On Bures fidelity of displaced squeezed thermal states

Xiang-Bin Wang *, C. H. Oh † and L. C. Kwek ‡

Department of Physics, Faculty of Science,

National University of Singapore, Lower Kent Ridge,

Singapore 119260, Republic of Singapore.

Fidelity plays a key role in quantum information and communication theory. Fidelity can be interpreted as the probability that a decoded message possesses the same information content as the message prior to coding and transmission. In this paper, we give a formula of Bures fidelity for displaced squeezed thermal states directly by the displacement and squeezing parameters and birefly discuss how the results can apply to quantum information theory.

An important tenet in classical information theory is the rigorous establishment of the Shannon Noiseless Coding Theorem, in which one shows that Shannon entropy can be interpreted as the average number of bits needed to code the output of a message source under ideal conditions. The analogous quantum version of Shannon's Coding Theorem is the Schumacher Quantum Coding Theorem [1]. In the quantum version, one introduces the idea of fidelity which can be interpreted as the probability that a decoded message carries the same information as the message prior to coding and replaces Shannon entropy with fidelity. More specifically, one can proves the Schumacher Noiseless Coding Theorem which states that if M is a quantum signal source with signal ensemble described by the density operator ρ then $\forall\ \ \delta, \epsilon > 0$,

*E-mail address: scip7236@leonis.nus.edu.sg

†E-mail address: phyohch@leonis.nus.edu.sg

‡E-mail address: scip6051@leonis.nus.edu.sg

- if $S(\rho)+\delta$ qubits are available per M signal, then, for sufficiently large N, groups of N signal from the signal source M can be transposed through the available qubits with fidelity $F > 1-\epsilon$.

- if $S(\rho)-\delta$ qubits are available per M signal, then, for sufficiently large N, groups of N signals from the signal source A can be transposed through the avaiable qubits with fidelity $F < \epsilon$.

Suppose a quantum signal source, M, generates a signal state $|i_A>$ with probability $p(a)$ and the density operator, ρ, is described by the equation

$$\rho = \sum_a p(a)|a_M >< a_M|, \tag{1}$$

one can define the Schumacher fidelity, F, as the overall probability that a signal from an ensemble M can be transmitted to M' using the relation [1, 2]

$$F = \sum_a p_i \mathrm{Tr}(\pi_a \rho'_a), \tag{2}$$

where $p_i \equiv |a_M >< a_M|$ and ρ'_a denotes the density operator of the final signal in M'. This definition applies strictly to pure states and it is generally not clear how it can be applied to mixed states.

Closely related to the problem of coding is the process of entanglement purification protocol (EPP) and quantum error-correction codes (QECC) [3, 4]. These protocols essentially shield quantum states from the environment. In EPP, maximally entangled states are extracted (or purified) from a mixed states while in QECC, an arbitrary quantum state is transmitted at some rate through a noisy channel with miminmal degradation. Central to the idea of entanglement is the need to define a measure of entanglement. Bennett and others have proposed a measure of entanglement using the von Neumann entropy. However, it is sometimes difficult to compute and obtain a closed form using their definition. Recently, Vedral and others have studied a wdie class of measures suitable for entanglement and they have proposed the Bures metric as an example of a possible means of quantifying entanglement or fidelity [5].

Finally, we note that experimentally, it is well-known that a squeezed electromagnetic field [6] provides a means to overcome the standard quantum limit for noise imposed by vacuum fluctuations. Furthermore, although the number-state channel is an optimal channel for quantum communication theory, it is often more realistic to consider the quadrature-squeezed channel [7] experimentally for several reasons. Firstly, one cannot faithfully reproduce the number eigenstates easily and secondly amplification of quadrature-squeezed channel can be realized experimentally using a phase-sensitive amplifier. Clearly, one should therefore investigate the plausibility of applying squeezed or displaced squeezed thermal states to quantum information and communication theory.

Indeed, Bures fidelity has been an important concept in the field of quantum optics. Recently, Twamley [8] has calculated the Bures fidelity for squeezed thermal states. Due to some technical difficulties, the displaced squeezed states was not considered in his article. Very recently, Scutario [9] proposed an approach to calculate the Bures fidelity for systems with quadratic Hamiltonian. However, a closed form for the matrix elements of the density operator is not explicitly given and the final result does not relate fidelity directly with the squeezing and displacement parameters. However, in a more recent paper [10], he has obtained an explicit form for the Bures fidelity for two displaced thermal states. In this article, we show an alternative method in which one can actually calculate the fidelity of displaced squeezed thermal states by simply using the Baker-Compell-Hausdorf(BCH) formula. A closed form result for displaced squeezed thermal states expressed in squeezing and displacement parameters is also obtained.

Squeezed states occur in a myriad of non-linear optical phenomena like optical parametric oscillation and four-wave mixing [11]. The single-mode squeezed states can be generated from the vaccuum by the action of the squeezed operator S,

$$S(\zeta) = \exp(\frac{1}{2}(\zeta^* a^2 - \zeta a^{\dagger 2})), \tag{3}$$

where $\zeta = r e^{i\phi}$ is a complex number with modulus r and argument ϕ, representing the squeezing parameter. The density operator of displaced squeezed thermal states can be defined as:

$$\rho = DS\Lambda S^{+}D^{+} \tag{4}$$

where $D = \exp\left[(a^{+}, a)\begin{pmatrix} k \\ -k^{*}\end{pmatrix}\right]$, $S = \exp\left[\frac{1}{2}r\left((a^2 - a^{+2}\right)\right)$ and $\Lambda = \exp[-\frac{1}{2}(aa^{+} + a^{+}a)]$. Note that we have considered the squeezing parameter to be real since the most important parameter of a squeezed state is the squeezed factor r and not its argument ϕ [12]. The general case in which the argument ϕ is nonzero can be treated similarly. We next recall that the Bures fidelity, F, can be defined by the relation

$$F = \left(\mathrm{tr}\sqrt{\rho_1{}^{\frac{1}{2}}\rho_2\rho_1{}^{\frac{1}{2}}}\right)^2. \tag{5}$$

For two displaced squeezed thermal states, one easily sees that the Bures fidelity can be expressed as

$$F = \left(\mathrm{tr}\sqrt{(D_1S_1\Lambda_1^{1/2}S_1^{\dagger}D_1^{\dagger})(D_2S_2\Lambda_2S_2^{\dagger}D_2^{\dagger})(D_1S_1\Lambda_1^{1/2}S_1^{\dagger}D_1^{\dagger})}\right)^2. \tag{6a}$$

$$= \left(\mathrm{tr}\sqrt{\Lambda_1^{1/2}S_1^{\dagger}D_1^{\dagger}D_2S_2\Lambda_2S_2^{\dagger}D_2^{\dagger}D_1S_1\Lambda_1^{1/2}}\right)^2 \tag{6b}$$

To simplify eq(6) [13], we need to rewrite $D_1{}^{+}D_2$ as

$$D_1{}^{+}D_2 = c \cdot D_0 = c \cdot \exp\left[(a^{+}, a)\begin{pmatrix} g \\ -g^{*}\end{pmatrix}\right] \text{ and } \quad D_2^{\dagger}D_1 = \frac{1}{c}\cdot D_0^{\dagger} \tag{7a}$$

where $\begin{pmatrix} g \\ -g^{*}\end{pmatrix} = \begin{pmatrix} k_2 - k_1{}^{*} \\ -\left(k_2 - k_1{}^{*}\right)^{*}\end{pmatrix}$ and c is a number. Thus, one can rewrite the formula for Bures fidelity of displaced squeezed thermal states appears as

$$F = \left(\mathrm{tr}\sqrt{\Lambda_1{}^{\frac{1}{2}}S_1{}^{+}D_0{}^{+}S_2\Lambda_2D_0S_1\Lambda_1{}^{\frac{1}{2}}}\right)^2 \tag{8}$$

with $D_0 = D_1{}^{+}D_2$. Eq(8) needs some simplication before we can actually proceed with the detailed calculations. Before we do this, we need to invoke the BCH relation,

$$S(a^{+}, a)S^{+} = (a^{+}, a)M; \quad S^{+}(a^{+}, a)S = (a^{+}, a)M^{-1}; \tag{9}$$

where $M = \begin{pmatrix} \mathrm{ch}r & -\mathrm{sh}r \\ -\mathrm{sh}r & \mathrm{ch}r \end{pmatrix}$ and

$$\Lambda(a^+, a)\Lambda^{-1} = (a^+, a)B. \tag{10}$$

Note that in eq(10), we have introduced the matrix $B \equiv \begin{pmatrix} \exp(-\beta) & 0 \\ 0 & \exp(\beta) \end{pmatrix}$. Let us define the matrix Ω as $\Lambda_1{}^{\frac{1}{2}} S_1{}^+ D_0{}^+ S_2 \Lambda_2 S_2^\dagger D_0 S_1 \Lambda_1{}^{\frac{1}{2}}$ in eq(8). It is instructive to note that, by using BCH formula, one can readily express the matrix Ω in a more convenient form as

$$\Omega = \Lambda_1{}^{\frac{1}{2}} S_1{}^+ S_2 \Lambda_2{}^{\frac{1}{2}} \exp\left[(a^+, a) B_2{}^{-\frac{1}{2}} M_2{}^{-1} \begin{pmatrix} g \\ -g* \end{pmatrix}\right]$$

$$\times \exp\left[-(a^+, a) B_2{}^{\frac{1}{2}} M_2{}^{-1} \begin{pmatrix} g \\ -g* \end{pmatrix}\right] \Lambda_2{}^{\frac{1}{2}} S_2^\dagger S_1 \Lambda_1{}^{\frac{1}{2}} \tag{11}$$

The linear terms within the exponential factor in the abovee formula (11) can be collapsed into a simpler term by using the following results (see Appendix A for a detailed proof):

$$\exp\left[(a^+, a) N_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] \exp\left[(a^+, a) N_2 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}\right]$$

$$= \exp\left[-\frac{1}{2}(z_1, z_2) \widetilde{N_1} \Sigma N_2 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}\right] \exp\left[(a^+, a) N_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + (a^+, a) N_2 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}\right] \tag{12}$$

where N_1, N_2 are arbitrary 2×2 complex matrices, Σ is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and z is arbitrary complex number. In this mannner, one sees that

$$\Omega = \delta_1 \rho_+ \exp\left[(a^+, a)(B_2{}^{-\frac{1}{2}} - B_2{}^{\frac{1}{2}}) M_2{}^{-1} \begin{pmatrix} g \\ -g^* \end{pmatrix}\right] \rho_- \tag{13}$$

and

$$\delta_1 = \exp\left[\frac{1}{2}(g, -g^*) \widetilde{M_2}^{-1} B_2{}^{-\frac{1}{2}} \Sigma B_2{}^{\frac{1}{2}} M_2{}^{-1} \begin{pmatrix} g \\ -g* \end{pmatrix}\right] \tag{14}$$

$\rho_+ = \Lambda_1{}^{\frac{1}{2}} S_1{}^+ S_2 \Lambda_2{}^{\frac{1}{2}},\ \rho_- = \Lambda_2{}^{\frac{1}{2}} S_2^\dagger S_1 \Lambda_1{}^{\frac{1}{2}}$

Let us now consider another operator

$$\Omega' = U\rho_+\rho_- U^+ \tag{15}$$

where $U = \exp\left[(a^+, a)\begin{pmatrix} l \\ -l^* \end{pmatrix}\right]$. If we apply BCH formula again, we shall see that

$$\Omega' = \rho_+ \exp\left[(a^+, a) B_2{}^{-\frac{1}{2}} M_2{}^{-1} M_1 B_1{}^{-\frac{1}{2}} \begin{pmatrix} l \\ -l^* \end{pmatrix}\right]$$

$$\times \exp\left[-(a^+, a) B_2{}^{\frac{1}{2}} M_2{}^{-1} M_1 B_1{}^{\frac{1}{2}} \begin{pmatrix} l \\ -l^* \end{pmatrix}\right] \rho_- \tag{16}$$

$$\Rightarrow \Omega' = \delta_2 \rho_+ \exp\left[(a^+, a)(B_2{}^{-\frac{1}{2}} M_2{}^{-1} M_1 B_1{}^{-\frac{1}{2}} - B_2{}^{\frac{1}{2}} M_2{}^{-1} M_1 B_1{}^{\frac{1}{2}}) \begin{pmatrix} l \\ -l^* \end{pmatrix}\right] \rho_- \tag{17}$$

and

$$\delta_2 = \exp\left[\frac{1}{2}(l, -l^*) B_1{}^{-\frac{1}{2}} \widetilde{M_1} \widetilde{M_2}^{-1} B_2{}^{-\frac{1}{2}} \Sigma B_2{}^{\frac{1}{2}} M_2{}^{-1} M_1 B_1{}^{\frac{1}{2}} \begin{pmatrix} l \\ -l^* \end{pmatrix}\right] \tag{18}$$

Setting

$$(B_2{}^{-\frac{1}{2}} M_2{}^{-1} M_1 B_1{}^{-\frac{1}{2}} - B_2{}^{\frac{1}{2}} M_2{}^{-1} M_1 B_1{}^{\frac{1}{2}}) \begin{pmatrix} l \\ -l^* \end{pmatrix} = (B_2{}^{-\frac{1}{2}} - B_2{}^{\frac{1}{2}}) M_2{}^{-1} \begin{pmatrix} g \\ -g^* \end{pmatrix} \tag{19}$$

we get

$$\Omega = \frac{\delta_1}{\delta_2} \Omega' \tag{20}$$

$$\left(\mathrm{tr}\sqrt{\Omega}\right)^2 = \frac{\delta_1}{\delta_2}\left(\mathrm{tr}\sqrt{U\rho_+\rho_- U^+}\right)^2 = \frac{\delta_1}{\delta_2}(\mathrm{tr}\sqrt{\rho_+\rho_-})^2 \tag{21}$$

Since $\left(\mathrm{tr}\sqrt{\rho_+\rho_-}\right)^2$ has already been computated in Ref [8], we can solved the whole problem by considering the reduced calculation of $\frac{\delta_1}{\delta_2}$. Following Twamley paper, one notes that the quantity $(\mathrm{tr}\sqrt{\rho_+\rho_-})^2$ in eq(21) can be written as

$$(\mathrm{tr}\sqrt{\rho_+\rho_-})^2 = \frac{2\sinh\frac{\beta_1}{4}\sinh\frac{\beta_2}{4}}{\sqrt{\sqrt{Y}-1}} \tag{22}$$

where $Y = \cosh^2(r_1-r_2)\cosh^2\frac{\beta_1+\beta_2}{4}+\cosh^2(r_1+r_2)\cosh^2\frac{\beta_1+\beta_2}{4}-\sinh^2(r_1-r_2)\cosh^2\frac{\beta_1-\beta_2}{4}-\cosh^2(r_1+r_2)\cosh^2\frac{\beta_1-\beta_2}{4}$. From eqs (18) and (19), one quickly get:

$$\delta_2 = \exp\Big\{(l,-l^*)B_1{}^{-\frac{1}{2}}\widetilde{M_1}\widetilde{M_2}^{-1}B_2{}^{-\frac{1}{2}}\cdot\Sigma\cdot$$
$$\times\left[B_2{}^{-\frac{1}{2}}M_2{}^{-1}M_1B_1{}^{-\frac{1}{2}}\begin{pmatrix} l \\ -l^* \end{pmatrix} - (B_2{}^{-\frac{1}{2}}-B_2{}^{\frac{1}{2}})M_2{}^{-1}\begin{pmatrix} g \\ -g^* \end{pmatrix}\right]\Big\} \tag{23}$$

It is instructive to note that the matrix B and M are all sympletic matrices, so that we have

$$(l,-l^*)B_1{}^{-\frac{1}{2}}\widetilde{M_1}\widetilde{M_2}^{-1}B_2{}^{-\frac{1}{2}}\Sigma B_2{}^{-\frac{1}{2}}M_2{}^{-1}M_1B_1{}^{-\frac{1}{2}}\begin{pmatrix} l \\ -l^* \end{pmatrix} = (l,-l^*)\Sigma\begin{pmatrix} l \\ -l^* \end{pmatrix} = 0. \tag{24}$$

With this observation, it is straightforward to see that eq(18) can be simplified as

$$\delta_2 = \exp\left[-(l,-l^*)B_1{}^{-\frac{1}{2}}\widetilde{M_1}\widetilde{M_2}^{-1}B_2{}^{-\frac{1}{2}}\Sigma(B_2{}^{-\frac{1}{2}}-B_2{}^{\frac{1}{2}})M_2{}^{-1}\begin{pmatrix} g \\ -g^* \end{pmatrix}\right]. \tag{25}$$

To obtain the final explicit form of δ_2, one notes that from eq(19), we have

$$(l,-l^*) = (g,-g^*)\widetilde{M_2}^{-1}(B_2{}^{-\frac{1}{2}}-B_2{}^{\frac{1}{2}})\widetilde{P}^{-1} \tag{26}$$

where the matrix $P \equiv (B_2{}^{-\frac{1}{2}}M_2{}^{-1}M_1B_1 - B_2{}^{\frac{1}{2}}M_2{}^{-1}M_1B_1{}^{\frac{1}{2}})$. If we plug eq(26) into the eq(24), we arrive at the following formula for calculation of δ_2

$$\delta_2 = \exp\Big[-\frac{1}{2}(g,-g^*)\widetilde{M_2}^{-1}(B_2{}^{-\frac{1}{2}}-B_2{}^{\frac{1}{2}})\widetilde{P}^{-1}$$
$$\times B_1{}^{-\frac{1}{2}}\widetilde{M_1}\widetilde{M_2}^{-1}B_2{}^{-\frac{1}{2}}\Sigma(B_2{}^{-\frac{1}{2}}-B_2{}^{\frac{1}{2}})M_2{}^{-1}\begin{pmatrix} g \\ -g^* \end{pmatrix}\Big] \tag{27}$$

In our case, it is not difficult to evaluate the expression for δ_1 and δ_2 explicitly. To do this, we note that if we denote

$$\delta_1 = \exp\left[\frac{1}{2}(g,-g^*)Q_1\begin{pmatrix} g \\ -g^* \end{pmatrix}\right], \tag{28}$$

then the matrix Q_1 is simply

$$Q_1 = \begin{pmatrix} \sinh\beta_2 \sinh(2r_2) & \cosh\beta_2 + \sinh\beta_2\cosh(2r_2) \\ -\cosh\beta_2 + \sinh\beta_2\cosh(2r_2) & \sinh\beta_2\sinh(2r_2) \end{pmatrix}. \tag{29}$$

For δ_2, one notes that a straightforward computation for the matrix P yields

$$P = \frac{1}{\Delta}\begin{pmatrix} \sinh\frac{\beta_2+\beta_1}{2}\cosh(r_1-r_2) & \sinh\frac{\beta_2-\beta_1}{2}\sinh(r_1-r_2) \\ -\sinh\frac{\beta_2-\beta_1}{2}\sinh(r_1-r_2) & -\sinh\frac{\beta_2+\beta_1}{2}\cosh(r_1-r_2) \end{pmatrix} \tag{30}$$

with $\Delta = \cosh\beta_1\cosh\beta_2 + \sinh\beta_1\sinh\beta_2\cosh 2(r_1-r_2) - 1$, so that if we denote

$$\frac{\delta_1}{\delta_2} = \exp\left\{\frac{1}{2}(g, -g^*) R \begin{pmatrix} g \\ -g^* \end{pmatrix}\right\}, \tag{31}$$

then a straightforward, albeit tedious, calculation yields

$$\begin{aligned} R = & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{2}{\Delta}\sinh\beta_1\sinh^2\frac{\beta_2}{2}\begin{pmatrix} \sinh(2r_1) & \cosh(2r_1) \\ \cosh(2r_1) & \sinh(2r_1) \end{pmatrix} \\ & + \frac{2}{\Delta}\sinh^2\frac{\beta_1}{2}\sinh\beta_2\begin{pmatrix} \sinh(2r_2) & \cosh(2r_2) \\ \cosh(2r_2) & \sinh(2r_2) \end{pmatrix} \end{aligned} \tag{32}$$

so that the factor $\dfrac{\delta_1}{\delta_2}$ works out explicitly into

$$\exp\left\{\frac{1}{\Delta}(\epsilon_1 + \epsilon_2)\right\} \tag{33}$$

where

$$\epsilon_1 = \sinh\beta_1\sinh^2\frac{\beta_2}{2}\left[\left(g^2 + g^{*2}\right)\sinh 2r_1 - 2|g|^2\cosh 2r_1\right] \tag{34a}$$

$$\epsilon_2 = \sinh^2\frac{\beta_1}{2}\sinh\beta_2\left[\left(g^2 + g^{*2}\right)\sinh 2r_2 - 2|g|^2\cosh 2r_2\right]. \tag{34b}$$

One can easily show that that $\frac{\delta_1}{\delta_2} < 1$ as it should be and that in the limit $g = g^* = 0$, the ratio reduces to unity so that one obtains the Bures fidelity for the undisplaced squeezed states as shown in ref [8]. Further, one should also note that in the limit when $r \to 0$, one gets the Bures fidelity for the displaced thermal coherent states. This Bures fidelity is the same as the result previously obtained by Scutaru [10].

[1] B. Schumacher, Phys. Rev. A, **51**, 2738 (1995)

[2] H. Barnum, C. Fuchs, R. Josza and B. Schumacher, *Sumitted to Phys. Rev. A*, quant-ph/9603014.

[3] C.H. Bennett, D. P. DiVencenzo, J.A. Smolin and W. K. Wooters, Phys. Rev. A, **54**, 3824 (1996)

[4] C.H. Bennett, G. Brassard, S. Popescu, B. schumacher, J.A. Smolin and W. K. Wooters, Phys. Rev. Lett., **76**, 722 (1996), quant-ph/9511027.

[5] V. Vedral, M.B. Plenio, M.A. Rippin and P.L. Knight, Phys. Rev. Lett., **78**, 2275 (1997), quantu-ph/9702027.

[6] G.A. Garrett, A.G. Rojo, A.K. Sood, J.F. Whitaker, and R. Merlin, Science, **275**, 1638 (1997).

[7] G.M. D'Ariano and M.F. Sacchi, quant-ph/9704033.

[8] J. Twamley, J. Phys. A:Math. Gen., **29**, 3723 (1996)

[9] H. Scutaru, J. Phys. A:Math. Gen., **31**, 3659

[10] Gh.-S. Paraoanu and H. Scutaru, Phys. Rev. A, *In Press*, (1998), quant-ph/9703051.

[11] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics*, Oxford Science Publication, Oxford, 1997.

[12] B.L. Schumaker, Phys. Reps., **135**, 317 (1986).

[13] X. Wang, C.H. Oh and L.C. Kwek, J. Phys. A:Math. Gen., **31**, 4329

APPENDIX A:

In this appendix, we shall explicitly show the proof for eq(12). For simplicity and convenience, we shall define Ω_i as the expression

$$\Omega_i = (a^\dagger, a) N_i \begin{pmatrix} z_{2i-1} \\ z_{2i} \end{pmatrix}, \quad \text{for} \quad \text{i= 1,2}. \tag{A1}$$

To show eq(12), we need to compute $\mathrm{e}^{\Omega_1}\mathrm{e}^{\Omega_2}$. Since N_1 and N_2 are simply two arbitrary 2×2 matrices, one can always write in all generality

$$N_1 = \begin{pmatrix} a & d \\ b & c \end{pmatrix}, \quad N_2 = \begin{pmatrix} e & h \\ f & g \end{pmatrix} \tag{A2}$$

We next compute the commutator for Ω_1 and Ω_2.

$$[\Omega_1, \Omega_2] = -(az_1 + dz_2)(fz_3 + gz_4) + (bz_1 + cz_2)(ez_3 + hz_4). \tag{A3}$$

On the other hand, we should note that

$$\begin{aligned}
(z_1, z_2)\widetilde{N_1}\Sigma N_2 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} &= (z_1, z_2) \begin{pmatrix} a & b \\ d & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e & h \\ f & g \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \\
&= (az_1 + dz_2, bz_1 + cz_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} ez_3 + hz_4 \\ fz_3 + gz_4 \end{pmatrix} \\
&= (-bz_1 - cz_2, az_1 + dz_2) \begin{pmatrix} ez_3 + hz_4 \\ fz_3 + gz_4 \end{pmatrix} && \text{(A4a)} \\
&= -(bz_1 + cz_2)(ez_3 + hz_4) + (az_1 + dz_2)(fz_3 + gz_4) && \text{(A4b)}
\end{aligned}$$

Consequently, using Baker-Campbell-Hausdorff formula, on gets

$$\begin{aligned}
\mathrm{e}^{\Omega_1}\mathrm{e}^{\Omega_2} &= \mathrm{e}^{\frac{1}{2}[\Omega_1,\Omega_2]}\mathrm{e}^{\Omega_1+\Omega_2} && \text{(A5a)} \\
&= \exp\left[-\frac{1}{2}(z_1, z_2)\widetilde{N_1}\Sigma N_2 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}\right] \\
&\quad \times \exp\left[(a^\dagger, a)N_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + (a^\dagger, a)N_1 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}\right]. && \text{(A5b)}
\end{aligned}$$