat{\varrho}_i^\perp$. We denote the primitive generator of $\widehat{\eta}_i$ by $\widehat{w}_i$, and its image in K by $w_i := Q(\widehat{w}_i)$.

Every ray ϱ_i determines an invariant prime divisor D_i in X. There is a canonical injection mapping E into the lattice $\mathrm{WDiv}^T(X)$ of invariant Weil divisors on X:

$$\mathfrak{D} \colon E \to \mathrm{WDiv}^T(X), \qquad \widehat{w} \mapsto \mathfrak{D}(\widehat{w}) := \sum_{i=1}^r \widehat{w}(\widehat{v}_i) D_i.$$

By construction, an element $u \in M$ is mapped to the principal divisor $\mathrm{div}(\chi^u)$ of X. Hence, we obtain, compare [10]:

Proposition 9.1. *There is a commutative diagram with exact rows and injective upwards arrows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{PDiv}^T(X) & \longrightarrow & \mathrm{WDiv}^T(X) & \longrightarrow & \mathrm{ClDiv}(X) & \longrightarrow & 0 \\
& & \Big\uparrow{\scriptstyle \cong} & & \Big\uparrow{\scriptstyle \mathfrak{D}} & & \Big\uparrow{\scriptstyle \overline{\mathfrak{D}}} & & \\
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & K & \longrightarrow & 0
\end{array}$$

Moreover, tensoring this diagram with $\mathbb{Q}$ provides isomorphisms of rational vector spaces:

$$\mathfrak{D} \colon E_\mathbb{Q} \to \mathrm{WDiv}^T_\mathbb{Q}(X), \qquad \overline{\mathfrak{D}} \colon K_\mathbb{Q} \to \mathrm{ClDiv}_\mathbb{Q}(X).$$

In the main result of this section, we study the group $\mathrm{Pic}_\mathbb{Q}(X) \subset \mathrm{ClDiv}_\mathbb{Q}(X)$ of rational Cartier divisor classes. We obtain very simple descriptions of the cone $\mathrm{C}^{\mathrm{sa}}(X) \subset \mathrm{Pic}_\mathbb{Q}(X)$ of semiample classes and the cone $\mathrm{C}^{\mathrm{a}}(X) \subset \mathrm{Pic}_\mathbb{Q}(X)$ of ample classes:

Theorem 9.2. *The map $\overline{\mathfrak{D}} \colon K_\mathbb{Q} \to \mathrm{ClDiv}_\mathbb{Q}(X)$ defines canonical isomorphisms:*

$$\mathrm{Pic}_\mathbb{Q}(X) \cong \bigcap_{\tau \in \Theta} \mathrm{lin}(\tau), \qquad \mathrm{C}^{\mathrm{sa}}(X) \cong \bigcap_{\tau \in \Theta} \tau, \qquad \mathrm{C}^{\mathrm{a}}(X) \cong \bigcap_{\tau \in \Theta} \tau^\circ.$$

Proof. For a rational Weil divisor $D \in \mathrm{WDiv}^T_{\mathbb{Q}}(X)$, set $\widehat{w} := \mathfrak{D}^{-1}(D) \in E_{\mathbb{Q}}$. Our task is to characterize the properties of being a $\mathbb{Q}$-Cartier, a semiample or an ample divisor in terms of the image $w := Q(\widehat{w})$ in $K_{\mathbb{Q}}$.

In order to verify the first isomorphism, assume that D is $\mathbb{Q}$-Cartier. Let $(u_\sigma)_{\sigma \in \Delta^{\max}}$ denote the support function corresponding to D, compare [10, p. 66]. Concretly this means that $u_\sigma(v_i) = \widehat{w}(\widehat{v}_i)$ holds for every $v_i \in \sigma^{(1)}$. Define $\ell_\sigma := \widehat{w} - P^*(u_\sigma)$ and denote the cone above σ by $\widehat{\sigma}$. Then ℓ_σ lies in $\widehat{\sigma}^{\perp} = \lim(\widehat{\sigma}^*)$, and hence $Q(\widehat{w}) = Q(\ell_\sigma)$ lies in $Q(\lim(\widehat{\sigma}^*)$. Since this applies for all $\sigma \in \Delta^{\max}$, we obtain

$$w = Q(\widehat{w}) \in \bigcap_{\gamma_0 \in \mathrm{cov}(\Theta)} \lim(Q(\gamma_0)) = \bigcap_{\tau \in \Theta} \lim(\tau).$$

For the converse, suppose that $w = Q(\widehat{w})$ lies in the right hand side intersection. Since for $\sigma \in \Delta^{\max}$ the image $Q(\widehat{\sigma}^*)$ contains an element of Θ, we find for each $\sigma \in \Delta^{\max}$ an $\ell_\sigma \in \lim(\widehat{\sigma}^*)$ such that $Q(\ell_\sigma) = w$ holds. Then the difference $u_\sigma := \widehat{w} - \ell_\sigma$ maps to zero and can therefore be viewed as an element of $M_{\mathbb{Q}}$. Hence, $(u_\sigma)_{\sigma \in \Delta^{\max}}$ provides the desired support function.

Now, for the descriptions of $\mathrm{C}^{\mathrm{sa}}(X)$ and $\mathrm{C}^{\mathrm{a}}(X)$, recall first that a $\mathbb{Q}$-Cartier divisor D is semiample (resp. ample) if and only if its support function (u_σ) is convex (resp. strictly convex), that means that $u_\sigma - u_{\sigma'}$ is nonnegative (resp. strictly positive) on $\sigma \setminus \sigma'$ for all $\sigma, \sigma' \in \Delta^{\max}$.

In terms of $\ell_\sigma := \widehat{w} - P^*(u_\sigma)$ these properties translate to nonnegativity (positivity) of $\ell_{\sigma'} - \ell_\sigma$ on $\widehat{\sigma} \setminus \widehat{\sigma}'$. Since $\ell_\sigma \in \widehat{\sigma}^{\perp}$ holds, this is equivalent to nonnegativity (positivity) of $\ell_{\sigma'}$ on $\widehat{\sigma} \setminus \widehat{\sigma}'$. But this is precisely the case if we have $\ell_\sigma \in \widehat{\sigma}^*$ (resp. $\ell_\sigma \in (\widehat{\sigma}^*)^{\circ}$) for all σ. This is the case if and only if for all $\sigma \in \Delta^{\max}$ we have

$$w = Q(\widehat{w}) = Q(\ell_\sigma) \in Q(\widehat{\sigma}^*) \quad (\text{resp. } w \in Q((\widehat{\sigma}^*)^{\circ})).$$

Now, the cones $\widehat{\sigma}^*$, where $\sigma \in \Delta^{\max}$ are just the cones of $\mathrm{cov}(\Theta)$. Since any interior $Q(\widehat{\sigma}^*)$ contains the interior of a cone of Θ, we can conclude that w lies in the respective right hand sides. $\qquad\square$

As immediate consequences, we obtain a quasiprojectivity criterion in the spirit of [8], and a result due to Oda and Park [19, Theorem 3.5]:

Corollary 9.3. *The variety X is quasiprojective if and only if the intersection of all relative interiors τ°, where $\tau \in \Theta$, is nonempty.* $\qquad\square$

Corollary 9.4. *Suppose that X is quasiprojective and $\mathbb{Q}$-factorial. Then every rational Weil divisor D of X admits a representation $D = D_1 - D_2$ with ample rational Cartier divisors D_1 and D_2.* $\qquad\square$

So far, we considered rational divisors and divisor classes. For toric varieties with free class group, we also obtain a simple picture for integral divisors. In the notation introduced at the beginning of this section, freeness means the following:

Remark 9.5. The bunch Θ is free if and only if $\widehat{w}_i(\widehat{v}_i) = 1$ holds for all $1 \le i \le r$.

In the free case, we have the following integral version of the corresponding statements in Proposition 9.1 and Theorem 9.2:

Proposition 9.6. *Assume that Θ is a free bunch. Then we have:*

$$K \cong \mathrm{ClDiv}(X), \qquad \mathrm{Pic}(X) \cong \bigcap_{\gamma_0 \in \mathrm{cov}(\Theta)} Q(\lim(\gamma_0) \cap E).$$

Proof. Using Remark 9.5, we infer from freeness of Θ that the map $\mathfrak{D} \colon E \to \mathrm{WDiv}^T(X)$ is an isomorphism. Using the 5-Lemma in the diagram of Proposition 9.1, we obtain that also the map $\overline{\mathfrak{D}} \colon K \cong \mathrm{ClDiv}(X)$ is an isomorphism. This gives the first part of the assertion.

The rest of the proof is similar to that of Proposition 9.2: Suppose that $w \in K$ lies in all $Q(\mathrm{lin}(\gamma_0) \cap E)$, where $\gamma_0 \in \mathrm{cov}(\Theta)$. Let $\widehat{w} \in E$ with $Q(\widehat{w}) = w$. Then for every $\sigma \in \Delta^{\max}$ we can choose ℓ_σ even in $\widehat{\sigma}^\perp \cap E$ with $Q(\ell_\sigma) = w$. But then all $u_\sigma := \widehat{w} - \ell_\sigma$ lie in M. Therefore $\mathfrak{D}(\widehat{w})$ is a Cartier divisor and thus we have $w \in \mathrm{Pic}(X)$.

Conversely, for a class $w \in K \cong \mathrm{Pic}(X)$, there is a $\widehat{w} \in E \cong \mathrm{WDiv}^T(X)$ representing w. As this $\widehat{w}$ is in fact a Cartier divisor, it is described by a support function $(u_\sigma)_{\sigma \in \Delta^{\max}}$ with $u_\sigma \in M$. But then all $\ell_\sigma = \widehat{w} - P^*(u_\sigma)$ are elements of E which proves the assertion. $\qquad\square$

We present some further applications. Recall that a toric variety is said to be $\mathbb{Q}$-*Gorenstein* if some multiple of its anticanonical divisor is Cartier. In terms of bunches, we have the following characterization.

Corollary 9.7. *Suppose that Θ is free. Then X is $\mathbb{Q}$-Gorenstein if and only if we have*

$$\sum_{i=1}^r w_i \ \in \ \bigcap_{\tau \in \Theta} \mathrm{lin}(\tau). \qquad\square$$

Similarly we can decide whether a toric variety is a Fano variety, i.e. its canonical divisor is Cartier and ample:

Corollary 9.8. (i) *Suppose that Θ is free. Then X is Fano if and only if we have*

$$\sum_{i=1}^r w_i \ \in \ \bigcap_{\gamma_0 \in \mathrm{cov}(\Theta)} Q(\gamma_0 \cap E)^\circ.$$

(ii) *Suppose that Θ is free and X is smooth. Then X is Fano if and only if we have*

$$\sum_{i=1}^r w_i \ \in \ \bigcap_{\tau \in \Theta} \tau^\circ. \qquad\square$$

We now apply our results to the case of simple bunches. Recall that these describe precisely the $\mathbb{Q}$-factorial 2-complete toric varieties. The first observation is that in this case we immediately obtain the Picard group.

Proposition 9.9. *Suppose that Θ is a simple bunch. Then $\mathfrak{D} \colon E \to \mathrm{WDiv}^T(X)$ induces an isomorphism $K \cong \mathrm{Pic}(X)$.*

Proof. As Θ is simple, Lemma 3.6 tells us that $(F \xrightarrow{P} N, \delta)$ and $\mathfrak{F}(\Theta)$ are the (unique) free Cox construction of the fan Δ of X. Hence, it suffices to show that the image of $\mathfrak{D}$ is the group of invariant Cartier divisors $\mathrm{CDiv}^T(X)$.

Now, for an element $\widehat{w} \in E$, consider the restrictions $\widehat{w}|_{\mathrm{lin}(\widehat{\sigma}) \cap F}$, where $\sigma \in \Delta^{\max}$. By the properties of a free Cox construction,

$$P \colon \mathrm{lin}(\widehat{\sigma}) \cap F \to N$$

is an isomorphism for each $\sigma \in \Delta^{\max}$. Hence, $\widehat{w}|_{\mathrm{lin}(\widehat{\sigma}) \cap F}$ defines a $\mathbb{Z}$-valued linear function on N. This shows that $\mathfrak{D}(\widehat{w})$ is in fact a Cartier divisor.

On the other hand, if D is a Cartier divisor, we have to show that $\widehat{w} := \mathfrak{D}^{-1}(D)$ is $\mathbb{Z}$-valued on F. As above, we see that the restriction of $\widehat{w}$ to any sublattice $\mathrm{lin}(\widehat{\sigma}) \cap F$ is $\mathbb{Z}$-valued. This implies the assertion, because the proof of Lemma 8.4 shows that we have

$$F = \sum_{\widehat{\sigma}} \mathrm{lin}(\widehat{\sigma}) \cap F. \qquad \square$$

For a $\mathbb{Q}$-factorial toric variety over $\mathbb{C}$, we can identify $H^2(X, \mathbb{Q})$ with $K_{\mathbb{Q}}$. Since $L_{\mathbb{Q}}$ is in a canonical way the dual vector space to $K_{\mathbb{Q}}$ we can identify $L_{\mathbb{Q}}$ with $H_2(X, \mathbb{Q})$. Under this duality, the Mori Cone, i.e., the cone $\mathrm{NE}(X)$ of numerically effective curve classes in $H_2(X, \mathbb{Q})$ is dual to the cone of numerically effective divisor classes $\mathcal{N}(X) \subset K_{\mathbb{Q}}$.

Reid proved that the Mori Cone of a $\mathbb{Q}$-factorial complete toric variety is precisely the cone generated by the classes of the invariant curves, see [21, Cor. 1.7]. Using the fact that on a $\mathbb{Q}$-factorial complete toric variety semiample and numerically effective divisors coincide, Theorem 9.2 gives a new description of the Mori Cone:

Corollary 9.10. *Suppose that X_Θ is complete and simplicial. Then the cone of numerically effective curve classes in $H_2(X, \mathbb{Q})$ is given by*

$$\mathrm{NE}(X_\Theta) \cong \sum_{\tau \in \Theta} \tau^\vee.$$

In particular, this cone is convex and polyhedral. Moreover, X_Θ is projective if and only if $\mathrm{NE}(X_\Theta)$ is strictly convex. $\qquad \square$

10. APPLICATIONS AND EXAMPLES

In this section, we present some applications and examples. First, we perform Kleinschmidt's classification in the setting of bunches. We use the visualition techniques introduced at the end of Section 2:

Proposition 10.1. *The 2-complete smooth toric varieties X with $\mathrm{Cl}(X) \cong \mathbb{Z}^2$ and $\mathcal{O}(X) \cong \mathbb{K}$ correspond to free bunches $\Theta = \{\mathrm{cone}(w_1, w_n)\}$ given by*

- *weight vectors $w_1 := (1, 0)$ and $w_i := (b_i, 1)$, with $0 = b_n < b_{n-1} < \ldots < b_2$,*
- *multiplicities $\mu_i := \mu(w_i)$ with $\mu_1 > 1$, $\mu_n > 0$ and $\mu_2 + \ldots + \mu_n > 1$.*
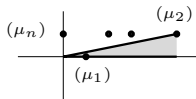
Moreover, the toric variety X defined by such a bunch Θ is always projective, and it is Fano if and only if we have

$$b_2(\mu_3 + \ldots + \mu_n) < \mu_1 + b_2\mu_3 + \ldots + b_{n-1}\mu_{n-1}.$$

Proof. We only have to show that every smooth 2-complete toric variety X with $\mathrm{Cl}(X) \cong \mathbb{Z}^2$ and $\mathcal{O}(X) \cong \mathbb{K}$ arises from a bunch as in the assertion. ¿From Theorem 6.3 (ii), we infer that the toric variety X arises from a free bunch Θ in a projected cone $(\mathbb{Z}^m \xrightarrow{Q} \mathbb{Z}^2, \gamma)$ with $\gamma = \mathrm{cone}(e_1, \ldots, e_m)$. Let $\{w_1, \ldots, w_n\}$ be the set of weight vectors, and let μ_i be the multiplicity of w_i.

Smoothness of X means that every image $Q(\gamma_0)$, where $\gamma_0 \in \mathrm{cov}(\Theta)$, is a regular twodimensional cone in $\mathbb{Z}^2$, see Proposition 7.3. Since we have $\mathcal{O}(X) = \mathbb{K}$, Proposition 7.4, tells us that the weight vectors (counted with multiplicities) generate a strictly convex cone $\vartheta \subset K_{\mathbb{Q}}$.

Now we make essentially use of the fact that Θ lives in a twodimensional space: Consider the intersection τ_0 of all $\tau \in \Theta$. Then τ_0 is a weight cone. Moreover, τ_0 is of dimension two, because for any two cones of Θ their relative interiors intersect. Thus the defining property of a bunch implies $\Theta = \{\tau_0\}$.

As a strictly convex twodimensional weight cone containing τ_0, also ϑ occurs among the images $Q(\gamma_0)$, where $\gamma_0 \in \mathrm{cov}(\Theta)$. Consequently, ϑ is a regular cone in $\mathbb{Z}^2$. Hence, using Remark 2.10, we may assume that ϑ is generated by the weight vectors w_1 and w_n.

We claim that the cone τ_0 has at least one ray in common with ϑ. Indeed, otherwise one of the cones generated by $\tau_0 \cup \mathbb{Q}_{\geq 0} e_1$ or $\tau_0 \cup \mathbb{Q}_{\geq 0} e_2$ would be non regular. But both cones occur among $Q(\gamma_0)$, where $\gamma_0 \in \mathrm{cov}(\Theta)$. A contradiction. Again by Remark 2.10, we may assume that τ_0 contains e_1, and hence is generated by vectors e_1 and $b_2 e_1 + e_2$, where $b_2 \geq 0$.

Any twodimensional cone generated by e_1 and a weight vector w_i is an image $Q(\gamma_0)$ of some $\gamma_0 \in \mathrm{cov}(\Theta)$, and hence it is regular. We can conclude that each w_i is of the form $b_i e_1 + e_2$. So we arrive at the desired picture. Note that the conditions $\mu_1 > 1$ and $\mu_2 + \ldots + \mu_n > 1$ are due to Property 6.1 (iii).

The fact that the toric variety X associated to Θ is projective is clear by Proposition 7.6 and Corollary 9.3. Moreover, Corollary 9.8 tells us that X is Fano if and only if the weighted sum $\mu_1 w_1 + \ldots + \mu_n w_n$ lies in the relative interior of the cone τ. But this holds if and only if the condition stated in the assertion is valid. $\quad\square$

Let us illustrate the quasiprojectivity criterion 9.3 by means of the following version of a classical example taken from Oda's book [18, p. 84]:
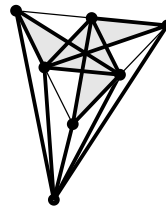
Example 10.2. The simplest nonprojective complete simplicial fans Δ live in $\mathbb{Z}^3$ and are of the following type: Consider a prism $P \subset \mathbb{Q}^3$ over a 2-simplex such that 0 lies in the relative interior of P. Here we take P with the vertices

$$(-1,0,0), \quad (0,-1,0), \quad (0,0,-1), \quad (0,1,1), \quad (1,0,1), \quad (1,1,0).$$

Then subdivide the facets of P according to the picture below, and let Δ be the fan generated by the cones over the simplices of this subdivision.
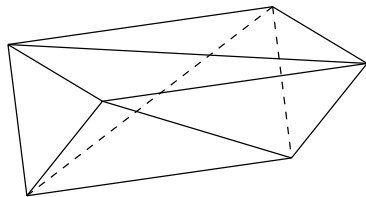
Defining polytope subdivision of Δ Corresponding bunch Θ

The bunch Θ corresponding to Δ is free and lives in a 3-dimensional lattice $K \cong \mathbb{Z}^3$. Combinatorially, it looks as indicated above. Explicitly it is given by the weight vectors e_1, e_2, e_3, $w_1 := e_1 + e_2$, $w_2 := e_1 + e_3$, and $w_3 = e_2 + e_3$ in $\mathbb{Z}^3$, and the four cones

$$\begin{array}{ll} \mathrm{cone}(e_3, w_1, w_2), & \mathrm{cone}(e_1, w_1, w_3) \\ \mathrm{cone}(e_2, w_2, w_3), & \mathrm{cone}(e_1, e_2, e_3) \end{array}$$

Using the Propositions 7.1, 7.6, and Corollary 9.3 it is immediately clear from the picture that $X := X_\Theta$ is $\mathbb{Q}$-factorial and nonprojective. Moreover, by Theorem 9.2,

the cone $C^{sa}(X)$ of semiample divisors is spanned by the class of the canonical divisor.

The last example concerns the problem whether or not the Betti numbers of a toric variety are determined by the combinatorial type of the defining fan. A first counterexample was given by McConnell, see [16]. Later Eikelberg [7] gave the following simpler one:

Example 10.3. Let $P \subset \mathbb{Q}^3$ be a prism over a 2-simplex such that 0 lies in the relative interior of P, and define Δ to be the fan generated the cones over the facets of P. As vertices of P we take:

$$v_1 := (1, 0, 1), \quad v_2 := (0, 1, 1), \quad v_3 := (-1, -1, 1),$$
$$v_4 := (1, 0, -1), \quad v_5 := (0, 1, -1), \quad v_6 := (-1, -1, -1).$$

Eikelberg gains a nonprojective fan Δ' from Δ by moving the ray through v_2 into the ray through $v_2' := (1, 2, 3)$. The picture is the following:
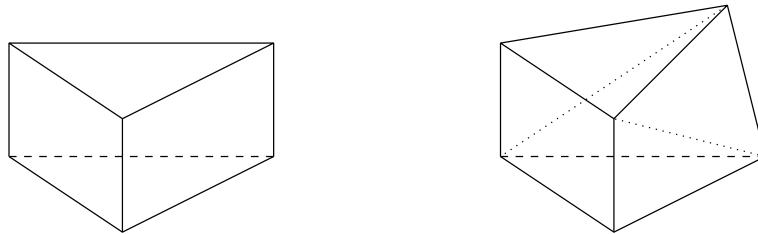
Ordinary prism P One ray moved

What does this mean in terms of bunches? First note that the bunches Θ and Θ' corresponding to Δ and Δ' contain 2-dimensional cones, because Δ as well as Δ' have nonsimplicial cones. The loss of projectivity is reflected in terms of bunches as follows:
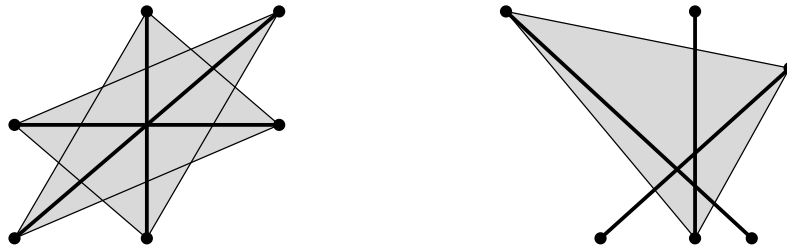
Bunch Θ corresponding to Δ Bunch Θ' corresponding to Δ'

Here we draw the intersection of Θ and Θ' with a plane orthogonal to an inner vector of the (strictly) convex hulls $|\Theta|$ and $|\Theta'|$ of the unions of the respective bunch cones. So the fat lines correspond to 2-dimensional bunch cones, whereas the colored simplices represent 3-dimensional bunch cones.

Now, both fans Δ and Δ' have the same combinatorial type. In terms of bunches, we see immediately that the associated toric varieties X and X' have different Betti numbers: The second Betti number $b_2(X)$ equals the dimension of $\mathrm{Pic}_\mathbb{Q}(X)$. Hence Theorem 9.2 gives us $b_2(X) = 1$. For X', we obtain $b_2(X') = 0$ by the same reasoning.

We conclude with a remark concerning Ewald's construction of "canonical extensions" of a given toric variety presented in [8, Section 3], compare also [5]. The bunch theoretical analogue is the following:

Construction 10.4. Consider a free bunch, represented in the sense of Construction 2.9 by a set weight vectors $\{w_1, \dots, w_n\}$ in a lattice K, multiplicities $\mu_1, \dots, \mu_n$ and a collection Θ of weight cones. Define a new free bunch by setting

$$K' := K, \qquad w_i' := w_i, \qquad \Theta' := \Theta$$

and replacing the multiplicities μ_i with bigger ones, say μ_i'. For the toric varieties X and X' associated to these bunches, we have

$$\dim(X') - \dim(X) = \sum_{i=1}^n (\mu_i' - \mu_i), \qquad \mathrm{Pic}(X') \cong \mathrm{Pic}(X).$$

Moreover, applying the respective characterizations of Section 7 and 9, one immediately verifies that X' is non quasiprojective (complete, $\mathbb{Q}$-factorial, smooth) if X was so.

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