Commun. Math. Phys. 115, 649-690 (1988)

Communications in Mathematical Physics

$C=1$ Conformal Field Theories on Riemann Surfaces

Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde

Institute for Theoretical Physics, Princetonplein 5, P.O. Box 80.006, NL-3508 TA Utrecht, The Netherlands

Abstract. We study the theory of $c=1$ torus and $\mathbb{Z}_2$-orbifold models on general Riemann surfaces. The operator content and occurrence of multi-critical points in this class of theories is discussed. The partition functions and correlation functions of vertex operators and twist fields are calculated using the theory of double covered Riemann surfaces. It is shown that orbifold partition functions are sensitive to the Torelli group. We give an algebraic construction of the operator formulation of these nonchiral theories on higher genus surfaces. Modular transformations are naturally incorporated as canonical transformations in the Hilbert space.

1. Introduction

Both in the study of two-dimensional critical phenomena and in string theory one of the major goals is to find a complete description of all conformal field theories [1]. In the first context these describe all universality classes of critical models [2], while in string theory they are known to correspond to all possible compactifications [3]. The conformal field theories with central charge c less than 1 have been successfully classified. The combined constraint of unitarity and one loop modular invariance selects a discrete set of models [4]. The extension of the analysis to c values larger than 1 appears to be a much harder problem and will surely reveal a quite different structure. For $c \geqq 1$ there is the possibility of a continuum of inequivalent models, since marginal operators can be present in the spectrum that generate continuous deformations of the conformal field theory.

In [5] an elegant formulation of two-dimensional conformal field theory has been given in terms of the behaviour of the partition function on the space of all inequivalent compact Riemann surfaces. The basic characteristics of the theory are translated into consistency conditions on the partition function, defined as the hermitian norm on a flat holomorphic vector bundle over this moduli space. The correlation functions are obtained through factorization of the partition function at the boundary of moduli space. Crossing symmetry of the amplitudes is then a consequence of modular invariance.

Related to this approach is the proposal to set up an operator formalism for conformal field theories on general Riemann surfaces [6] using techniques developed in the theory of soliton equations, notably the KP-hierarchy [7]. In this program one associates to any punctured Riemann surface a state in the Hilbert space, which encodes all information of the correlation functions on the surface. It provides a natural and powerful method for the analysis of the behaviour of partition functions on degenerate surfaces. On the other hand modular invariance seems much more difficult to formulate in this approach.

In this paper we will study with the above techniques the structure of the class of gaussian $c=1$ conformal field theories which consists of the torus and $\mathbb{Z}_2$-orbifold models. The torus model describes a free massless scalar field $\phi(z,\bar{z})$ compactified on the circle $\mathbb{R}/2\pi R\mathbb{Z}$ with action

$$S[\phi]=\frac{1}{2\pi}\int d^2z\,\partial\phi\,\bar{\partial}\phi\,. \tag{1.1}$$

The line of models parametrized by the compactification radius R describe $O(2)$ invariant statistical systems at criticality [8]. The $\mathbb{Z}_2$-orbifold models [9–11] are obtained from the corresponding torus model by identifying ϕ with $-\phi$. This line is known to describe the critical line of the Ashkin-Teller model [12–14], i.e. two Ising models coupled by a four spin interaction. Both lines are invariant under an electric-magnetic duality transformation relating the models at R and $2/R$. The torus model at $R=\frac{1}{2}\sqrt{2}$ and the orbifold model at $R=\sqrt{2}$ can be seen to correspond to one and the same theory. This equivalence is a well-known phenomenon in the context of string compactifications [9], and will turn out to be

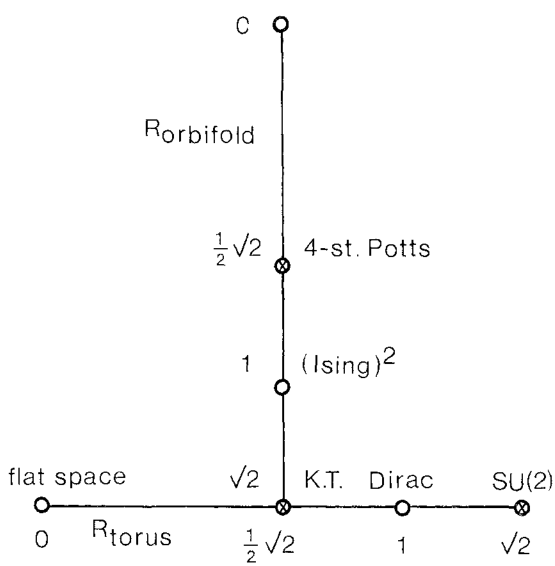

Fig. 1. The moduli space of gaussian $c=1$ conformal field theories. The crosses indicate multicritical points

a useful tool in the analysis of the orbifold model. In terms of critical models this point corresponds to the Kosterlitz-Thouless transition point of the XY-model [12]. Thus the space of conformal field theories we consider has the structure as displayed in Fig. 1. We will show it to be complete in the sense that there exists no marginal deformation of any of these theories that brings it out of this class of models.

At many special values of the parameters, in fact one can argue at any rational value of R^2, the model will have an enhanced symmetry. This will show both in the higher multiplicities in the spectrum and the finite dimensionality of the corresponding vector bundle over moduli space. We mention here the $R=\sqrt{2}$ torus model, which has a $SU(2)\times SU(2)$ symmetry, and the $R=1$ orbifold model, which is equivalent to two decoupled Ising models and consequently carries a representation of two $c=\frac{1}{2}$ Virasoro algebras [14, 15].

The main objective of this paper is to gain insight in the structure of these gaussian theories on higher genus surfaces, and especially we will investigate how this structure is influenced by the local properties of the theory, i.e. its operator content, symmetries, etc. To analyse the $\mathbb{Z}_2$-orbifold models we will use the theory of double coverings of Riemann surfaces and of theta-functions defined on Prym varieties [16, 17]. The equivalence of the $R=\sqrt{2}$ orbifold and $R=\frac{1}{2}\sqrt{2}$ torus model provides the necessary equations to solve for the quantum contributions to the partition and correlation functions, and gives a physical interpretation of various nontrivial mathematical identities known as Schottky relations [16]. We will show that the absence of momentum conservation in the orbifold theory has important consequences for the behaviour of the partition function under the modular group and near the boundary of moduli space. In particular in contrast with toroidal models the Torelli group acts nontrivially on the chiral constituents of the partition function.

The gaussian $c=1$ theories naturally admit an operator formulation on general Riemann surfaces. The Hilbert states representing these nonchiral theories on the surface can be constructed both within the path-integral formalism and in an abstract algebraic way, an approach which up to now was only well understood for chiral fermionic theories. The discussion of modular invariance in this context will give rise to an identification of modular transformations with a special set of canonical transformations in the Hilbert space.

The paper is organized as follows. In Sect. 2, after a short review of the basic properties of marginal operators and multi-critical points, we give the analysis leading to Fig. 1 and discuss the operator content of the various models and their mutual relations. In Sect. 3 we will analyse the partition functions on arbitrary Riemann surfaces and their behaviour under the modular group. The correlation functions of vertex operators and the factorization properties of the torus and orbifold model on higher genus surfaces are described in Sect. 4. In Sect. 5, we present the twist field correlation functions. The operator formalism on Riemann surfaces for both the torus and orbifold models is constructed in Sect. 6. We further give an extension of this formalism to include twist operators and discuss the relation with the tau-functions of the KP and the BKP-hierarchy. Finally, Sect. 7 contains some concluding remarks.

2. Operator Content and Multi-Critical Points in $c=1$ Gaussian Models

In this section we discuss the structure of the operator content of the lines of torus and $\mathbb{Z}_2$-orbifold models. We will analyse the multi-critical points on these lines and their mutual relations. To this end we first briefly review some basic properties of marginal operators and multi-critical points.

2a. *Marginal Operators and Multi-Critical Points*

A two-dimensional conformal field theory (CFT) [1] is fully specified by the value of the central charge c, the weights $(h_\alpha, \bar{h}_\alpha)$ and multiplicities of the primary fields ψ_α and their operator product relations. The two-point functions can be normalized as

$$\langle \psi_\alpha(z,\bar{z})\psi_\beta(w,\bar{w})\rangle = \delta_{\alpha\beta}(z-w)^{-2h_\alpha}(\bar{z}-\bar{w})^{-2\bar{h}_\alpha}. \tag{2.1}$$

The operator product coefficients $c_{\alpha\beta\gamma}$ defined by

$$\psi_\alpha(z,\bar{z})\psi_\beta(w,\bar{w}) \sim \sum_\gamma c_{\alpha\beta\gamma}(z-w)^{h_\gamma-h_\alpha-h_\beta}(\bar{z}-\bar{w})^{\bar{h}_\gamma-\bar{h}_\alpha-\bar{h}_\beta}\psi_\gamma(w,\bar{w}) \tag{2.2}$$

are then symmetric in all indices.

In general one can consider deformations of a conformal field theory, preserving the infinite conformal symmetry and the value of the central charge. In first order these perturbations are generated by the marginal operators ψ_i, i.e. the primary fields with conformal weight $(1,1)$ [12]. In a path-integral formulation of the theory these perturbations can be represented by an additional term in the action

$$\delta S = \sum_i \frac{\delta g_i}{2\pi} \int d^2z\, \psi_i(z,\bar{z}). \tag{2.3}$$

Equivalently, the correlation function of any product of operators $\mathcal{O}$ is modified according to

$$\frac{\delta}{\delta g_i}\langle \mathcal{O}\rangle = \frac{1}{2\pi}\int d^2z \langle \psi_i(z,\bar{z})\mathcal{O}\rangle. \tag{2.4}$$

In string theory this corresponds to a condensate of on-shell string modes described by the background field δg_i. We will call those weight $(1,1)$ operators $\psi_i(z,\bar{z})$ for which Eq. (2.4) can be integrated to finite perturbation integrable marginal operators. Locally, their coupling constants g_i can serve as a coordinate system for the space of conformal field theories in the neighbourhood of the unperturbed theory. In the string context this "theory-space" can be interpreted as the moduli space of classical solutions to the string equations of motion. One can visualize the motion in CFT-space generated by these marginal operators as a flow of the weights and operator product coefficients of the primary fields. The change in the conformal weights can be derived from the variation of the two-point functions. Combining (2.4) and (2.2) one finds [12]

$$\frac{\delta}{\delta g_i}\langle \psi_\alpha(z,\bar{z})\psi_\alpha(w,\bar{w})\rangle = 2c_{i\alpha\alpha}(z-w)^{-2h_\alpha}(\bar{z}-\bar{w})^{-2\bar{h}_\alpha}\log|z-w|^2. \tag{2.5}$$

where we absorbed an infinite term in a wavefunction renormalization of the ψ_α. Comparing with (2.1) we see that the weights $(h_\alpha, \bar{h}_\alpha)$ are shifted by

$$\delta h_\alpha = \delta \bar{h}_\alpha = -\sum_i c_{i\alpha\alpha} \delta g_i . \tag{2.6}$$

Note that in case there are several operators with the same weight one has to diagonalize the matrix $c_{i\alpha\beta}$ to find the variation of the weights. In a similar way one can derive variational formulas for the operator product coefficients $c_{\alpha\beta\gamma}$.

Following [12] we can now formulate the conditions a marginal operator ψ_i has to satisfy in order to be integrable to first order in g_i. First of all, it is evident that its weight must not be changed by the perturbation generated by ψ_i itself. So from (2.6) we see that at least c_{iii} must vanish. An additional constraint arises if there are more weight $(1,1)$ primary operators ψ_j. An easy application of degenerate perturbation theory shows that if $c_{iij} \neq 0$ for some of these ψ_j then in the perturbed CFT ψ_i becomes a linear combination of primary fields with different weights and its marginality is destroyed by the perturbation. So, in summary, we have the following necessary conditions for an integrable marginal operator ψ_i:

i) $h_i = \bar{h}_i = 1$,

ii) $c_{iij} = 0$, for any primary field ψ_j of weight $(1,1)$. Note that, because of condition ii), linear combinations of integrable marginal operators need in general not be integrable.

In a local neighbourhood of a generic conformal field theory the space of CFT's connected to this theory has the structure of a smooth manifold, which can be parametrized by the couplings g_i of the marginal operators. It can happen, however, that at some value of the g_i the number of integrable marginal operators in the theory, through accidental degeneracy, jumps to a larger value. The points where this happens are called multi-critical. In general, the extra integrable marginal operators signal the presence of new independent directions in which the theory can be deformed. In this case the multi-critical point lies at the intersection locus of two or more submanifolds of CFT space. However, it often happens that in such a point the conformal field theory has an enhanced symmetry. This symmetry group can reduce by its action on the marginal operators the number of inequivalent deformations of the theory. The coupling constants g_i then give a redundant coordinatization of the CFT space, which can have an orbifold type of singularity at the multi-critical point.

2b. Structure of Torus and $\mathbb{Z}_2$-Orbifold Models

We will now apply the concepts of integrable marginal deformations and multi-critical points to the free scalar field compactified on a circle with radius R. We first give a short description of the spectrum of primary operators in this model [12, 14, 18].

The easiest way to determine this operator content is to compute the one loop partition function,

$$Z = \mathrm{tr}[q^{L_0 - 1/24} \bar{q}^{\bar{L}_0 - 1/24}], \tag{2.7}$$

and to decompose it into characters of irreducible representations of the $c=1$ Virasoro algebra. The partition function of a compactified scalar field is well-known to be

$$Z(R)=|\eta(q)|^{-2}\sum_{(p,\bar{p})\in\Gamma_R} q^{1/2p^2}\bar{q}^{1/2\bar{p}^2};\qquad \eta(q)=q^{1/24}\prod_{n\geq 1}[1-q^n],\tag{2.8}$$

where the momentum summation runs over the lattice

$$\Gamma_R=\left\{(p,\bar{p})=\left(\frac{n}{R}+\tfrac{1}{2}mR,\ \frac{n}{R}-\tfrac{1}{2}mR\right);\ n,m\in\mathbb{Z}\right\}.\tag{2.9}$$

Γ_R is an even self-dual lattice if we adopt a lorentzian metric. This property of Γ_R ensures that (2.8) is modular invariant [19]. We can decompose Z into characters $\chi_h\bar{\chi}_{\bar{h}}$, using [20]

$$\eta(q)^{-1}q^{1/2p^2}=\begin{cases}\chi_{1/2p^2}, & p\notin\mathbb{Z}/\sqrt{2},\\ \sum\limits_{k\geq 0}\chi_{1/2(p+\sqrt{2}k)^2}, & p\in\mathbb{Z}/\sqrt{2}.\end{cases}\tag{2.10}$$

The resulting spectrum can be described in terms of the quantum field $\phi(z,\bar{z})=\varphi(z)+\bar{\varphi}(\bar{z})$ as follows. We have normalized vertex operators

$$\begin{aligned}V_{nm}^{+}(z,\bar{z})&=\sqrt{2}\cos[p\varphi(z)+\bar{p}\bar{\varphi}(\bar{z})],\\ V_{nm}^{-}(z,\bar{z})&=\sqrt{2}\sin[p\varphi(z)+\bar{p}\bar{\varphi}(\bar{z})],\end{aligned}\tag{2.11}$$

where $(p,\bar{p})$ is related to (n,m) by (2.9). The combinations $V_{nm}^{+}\pm iV_{nm}^{-}$ create states with momentum $\pm\frac{1}{2}(p+\bar{p})$ and winding number $\pm(p-\bar{p})$. The allowed values of the momenta $p,\bar{p}$ follow from the requirements that V_{nm} has to be invariant under a shift $2\pi R$ of $\phi(z,\bar{z})$ and that $\phi(z,\bar{z})$ must be single-valued (modulo $2\pi R$) in the presence of a vertex operator. The operators V_{no} and V_{om} are usually called electric and magnetic respectively.

If p^2 equals $\frac{1}{2}k^2$ for some integer k extra primary fields of the form $f(\partial\varphi,\partial^2\varphi,\ldots)V_{nm}$ are present where the polynomials f are given by Schur polynomials [20]. Similar considerations can be made with respect to $\bar{p}$. In particular we always have the $(1,0)$ and $(0,1)$ conformal fields $\partial\varphi$ and $\overline{\partial\varphi}$ which can be integrated to give the conserved left and right momenta. These chiral currents generate a $U(1)\times U(1)$ symmetry; the full symmetry group of the torus model is extended to $O(2)\times O(2)$ by the discrete $\mathbb{Z}_2$ symmetries $(\varphi,\bar{\varphi})\to(-\varphi,-\bar{\varphi})$ and $(\varphi,\bar{\varphi})\to(\bar{\varphi},\varphi)$. The vertex operators V_{nm} are singled out as the primary fields of the $U(1)$ current algebra.

For all R we have the primary $(1,1)$ tensor $\partial\varphi\overline{\partial\varphi}$, which can easily be seen to satisfy the conditions of an integrable marginal operator. In fact an additional term in the action of the form

$$\delta S=\frac{\delta g}{2\pi}\int d^2z\,\partial\phi\bar{\partial}\phi\tag{2.12}$$

can be absorbed by a redefinition of ϕ changing its compactification radius as $\delta R^2=\delta g R^2$. Indeed according to (2.6) the resulting shift in the weights h_{nm} of the vertex operators V_{nm} is [12]

$$\frac{\delta}{\delta g}h_{nm}=-\left(\frac{n^2}{R^2}-\frac{m^2R^2}{4}\right)=\tfrac{1}{2}R\frac{\delta}{\delta R}h_{nm},\tag{2.13}$$

where we used the operator product relation

$$[\partial\varphi\bar{\partial}\bar{\varphi}]\cdot[V_{nm}]\sim\left(\frac{n^2}{R^2}-\frac{m^2R^2}{4}\right)[V_{nm}]\,. \tag{2.14}$$

The motion induced on the momentum lattice Γ_R corresponds to a $SO(1,1)$ boost [19].

The different radii R do not all correspond to different theories. As can be seen e.g. from the partition function, there exists an electric-magnetic duality [18] relating

$$\begin{aligned} R &\Leftrightarrow 2/R\,,\\ V_{nm} &\Leftrightarrow V_{mn}\,,\\ \partial\varphi\,\bar{\partial}\bar{\varphi} &\Leftrightarrow -\partial\varphi\,\bar{\partial}\bar{\varphi}\,. \end{aligned} \tag{2.15}$$

All correlation functions and operator product relations are invariant under this duality transformation. From now on we will choose $R\leqq\sqrt{2}$.

Let us now look for the multi-critical points on the line of torus models. For R or $1/R$ equal to a multiple of $\sqrt{2}$ extra marginal operators appear in the form of vertex operators. However, because of the operator product relation (2.14) we find that the condition ii) of integrability can only be met if there are both electric and magnetic weight $(1,1)$ vertex operators present. This condition leaves as only possible multi-critical points $R=\frac{1}{2}\sqrt{2}$ and $R=\sqrt{2}$.

The model at $R=\sqrt{2}$, which in some sense can be regarded as the limit $c\to1$ of the discrete unitary series, is the fixed point of the duality transformation. It has an enhanced symmetry group $SU(2)\times SU(2)$. The associated conserved chiral currents

$$\begin{aligned} j_1&=\cos\sqrt{2}\varphi\,,\\ j_2&=\sin\sqrt{2}\varphi\\ j_3&=i\tfrac{1}{2}\sqrt{2}\partial\varphi\,, \end{aligned} \tag{2.16}$$

generate the $k=1$ $A_1^{(1)}$ Kac-Moody algebra

$$j_i(z)j_j(w)\sim\tfrac{1}{2}\delta_{ij}\frac{1}{(z-w)^2}-i\varepsilon_{ijk}j_k(w)\frac{1}{z-w}\,. \tag{2.17}$$

As is well-known, this $R=\sqrt{2}$ torus model can be identified with the $k=1$ $SU(2)$ Wess-Zumino-Witten model. For each weight there is an enlarged set of primary fields which can be arranged into one $SU(2)\times SU(2)$ multiplet of dimension $(2s+1)(2\bar{s}+1)$, where $(h,\bar{h})=(s^2,\bar{s}^2)$. Modular invariance of the one loop partition function implies that $(s,\bar{s})\in\mathbb{Z}^2$ or $(\mathbb{Z}+\frac{1}{2})^2$ [21]. In total there are 9 marginal operators $j_i(z)\bar{j}_k(\bar{z})$. Imposing the conditions of integrability and using the OPE (2.17) one finds that all linear combinations of the form

$$\left(\sum_{k=1}^{3}\alpha_kj_k\right)\left(\sum_{k=1}^{3}\bar{\alpha}_k\bar{j}_k\right) \tag{2.18}$$

generate a continuous deformation of the $SU(2)$ model. But, since all these operators are related by the symmetry group to $j_3\bar{j}_3 = \frac{1}{2}\partial\varphi\overline{\partial\varphi}$, every marginal perturbation is equivalent to a modification of the compactification radius R. Note that the electric-magnetic duality can also be understood as a consequence of the $SU(2)\times SU(2)$ symmetry, since $\pm\partial\varphi\overline{\partial\varphi}$ are equivalent marginal operators at $R=\sqrt{2}$.

Our second candidate multi-critical point is at $R=\frac{1}{2}\sqrt{2}$, i.e. the continuum limit of the XY-model at the Kosterlitz-Thouless point [12]. Let us analyse this model using its relationship with the $SU(2)$ model. Quite generally one implements the transformation $R\to\frac{1}{2}R$ on the spectrum by projecting onto even momentum states and adding extra sectors with half-integer winding numbers. These sectors have ground states created out of the vacuum by the magnetic operators $V^{\pm}_{0\frac{1}{2}}$. In the case $R=\sqrt{2}$ we can be more explicit about this. Let us introduce the projection operators

$$\Pi_k = \tfrac{1}{2}(1+\theta_k\bar{\theta}_k), \tag{2.19a}$$

$$\theta_k = \exp\left[\tfrac{1}{2}\oint dz\, j_k(z)\right]. \tag{2.19b}$$

The operators θ_k, which satisfy

$$\theta_i\theta_j = \delta_{ij} + |\varepsilon_{ijk}|\theta_k, \tag{2.20}$$

generate the following involutive transformations of $\varphi(z)(\mathrm{mod}\sqrt{2}\pi)$:

$$\theta_1\varphi\theta_1 = -\varphi, \qquad \theta_2\varphi\theta_2 = -\varphi+\tfrac{1}{2}\sqrt{2}\pi, \qquad \theta_3\varphi\theta_3 = \varphi+\tfrac{1}{2}\sqrt{2}\pi. \tag{2.21}$$

The appropriate projection operator in our context is Π_3, since it evidently reduces the compactification scale to $R=\frac{1}{2}\sqrt{2}$. The half-integer winding number sectors are created by the weight $(\frac{1}{16},\frac{1}{16})$ vertex operators $\sqrt{2}\cos\frac{1}{4}\sqrt{2}(\varphi-\bar{\varphi})$ and $\sqrt{2}\sin\frac{1}{4}\sqrt{2}(\varphi-\bar{\varphi})$. The projection Π_3 leaves a total of 5 independent marginal operators, out of which the following integrable – at least to first order – combinations can be formed

$$j_3\bar{j}_3\,;\ \left(\sum_{k=1}^{2}\alpha_k j_k\right)\left(\sum_{k=1}^{2}\bar{\alpha}_k\bar{k}_k\right). \tag{2.22}$$

All the elements of the second set are related by the $U(1)\times U(1)$ symmetry generated by the two chiral currents j_3 and $\bar{j}_3$. These marginal deformations, which in a moment we will show to be integrable to all orders, are inequivalent to the one induced by $j_3\bar{j}_3$. This implies that at this particular point $R=\frac{1}{2}\sqrt{2}$ a new continuous deformation of the theory exists that changes the nature of the compactification. The meaning of this new direction becomes clear when we use the obvious invariance under permutation of the three projections Π_i. In particular we can repeat the above construction with Π_1 as the projection operator. As seen from the action of θ_1 on φ, this construction results in an identification of φ with $-\varphi$. So in fact we have constructed a $\mathbb{Z}_2$-orbifold compactification with $R=\sqrt{2}$. This radius is varied by the action of the marginal operator $j_3\bar{j}_3$. Thus the new marginal direction is seen to correspond to the line of $\mathbb{Z}_2$-orbifold models. The equivalence of torus and orbifold models has been noted previously [9], see also [22]. It will proof to be very useful in the subsequent analysis.

We now briefly review some basic facts of the conformal field theory of $\mathbb{Z}_2$-orbifold models [9–11]. The spectrum of operators of a scalar field on a $\mathbb{Z}_2$-orbifold consists of a twisted and untwisted part. The untwisted states are found by projecting the spectrum of the corresponding torus theory onto even states under $\phi \to -\phi$ using the projection operator $\Pi = \frac{1}{2}(1+\Theta)$. Here Θ generates the $\mathbb{Z}_2$ symmetry

$$\Theta\phi(z,\bar{z})\Theta = -\phi(z,\bar{z}). \tag{2.23}$$

Of the vertex operators only the V^+_{nm} are left. In addition we have a twisted sector, corresponding to antiperiodic field configurations. The two-fold degenerate twisted ground state is created out of the vacuum by the twist fields σ_1 and σ_2 with conformal weights $(\frac{1}{16}, \frac{1}{16})$. These σ_i and their partners τ_i defined by

$$\partial\varphi\,\overline{\partial\varphi}(z,\bar{z})\sigma_i(w,\bar{w}) \sim \frac{1}{|z-w|}\tau_i(w,\bar{w}) \tag{2.24}$$

are the only relevant operators in the twisted sectors. The weights of the τ_i are $(\frac{9}{16}, \frac{9}{16})$. In fact, the weights of all twisted operators are independent over the radius R. One way to see this is to note that the OPE of the marginal operator $\partial\varphi\overline{\partial\varphi}$ with any twisted field ψ only contains operators with weights which differ from that of ψ by a half-integer, since both $\partial\varphi$ and $\overline{\partial\varphi}$ acquire a branch cut starting at ψ. This forces the three point function $\langle\partial\varphi\overline{\partial\varphi}(z_1)\psi(z_2)\psi(z_3)\rangle$ to be zero, and consequently $\partial\varphi\overline{\partial\varphi}$ does not change the weight of ψ. The scale independence of the twisted sector implies that the one loop partition function of the orbifold models is of the form

$$Z_{\mathrm{orbifold}}(R) = \tfrac{1}{2}Z(R) + Z_{\mathrm{twist}}. \tag{2.25}$$

Here $Z(R)$ is the torus partition function (2.8). The twisted part Z_{twist} can be determined at the multi-critical point $R=\sqrt{2}$, using the equivalence with the $R=\frac{1}{2}\sqrt{2}$ torus model:

$$Z_{\mathrm{twist}} = Z(\tfrac{1}{2}\sqrt{2}) - \tfrac{1}{2}Z(\sqrt{2}), \tag{2.26}$$

in accordance with the results of [14].

There also exists an electric-magnetic duality for the line of $\mathbb{Z}_2$-orbifold models. As for the untwisted part of the spectrum this transformation acts similarly as in the torus model. The action of duality on the twisted part can most easily be understood at the selfdual point $R=\sqrt{2}$. Here the twist fields σ_1 and σ_2 are equivalent to the half-integer magnetic operators $V^+_{0\frac{1}{2}}$ and $V^-_{0\frac{1}{2}}$ of the $SU(2)$ model. In this correspondence duality translates into shifting $\bar{\varphi} \to \bar{\varphi} + \frac{1}{2}\sqrt{2}\pi$, $\varphi \to \varphi$, since that changes the sign of the marginal operator $j_1\bar{j}_1 = \cos\sqrt{2}\varphi\,\cos\sqrt{2}\bar{\varphi}$. The effect of this shift is easily calculated and the resulting duality transformation is seen to act on the twist fields as

$$\sigma_1 \Leftrightarrow \frac{1}{\sqrt{2}}(\sigma_1+\sigma_2), \qquad \sigma_2 \Leftrightarrow \frac{1}{\sqrt{2}}(\sigma_1-\sigma_2). \tag{2.27}$$

The two twist fields σ_1 and σ_2 and their respective sectors correspond to the two conjugacy classes (or equivalently fixed points) of the $\mathbb{Z}_2$-twist: $\phi \to -\phi \,(\mathrm{mod}\, 4\pi R)$ respectively $\phi \to -\phi + 2\pi R \,(\mathrm{mod}\, 4\pi R)$. From this we conclude that the product of two operators in the same twisted sector produces operators which create an even winding number, whereas the product of two operators in different twist sectors creates odd winding numbers. In combination with electric-magnetic duality this implies the following form of the operator product relations:

$$\begin{aligned} [\sigma_1]\cdot[\sigma_1] &\sim \sum_{n,m} C^{2n,2m}[V^+_{2n,2m}] + \sum_{n,m} C^{2n+1,2m}[V^+_{2n+1,2m}]\,, \\ [\sigma_2]\cdot[\sigma_2] &\sim \sum_{n,m} C^{2n,2m}[V^+_{2n,2m}] - \sum_{n,m} C^{2n+1,2m}[V^+_{2n+1,2m}]\,, \\ [\sigma_1]\cdot[\sigma_2] &\sim \sum_{n,m} C^{2n,2m+1}[V^+_{2n,2m+1}]\,. \end{aligned} \tag{2.28}$$

The numerical values of the coefficients C^{nm} are given by [23, 11]:

$$\begin{aligned} (C^{oo})^2 &= 1\,, \\ (C^{nm})^2 &= 2\cdot 16^{-(h_{nm}+\bar{h}_{nm})}\,; \qquad (n,m) \neq (o,o)\,. \end{aligned} \tag{2.29}$$

The symmetry group of the $\mathbb{Z}_2$-orbifold model is the discrete group $\mathbb{D}_4$ (the symmetry group of the square) generated by

$$\begin{aligned} (\sigma_1, \sigma_2, V_{nm}) &\to (-\sigma_1, \sigma_2, (-)^m V_{nm})\,, \\ (\sigma_1, \sigma_2, V_{nm}) &\to (\sigma_2, \sigma_1, (-)^n V_{nm})\,. \end{aligned} \tag{2.30}$$

The invariance under these transformations follows from (2.28). Of course, the duality transformations (2.15), (2.27) should be read modulo $\mathbb{D}_4$.

The twisted sector does not contain any weight (1, 1) conformal fields, so we can copy the arguments applied to the torus model to show that besides $R=\sqrt{2}$ the only other multi-critical orbifold model is $R=\frac{1}{2}\sqrt{2}$. This model is known to correspond to the continuum limit of the 4-state Potts model [8, 18]. It contains the following relevant and marginal operators: $3\times(\frac{1}{16},\frac{1}{16})$, $1\times(\frac{1}{4},\frac{1}{4})$, $3\times(\frac{9}{16},\frac{9}{16})$, and $3\times(1,1)$. The frequent occurrence of the multiplicity 3 can be understood from the fact that this model can be obtained from the $SU(2)$-model by twisting with all three θ_i of Eq. (2.19). The integrable marginal operators which are not projected out are $j_i\bar{j}_i\,(i=1,2,3)$. It is clear from this construction that the $SU(2)$ invariance is broken to a discrete symmetry which permutes the three operators. In this picture we have, besides the projected $SU(2)$ spectrum, three extra sectors generated by the operators σ_1, σ_2, and $\sigma_3=\sqrt{2}\cos\frac{1}{4}\sqrt{2}(\varphi-\bar{\varphi})$ with OPE

$$\sigma_1(z,\bar{z})\sigma_2(w,\bar{w}) \sim 2^{-1/4}|z-w|^{-1/8}\sigma_3(w,\bar{w}) \quad \text{(and cyclic)}\,. \tag{2.31}$$

These three twist operators correspond to the spins S_1, S_2, and S_1S_2 of the Ashkin-Teller model. The full symmetry is $\mathbb{S}_4$ (the permutation group of 4 elements) which acts on the σ_i according to

$$(\sigma_1, \sigma_2, \sigma_3) \to ((-)^p\sigma_{i_1}, (-)^q\sigma_{i_2}, (-)^{p+q}\sigma_{i_3})\,. \tag{2.32}$$

Indeed, $\mathbb{S}_4$ is the symmetry group of the 4-state Potts model. Before the projections, each twisted sector is an irreducible representation of the (twisted) $A_1^{(1)}$ algebra. The fact that all three representations are isomorphic is known in the mathematical literature as triality of $A_1^{(1)}$ [24].

The orbifold theory with $R=1$ corresponds to the decoupling point of the Ashkin-Teller model, where the model reduces to two independent Ising systems [14, 15]. The continuum limit of the Ising model is described in terms of a free massless Majorana fermion $(\psi(z), \bar{\psi}(\bar{z}))$ and its spinfield $\sigma(z, \bar{z})$. The analytic part of the stress-energy tensor

$$T(z) = -\tfrac{1}{2}\psi(z)\partial\psi(z) \tag{2.33}$$

satisfies the Virasoro algebra with $c=\frac{1}{2}$. A mutually local set of operators is given by [1],

$$\{[\mathbb{1}], [\sigma], [\varepsilon]\}, \tag{2.34}$$

where ε is the energy operator $\varepsilon(z, \bar{z}) = \bar{\psi}(\bar{z})\psi(z)$. The OPE's are

$$[\sigma]\cdot[\sigma] \sim [\mathbb{1}] + \tfrac{1}{2}[\varepsilon], \qquad [\varepsilon]\cdot[\varepsilon] \sim [\mathbb{1}]. \tag{2.35}$$

In the orbifold model we can indeed identify in the spectrum two sets of the above operators. The identifications are

$$\begin{aligned}
&T^{(1)} = -\tfrac{1}{4}(\partial\varphi)^2 + \tfrac{1}{2}\cos 2\varphi, \qquad && T^{(2)} = -\tfrac{1}{4}(\partial\varphi)^2 - \tfrac{1}{2}\cos 2\varphi,\\
&\bar{T}^{(1)} = -\tfrac{1}{4}(\bar{\partial}\bar{\varphi})^2 + \tfrac{1}{2}\cos 2\bar{\varphi}, && \bar{T}^{(2)} = -\tfrac{1}{4}(\bar{\partial}\bar{\varphi})^2 - \tfrac{1}{2}\cos 2\bar{\varphi},\\
&\sigma^{(1)} = \sigma_1, && \sigma^{(2)} = \sigma_2,\\
&\varepsilon^{(1)} = -2\sin\varphi\sin\bar{\varphi}, && \varepsilon^{(2)} = 2\cos\varphi\cos\bar{\varphi},\\
&\qquad \sigma^{(1)}\cdot\sigma^{(2)} = \sqrt{2}\cos\tfrac{1}{2}(\varphi - \bar{\varphi}),\\
&\qquad \varepsilon^{(1)}\cdot\varepsilon^{(2)} = -\partial\varphi\,\bar{\partial}\varphi,\\
&\qquad \sigma^{(1)}\cdot\varepsilon^{(2)} = \tau_1,\\
&\qquad \varepsilon^{(1)}\cdot\sigma^{(2)} = \tau_2.
\end{aligned} \tag{2.36}$$

One easily checks with (2.28) that the two sets of operators have the correct operator product relations. All other primary fields of the orbifold model are descendants of these operators with respect to the two Virasoro algebras generated

Table 1. Some properties of the multi-critical models in the $c=1$ spectrum. The notation is explained in (2.18) and (2.19)

Model		Projection	Currents	Integrable Marginal Operators	Symmetry
Torus	$R=\sqrt{2}$		j_1, j_2, j_3	$(\sum \alpha_k j_k)(\sum \bar{\alpha}_k \bar{j}_k)$ $(k=1,2,3)$	$SU(2)\times SU(2)$
	$R=\frac{1}{2}\sqrt{2}$	Π_3	j_3	$j_3\bar{j}_3$, $(\sum \alpha_k j_k)(\sum \bar{\alpha}_k \bar{j}_k)$ $(k=1,2)$	$O(2)\times O(2)$
Orbifold	$R=\sqrt{2}$	Π_1	j_1	$j_1\bar{j}_1$, $(\sum \alpha_k j_k)(\sum \bar{\alpha}_k \bar{j}_k)$ $(k=2,3)$	$O(2)\times O(2)$
	$R=\frac{1}{2}\sqrt{2}$	Π_1, Π_2, Π_3	–	$j_1\bar{j}_1, j_2\bar{j}_2, j_3\bar{j}_3$	$\mathbb{S}_4$

by $T^{(1)}(z)$ and $T^{(2)}(z)$. The identifications (2.36) can be used to calculate Ising correlation functions within the $c=1$ orbifold model (see also [39]).

As a summary, we have collected in Table 1 the relevant quantities of the three multi-critical points that occur in the class of $c=1$ gaussian models.

3. Partition Functions for Arbitrary Genus

In this section we will analyse the partition functions for the $c=1$ models on arbitrary Riemann surfaces. In principle, the partition function encodes through its dependence on the moduli parameters all the information of the conformal field theory. Whereas at one loop it gives us the weights and multiplicities of the primary fields, all operator product coefficients and correlation functions can be obtained through factorization of higher genus partition functions at the boundary of moduli space [5].

We consider a genus g surface Σ with canonical homology cycles A_i, B_i $(i=1, \ldots, g)$ normalized with respect to the intersection product

$$\#(A_i, A_j) = \#(B_i, B_j) = 0\,, \qquad \#(A_i, B_j) = -\#(B_j, A_i) = \delta_{ij}\,. \tag{3.1}$$

The computation of the partition function of a compactified scalar field on such a Riemann surface has become quite standard [25, 26]. It is given by the product of a factor Z_0^{qu}, representing the quantum fluctuations of ϕ, times a classical part Z^{cl} given by the partition sum over the classical solutions in the different winding sectors. These solutions are given essentially by the integrals of the holomorphic and anti-holomorphic one forms $\omega_i = \omega_i(z)dz$, $\bar{\omega}_i = \bar{\omega}_i(\bar{z})d\bar{z}$ $(i=1, \ldots, g)$ on Σ. Their classical action can be expressed in terms of the period matrix τ defined by

$$\oint_{A_i} \omega_j = \delta_{ij}\,, \qquad \oint_{B_i} \omega_j = \tau_{ij}\,. \tag{3.2}$$

Referring for the details of the calculation to [25], we immediately give the result for Z^{cl},

$$Z^{cl}(R) = \sum_{(p,\bar{p}) \in \Gamma_R^g} \exp[i\pi(p \cdot \tau \cdot p - \bar{p} \cdot \bar{\tau} \cdot \bar{p})]\,. \tag{3.3}$$

The $p_i, \bar{p}_i$ $(i=1, \ldots, g)$ have a natural interpretation as momenta running through the g loops. For the quantum part $Z_0^{qu} = |\det \bar{\partial}_0|^{-1}$ an expression has been given in [26]. It is the modulus of a holomorphic function on moduli space, which transforms as a modular form of weight -1. The total partition function $Z = Z_0^{qu} Z^{cl}(R)$ is modular invariant.

For rational values of R^2 the momentum lattice can be built up from a finite number of square sublattices. As a consequence the classical part of the partition function can be expressed as a finite sum of ϑ-functions defined as

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau) = \sum_{n \in \mathbb{Z}^g} \exp[i\pi(n+\alpha) \cdot \tau \cdot (n+\alpha) + 2\pi i(n+\alpha) \cdot (z+\beta)]\,. \tag{3.4}$$

If $\frac{1}{2}R^2=p/q$ the result reads

$$Z^{cl}(R)=\sum_{\alpha,\beta,\gamma}\vartheta\begin{bmatrix}\frac{1}{2}\alpha+\frac{1}{2}\beta+\gamma\\0\end{bmatrix}(0|2pq\tau)\,\overline{\vartheta\begin{bmatrix}\frac{1}{2}\alpha-\frac{1}{2}\beta+\gamma\\0\end{bmatrix}(0|2pq\tau)} \tag{3.5a}$$

$$=2^{-g}\sum_{\alpha,\beta,\gamma}e^{4\pi i\beta\cdot\gamma}\vartheta\begin{bmatrix}\alpha+\beta\\\gamma\end{bmatrix}(0|\tfrac{1}{2}pq\tau)\,\overline{\vartheta\begin{bmatrix}\alpha-\beta\\\gamma\end{bmatrix}(0|\tfrac{1}{2}pq\tau)}, \tag{3.5b}$$

where the summation is over $\alpha\in\left(\frac{1}{p}\mathbb{Z}_p\right)^g$, $\beta\in\left(\frac{1}{q}\mathbb{Z}_q\right)^g$, $\gamma\in(\frac{1}{2}\mathbb{Z}_2)^g$. Note the manifest duality $p\leftrightarrow q$. The latter of these two equations can be seen as a generalized spin model construction [27]. One starts with a chiral spin-$\frac{1}{4}pq$ "fermion" $\psi(z)=V_{1/2p,1/2q}(z)$, extends it to an intermediate nonlocal system by adding $p\times q$ new sectors and finally projects on even fermion parity. (We should mention that for rational values of R, not R^2, it is also possible to give a construction solely in terms of spin $\frac{1}{2}$ fermions [29].) Cases of particular relevance to us are

$$Z^{cl}(1)=2^{-g}\sum_{\alpha,\beta}\left|\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(0|\tau)\right|^2, \tag{3.6a}$$

$$Z^{cl}(\sqrt{2})=\sum_{\alpha}\left|\vartheta\begin{bmatrix}\alpha\\0\end{bmatrix}(0|2\tau)\right|^2, \tag{3.6b}$$

$$Z^{cl}(\tfrac{1}{2}\sqrt{2})=2^{-g}\sum_{\alpha,\beta,\gamma}\left|\vartheta\begin{bmatrix}\alpha+\frac{1}{2}\gamma\\\beta\end{bmatrix}(0|2\tau)\right|^2. \tag{3.6c}$$

Here the summations are over half-integer characteristics only.

Now let us consider the partition function of a $\mathbb{Z}_2$-orbifold model. It is represented by a functional integral over fields ϕ which can be double-valued on Σ. These field configurations fall into 2^{2g} distinct topological sectors corresponding to the elements of $H_1(\Sigma,\mathbb{Z}_2)$. The partition function is written as the sum over all sectors

$$Z_{\text{orbifolds}}(R)=2^{-g}\sum_{\varepsilon,\delta}Z_{\varepsilon,\delta}(R), \tag{3.7}$$

where $\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}=\begin{bmatrix}\varepsilon_1\dots\varepsilon_g\\\delta_1\dots\delta_g\end{bmatrix}(\varepsilon_i,\delta_i=0,\frac{1}{2})$ labels the contribution of the field configurations having a branch cut along the cycle $\Sigma_i 2(\delta_i A_i+\varepsilon_i B_i)$. The untwisted part is simply $Z_{0,0}=Z_{\text{torus}}$ and is by itself modular invariant. The other $Z_{\varepsilon,\delta}$ with $\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$ are all permuted by the modular group, which means that only the sum over all twisted sectors is modular invariant.

We will calculate the contribution of the twisted sectors using the theory of unramified coverings of Riemann surfaces [16] following the general strategy as advocated in [10, 11]. (For a related calculation, see [28].) Let us concentrate on one of the terms $Z_{\varepsilon,\delta}$. Since $\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$ we can always choose a new homology basis such that the branch cut is along A_g, i.e. $\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}=\begin{bmatrix}0\dots0\\0\dots\frac{1}{2}\end{bmatrix}$. This branch cut defines an unramified double cover $\pi:\hat{\Sigma}\to\Sigma$ (as depicted in Fig. 2 for the $g=2$ case).

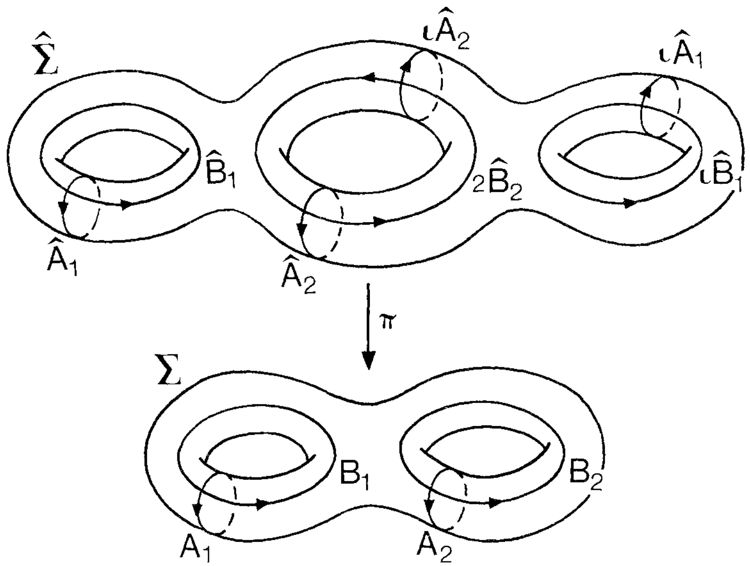

Fig. 2. The double cover $\hat{\Sigma}$ of a genus 2 Riemann surface Σ with canonical homology bases. The branched cycle is A_2

The surface $\hat{\Sigma}$ is obtained by taking two copies of the surface Σ, cutting each open along the cycle A_g and then pasting the two copies together as in Fig. 2. This defines a natural complex structure on $\hat{\Sigma}$. By the Riemann-Hurwitz theorem it is a compact Riemann surface of genus $2g-1$ and admits an involutive conformal automorphism $\iota:\hat{\Sigma}\to\hat{\Sigma}$ satisfying $\pi\circ\iota=\pi$, which is simply sheet interchange.

Once we have chosen a basis for $H_1(\Sigma,\mathbb{Z})$ and specified the twist characteristic $\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}$ the double cover $\hat{\Sigma}$ is uniquely determined. A convenient choice for the homology basis on $\hat{\Sigma}$ is one that projects onto the homology basis on Σ, i.e. $H_1(\hat{\Sigma},\mathbb{Z})$ is generated by $\hat{A}_1,\hat{B}_1,\ldots,\hat{A}_{g-1},\hat{B}_{g-1}$, their images under ι and $\hat{A}_g, 2\hat{B}_g$, with $\pi(\hat{A}_1)=A_1$ etc. as depicted in Fig. 2. Note that $\iota(\hat{A}_g)=\hat{A}_g$ and $\iota(2\hat{B}_g)=2\hat{B}_g$. We should mention here that such an homology basis is not uniquely fixed by these conditions. Different choices are related by modular transformations on $\hat{\Sigma}$ which projected onto Σ leave the homology basis fixed.

The Prym differentials $v_i=v_i(\hat{z})d\hat{z}$ $(i=1,\ldots,g-1)$ are the holomorphic 1-forms on $\hat{\Sigma}$, which are odd under the defining involution $v_i(\iota(\hat{z}))=-v_i(\hat{z})$. They are normalized with respect to the A-cycles

$$\begin{aligned}\oint_{\hat{A}_i} v_j &= -\oint_{\iota(\hat{A}_i)} v_j=\delta_{ij},\\ \oint_{\hat{B}_i} v_j &= -\oint_{\iota(\hat{B}_i)} v_j=\Pi_{ij}\qquad (i,j=1,\ldots,g-1).\end{aligned}\tag{3.8}$$

Π_{ij} is the period matrix of the Prym differentials and is a symmetric $(g-1)\times(g-1)$ matrix with positive definite imaginary part [16]. The Prym differentials have no periods around $\hat{A}_g$ and $2\hat{B}_g$. In terms of the underlying surface Σ the Prym differentials are double-valued holomorphic 1-forms which are anti-periodic around B_g. The complex torus $\mathbb{C}^{g-1}/(\mathbb{Z}^{g-1}+\Pi\mathbb{Z}^{g-1})$ is called the Prym variety and is isomorphic to the set of degree zero divisor classes on $\hat{\Sigma}$ odd under $\imath$. The isomorphism is given by the Abel map

$$Q-\imath(Q) \rightarrow \frac{1}{2}\int_{\imath(Q)}^{Q} v_i\,. \tag{3.9}$$

Divisors on the surface Σ have (up to a minus sign) a unique image in the Prym variety.

The contribution of the twisted sector to the orbifold partition function is now given by the functional integral over fields $\hat{\phi}$ defined on $\hat{\Sigma}$ which are odd under $\imath:\hat{\phi}(\imath(\hat{z}))=-\hat{\phi}(\hat{z})\,(\mathrm{mod}\,2\pi R)$. Furthermore, because of the double area of $\hat{\Sigma}$, the action of $\hat{\phi}$ has to be rescaled with a factor of $\frac{1}{2}$ compared to (1.1).

The computation of the soliton contribution to the partition function is analogous to the untwisted case. We split the scalar field $\hat{\phi}$ into a classical part and a quantum part $\hat{\phi}=\hat{\phi}^{cl}+\hat{\phi}^{qu}$. The classical part is now given by the integral of the Prym differentials,

$$\hat{\phi}^{cl}=-\tfrac{1}{2}i\pi R(m-\bar{\Pi}n)\cdot(\mathrm{Im}\,\Pi)^{-1}\cdot\int_{\imath(\hat{z})}^{\hat{z}} v+\text{c.c.}\,. \tag{3.10}$$

Its action can be expressed in terms of the period matrix Π

$$S[\hat{\phi}^{cl}]=\tfrac{1}{2}\pi R^2(n-\bar{\Pi}m)\cdot(\mathrm{Im}\,\Pi)^{-1}\cdot(n-\Pi m)\,. \tag{3.11}$$

After a Poisson resummation of the soliton sum the expression for the contribution $Z_{\varepsilon,\delta}$ to the partition function reads

$$Z_{\varepsilon,\delta}(R)=Z^{qu}_{\varepsilon,\delta}\sum_{(p,\bar{p})\in\Gamma_R^{g-1}}\exp[i\pi(p\cdot\Pi\cdot p-\bar{p}\cdot\bar{\Pi}\cdot\bar{p})]\,. \tag{3.12}$$

For rational R^2 this expression can of course also be rewritten in in terms of ϑ-functions defined on the Prym variety using (3.5) with τ replaced by Π.

The quantum contribution to the partition function seems much more difficult to determine. However, since it is independent of the compactification radius, we can choose the multi-critical value $R=\sqrt{2}$, where we can equate the partition function of the orbifold to that of the $R=\frac{1}{2}\sqrt{2}$ torus model:

$$Z_{\text{orbifold}}(R=\sqrt{2})=Z_{\text{torus}}(R=\tfrac{1}{2}\sqrt{2})\,. \tag{3.13}$$

Combining Eqs. (3.6b) and (3.6c) we find for the twisted part

$$2^{-g}\sum_{\varepsilon,\delta,\gamma}Z^{qu}_{\varepsilon,\delta}\left|\vartheta\begin{bmatrix}\gamma\\0\end{bmatrix}(0|2\Pi_{\varepsilon,\delta})\right|^2=2^{-g}\sum_{\varepsilon,\delta,\gamma}Z^{qu}_0\left|\vartheta\begin{bmatrix}\frac{1}{2}\varepsilon+\gamma\\\delta\end{bmatrix}(0|2\tau)\right|^2. \tag{3.14}$$

Here on both sides the summation runs over $(\varepsilon,\delta)\neq(0,0)$. To interpret the separate terms in this equation, let us conceive an operator formalism in which the partition function is obtained by summing in each loop over all states in the Hilbert space. The characteristics ε, δ, and γ then indicate which sector of the spectrum contributes at the different loops. Both theories correspond to a twisted $SU(2)$ model; so, in order to identify these sectors, let us first compare (3.14) with the untwisted $SU(2)$ partition function

$$Z_{SU(2)}=\sum_{\gamma} Z_0^{qu}\left|\vartheta\begin{bmatrix}\gamma\\0\end{bmatrix}(0|2\tau)\right|^2. \tag{3.15}$$

Here $\gamma_i=0,\frac{1}{2}$ can be seen to distinguish the integer respectively half-integer spin representations of $SU(2)$ in the i^{th} loop. Indeed for $g=1$ the two terms are given by the $k=1$ $A_1^{(1)}$ characters $|\chi_0|^2$ and $|\chi_1|^2$ [22]. It is evident that γ has the same function for the untwisted loops in Eq. (3.14). The interpretation of ε and δ is now also clear. On the right-hand side the sum over δ corresponds to a projection on even momentum states, while ε labels the integer respectively half-integer winding number sectors. On the left-hand side ε and δ have a comparable function. The untwisted and twisted sectors are indexed by $\varepsilon=0,\frac{1}{2}$ whereas the summation over δ projects onto $\mathbb{Z}_2$-even states. This can be neatly demonstrated in the genus 1 case. Here we can explicitly evaluate the twisted contributions to the orbifold partition function using an oscillator representation:

$$\begin{aligned}
Z_{0,1/2}&=\mathrm{tr}_0[\Theta q^{L_0-1/24}\bar{q}^{\bar{L}_0-1/24}]=|q^{1/24}\prod_{n=1}^{\infty}[1+q^n]|^{-2}\\
&=|\eta(q)|^{-2}\left|\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(0|2\tau)\right|^2,\\
Z_{1/2,0}&=\mathrm{tr}_{1/2}[q^{L_0-1/24}\bar{q}^{\bar{L}_0-1/24}]=2|q^{-1/48}\prod_{n=1}^{\infty}[1-q^{n+1/2}]|^2\\
&=2|\eta(q)|^{-2}\left|\vartheta\begin{bmatrix}\frac{1}{4}\\0\end{bmatrix}(0|2\tau)\right|^2,\\
Z_{1/2,1/2}&=\mathrm{tr}_{1/2}[\Theta q^{L_0-1/24}\bar{q}^{\bar{L}_0-1/24}]=2|q^{-1/48}\prod_{n=1}^{\infty}[1+q^{n+1/2}]|^{-2}\\
&=2|\eta(q)|^{-2}\left|\vartheta\begin{bmatrix}\frac{1}{4}\\\frac{1}{2}\end{bmatrix}(0|2\tau)\right|^2.
\end{aligned} \tag{3.16}$$

Here Θ is the twist operator (2.23) and the suffix $0,\frac{1}{2}$ attached to the trace denotes the restriction to the untwisted respectively twisted Hilbert space. So we see that for $g=1$ the separate terms in (3.14) can indeed be equated. Generalizing this observation to higher genus surfaces we can now solve the twisted partition function $Z_{\varepsilon,\delta}^{qu}$ as

$$Z_{\varepsilon,\delta}^{qu}=\left|c\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}\right|^{-2} Z_0^{qu}. \tag{3.17}$$

where $c\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}$ is defined as the ratio of the classical contributions. For our canonical twist $\begin{bmatrix}\varepsilon\\\delta\end{bmatrix}=\begin{bmatrix}0\dots0\\0\dots\frac{1}{2}\end{bmatrix}$

$$c\begin{bmatrix}0\dots0\\0\dots\frac{1}{2}\end{bmatrix}=\frac{\vartheta\begin{bmatrix}\gamma\\0\end{bmatrix}(0|2\Pi)}{\vartheta\begin{bmatrix}\gamma & 0\\0 & \frac{1}{2}\end{bmatrix}(0|2\tau)}. \tag{3.18}$$

As we have just argued, c is independent of the characteristic

$$\begin{bmatrix} \gamma \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma_1 \ldots \gamma_{g-1} \\ 0 \ldots 0 \end{bmatrix},$$

a fact that is indeed known in the mathematical literature as one of the Schottky relations [16]. The other $c\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}$ are related to (3.18) by a modular transformation.

Finally, combining Eqs. (3.7), (3.12), and (3.17) we arrive at the following general result for the orbifold partition function on a genus g Riemann surface:

$$\begin{aligned} Z_{\text{orbifold}}(R) &= 2^{-g} Z_{\text{torus}}(R) + Z_{\text{twist}}(R), \\ Z_{\text{twist}}(R) &= Z_0^{qu} 2^{-g} \sum_{\varepsilon,\delta} \left| c\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} \right|^{-2} \\ &\times \sum_{(p,\bar{p}) \in \Gamma_R^{g-1}} \exp[i\pi(p \cdot \Pi_{\varepsilon,\delta} \cdot p - \bar{p} \cdot \bar{\Pi}_{\varepsilon,\delta} \cdot \bar{p})]. \end{aligned} \tag{3.19}$$

From this result one can now extract all correlation functions of twist fields, vertex operators, etc. by factorization at the compactification divisor and projecting onto the relevant sector, indexed by the twist structure (ε, δ) and loop momenta $(p, \bar{p})$. This will be worked out in detail in the subsequent sections.

Let us make some comments on the case $R=1$. As mentioned in Sect. 2 the model can be described as two decoupled Ising systems, i.e. two Majorana fermions. This should be compared with the corresponding $R=1$ torus model which is well-known to be equivalent to the spin model of a complex Dirac fermion $\psi = e^{i\varphi}$. On a higher genus Riemann surface the construction of a spin model implies a summation over all spin structures. In the orbifold model one introduces extra twisted sectors. In terms of the Dirac fermion the twist $\phi \to -\phi$ introduces a difference between the spin structures of its real and imaginary component. So the $R=1$ orbifold model is given by the spin model of two uncorrelated Majorana fermions [15]. Indeed using ϑ-function doubling formulas we can rewrite the Schottky relation (3.18) as

$$c^2 \begin{bmatrix} 0 \ldots 0 \\ 0 \ldots \frac{1}{2} \end{bmatrix} = \frac{\vartheta^2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0|\Pi)}{\vartheta \begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix} (0|\tau) \vartheta \begin{bmatrix} \alpha & 0 \\ \beta & \frac{1}{2} \end{bmatrix} (0|\tau)}. \tag{3.20}$$

Hence the twisted contribution to the $R=1$ orbifold partition function can be re-expressed as

$$\begin{aligned} Z_{\text{twist}}(R=1) &= Z_0^{qu} 2^{-2g} \sum_{\varepsilon,\delta} \left| c\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} \right|^{-2} \sum_{\alpha,\beta} \left| \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0|\Pi) \right|^2 \\ &= Z_0^{qu} 2^{-2g} \sum_{\substack{\alpha,\beta,\mu,\nu \\ (\alpha,\beta) \neq (\mu,\nu)}} \left| \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0|\tau) \vartheta \begin{bmatrix} \mu \\ \nu \end{bmatrix} (0|\tau) \right|. \end{aligned} \tag{3.21}$$

Adding the untwisted part we obtain

$$Z_{\text{orbifold}}(R=1)=(Z_{\text{Ising}})^2\,, \tag{3.22}$$

with

$$Z_{\text{Ising}}=(Z_0^{qu})^{1/2}2^{-g}\sum_{\alpha,\beta}\left|\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\tau)\right|, \tag{3.23}$$

confirming the identification with the Ashkin-Teller model at the decoupling point. The relation (3.20) was used in [28] to arrive at the result (3.19).

We close this section with a discussion of the action of the mapping class (or modular) group Γ_g [30], i.e. the group of all disconnected diffeomorphisms of Σ. The mapping class group is generated by Dehn twists around cycles. The action of such a twist D_C can be represented by cutting the Riemann surface along the cycle C and glueing it together after rotating one of the boundaries over 2π. The effect of this transformation on the elements γ of the homology group is

$$D_C:\gamma\to\gamma+\#(\gamma,C)C\,. \tag{3.24}$$

Note that this transformation does not depend on the orientation of either C or γ. Dehn twists of the form (3.24) are called positive, the inverse twists D_C^{-1} are called negative. The mapping class group leaves the intersection product invariant, hence it acts on the homology basis A_i, B_i by $Sp(2g,\mathbb{Z})$ transformations. Moreover, all elements of the symplectic group $Sp(2g,\mathbb{Z})$ correspond to modular transformations. The subgroup of Γ_g that leaves the homology fixed is known as the Torelli group and is generated by the Dehn twists around the homologically trivial cycles on Σ. So, in summary, we have the following exact sequence:

$$1\to Torelli\to\Gamma_g\to Sp(2g,\mathbb{Z})\to 1\,. \tag{3.25}$$

We have seen that for rational R^2 both the torus and the orbifold partition function can be written as a finite sum of products of a holomorphic times an anti-holomorphic function on moduli space. This total sum is invariant under the mapping class group, but the individual analytic parts are not. For the torus models these are – apart from an overall factor $(\det\bar{\partial}_0)^{-1/2}$ – all functions of the period matrix τ, on which Γ_g acts by the $Sp(2g,\mathbb{Z})$ transformations

$$\tau\to(A\tau+B)(C\tau+D)^{-1}\,. \tag{3.26}$$

Hence all torus models are insensitive to the Torelli group.

The chiral orbifold partition functions, on the other hand, are functions of the period matrix Π of the Prym differentials and in general they do feel the Torelli group. To explain this let us consider a zero homology cycle C on Σ and a twist structure with two branched cycles, one on each side of C (see Fig. 3). For this case the lifts $\hat{C}$ and $\iota(\hat{C})$ to the double cover $\hat{\Sigma}$ are homologically nontrivial. The positive Dehn twist D_C on Σ lifts to the composite transformation $D_{\hat{C}}\circ D_{\iota(\hat{C})}$ on $\hat{\Sigma}$, which has a nontrivial action on the homology of the double cover. More specifically, it transforms the elements of $H_1(\hat{\Sigma},\mathbb{Z})$ odd under the involution ι, whereas the even elements are inert. Consequently the period matrix Π will not be invariant under the Dehn twist D_C. The Torelli group projects onto a subgroup of the symplectic

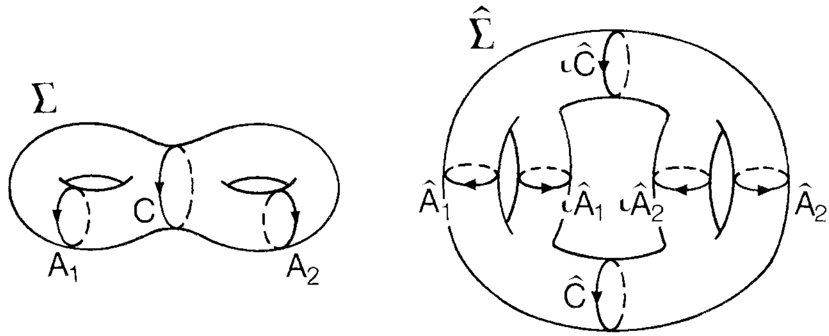

Fig. 3. A genus 2 surface Σ with branch cut along A_1+A_2 and its double cover $\hat{\Sigma}$. The lifts $\hat{C}$ and $\iota(\hat{C})$ of the zero-homology cycle C are homologically nontrivial

group $Sp(2g-2,\mathbb{Z})$ that acts on Π as in (3.26). For example, in the case considered in Fig. 3 the transformation reads $\Pi\to\Pi+4$.

To give a physical interpretation [31] for this nontrivial behaviour of the twisted chiral partition functions under the Torelli group, we consider the $R=1$ orbifold model. Here the chiral components are given (in a convenient homology basis) by

$$\frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\Pi)}{c\begin{bmatrix}0\dots 0\\ 0\dots \frac{1}{2}\end{bmatrix}}=\left\{\vartheta\begin{bmatrix}\alpha & 0\\ \beta & 0\end{bmatrix}(0|\tau)\vartheta\begin{bmatrix}\alpha & 0\\ \beta & \frac{1}{2}\end{bmatrix}(0|\tau)\right\}^{1/2}. \tag{3.27}$$

Let us analyse both these expressions near the boundary component of moduli space describing the degeneration of our dividing cycle C, and concentrate on the dependence on the pinching parameter t (i.e. the length of C). The Dehn twist D_C acts on t as

$$D_C: t\to e^{2\pi i}t. \tag{3.28}$$

If we choose the (even) spin structure $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ on the left-hand side of (3.27) such that it splits into two odd spin structures for $t\to 0$, then the same is true for one of the spin structures on the right-hand side. It is quite easy to see that for this situation the chiral partition function behaves to first order as $t^{1/2}$ and hence is not invariant under (3.28). The fermionic explanation is that the partition function of one of the chiral Majorana fermions factorizes on the 1-point functions of two ψ fields absorbing the zero mode on each side. The power of t equals the conformal dimension of ψ. In the orbifold picture (3.27) is seen to factorize on the 1-point function of the corresponding chiral vertex operator $\cos\varphi$. The key observation here is that, as we will show explicitly in the next section, vertex operators indeed can have vacuum expectation values in orbifold models. This is a consequence of the fact that φ-charge is no longer conserved around twisted cycles. This argumentation makes it clear that also for other radii R the chiral orbifold partition functions are not invariant under the Torelli group, as long as the corresponding chiral theories contain vertex operators with noninteger conformal weights. Of course, the total partition function (3.19) is invariant, since the full theory only contains operators with integer spin $h-\bar{h}$.

4. Vertex Operators and Factorization

In this section we compute the correlation functions of vertex operators in both the torus and the orbifold model on the genus g Riemann surface Σ. In these calculations we will employ the factorization expansion of the partition function to determine the analytic structure of the correlators. The results of this section will be used for the construction of the operator formalism on Riemann surfaces, as described in Sect. 6.

The n-point functions of vertex operators

$$A(z_i,\bar{z}_i;q_i,\bar{q}_i)=\left\langle\prod_{i=1}^{n}\exp[iq_i\varphi(z_i)+i\bar{q}_i\bar{\varphi}(\bar{z}_i)]\right\rangle \tag{4.1}$$

can be obtained by considering the partition function near the compactification divisor, where an appropriate number of handles is pinched, and projecting on loop momenta $q_i,\bar{q}_i$ in the degenerate channels. It is clear that the resulting expressions will reveal the same analytic structure as the partition function. So in the case of a torus compactification we expect that the amplitudes (4.1) can be written as a sum over the g-th power of the momentum lattice Γ_R and each contribution is the product of a meromorphic times an anti-meromorphic function of the positions z_i. We have the following factorization formulas for the degeneration of a nonzero homology cycle [16]

$$i\pi[p\cdot\tau\cdot p]_{g+1}\to\tfrac{1}{2}p_0^2\log t+i\pi(p\cdot\tau\cdot p)+2\pi ip\cdot p_0\int_a^b\omega$$
$$-p_0^2\log E(a,b)+\mathcal{O}(t), \tag{4.2a}$$

$$\omega_0(z)\to\partial_z\log\left[\frac{E(a,z)}{E(b,z)}\right]+\mathcal{O}(t), \tag{4.2b}$$

while all other quantities factorize trivially, to first order. By repeatedly applying these formulas one deduces the following expression for the unnormalized n-point function in terms of the prime form $E(z,w)$ and the holomorphic 1-forms $\omega_i(z)$ $(i=1,\ldots,g)$ of the Riemann surface Σ:

$$\begin{aligned}A_0(z_i,\bar{z}_i;q_i,\bar{q}_i)&=Z_0^{qu}\sum_{(p,\bar{p})\in\Gamma_R^g}A_0^p(z_i;q_i)\overline{A_0^p(z_i;q_i)},\\ A_0^p(z_i;q_i)&=\prod_{i<j}E(z_i,z_j)^{q_iq_j}\\ &\quad\times\exp\left[i\pi(p\cdot\tau\cdot p)+2\pi i\left(p\cdot\sum_i q_i\int^{z_i}\omega\right)\right].\end{aligned} \tag{4.3}$$

In this expression complex conjugation also takes $p\to\bar{p}$ and $q\to\bar{q}$. The momenta $q_i,\bar{q}_i$ are forced by crossing symmetry of the amplitude to be elements of the lattice

Γ_R. Factorization automatically gives momentum conservation:

$$\sum_i (q_i, \bar{q}_i) = (0, 0). \tag{4.4}$$

The expressions for all other correlation functions follow from (4.3) by taking suitable operator products, i.e. limits of the positions z_i.

The result (4.3) can also be obtained by a generalization of the path-integral method of [26]. Of course the expression (4.1) cannot be directly inserted into the path-integral, since it is not a function of the integration variable $\phi(z, \bar{z})$. Instead for each vertex operator in (4.1) we insert $\exp[i\frac{1}{2}(q_i + \bar{q}_i)\phi(z_i, \bar{z}_i)]$ and integrate over field configurations with winding number $q_i - \bar{q}_i$ around the point z_i. This is achieved by adding to the classical solitons the extra piece

$$\sum_i -\tfrac{1}{2}i(q_i - \bar{q}_i)\left[\log E(z, z_i) - \tfrac{1}{2}\pi \int_{z_i}^{z} \omega \cdot (\operatorname{Im}\tau)^{-1} \cdot \int_{z_i}^{z} \omega\right] + \text{c.c.}.$$

A straightforward application of Wick's theorem and the Poisson resummation formula then yields (4.3).

Let us now turn to the correlators of vertex operators in the $\mathbb{Z}_2$-orbifold theory. On the sphere or complex plane these are the same as in the torus model; for higher genus, however, some new features appear due to the fact that we can have twisted states in the loops. In particular, momentum conservation is lost, since the chiral currents $\partial\varphi$ and $\overline{\partial\varphi}$ are no longer allowed operators in the orbifold model. However, we still have a selection rule as a result of the $\mathbb{D}_4$ symmetry (2.30). It reads

$$\sum_i (q_i, \bar{q}_i) \in 2\Gamma_R. \tag{4.5}$$

The analytic structure of the amplitudes again follows from factorization. So, just like the partition function, they will be a sum of two terms which are by itself single-valued and modular invariant. The first equals 2^{-g} times the corresponding untwisted expression, while the latter, which we will denote as A_t, contains the contribution of the twisted intermediate states. If we choose a particular twisted cycle and project onto definite loop momenta in the untwisted loops, the amplitudes will factorize into an analytic times an anti-analytic part. Because there is no explicit factorization formula available for the period matrix of the Prym differentials, we will compute A_t via the path-integral method described above. To carry out this calculation, it will again be convenient to consider the double cover $\hat{\Sigma}$ of the multi-loop surface Σ. This enables us to express the amplitudes in terms of the prime form $\hat{E}(\hat{z}, \hat{w})$ on $\hat{\Sigma}$ and the Prym differentials v_i. Note that we have to insert vertex operators on both sheets of the double cover with opposite momenta as depicted in Fig. 4. This automatically guarantees momentum conservation on $\hat{\Sigma}$.

The propagator of an uncompactified double-valued scalar field twisted around the cycle B_g is given by

$$\langle \phi(z, \bar{z})\phi(w, \bar{w})\rangle_{\text{twisted}} = \log F_t(z, w), \tag{4.6}$$

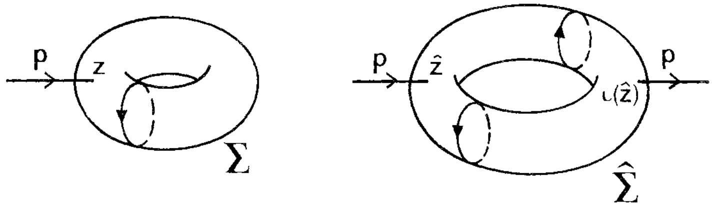

Fig. 4. The insertion of a vertex operator on the surface Σ corresponds on the double cover $\hat{\Sigma}$ to the insertion of a conjugate pair of operators

where

$$F_t(z,w)=|E_t(z,w)|^2 \exp\left[2\pi \operatorname{Im} \int_{\iota(\hat{z})}^{\hat{z}} v\cdot(\operatorname{Im}\Pi)^{-1}\cdot \operatorname{Im}\int_{\iota(\hat{w})}^{\hat{w}} v\right], \tag{4.7}$$

$$E_t(z,w)=\frac{\hat{E}(\hat{z},\hat{w})}{\hat{E}(\hat{z},\iota(\hat{w}))}e^{-1/2i\pi\tau_{gg}}. \tag{4.8}$$

F_t is single-valued around all cycles except for the twisted cycle B_g, around which it transforms into $1/F_t$. The above result can be proved by a similar method as described in [26] Sect. 5. The expression for the amplitude (4.1) for $R=0$ (or ∞) now follows directly from Wick's theorem. The modification at finite R due to the soliton contributions is computed completely analogous to the torus case. Again skipping the details of the calculation we proceed with the final expression for the twisted contribution to the n-point function (4.1),

$$A_t(z_i,\bar{z}_i;q_i,\bar{q}_i)=Z_0^{qu}2^{-g}\sum_{\varepsilon,\delta}\sum_{(p,\bar{p})\in\Gamma_R^{g-1}} A^p_{\varepsilon,\delta}(z_i;q_i)\overline{A^{\bar{p}}_{\varepsilon,\delta}(z_i;q_i)}, \tag{4.9}$$

where the meromorphic contributions are given by

$$A^p_{\varepsilon,\delta}(z_i;q_i)=c\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}^{-1}\prod_{i<j}E_t(z_i,z_j)^{q_iq_j}\prod_i \varepsilon(z_i)^{1/2q_i^2}$$
$$\times\exp\left[i\pi(p\cdot\Pi\cdot p)+2\pi i\left(p\cdot\sum\tfrac{1}{2}q_i\int_{\iota(\hat{z}_i)}^{z_i} v\right)\right]. \tag{4.10}$$

Here the presence of the factors

$$\varepsilon(z)=\hat{E}(\hat{z},\iota(\hat{z}))^{-1}e^{-1/2i\pi\tau_{gg}} \tag{4.11}$$

is due to the self-energy of the vertex-operators. Of course, all quantities on the right-hand side of (4.10) are understood to be those related to the twist ε,δ. The above expression for the amplitude A is not yet a single-valued function of the positions of all vertex operators; only the combinations $V^+_{nm}(z,\bar{z})$ [see Eq. (2.11)] have single-valued correlators.

We have shown that vertex operators of the form $V^+_{2n,2m}$ indeed have vacuum expectation values in the orbifold theory. As we have argued this implies directly that the twisted chiral partition functions of the $\mathbb{Z}_2$-orbifold theories are sensitive to the Torelli group. We are now in a position to make this relation more explicit.

So again let us consider a Dehn twist around a homologically trivial cycle C, and the behaviour of the partition function near the boundary of moduli space corresponding to the degeneration of C. Here we can compare the expression (3.19) for the partition function with the factorization expansion

$$Z_{g_1+g_2} \to Z_{g_1} Z_{g_2}\left[1 + \sum_{n,m\in 2\mathbb{Z}} t^{h_{nm}} \bar{t}^{\bar{h}_{nm}} \langle V^+_{nm}\rangle_{g_1} \langle V^+_{nm}\rangle_{g_2} + (\text{descendants})\right], \quad (4.12)$$

where g_1 and g_2 are the genera of the two parts of Σ separated by C, and $\langle V^+_{nm}\rangle_{g_i}$ denotes the 1-point function of the vertex operator V^+_{nm} on each part. The chiral partition functions on Σ with twist characteristics which are divided by C into two nontrivial parts will receive non-vanishing contributions of these 1-point functions. Equating the separate momentum contributions on both sides of (4.12) and isolating the part quadratic in the loop momenta we read off that, if C degenerates, the Π matrix of such a twist characteristic factorizes as

$$\begin{aligned} i\pi[p\cdot\Pi\cdot p]_{g_1+g_2} &\to 2p_0^2 \log t + i\pi[p\cdot\Pi\cdot p]_{g_1} + i\pi[p\cdot\Pi\cdot p]_{g_2} \\ &+ 2\pi i p_1\cdot p_0 \int_{\iota(\hat{Q}_1)}^{\hat{Q}_1} v_1 + 2\pi i p_2\cdot p_0 \int_{\iota(\hat{Q}_2)}^{\hat{Q}_2} v_2 + 2p_0^2 \log[\varepsilon_1(Q_1)\varepsilon_2(Q_2)] + \mathcal{O}(t). \quad (4.13)\end{aligned}$$

The first term on the right-hand side signals the momentum $2p_0$ running through the tube enclosed by C and is clearly not invariant under $t\to e^{2\pi i}t$. This factorization behaviour of the Π matrix should be contrasted with that of the period matrix τ:

$$i\pi[p\cdot\tau\cdot p]_{g_1+g_2} \to i\pi[p\cdot\tau\cdot p]_{g_1} + i\pi[p\cdot\tau\cdot p]_{g_2} + \mathcal{O}(t). \quad (4.14)$$

where $\mathcal{O}(t)$ is a single-valued function of t.

For the specific models in the $c=1$ spectrum many relevant correlation functions can now be calculated using (4.9)–(4.11). As examples let us determine in the $R=1$ orbifold model the two-point function of the magnetic vertex operator $\sqrt{2}\cos\frac{1}{2}(\varphi-\bar{\varphi})$. This operator according to (2.36) equals the composite operator $\sigma^{(1)}\sigma^{(2)}(z,\bar{z})$ in the doubled Ising system, so the result should be equal to the square of the spin-spin correlator of the Ising model [15, 35]. Let us see how this comes about. By lack of momentum conservation we have two different twisted contributions to this correlation function, with holomorphic components:

$$\begin{aligned}\langle e^{1/2i\varphi(z)} e^{-1/2i\varphi(w)}\rangle &= c\begin{bmatrix}0\ldots 0\\ 0\ldots\frac{1}{2}\end{bmatrix}^{-1} E_t(z,w)^{-1/4}\varepsilon(z)^{1/8}\varepsilon(w)^{1/8} \\ &\times \iota\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{1}{4}\int_{\iota(\hat{z})}^{\hat{z}} v - \frac{1}{4}\int_{\iota(\hat{w})}^{\hat{w}} v \middle| \Pi\right), \quad (4.15\text{a})\end{aligned}$$

$$\begin{aligned}\langle e^{1/2i\varphi(z)} e^{1/2i\varphi(w)}\rangle &= c\begin{bmatrix}0\ldots 0\\ 0\ldots\frac{1}{2}\end{bmatrix}^{-1} E_t(z,w)^{1/4}\varepsilon(z)^{1/8}\varepsilon(w)^{1/8} \\ &\times \iota\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{1}{4}\int_{\iota(\hat{z})}^{\hat{z}} v + \frac{1}{4}\int_{\iota(\hat{w})}^{\hat{w}} v \middle| \Pi\right). \quad (4.15\text{b})\end{aligned}$$

After some manipulations using a generalized version of the Schottky relation [16] the right-hand sides can be rewritten as:

$$\langle e^{1/2i\varphi(z)}e^{-1/2i\varphi(w)}\rangle = E(z,w)^{-1/4}\left\{ \vartheta\begin{bmatrix}\alpha & 0\\ \beta & 0\end{bmatrix}\left(\frac{1}{2}\int_w^z \omega\,|\,\tau\right) \right.$$

$$\left. \times\, \vartheta\begin{bmatrix}\alpha & 0\\ \beta & \frac{1}{2}\end{bmatrix}\left(\frac{1}{2}\int_w^z \omega|\tau\right)\right\}^{1/2}, \qquad (4.16a)$$

$$\langle e^{1/2i\varphi(z)}e^{1/2i\varphi(w)}\rangle = E(z,w)^{-1/4}\left\{ \vartheta\begin{bmatrix}\alpha & \frac{1}{2}\\ \beta & 0\end{bmatrix}\left(\frac{1}{2}\int_w^z \omega\,|\,\tau\right) \right.$$

$$\left. \times\, \vartheta\begin{bmatrix}\alpha & \frac{1}{2}\\ \beta & \frac{1}{2}\end{bmatrix}\left(\frac{1}{2}\int_w^z \omega|\tau\right)\right\}^{1/2}. \qquad (4.16b)$$

In terms of the two Majorana fermions we see that the first term gives the contribution of the [even] × [even] or [odd] × [odd] spin structures (depending on whether $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ in (4.15) is even or odd), whereas the second corresponds to [even] × [odd]. This identification is in agreement with the leading behaviour of (4.15) for $z \to w$. Performing the summation over all sectors and taking the square root, we obtain for the unnormalized spin 2-point function

$$\langle\sigma(z,\bar{z})\sigma(w,\bar{w})\rangle_{\text{Ising}} = 2^{-1/2g}[Z_0^{qu}]^{1/2}|E(z,w)|^{-1/4}\sum_{\alpha,\beta}\left|\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{1}{2}\int_w^z \omega|\tau\right)\right|. \qquad (4.17)$$

Of course, this expression can also be obtained by direct factorization of the Ising partition function (3.23). The result (4.17) agrees with that of [32, 35] for $g=1$.

Finally, we notice that the equivalence of the $R=\sqrt{2}$ and $R=1$ orbifold theory with the $R=\frac{1}{2}\sqrt{2}$ Gaussian model respectively squared Ising model may be exploited to derive various nontrivial identities relating geometrical objects on the multi-loop surface Σ to objects defined on its double cover. To give a simple example: the equality of the two vacuum expectation values

$$\langle \tfrac{1}{2}\sqrt{2}\, i\,\partial\varphi(z)\rangle_{\text{torus},\,R=1/2\sqrt{2}} = \langle \cos\sqrt{2}\varphi(z)\rangle_{\text{orbifold},\,R=\sqrt{2}} \qquad (4.18)$$

implies, when projected on the relevant subsectors, the identity [16, p. 83]:

$$\sum_j i\partial_j\vartheta\begin{bmatrix}\alpha & \frac{1}{2}\\ \beta & \frac{1}{2}\end{bmatrix}(0|2\tau)\omega_j(z) = c\begin{bmatrix}0\dots 0\\ 0\dots \frac{1}{2}\end{bmatrix}^{-1}\varepsilon(z)\vartheta\begin{bmatrix}\alpha\\ 0\end{bmatrix}\left(\int_{\iota(\hat{z})}^{\hat{z}} v|2\Pi\right). \qquad (4.19)$$

Pursuing this line much further would lead us deep into the mathematics of unramified coverings of Riemann surfaces, which is of course not the intent of this paper. We would like to emphasize, however, that many existing mathematical identities can be of great help in the analysis of the correspondences between the various physical models.

5. Correlation Functions of Twist Operators

As a further application of the methods developed in the previous sections we will now proceed to calculate correlation functions including orbifold twist operators

$$\langle \sigma_1(z_1) \dots \sigma_1(z_m)\sigma_2(z_{m+1}) \dots \sigma_2(z_n)V_1(w_1) \dots V_l(w_l)\rangle . \tag{5.1}$$

To evaluate this function we will again make use of the theory of double covers of Riemann surfaces. We will describe the calculation for arbitrary genus, and then turn to the sphere for more explicit results. A somewhat different approach is given by Miki [28] and for $g=0$ in [11, 33, 34].

Twist field correlation functions are produced by factorization of the partition function when a non-zero homology cycle is pinched. The different constituents $Z_{\varepsilon,\delta}$ of the genus $g+1$ partition function factorize in lowest order of the pinching parameter t either on the genus g partition function or on the unnormalized twist field two-point function depending on whether or not the pinched loop is twisted. Schematically

$$Z_{g+1} \to Z_g + \dots + |t|^{1/8} \sum_{i=1}^{2} \langle \sigma_i(z_1)\sigma_i(z_2)\rangle_g + \dots . \tag{5.2}$$

In general we see that pinching n twisted loops of a genus $g+n$ surface leaves a genus g Riemann surface Σ with $2n$ twist field insertions. In this factorization process the genus of the double cover changes from $2g+2n-1$ to $2g+n-1$. The resulting cover $\hat{\Sigma}$ of Σ is ramified over $2n$ branch points, i.e. fixed points of the defining involution ι, at the positions of the twist fields. Of course, besides the twists around the operators $\sigma(z_k, \bar{z}_k)$ we should also account for possible twists along the nontrivial cycles of Σ indexed by the characteristics ε_i, δ_i $(i=1, \dots, g)$. Modular invariance at genus $g+n$ translates into modular invariance at genus g and crossing symmetry of the σ's [5, 23]. Indeed, by transporting a twist operator around an untwisted loop one converts it into a twisted loop, and vice versa. So any twist characteristic $\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}$ can be mapped to $\begin{bmatrix}0\\ 0\end{bmatrix}$ by the crossing transformation

$$\int^{z_k} \omega \to \int^{z_k} \omega + 2\varepsilon \cdot \tau + 2\delta . \tag{5.3}$$

As a homology basis of the cover $\hat{\Sigma}$ we choose the cycles $\hat{A}_i$, $\hat{B}_i$, $\iota(\hat{A}_i)$, $\iota(\hat{B}_i)$ $(i=1, \dots, g)$ and the $2(n-1)$ extra cycles $\hat{A}_\lambda, \hat{B}_\lambda$ $(\lambda=1, \dots, n-1)$. The latter correspond on the underlying surface Σ to loops encircling an even number of twist fields.

The number of Prym differentials does not change by the factorization; it stays $g+n-1$. We will normalize them with respect to the $\hat{A}_i$ and $\hat{A}_\lambda$-cycles of $\hat{\Sigma}$. All the Prym differentials v_i, v_λ have square root singularities at the branch points z_k. It will be convenient to define their period matrix Π around the cycles $\hat{B}_i$ and $\frac{1}{2}\hat{B}_\lambda$. This normalization is consistent with factorization.

The essential idea now is to equate any multi-point correlation function with twist field insertions to the corresponding quantity without the twist fields calculated on the ramified double cover $\hat{\Sigma}$ with the field $\hat{\phi}$ odd under ι. The dependence on the positions of the twist operators $\sigma(z_k)$ is incorporated in the definition of $\hat{\Sigma}$. Note however, that because of the conformal anomaly all these quantities depend on the coordinatization we choose. The prescription is to use the coordinate z of the Riemann surface Σ. The conformal dimension $(\frac{1}{16},\frac{1}{16})$ of the twist operators is due to the fact that z is a singular coordinate at the positions of the twist fields.

The field $\hat{\phi}$ cannot have arbitrary winding numbers along the cycles $\hat{A}_\lambda, \hat{B}_\lambda$ of $\hat{\Sigma}$. Around any cycle $\mathscr{C}$ that projected on Σ encircles an even number of σ's, the operator product relations (2.28) imply the monodromy condition [11]

$$\oint_{\mathscr{C}} \frac{dz}{2\pi R}\,\partial\hat{\varphi}(z)+\text{c.c.}=Q\,(\text{mod}\,2)\,, \tag{5.4}$$

where Q is a $\mathbb{Z}_2$-charge defined as

$$Q=\#\sigma_1=\#\sigma_2\,(\text{mod}\,2)\,. \tag{5.5}$$

We define $\mathbb{Z}_2$-charges a_λ, b_λ for the cycles $\hat{A}_\lambda, \hat{B}_\lambda$ as in (5.4).

Each quantity can now be written as a sum over ε_i and δ_i, numbering the possible twists around the cycles A_i, B_i, and loop momenta $(p_i, p_\lambda; \bar{p}_i, \bar{p}_\lambda)$. The appropriate momentum lattice is fixed by the configuration of twist fields σ_1 and σ_2. To investigate this relation let us analyse the pure twist correlation function

$$A(z_k,\bar{z}_k)=\left\langle \prod_{k=1}^{2n_1}\sigma_1(z_k,\bar{z}_k)\prod_{k=1}^{2n_2}\sigma_2(z_k,\bar{z}_k)\right\rangle, \tag{5.6}$$

i.e. the partition function $\hat{Z}$ on $\hat{\Sigma}$. (Note that the discrete symmetry group $\mathbb{D}_4$ forces correlation functions of an odd number of σ_1's or σ_2's to vanish.) This partition function is again given by a summation over all twisted sectors. Each term is the product of a quantum contribution $\hat{Z}^{qu}$ times a sum over winding sectors. The soliton sum $\hat{Z}^{cl}_{a,b}$ is restricted by the monodromy conditions labelled by a_λ, b_λ. The calculation is quite analogous to the one described in Sect. 3. The classical solutions $\hat{\phi}^{cl}$ are labelled by $m_i, n_i \in \mathbb{Z}$, $m_\lambda \in \mathbb{Z}+\frac{1}{2}b_\lambda$, $n_\lambda \in 2\mathbb{Z}+a_\lambda$ and have action

$$S[\hat{\phi}^{cl}]=\tfrac{1}{2}\pi R^2(n-\bar{\Pi}m)\cdot(\text{Im}\,\Pi)^{-1}\cdot(n-\Pi m)\,. \tag{5.7}$$

The soliton summation yields [28]

$$\hat{Z}^{cl}_{a,b}(R)=\sum_{p,\bar{p}}(-)^{b\cdot n}\exp[i\pi(p\cdot\Pi\cdot p-\bar{p}\cdot\bar{\Pi}\cdot\bar{p})]\,, \tag{5.8}$$

where

$$\begin{gathered} p_i,\bar{p}_i\in\Gamma_R \qquad (i=1,\ldots,g)\,,\\ p_\lambda,\bar{p}_\lambda\in\left\{\left[\frac{n}{R}+(m+\tfrac{1}{2}a_\lambda)R,\ \frac{n}{R}-(m+\tfrac{1}{2}a_\lambda)R\right];n,m\in\mathbb{Z}\right\} \qquad (\lambda=1,\ldots,n-1)\,. \end{gathered} \tag{5.9}$$

As for the quantum part we again make use of the multi-critical point $R=\sqrt{2}$. Namely, as we have seen, in this point the twist fields σ_1 and σ_2 are equivalent to the magnetic vertex operators $\sqrt{2}\cos\frac{1}{4}\sqrt{2}(\varphi-\bar{\varphi})$ and $\sqrt{2}\sin\frac{1}{4}\sqrt{2}(\varphi-\bar{\varphi})$ of the $R=\frac{1}{2}\sqrt{2}$ torus model, whose correlation functions are simply a sum of generalized

Koba-Nielsen amplitudes (4.3),

$$A(z_k,\bar{z}_k)|_{R=\sqrt{2}}=2^{-g-n+1}Z_0^{qu}\sum_{\gamma,\delta,\varepsilon}\sum_{(x|y)}(-)^{Q(x)}\left|\frac{\prod_{i<j}E(x_i,x_j)E(y_i,y_j)}{\prod_{i,j}E(x_i,y_j)}\right|^{1/4}$$
$$\times\left|\vartheta\begin{bmatrix}\gamma+\frac{1}{2}\varepsilon\\ \delta\end{bmatrix}\left(\frac{1}{2}\sum_{i=1}^{n}\int_{y_i}^{x_i}\omega|2\tau\right)\right|^2. \tag{5.10}$$

Here we sum over all $\frac{1}{2}\binom{2n}{n}$ partitions $(x|y)$ of the branch points z_k into two subsets $\{x_k\}$ and $\{y_k\}$ of n elements. The charge $Q(x)$ is the total $\mathbb{Z}_2$-charge in the subset $\{\sigma(x_k)\}$. Note that we regard $(x|y)$ and $(y|x)$ as equivalent. On the other hand we can express the partition function on $\hat{\Sigma}$ in terms of ϑ-functions as explained in Sect. 3,

$$\hat{Z}_{a,b}(\sqrt{2})=2^{-g-n+1}\sum_{\varepsilon,\delta}\hat{Z}^{qu}_{\varepsilon,\delta}\sum_{\mu,\nu,\gamma}(-)^{(2\gamma\cdot b+b\cdot a)}\left|\vartheta\begin{bmatrix}\gamma & \mu\\ 0 & \nu\end{bmatrix}(0|2\Pi_{\varepsilon,\delta})\right|^2. \tag{5.11}$$

Here the summation is over the half-integer characteristics

$$\gamma,\varepsilon,\delta\in(\tfrac{1}{2}\mathbb{Z}_2)^g,\qquad \mu,\nu\in(\tfrac{1}{2}\mathbb{Z}_2)^{n-1}. \tag{5.12}$$

Remarkably, it turns out that again one can identify the separate holomorphic terms in (5.10) and (5.11). For $\gamma,\delta,\varepsilon$ we can repeat the argument of Sect. 3. As for the characteristics μ,ν we should note that not all even spin structures contribute in (5.11). In fact there are only $\frac{1}{2}\binom{2n}{n}$ non-singular even spin structures, i.e. with non-vanishing ϑ-function, which exactly correspond to the partitions $(x|y)$ of the branch points. This is explained in all detail in [16]. This allows us to solve for $\hat{Z}^{qu}_{\varepsilon,\delta}=\left|c\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}\right|^{-2}Z_0^{qu}$ as the ratio of the two corresponding contributions in (5.10) and (5.11). The result is the modulus squared of an holomorphic function of the positions of the branch points. So for example for the two-point function we find

$$\langle\sigma(z,\bar{z})\sigma(w,\bar{w})\rangle=2^{-g}Z_0^{qu}\sum_{\varepsilon,\delta}\left|c\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}\right|^{-2}\hat{Z}^{cl}(R,\Pi_{\varepsilon,\delta}), \tag{5.13}$$

where

$$c\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}^{-1}=E(z,w)^{-1/8}\frac{\vartheta\begin{bmatrix}\gamma+\frac{1}{2}\varepsilon\\ \delta\end{bmatrix}\left(\frac{1}{2}\int_z^w\omega|2\tau\right)}{\vartheta\begin{bmatrix}\gamma\\ 0\end{bmatrix}(0|2\Pi_{\varepsilon,\delta})}, \tag{5.14}$$

which is independent of the characteristic γ. For $R=1$ this expression is consistent with the result (4.17), since we can rewrite (5.14) as

$$c\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}^{-1}=E(z,w)^{-1/8}\frac{\left[\vartheta\begin{bmatrix}\alpha+\varepsilon\\ \beta+\delta\end{bmatrix}\left(\frac{1}{2}\int_z^w\omega|\tau\right)\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\tau)\right]^{1/2}}{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\Pi_{\varepsilon,\delta})} \tag{5.15}$$

for arbitrary α,β.

For higher n-point functions the resulting expressions have the same structure as (5.13). An unpleasant feature is that they are not manifestly independent of the choice of the spin structure $\begin{bmatrix}\mu\\ \nu\end{bmatrix}$ or partition $(x|y)$. It would be very interesting to find such a manifest independent formulation.

The computation of correlation functions including vertex operators is completely analogous to the one described in the previous section, provided one uses the branched double cover $\hat{\Sigma}$ and the momentum lattice (5.9).

Now let us specialize to the case of twist operator correlation functions on the sphere. Here we can explicitly write down the double cover. It is the hyperelliptic Riemann surface

$$\hat{\Sigma}=\left\{(w,z)|w^2=\prod_{i=1}^{2n}(z-z_i)^2\right\}. \tag{5.16}$$

The involution ι is simply $(w,z)\to(-w,z)$. The period matrix τ of $\hat{\Sigma}$ equals 2Π. For the case $g=0$ Eqs. (5.10) and (5.11) simplify considerably

$$A(z_k,\bar{z}_k)|_{R=\sqrt{2}}=2^{-n-1}\sum_{(x|y)}(-)^{Q(x)}\left|\frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}(x_i-y_i)}\right|^{1/4}, \tag{5.17}$$

$$\hat{Z}_{a,b}(\sqrt{2})=2^{-n-1}\hat{Z}^{qu}\sum_{\mu,\nu}(-)^{(2\gamma\cdot b+b\cdot a)}\left|\vartheta\begin{bmatrix}\mu\\ \nu\end{bmatrix}(0|\tau)\right|^2. \tag{5.18}$$

The correspondence between the non-singular even spin structures $\begin{bmatrix}\mu\\ \nu\end{bmatrix}$ and the partitions $(x|y)$ is expressed by the Thomae identity valid for hyperelliptic ϑ-functions [16]

$$\vartheta\begin{bmatrix}\mu\\ \nu\end{bmatrix}^8(0|\tau)=(\det M)^{-4}\prod_{i<j}(z_i-z_j)\frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}(x_i-y_i)}. \tag{5.19}$$

The matrix M is related to the canonical 1-forms ω_i on $\hat{\Sigma}$ by

$$\omega_i(z)=\sum_{i=1}^{n-1}M_{ij}\frac{z^{j-1}}{y}dz. \tag{5.20}$$

Thus we find as a final answer for the twist correlator

$$A(z_k,\bar{z}_k)=|\det M|\prod_{i<j}|z_i-z_j|^{-1/4}\sum_{p,\bar{p}}(-)^{b\cdot n}\exp[\tfrac{1}{2}i\pi(p\cdot\tau\cdot p-\bar{p}\cdot\bar{\tau}\cdot\bar{p})], \tag{5.21}$$

where the momentum summation is as in (5.9) with $g=0$. This reproduces the results obtained in [11, 33, 34].

Some comments on (5.21) are due here. First we see that $R\to 1/R$ leaves the correlation function invariant. (Note however that this symmetry is only valid for genus $g=0$.) So in particular the twist field n-point function at $R=\frac{1}{2}\sqrt{2}$ (the 4 state Potts model) is also given by the Koba-Nielsen amplitude (5.17). This is in accordance with the permutational symmetry $\mathbb{S}_3$ between the two twist operators σ_1 and σ_2 and the magnetic vertex operator $\sigma_3=\sqrt{2}\cos\frac{1}{4}\sqrt{2}(\varphi-\bar{\varphi})$ at this special point.

At first sight this invariance under $R\to 1/R$ seems hard to reconcile with the duality $R\to 2/R$. We must realize however that the latter transformation also acts on the twist fields. This can serve as a check on our transformation rule (2.27). Let us consider the simple case $n_2=0$,

$$\left\langle \prod_{k=1}^{2n} \sigma_1(z_k,\bar{z}_k)\right\rangle_R = \left\langle \prod_{k=1}^{2n} \sigma_1(z_k,\bar{z}_k)\right\rangle_{1/R} = \hat{Z}_{0,0}(1/R) = 2^{1-n} \sum_{a,b\in\mathbb{Z}_2} \hat{Z}_{a,b}(2/R)$$
$$= 2^{-n}\left\langle \prod_{k=1}^{2n} [\sigma_1(z_k,\bar{z}_k)+\sigma_2(z_k,\bar{z}_k)]\right\rangle_{2/R} . \tag{5.22}$$

Here we used that the sum over all monodromy conditions is equivalent to the transformation $2/R\to 1/R$. Thus the transformation rule (2.27) is confirmed.

6. Operator Formalism on Riemann Surfaces

In this section we describe the operator formulation of the compactified scalar field on the Riemann surface Σ. More specifically, this involves the construction of a state $|\Sigma\rangle$ in the Hilbert space of the scalar field ϕ describing the theory on Σ [6]. After a short discussion of the action of the Virasoro algebra we construct the state $|\Sigma\rangle$ for the torus models using the previous results for the correlators of vertex operators. Next we give an independent derivation based on a more algebraic approach which uses only some basic facts concerning the geometry of the surface. We end the section with a discussion of the operator formulation of orbifold models.

6a. Operator Formalism and the Virasoro Algebra

Given a base point Q on Σ and a choice of an analytic coordinate z near Q such that $Q=\{z=\infty\}$, one can associate a state $|\Sigma\rangle$ to any conformal field theory defined on Σ. This state $|\Sigma\rangle$ is defined to satisfy the condition that for any local set of operators $A_n(z_n,\bar{z}_n)$ we have

$$\langle 0|\prod_n A_n(z_n,\bar{z}_n)|\Sigma\rangle = \left\langle \prod_n A_n(z_n,\bar{z}_n)\right\rangle , \tag{6.1}$$

where $\langle\ \rangle$ denotes the unnormalized expectation value on the surface Σ. In particular the partition function is given by $Z=\langle 0|\Sigma\rangle$. In this formalism the only reference to the global geometry of the Riemann surface is contained in the boundary conditions imposed by the state $|\Sigma\rangle$. Condition (6.1) determines $|\Sigma\rangle$ uniquely. Note however that it depends on the choice of the point Q and the coordinatization z.

In general, for any chiral primary field $\psi(z)$ whose correlation functions are sections of a holomorphic line bundle $\mathscr{L}$ we have

$$\oint \frac{dz}{2\pi i}\,\xi(z)\psi(z)|\Sigma\rangle = 0\,, \tag{6.2}$$

whenever $\xi\in H^0(\Sigma-Q, K\otimes\mathscr{L}^{-1})$, i.e. ξ is a section of the bundle $K\otimes\mathscr{L}^{-1}$ that extends holomorphically to $\Sigma-Q$. Here K is the canonical line bundle on Σ and the contour integral is taken around Q. Equation (6.2) follows from the fact that the integral can be deformed and pulled off the surface without leaving any residues.

This fact is very useful in discussing the action of the stress-energy tensor $T(z)$ on the state $|\Sigma\rangle$. We adjoin to vector fields $\xi(z)$ holomorphic in the local coordinate patch except for possible poles in Q the Virasoro generators

$$L[\xi(z)]=\oint\frac{dz}{2\pi i}\xi(z)T(z), \tag{6.3}$$

satisfying [1]

$$[L[\xi],L[\eta]]=L[\xi\partial\eta-\eta\partial\xi]+\frac{c}{12}\oint\frac{dz}{2\pi i}\xi(z)\partial^3\eta(z). \tag{6.4}$$

The different components of $T(z)$ can now be classified in terms of the dual objects $\xi(z)$. If $\xi(z)$ extends holomorphically to $\Sigma-Q$ then in view of the above one concludes that $L[\xi]$ annihilates the state $|\Sigma\rangle$,

$$L[\xi]\,|\Sigma\rangle=0 \quad \text{for all} \quad \xi\in H^0(\Sigma-Q,K^{-1}). \tag{6.5}$$

However, there are some subtleties here, since $T(z)$ is not a conformal tensor. In fact $T(z)$ is a projective connection, i.e. it transforms with a schwarzian derivative [1, 36]. The problems associated with this are avoided by using a coordinate covering $\{U_\alpha, z_\alpha\}$ on Σ with transition functions $z_\alpha\circ z_\beta^{-1}\in Sl(2,\mathbb{C})$, since for these the schwarzian derivative vanishes. The equivalence class of such coverings is called a projective structure and always exists by the uniformization theorem. So if we choose the local coordinate z compatible with the projective structure on Σ then (6.5) is correct. The Virasoro operators L_n $(n\leqq 1)$ which correspond to vector fields nonsingular at Q generate analytic coordinate transformations in the local patch and will in general modify the property (6.5). When $\xi(z)$ extends neither to Q nor to $\Sigma-Q$ the corresponding components of $T(z)$ change the moduli parameters $\{m_k; k=1,\dots,3g-3\}$ of the surface Σ. More precisely, we can choose $3g-3$ elements ξ_k dual to the quadratic differentials such that for the partition function Z on Σ we have

$$\frac{\partial}{\partial m_k}Z=\langle 0|L[\xi_k]\,|\Sigma\rangle\,, \qquad (k=1,\dots,3g-3). \tag{6.5}$$

The operator formulation of conformal field theory is the natural language to describe the factorization expansion of the partition function and other quantities at the boundary of moduli space. For example, in the case of the degeneration of a dividing cycle C the behaviour of the partition function as a function of the pinching parameter t is exactly described by [5]

$$Z_{g_1+g_2}(t,\bar{t})=\langle\Sigma_1|t^{L_0}\bar{t}^{\bar{L}_0}|\Sigma_2\rangle\,. \tag{6.6}$$

where $|\Sigma_1\rangle$ and $|\Sigma_2\rangle$ describe the conformal field theory on the left respectively right half of the surface Σ. In a similar way the pinching and creation of handles can be dealt with. One chooses two points on Σ and and attributes a density matrix ϱ_Σ to this twice punctured Riemann surface. The partition function of the surface with the extra handle connecting the two points is now calculated as

$$Z_{g+1}(t,\bar{t})=\mathrm{tr}(\varrho_\Sigma t^{L_0}\bar{t}^{\bar{L}_0})\,. \tag{6.7}$$

6b. The Torus Case

Let us now turn to the explicit construction of the state $|\Sigma\rangle$ for the compactified conformal scalar field theories [6]. In the operator language $\phi(z,\bar{z})$ can be expanded as:

$$\begin{aligned}\phi(z,\bar{z}) &= \varphi(z)+\bar{\varphi}(\bar{z}),\\ \varphi(z) &= \tfrac{1}{2}q + a_0 \log z + \sum_{n=1}^{\infty}\left[z^{-n}\frac{a_n}{\sqrt{n}} + z^n \frac{a_n^\dagger}{\sqrt{n}}\right],\end{aligned} \tag{6.8}$$

with commutation relations $[a_0, q] = [a_n, a_n^\dagger] = 1$. The Hilbert space consists of all states of the form

$$|\Phi\rangle = \sum_{(p_0,\bar{p}_0)\in\Gamma_R} \Phi^{p_0,\bar{p}_0}[\varphi_+(z);\bar{\varphi}_+(\bar{z})]e^{ip_0 q}e^{-i\bar{p}_0\bar{q}}|0\rangle\,, \tag{6.9}$$

where $\varphi_+(z)$ is the creation part of $\varphi(z)$. The functionals $\Phi^{p_0,\bar{p}_0}$ are obtained from the state $|\Phi\rangle$ by taking the inner product with coherent states

$$\begin{aligned}\Phi^{p_0,\bar{p}_0}[\lambda(z);\bar{\lambda}(\bar{z})] = \langle 0|\exp\Bigg(&-ip_0 q + i\bar{p}_0\bar{q} + i\oint\frac{dz}{2\pi i}\lambda(z)\partial\varphi(z)\\ &-i\overline{\oint\frac{dz}{2\pi i}\lambda(z)\partial\varphi(z)}\Bigg)|\Phi\rangle\,.\end{aligned} \tag{6.10}$$

Here and in the subsequent equations the contour will be understood to encircle the point $Q=\{z=\infty\}$ unless otherwise stated. The action of the creation and annihilation operators $a_n^\dagger$ and a_n on the functionals Φ is given by the identification

$$a_n \leftrightarrow (1/i\sqrt{n})\partial/\partial\lambda_n, \qquad a_n^\dagger \leftrightarrow (i\sqrt{n})\lambda_n\,, \tag{6.11}$$

where λ_n is the n-th Laurent coefficients of $\lambda(z)$.

Combining Eqs. (6.1) and (6.10), it is clear that, with a straightforward application of the path-integral method described in Sect. 4, we can immediately obtain the answer for our state $|\Sigma\rangle$ for the torus model. It has zero momentum p_0 because of φ charge conservation. Projected onto a given set of loop momenta it factorizes into a left-moving times a right-moving state.

$$|\Sigma\rangle_{\text{torus}} = Z_0^{qu} \sum_{(p,\bar{p})\in\Gamma_R^g} A^p[\varphi_+(z)]\overline{A^p[\varphi_+(z)]}\,|0\rangle\,, \tag{6.12a}$$

where

$$\begin{aligned}A^p[\lambda(z)] = \exp&\left[\tfrac{1}{2}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\lambda(z)\lambda(w)\partial_z\partial_w \log E(z,w)\right]\\ &\times\exp[i\pi(p\cdot\tau\cdot p)+(p\cdot\oint dz\lambda(z)\omega(z))]\,.\end{aligned} \tag{6.12b}$$

This formula may be considered as a generating functional of all correlation functions on the surface Σ. In particular, by acting with vertex operators $V(\lambda_n, \partial/\partial\lambda_n)$ onto (6.12) one may obtain expression (4.3).

For rational values of R^2 the momentum summation in (6.12) can be replaced by a finite sum of ϑ-functions, using (3.5). For example, for the $R=1$ model the chiral components of $|\Sigma\rangle$ are given by the famous τ-function of the KP-hierarchy [7],

$$\tau[\lambda(z)] = \exp\left[\frac{1}{2}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\lambda(z)\lambda(w)\partial_z\partial_w \log E(z,w)\right] \times \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\oint\frac{dz}{2\pi i}\lambda(z)\omega(z)|\tau\right). \tag{6.13}$$

This τ-function describes a vacuum state of a chiral Dirac fermion. The space of such fermionic vacuum states, which can be obtained from the standard vacuum by a Bogoliubov transformation, is called the universal grassmannian manifold [7]. Also for general rational R^2 we can try to interpret the state $|\Sigma\rangle$ in terms of this grassmannian. Namely, we can associate to each ϑ-function in the sum (3.4b) a (projective) line bundle $\mathscr{L}$ on Σ with meromorphic sections given by the correlation functions of the chiral spin-$\frac{1}{4}pq$ "fermion" $\psi(z) = V_{1/2p,1/2q}(z)$. The corresponding chiral component of $|\Sigma\rangle$, in view of (6.2), distinguishes the positive and negative frequencies of the field ψ and can thus be identified with an element of the universal grassmannian manifold.

6c. Algebraic Construction

In this subsection we give an alternative derivation of the result (6.12) in terms of (generalized) Bogoliubov transformations, without using the path-integral. This more algebraic approach will clarify a great deal of the structure of the state $|\Sigma\rangle$ and furthermore lead to an intriguing relation between modular and canonical transformations.

We introduce the chiral creation/annihilation operators

$$a[f(z)] = \oint\frac{dz}{2\pi i} f(z)\partial\varphi(z), \tag{6.14}$$

with commutation relations

$$[a[f], a[g]] = \oint\frac{dz}{2\pi i} f(z)\partial g(z). \tag{6.15}$$

The key observation that enables us to reconstruct the state $|\Sigma\rangle$ is that $\partial\varphi(z)$ is a chiral primary field, and hence the state representing Σ is annihilated by the operators $a[f]$ precisely for those $f(z)$ that can be extended to a holomorphic function on $\Sigma - Q$, i.e. $f \in H^0(\Sigma - Q)$. It is clear from (6.15) that these modified annihilation operators $a[f]$ mutually commute. A basis $\{f_n\}$ for $H^0(\Sigma - Q)$ can be chosen of the form $f_n(z) = z^n +$(regular at Q). However, by the Weierstrass gap theorem [17], not all $n \geqq 0$ occur: there are g values of n between 1 and $2g$ missing. For generic positions of the point Q these values are $n = 1, \ldots, g$. The Weierstrass gap forms an obstruction to view the state $|\Sigma\rangle$ as a genuine Bogoliubov transform of the standard vacuum $|0\rangle$. As we will see many nontrivial features are due to the presence of this gap.

The state $|\Sigma\rangle$ is not uniquely determined by the condition that it is annihilated by all the elements of the set $\{a[f]; f \in H^0(\Sigma - Q)\}$. In fact there is an infinite dimensional space $\mathscr{H}_\Sigma$ of such vacuum states. To further analyse this space $\mathscr{H}_\Sigma$ we need additional operators that commute with all the annihilation operators $a[f]$. An obvious way to find such operators is to extend the set $\{a[f]; f\in H^0(\Sigma-Q)\}$ to a complete set of annihilation and creation operators. Because of the Weierstrass gap theorem such a complete set will contain $2g$ additional modes $a[g]$ satisfying

$$[a[f],a[g]]=\oint\frac{dz}{2\pi i}f(z)\partial g(z)=0\,,\quad \text{for all}\quad f\in H^0(\Sigma-Q)\,. \tag{6.16}$$

The idea is now to use a commuting subset of these operators to decompose the space $\mathscr{H}_\Sigma$ into eigenspaces. In order to have an interpretation of the corresponding eigenvalues we need to know a bit more about the space of functions $g(z)$ satisfying (6.16). First we observe that this space is of course only defined modulo elements of $H^0(\Sigma-Q)$. So we are dealing with a cohomology problem. Furthermore, we see from (6.16) that $\partial g(z)$ must be an element of $H^0(\Sigma-Q,K)$, which implies that $g(z)$ is extendable to a multi-valued holomorphic function with constant shifts around the nontrivial cycles of the surface. The space of such functions modulo $H^0(\Sigma-Q)$ is naturally dual to $H_1(\Sigma,\mathbb{C})$, the space of cycles on Σ, and hence is indeed $2g$-dimensional. The duality is expressed by the map

$$(C,g)\to\oint_C dz\,\partial g(z)\,, \tag{6.17}$$

where C is a cycle on the surface. We can now use the intersection product to adjoin to any cycle C a function $g_C(z)$ by

$$\oint_D dz\,\partial g_C(z)=\#(D,C)\quad \text{for all}\quad D\in H_1(\Sigma,\mathbb{C})\,. \tag{6.18}$$

This relation defines a one-to-one map between the functions $g(z)\,(\mathrm{mod}\,H^0(\Sigma-Q))$ such that $a[g]$ commutes with all annihilation operators and the nontrivial cycles on the surface. A reformulation of the condition (6.18) is given by the requirement that for any 1-form $\Omega(z)$ holomorphic on $\Sigma-Q$

$$\oint dz\, g_C(z)\Omega(z)=\oint_C dz\,\Omega(z)\,. \tag{6.19}$$

Commutation relations among the $a[g_C]$ are translated into intersection products of the corresponding cycles

$$[a[g_C],a[g_D]]=\oint\frac{dz}{2\pi i}g_C(z)\partial g_D(z)=\frac{1}{2\pi i}\#(D,C)\,. \tag{6.20}$$

We can refine the correspondence between cycles $C\in H_1(\Sigma,\mathbb{C})$ and modes g_C to $C\in H_1(\Sigma,\mathbb{Z})$ by demanding $\exp(2\pi i g_C)$ to be extendable to a nowhere vanishing, single-valued holomorphic function on $\Sigma-Q$. A natural basis for $H_1(\Sigma,\mathbb{Z})$ is given by a canonical set of homology cycles A_i, B_i satisfying (3.1). The representatives $a[g_{A_i}]$ and $a[g_{B_i}]$ of these cycles then satisfy canonical commutation relations

$$\begin{aligned}&[a[g_{A_i}],a[g_{A_j}]]=[a[g_{B_i}],a[g_{B_j}]]=0\,,\\ &[a[g_{A_i}],a[g_{B_j}]]=\frac{i}{2\pi}\delta_{ij}\,.\end{aligned} \tag{6.21}$$

We can now define the chiral state $|\Sigma; p\rangle$ which is annihilated by the operators $a[f]$ and is a common eigenstate of the $a[g_{A_i}]$:

$$a[f]\,|\Sigma;p\rangle=0\,,\qquad f\in H^0(\Sigma-Q)\,,$$
$$a[g_{A_i}]\,|\Sigma;p\rangle=\oint_{A_i}\frac{dz}{2\pi i}\,\partial\varphi(z)\,|\Sigma;p\rangle=p_i|\Sigma;p\rangle\,. \tag{6.22}$$

The eigenvalues p_i can be interpreted as the loop momenta flowing through the cycles A_i. The dependence on the momenta is fixed by the action of the conjugated operators

$$a[g_{B_i}]\,|\Sigma;p\rangle=\oint_{B_i}\frac{dz}{2\pi i}\,\partial\varphi(z)\,|\Sigma;p\rangle=\frac{1}{2\pi i}\frac{\partial}{\partial p_i}|\Sigma;p\rangle\,. \tag{6.23}$$

In the following the operators $a[g_{A_i}]$ and $2\pi i a[g_{B_i}]$ are identified with p_i and $\partial/\partial p_i$.

The definition (6.22)–(6.23) is invariant under shifting the functions g_{A_i} and g_{B_i} with elements of $H^0(\Sigma-Q)$. However the state $|\Sigma;p\rangle$ depends on the choice of basis in $H_1(\Sigma,\mathbb{Z})$. Different homology bases are related by modular transformations. We now want to construct a modular invariant vacuum state $|\Sigma\rangle$ of the nonchiral theory. In general $|\Sigma\rangle$ will be a linear combination of tensor products of left-moving and right-moving states of definite loop momenta

$$|\Sigma\rangle=\int d^gpd^g\bar{p}\,N(p,\bar{p})\,|\Sigma;p\rangle\otimes\overline{|\Sigma;p\rangle}\,. \tag{6.24}$$

The multiplicities $N(p,\bar{p})$ will be determined by the constraint of modular invariance. The action of the symplectic modular group $Sp(2g,\mathbb{Z})$ on the homology basis translates into canonical transformations on the operators p_i and $\partial/\partial p_j$,

$$2\pi i p_i\to 2\pi i A_{ij}p_j+B_{ij}\frac{\partial}{\partial p_j}\,,$$
$$\frac{\partial}{\partial p_i}\to 2\pi i C_{ij}p_j+D_{ij}\frac{\partial}{\partial p_j}\,,\qquad \begin{bmatrix}A & B\\ C & D\end{bmatrix}\in Sp(2g,\mathbb{Z})\,. \tag{6.25}$$

It is well known that a canonical transformation induces a unitary transformation on the states in the Hilbert space

$$|\Sigma;p\rangle\to U|\Sigma;p\rangle\,,\qquad U=\exp[2\pi i G(p_i,\partial/\partial p_i)]\,, \tag{6.26}$$

where $G(p,\partial/\partial p)$ is the generating function of the infinitesimal canonical transformation which by exponentiation gives the transformation $p_i\to Up_iU^\dagger$, $\partial/\partial p_i\to U\partial/\partial p_iU^\dagger$. The generating function G for an element of $Sp(2g,\mathbb{Z})$ is in general very complicated. Fortunately for studying modular invariance we only need to know G for the Dehn twists D_C (3.24) that generate the mapping class group. Combining (3.24) and (6.20) gives:

$$D_C: a[g_\gamma]\to a[g_\gamma]-2\pi i[a[g_\gamma],a[g_C]]a[g_C]\,, \tag{6.27}$$

so that

$$G(D_C)=\tfrac{1}{2}(a[g_C])^2\,, \tag{6.28}$$

and in particular for the minimal set of generators D_{A_i}, D_{B_i}, and $D_{A_i^{-1}A_{i+1}}$ the generating functions are

$$G(D_{A_i})=\tfrac{1}{2}p_i^2\,,\qquad G(D_{B_i})=\frac{-1}{4\pi^2}\frac{\partial^2}{\partial p_i^2}\,,$$
$$G(D_{A_i^{-1}A_{i+1}})=\tfrac{1}{2}(p_i-p_{i+1})^2\,. \tag{6.29}$$

We are now in a position to impose the condition of modular invariance and determine which states of the form (6.22) are allowed. Clearly, for all generating functions of the $Sp(2g,\mathbb{Z})$ transformations (6.25) we should demand

$$\exp[2\pi i(G-\bar{G})]\,|\Sigma\rangle=|\Sigma\rangle\,, \tag{6.30}$$

where G and $\bar{G}$ act on the left-moving respectively right-moving states. This condition implies that $|\Sigma\rangle$ is a sum of eigenstates of $G-\bar{G}$ with only integer eigenvalues. It is sufficient to impose this only for the generating Dehn twists (6.29). This gives the following set of restrictions on the multiplicities $N(p,\bar{p})$:

$$N(p,\bar{p})\neq 0 \quad \text{only if} \quad \begin{cases}(p_i^2-\bar{p}_i^2)\in 2\mathbb{Z}\\ (p_ip_{i+1}-\bar{p}_i\bar{p}_{i+1})\in\mathbb{Z}\end{cases} \quad \text{for all } i\,,$$
$$\tilde{N}(q,\bar{q})\neq 0 \quad \text{only if} \quad (q_i^2-\bar{q}_i^2)\in 2\mathbb{Z} \quad \text{for all } i\,, \tag{6.31}$$

where

$$\tilde{N}(q,\bar{q})=\int d^gp\,d^g\bar{p}\;N(p,\bar{p})e^{2\pi i(p\cdot q-\bar{p}\cdot\bar{q})} \tag{6.32}$$

is the Fourier transform of $N(p,\bar{p})$. It is well-known that these conditions restrict the integral (6.24) to a summation where each pair of loop momenta $p_i,\bar{p}_i$ runs over the same lattice, which is required to be lorentzian even and self-dual [19]. It is easily verified that there is a one parameter family of such modular invariant vacuum states $|\Sigma\rangle$:

$$|\Sigma\rangle=\sum_{(p,\bar{p})\in\Gamma_R^g}|\Sigma;p\rangle\otimes|\Sigma;p\rangle\,, \tag{6.33}$$

with Γ_R as defined in (2.9). Note that the flat space vacuum ($R=\infty$) is even invariant under $Sp(2g,\mathbb{R})$ canonical transformations. The parameter R is varied by the $SO(1,1)$-boost generator

$$\tfrac{1}{2}R\frac{d}{dR}|\Sigma\rangle=-\tfrac{1}{2}\sum_i\left[p_i\frac{\partial}{\partial\bar{p}_i}+\bar{p}_i\frac{\partial}{\partial p_i}\right]|\Sigma\rangle\,. \tag{6.34}$$

This is in fact the only operator bilinear in p, $\bar{p}$, $\partial/\partial p$, and $\partial/\partial\bar{p}$ that commutes with all generating functions $G-\bar{G}$. It indeed corresponds to the marginal operator as can be seen from (6.22)–(6.23) and

$$\frac{1}{2\pi}\int_\Sigma d^2z\,\partial\varphi\bar{\partial}\varphi|\Sigma\rangle=\frac{i}{4\pi}\sum_i\left[\oint_{A_i}\partial\varphi\,\overline{\oint_{B_i}\partial\varphi}-\oint_{B_i}\partial\varphi\,\overline{\oint_{A_i}\partial\varphi}\right]|\Sigma\rangle\,. \tag{6.35}$$

We like to stress that up to this point we did not use the explicit form of $|\Sigma;p\rangle$ in terms of the oscillators $a_n^\dagger$. We will now show that the properties (6.22) and (6.23) are sufficient to determine this state up to normalization. To this end we first observe that the one-to-one correspondence between the modes $g_{A_i}(z)$, $g_{B_i}(z)$ and the homology basis A_i, B_i can be formulated equivalently by the following local conditions:

$$\oint dz\, g_{A_i}(z)\partial_z\partial_w \log E(z,w)=0\,, \tag{6.36a}$$

$$\oint dz\, g_{A_i}(z)\omega_i(z)=\delta_{ij}\,, \tag{6.36b}$$

$$\oint dz\, g_{B_i}(z)\partial_z\partial_w \log E(z,w)=2\pi i\omega_i(w)\,, \tag{6.36c}$$

$$\oint dz\, g_{B_i}(z)\omega_i(z)=\tau_{ij}\,, \tag{6.36d}$$

where the contour encircles both Q and w. Further all quantities are defined with respect to the homology cycles A_i, B_i. These equations follow directly from the fact that the space of 1-forms holomorphic on $\Sigma-Q$ is spanned by $\omega_i(z)$ and $\omega^{(n)}(z) =\partial_z\partial_Q^n \log E(Q,z)$ $(n\geqq 1)$. By the same observation it is straightforward to verify that the state $|\Sigma;p\rangle$ satisfies

$$\partial\varphi(z)|\Sigma;p\rangle=\left[\oint\frac{dw}{2\pi i}\varphi_+(w)\partial_w\partial_z\log E(w,z)+2\pi i\sum p_i\omega_i(z)\right]|\Sigma;p\rangle\,. \tag{6.38}$$

The right-hand side, which is manifestly holomorphic on all of $\Sigma-Q$, may serve as a definition of the operator $\partial\varphi(z)$ outside the coordinate patch. This equation can be read as a compact way of writing the (generalized) Bogoliubov transformation relating the state $|\Sigma;p\rangle$ to the standard vacuum $|0\rangle$. Now using (6.37) we find

$$\frac{\partial}{\partial p_i}|\Sigma;p\rangle=[2\pi i(\tau\cdot p)_i+\oint dz\,\varphi_+(z)\omega_i(z)]\,|\Sigma;p\rangle\,. \tag{6.39}$$

Equations (6.38)–(6.39) can be integrated to the following result for the chiral vacuum state $|\Sigma;p\rangle$

$$|\Sigma;p\rangle=CA^p[\varphi_+(z)]\,|0\rangle\,, \tag{6.40}$$

where $A^p[\lambda(z)]$ is given in (6.12). The normalization constant C is determined by using the fact that the modular dependence of $\langle 0|\Sigma;p\rangle$ is given by the action of the stress-energy tensor (6.6). Using the variational formulas of [26], this yields

$$C=(\det\bar\partial_0)^{-1/2}\,. \tag{6.41}$$

This concludes our algebraic derivation of the result (6.12).

6d. *Operator Formulation of Orbifold Models*

For the twisted scalar field the state $|\Sigma\rangle$ has a slightly more complicated structure. It is given by a sum of 2^{2g} states $|\Sigma\rangle_{\varepsilon,\delta}$ representing the different twist structures. All states $|\Sigma\rangle_{\varepsilon,\delta}$ lie in the untwisted sector of the Hilbert space. The contribution $|\Sigma\rangle_{0,0}$ equals 2^{-g} times the torus result (6.12). For the other characteristics there is no momentum conservation, so the corresponding states will have components with non-zero momentum p_0. Again using (6.1), (6.10) and the expressions (4.9)–(4.11) for the correlator of the vertex operators, we obtain after a straightforward calculation

$$|\Sigma\rangle_{\varepsilon,\delta}=2^{-g}Z_0^{qu}\sum_{p_0,\bar p_0}\sum_{p,\bar p}A^{p_0;p}_{\varepsilon,\delta}[\varphi_+(z)]\overline{A^{\bar p_0;\bar p}_{\varepsilon,\delta}[\varphi_+(z)]}e^{ip_0q}e^{-i\bar p_0\bar q}|0\rangle\,, \tag{6.42a}$$

where $(p_0, \bar{p}_0) \in 2\Gamma_R$ [see Eq. (4.5)], $(p, \bar{p}) \in \Gamma_R^{g-1}$ and

$$A_{\varepsilon,\delta}^{p_0;p}[\lambda(z)] = c\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}^{-1} \exp\left[\frac{1}{2}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\lambda(z)\lambda(w)\partial_z\partial_w \log E_t(z,w)\right]$$
$$\times \exp\left[i\pi(p\cdot\Pi\cdot p) + 2\pi i\left(p\cdot\tfrac{1}{2}p_0\int_{\iota(\hat{Q})}^{\hat{Q}} v\right) + \tfrac{1}{2}p_0^2\log\varepsilon(Q)\right.$$
$$\left. + \oint\frac{dz}{2\pi i}\lambda(z)\left[2\pi i p\cdot v(z) + p_0\partial_z\log E_t(z,Q)\right]\right], \tag{6.42b}$$

where Π_{ij}, $v_i(z)$, $\varepsilon(z)$, and $E_t(z,w)$ are defined with respect to the twist characteristic $\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}$. Note that $|\Sigma\rangle_{\varepsilon,\delta}$ is even under $\phi\to-\phi$, as it should be.

One of the striking features of Eq. (6.42) is the symmetric role of the state momentum p_0 and the loop momenta p_i, $i=1,\ldots,g-1$. An intuitive explanation for this is that on the double cover $\hat{\Sigma}$ we have in fact constructed a density matrix, i.e. an element in the tensor product of two Hilbert spaces, one at $\hat{Q}$ and one at $\iota(\hat{Q})$, whose quantum numbers are related by $\phi\to-\phi$. By taking the trace of this density matrix as in (6.8) one creates in a way an extra handle of $\hat{\Sigma}$ of which p_0 is the loop momentum. Comparing with (4.2) one can indeed recognize in (6.42) the factorization expansion of the partition function on the resulting surface of a scalar field which is odd under the involution ι.

What about an algebraic derivation of (6.42)? First we note that the chiral contributions to $|\Sigma\rangle_{\varepsilon,\delta}$ with $(\varepsilon,\delta)\neq(0,0)$ can be derived in an analogous fashion as in the torus case. Namely we can consider the modes $a[f]$, where $f(z)$ is a meromorphic function with half-integer characteristics $\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}$, i.e. with multipliers $(-)^{2\varepsilon_i}$ and $(-)^{2\delta_i}$ when continued analytically around the cycles A_i and B_i. The chiral states are annihilated by the holomorphically extendable modes $a[f]$ and can be chosen to be eigenstates of $a_0=\oint\partial\varphi$ and the loop momentum operators for $g-1$ untwisted cycles. Next, the invariance of $|\Sigma\rangle_{\varepsilon,\delta}$ under the Dehn twists around these untwisted cycles should restrict the loop momentum summation to the lattice Γ_R^{g-1}. In a similar way the constraint on the state momenta $(p_0,\bar{p}_0)\in 2\Gamma_R$ will arise. The factor 2 follows from considering the Dehn twist around the branched cycle $2(\delta\cdot A+\varepsilon\cdot B)$, because the momentum flow through this cycle is up to a sign equal to $(\frac{1}{2}p_0,\frac{1}{2}\bar{p}_0)$. Finally, invariance under the full modular group, since it is also generated by Dehn twists around cycles intersecting the branched cycle, requires the sum over all twist characteristics. We do not yet understand how these Dehn twists can be naturally incorporated in an algebraic operator formulation.

We end this section with an extension of the operator formulation on Riemann surfaces to twist operators. Since twist fields are intertwining operators between the two sectors of the Hilbert space, their action on the state $|\Sigma\rangle$ is, although perfectly well defined, not very easily expressed in the formalism as developed up to now. As we have seen, the presence of twist fields is responsible for a summation over the loop momenta $p_\lambda, \bar{p}_\lambda$. So they can be regarded as topological objects comparable to extra handles on the Riemann surface. Accordingly we will attribute a state $|\Sigma;\sigma^N\rangle$ in the Hilbert space of the orbifold theory to the punctured Riemann surface $\Sigma - Q$ with N twist field insertions. By definition we have

$$|\Sigma;\sigma^N\rangle = \prod_{k=1}^{N} \sigma(z_k,\bar{z}_k)|\Sigma\rangle\,, \tag{6.42}$$

i.e. the state $|\Sigma;\sigma^N\rangle$ satisfies for any set of operators $A_l(w_l,\bar{w}_l)$,

$$\langle 0|\prod_l A(w_l,\bar{w}_l)\,|\Sigma;\sigma^N\rangle = \left\langle \prod_l A(w_l,\bar{w}_l)\prod_{k=1}^{N}\sigma(z_k,\bar{z}_k)\right\rangle . \tag{6.43}$$

This state is an element of the twisted or untwisted sector depending on whether N is odd or even. The untwisted case is completely analogous to (6.41) with the appropriate double cover $\hat{\Sigma}$ and momentum lattice (5.9). As for the twisted case, we now expand $\varphi(z)$ in half-integer modes and write an arbitrary element in the twisted Hilbert space as

$$|\Phi\rangle = \sum_{i=1}^{2} \Phi^i[\varphi_+(z);\bar{\varphi}_+(\bar{z})]\sigma_i|0\rangle\,, \tag{6.44}$$

where $\varphi_+(z)$ is the creating part of $\varphi(z)$. (Note that the twisted states do not carry any definite momentum.) The functionals Φ^i are related to the state $|\Phi\rangle$ by

$$\Phi^i[\lambda(z);\bar{\lambda}(\bar{z})] = \langle 0|\sigma_i \exp\left[i\oint\frac{dz}{2\pi i}\lambda(z)\partial\varphi(z) - i\overline{\oint\frac{dz}{2\pi i}\lambda(z)\partial\varphi(z)}\right]|\Phi\rangle\,. \tag{6.45}$$

This expression is well defined, since both $\lambda(z)$ and $\varphi(z)$ are expanded in half-integer powers of z. Using the twisted chiral propagator,

$$E_t(z,w) = \frac{\hat{E}(\hat{z},\hat{w})}{\hat{E}(\hat{z},\imath(\hat{w}))}\,, \tag{6.46}$$

we find for $|\Sigma;\sigma^N\rangle$, N odd

$$|\Sigma;\sigma^N\rangle = Z_0^{qu}\sum_i\sum_{\varepsilon,\delta}\sum_{p,\bar{p}} A^{i,p}_{\varepsilon,\delta}[\varphi_+(z)]\overline{A^{i,\bar{p}}_{\varepsilon,\delta}[\varphi_+(z)]}\sigma_i|0\rangle\,, \tag{6.47a}$$

$$A^{i,p}_{\varepsilon,\delta}[\lambda(z)] = c\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}^{-1} \exp\left[\frac{1}{2}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\lambda(z)\lambda(w)\partial_z\partial_w \log E_t(z,w)\right]$$
$$\times\exp\left[i\pi(p\cdot\Pi\cdot p) + \left(p\cdot\oint\lambda(z)\nu(z)\,dz\right)\right], \tag{6.74b}$$

where all quantities on the right-hand side are those on the branched cover defined by the positions of the $N+1$ twist fields $\sigma(z_k,\bar{z}_k)$ and $\sigma_i(Q)$ and the twist structure ε,δ. The momenta are summed over (5.9).

An interesting special case is $R=1$, $N=1$, where $|\Sigma;\sigma\rangle$ is a finite sum with chiral components proportional to

$$\tau_{\mathrm{BKP}}[\lambda(z)] = \exp\left[\frac{1}{2}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\lambda(z)\lambda(w)\partial_z\partial_w\log E_t(z,w)\right]$$
$$\times\,\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\oint\frac{dz}{2\pi i}\lambda(z)\nu(z)\,|\Pi\right). \tag{6.48}$$

As discussed extensively in [7, 37], this function solves a hierarchy of differential equations known as the BKP-hierarchy and is called the BKP τ-function. It is furthermore shown in [7, 37] that it is related to the KP τ-function [see Eq. (6.13)] by

$$\tau_{\mathrm{BKP}}[\lambda(z)]^2 = \mathrm{const} \cdot \tau_{\mathrm{KP}}[\hat{\lambda}(\hat{z})], \tag{6.49}$$

where the KP τ-function is defined on $\hat{\Sigma}$. Here $\hat{\lambda}$ is the lift of λ to $\hat{\Sigma}$ and satisfies $\hat{\lambda}(\imath(\hat{z})) = -\hat{\lambda}(\hat{z})$. (Note that locally $z \sim \hat{z}^2$.) At first sight this relation seems very mysterious. However, it has a fairly simple physical explanation as follows. The chiral state corresponding to the BKP τ-function (6.48) describes a chiral Dirac fermion $\psi_D(z)$ on Σ with twisted boundary conditions: when $\psi_D(z)$ is moved around one of the twist fields or twisted cycles it is transformed (modulo a sign) into $\bar{\psi}_D(z)$. When we lift this situation to $\hat{\Sigma}$ we describe the theory of a "Majorana fermion" $\psi_M(\hat{z})$ living on $\hat{\Sigma}$, satisfying the reality condition

$$\bar{\psi}_M(\hat{z}) = \psi_M(\imath(\hat{z})). \tag{6.50}$$

The BKP τ-function is obtained from this fermion theory through

$$\tau_{\mathrm{BKP}}[\lambda(z)] = \left\langle \exp\left[\tfrac{1}{2} \oint \frac{d\hat{z}}{2\pi i} \hat{\lambda}(\hat{z}) \psi_M(\imath(\hat{z})) \psi_M(\hat{z}) \right] \right\rangle. \tag{6.51}$$

Thus (6.48) has an interpretation as a vacuum state in a real fermion Hilbert space [although with modified commutation relations due to the unusual reality conditions (6.50)] and is as such an element of the orthogonal grassmannian manifold [7, 37]. Note that the construction makes essential use of the fact that $\hat{\Sigma}$ has an involutive automorphism $\imath$. The KP τ-function, on the other hand, describes a chiral Dirac fermion $\psi_D(\hat{z})$ on $\hat{\Sigma}$ and can be expressed as the expectation value

$$\tau_{\mathrm{KP}}[\hat{\lambda}(\hat{z})] = \left\langle \exp\left[\oint \frac{d\hat{z}}{2\pi i} \hat{\lambda}(\hat{z}) \bar{\psi}_D(\hat{z}) \psi_D(\hat{z}) \right] \right\rangle. \tag{6.52}$$

Out of this Dirac fermion $\psi_D(\hat{z})$ we can construct two Majorana fermions of the type (6.50) by taking the combinations

$$\begin{aligned} \psi_{M_1}(\hat{z}) &= \frac{1}{\sqrt{2}} [\psi_D(\hat{z}) + \bar{\psi}_D(\imath(\hat{z}))], \\ \psi_{M_2}(\hat{z}) &= \frac{1}{i\sqrt{2}} [\psi_D(\hat{z}) - \bar{\psi}_D(\imath(\hat{z}))]. \end{aligned} \tag{6.53}$$

The relation (6.49) between (6.51) and (6.52) is now readily verified.

7. Conclusion

In this paper we have given a detailed analysis of $c=1$ gaussian conformal field theories on arbitrary compact surfaces both within the path-integral and the operator formalism. We have calculated their partition and correlation functions in terms of quantities developed in the study of Riemann surfaces and their double covers. The resulting expressions proved to be very useful for the study of the analytic structure and factorization properties of the theories.

We have applied the concepts of marginal deformations and multi-critical points to the lines of $c=1$ torus and $\mathbb{Z}_2$-orbifold theories and demonstrated the completeness of this connected set of models. This does not exclude the existence of other $c=1$ theories outside this set. Indeed three of such models have recently been constructed by twisting the $SU(2)$ model by a discrete polyhedral subgroup [38]. They do not admit marginal deformations and consequently correspond to isolated points in the spectrum of $c=1$ conformal field theories.

As we have seen the class of $c=1$ torus and orbifold models both display many nontrivial features and yet admit an explicit analysis. As such it can serve as a playing ground for developing ideas and techniques in conformal field theory in general.

Acknowledgements. We acknowledge useful discussions with O. Foda, B. Nienhuis, P. Ginsparg, S. Shenker, and J. B. Zuber. This work is financially supported by the Stichting voor Fundamenteel Onderzoek der Materie (F.O.M.).

References

1. Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetry in two dimensional quantum field theory. Nucl. Phys. B **241**, 333 (1984)
2. Kadanoff, L.P.: Phys. Rev. Lett. **23**, 1430 (1969)
   Polyakov, A.M.: ZhETP **57**, 271 (1969) [JETP **30**, 151 (1970)]
3. Friedan, D.: Ann. Phys. **163**, 318 (1985)
   Lovelace, C.: Nucl. Phys. B **273**, 413 (1986)
   Fradkin, E., Tseytlin, A.: Phys. Lett. **158B**, 316 (1985)
   Candelas, P., Horowitz, G., Strominger, A., Witten, E.: Nucl. Phys. B **258**, 46 (1985)
   Callan, C., Friedan, D., Martinec, E., Perry, M.: Nucl. Phys. B **262**, 593 (1985);
   Sen, A.: Phys. Rev. D **32**, 2102 (1985)
   Jain, S., Shankar, R., Wadia, S.R.: Phys. Rev. D **32**, 2713 (1985)
4. Friedan, D., Qiu, Z., Shenker, S.: Phys. Rev. Lett. **52**, 1575 (1984)
   Cardy, J.L.: Nucl. Phys. B **270** [FS16], 186 (1986)
   Gepner, D.: Nucl. Phys. B **287**, 111 (1987)
   Capelli, A., Itzykson, C., Zuber, J.B.: Nucl. Phys. B **275** [FS17], 445 (1987)
5. Friedan, D., Shenker, S.: The integrable analytical geometry of quantum string. Phys. Lett. **175B**, 287 (1986); The analytic geometry of two-dimensional conformal field theory. Nucl. Phys. B **281**, 509 (1987)
6. Ishibashi, N., Matsuo, Y., Ooguri, H.: Soliton equations and free fermions on Riemann surfaces. Tokyo preprint, UT-499, 1986
   Alvarez-Gaumé, L., Gomez, C., Reina, C.: Loop groups, grassmannians and string theory. Phys. Lett. **190B**, 55 (1987); New methods in string theory, preprint CERN TH-4775/87
   Vafa, C.: Operator formulation on Riemann surfaces. Phys. Lett. **190B**, 47 (1987)
   Witten, E.: Conformal field theory, grassmannians and algebraic curves, Princeton preprint PUTP-1057
   Matsuo, Y.: Moduli space, conformal algebra and operator formulation on a Riemann surface. Tokio preprint UT-511
7. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations. In: Non-linear integrable systems – Classical theory and quantum theory. Jimbo, M., Miwa, T. (eds.), Singapore: World Scientific 1983
   Segal, G., Wilson, G.: Loop groups and equations of the KdV type. Publ. I.H.E.S. **61**, 1 (1985)
8. Kadanoff, L.P.: J. Phys. A **11**, 1399 (1978)
   Nienhuis, B.: J. Stat. Phys. **34**, 731 (1984)

9. Dixon, L., Harvey, J.A., Vafa, C., Witten, E.: Strings on orbifolds. Nucl. Phys. B **261**, 620 (1985); Nucl. Phys. B **274**, 285 (1986)
10. Hamidi, S., Vafa, C.: Interactions on orbifolds. Nucl. Phys. B **279**, 465 (1987)
11. Dixon, L., Friedan, D., Martinec, E., Shenker, S.: The conformal field theory of orbifolds. Nucl. Phys. B **282**, 13 (1987)
12. Kadanoff, L.P.: Multicritical behavior at the Kosterlitz-Thouless critical point. Ann. Phys. **120**, 39 (1979)
Kadanoff, L.P., Brown, A.C.: Correlation functions on the critical lines of the Baxter and Askin-Teller models. Ann. Phys. **121**, 318 (1979)
13. Friedan, D., Shenker, S.: Supersymmetric critical phenomena and the two dimensional gaussian model (unpublished)
14. Yang, S.-K.: Modular invariant partition function of the Ashkin-Teller model on the critical line and $N=2$ superconformal invariance. Nucl. Phys. B **285** [FS19], 183 (1987)
Saleur, H.: Partition functions of the two dimensional Ashkin-Teller model on the critical line. Saclay preprint S. Ph-T/87-46
15. Elitzur, S., Gross, E., Rabinovici, E., Seiberg, N.: Aspects of bosonization in string theory. Nucl. Phys. B **283**, 431 (1987)
16. Fay, J.D.: Theta functions on Riemann surfaces. Lecture Notes in Mathematics, Vol. 352. Berlin, Heidelberg, New York: Springer 1973
17. Farkas, H.M., Kra, I.: Riemann surfaces. Berlin, Heidelberg, New York: Springer 1980
18. Di Francesco, P., Saleur, H., Zuber, J.B.: Modular invariance in non-minimal two-dimensional conformal theories. Nucl. Phys. B **285** [FS19] (1987) 454; Relations between the Coulomb gas picture and conformal invariance of 2 dimensional critical models. Saclay preprint S. Ph-T/87-31
19. Narain, K.S.: New heterotic string theories in uncompactified dimensions <10. Phys. Lett. **169**B, 41 (1986)
Narain, K.S., Sarmadi, M.H., Witten, E.: A note on toroidal compactification of heterotic string theory. Nucl. Phys. B **279**, 369 (1987)
20. Kac, V.G.: In: Lecture Notes in Physics, Vol. 94, p. 441. Berlin, Heidelberg, New York: Springer 1979
21. Gepner, D., Witten, E.: String theory on group manifolds. Nucl. Phys. B **278**, 493 (1986)
22. Narain, K.S., Sarmadi, M.H., Vafa, C.: Asymmetric orbifolds. Nucl. Phys. B **288**, 551 (1987)
Kobayashi, K.: Correspondence of Z_2-orbifold string and torus string. Phys. Rev. Lett. **58**, 2507 (1987)
23. Zamolodchikov, Al.B.: Two-dimensional conformal symmetry and critical four-spin correlation functions in the Ashkin-Teller model. ZhETP **90**, 1808 (1986); [JETP **63**, 1061 (1986)]
24. Frenkel, I.B., Lepowsky, J., Meurman, A.: A moonshine module for the monster. In: Vertex operators in mathematics and physics. Lepowsky, J., et al. (eds.). Berlin, Heidelberg, New York: Springer 1984
25. Alvarez-Gaumé, L., Moore, G., Vafa, C.: Theta functions, modular invariance, and strings. Commun. Math. Phys. **106**, 1 (1986)
Alvarez-Gaumé, L., Bost, J.B., Moore, G., Nelson, P., Vafa, C.: Bosonization on higher genus Riemann surfaces. Phys. Lett. B **178**, 41 (1986); Commun. Math. Phys. **112**, 503 (1987)
Schnitzer, H., Tsokos, K.: Partition functions and fermi-bose equivalence for simply-laced groups on compact Riemann surfaces. Nucl. Phys. B **291**, 429 (1987)
Ginsparg, P., Vafa, C.: Toroidal compactifications of non-supersymmetric heterotic string. Nucl. Phys. B **289**, 414 (1987)
26. Verlinde, E., Verlinde, H.: Chiral bosonization, determinants and the string partition function. Nucl. Phys. B **288**, 357 (1987)
27. Friedan, D., Qiu, Z., Shenker, S.: Two dimensional superconformal invariance and the tricritical Ising model. Phys. Lett. **151**B, 37 (1985)
28. Miki, K.: Vacuum amplitudes without twist fields for Z_N orbifold and correlation functions of twist fields for Z_2 orbifold. Phys. Lett. **191**B, 127 (1987)
Bernard, D.: Z_2-twisted fields and bosonization on Riemann surfaces, Meudon preprint 1987

29. Bagger, J., Nemeschansky, D., Seiberg, N., Yankielowicz, S.: Bosons, fermions, and Thirring strings. Nucl. Phys. B **289**, 53 (1987)
30. Birman, J.S.: Braids, Links, and mapping class groups. Annals of Mathematic Studies, Vol. 82. Princeton, NJ: Princeton University Press 1975
31. The following argumentation was developed in a discussion with S. Shenker
32. Atick, J.J., Sen, A.: Correlation functions of spin operators on a torus. Nucl. Phys. B **286**, 189 (1987)
33. Bershadsky, N., Radul, A.: Conformal field theories with additional Z_N symmetry. Int. J. Mod. Phys. A **2**, 165 (1987)
34. Zamolodchikov, Al.B.: Conformal scalar field on the hyperelliptic curve and critical Ashkin-Teller multipoint correlation functions. Nucl. Phys. B **285** [FS19], 481 (1987)
35. Di Francesco, P., Saleur, H., Zuber, J.B.: Critical Ising correlation functions in the plane and on the torus. Saclay preprint S. Ph-T/87-097
36. Gunning, R.C.: Lectures on Riemann surfaces. Princeton, NJ: Princeton University Press 1966
37. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Quasi-periodic solutions of the orthogonal KP equation. Publ. RIMS. Kyoto Univ. **18**, 1111 (1982)
38. Ginsparg, P.: Private communication
39. Kiritsis, E.B.: A bosonic representation of the Ising model. Preprint CALT-68-1442
40. Atick, J.J., Dixon, L.J., Griffin, P.A., Nemeschansky, D.D.: Multi-loop twist field correlation functions for Z_N orbifolds. Preprint SLAC-PUB-4273

Communicated by K. Gawedzki

Received August 14, 1987; in revised form September 16, 1987

Note added in proof. Twist field correlators have also been calculated in [40] by a generalization of the methods of [11, 33, 34] to arbitrary Riemann surfaces.