C^*-ALGEBRAS OF ALMOST PERIODIC PSEUDO-DIFFERENTIAL OPERATORS

BY

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0. Introduction

Our basic goal is to develop an index theory for almost periodic pseudo-differential operators on $\mathbf{R}^n$. The prototype of this theory is [5] which has direct application to the almost periodic Toeplitz operators. Here, we study index theory for a C^*-algebra of operators on $\mathbf{R}^n$ which contains most almost periodic pseudo-differential operators such as those arising in the study of elliptic boundary value problems for constant coefficient elliptic operators on a half space with almost periodic boundary conditions.

Our program is as follows: We begin with a discussion of a C^*-algebra with symbol which contains all of the classical pseudo-differential operators on $\mathbf{R}^n$. Precisely, if A is a bounded operator on $L^2(\mathbf{R}^n)$ and $\lambda \in \mathbf{R}^n$, let $\varepsilon_\lambda(A)$ denote the conjugate of A with the function $e^{i\lambda\cdot x}$ acting as a multiplier denoted e_λ. We first study the C^*-algebra of those A for which the function $\lambda \mapsto \varepsilon_\lambda(A)$ has a strongly continuous extension to the radial compactification of $\mathbf{R}^n$. The restriction of this function to the complement of $\mathbf{R}^n$ then gives the usual (principal) symbol $\sigma(A)$ when A is a pseudo-differential operator of order zero (of a suitable type). We characterize the Fourier multipliers in this algebra and the image of the symbol map. We give sufficient conditions for the usual construction of a pseudo-differential operator as well as one of Friedrichs' constructions to give an element of this algebra. In particular, the latter gives a positive linear right inverse for the symbol map—at least when the symbol is sufficiently smooth. In fact, we show in § 3 that the Friedrichs map is a right inverse to the symbol map in the almost periodic case. We expect this to be true in the general case also.

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Our next step is to discuss almost periodic operators. In a general C^*-algebra, we think of these as continuous almost periodic maps from a subgroup of the automorphism group to the C^*-algebra. In the case of operators on $L^2(\mathbf{R}^n)$, we use the automorphisms τ_μ obtained by conjugation with translation by μ, $\mu \in \mathbf{R}^n$. Indeed, Fourier multipliers and almost periodic multipliers are almost periodic in this sense.

For an almost periodic operator A on $L^2(\mathbf{R}^n)$, $\mu \mapsto \tau_\mu(A)$ has a continuous extension to the Bohr compactification. Its Haar integral $E(A)$ defines an expectation with values in the algebra of Fourier multipliers. Since this is essentially $L^\infty(\mathbf{R}^n)$, the image of E is easy to understand. In particular, we are able to distinguish closed translation invariant *-ideals in C^*-algebras of almost periodic operators by their image under E. One also has the analogue of a Fourier transform by considering the function $\lambda \mapsto E(e_\lambda A)$ which can be used to distinguish almost periodic operators. Finally, the usual trace on $L^\infty(\mathbf{R}^n)^+$ composed with E provides a trace for the algebra of almost periodic operators which is the cornerstone of the representation and index theories of § 4.

By combining the notions of symbol and almost periodicity, we arrive at our algebra $\mathcal{A}$—the C^*-algebra of almost periodic operators on $L^2(\mathbf{R}^n)$ which have symbol in the above sense. The image of $\mathcal{A}$ under σ is CAP $\otimes$ $C(S^{n-1})$, where CAP is the algebra of continuous almost periodic functions on $\mathbf{R}^n$. The Friedrichs construction gives a positive linear right inverse defined on all of CAP $\otimes$ $C(S^{n-1})$.

We close § 3 with a simple structure theorem for $\mathcal{A}$ and a proof of the fact that the commutator ideal in $\mathcal{A}$ is the kernel of σ.

In § 4 we discuss the Π_∞-factor representations of $\mathcal{A}$ and the resulting analytic index for relative Fredholm operators ([2, 3]). We first review the group-measure space construction of Murray and von Neumann to obtain a faithful representation of $\mathcal{A}$ in a Π_∞-factor. We then show that this is unique in the sense that any faithful Π_∞-factor representation factors through a faithful representation of this Π_∞-factor with equivalence of trace. We would like to compute the analytic index of the relative Fredholm operators in the image of $\mathcal{A}$, but the algebra seems to be too large for us.

We therefore restrict our attention to a subalgebra $\mathcal{A}_0$ which contains all practical examples. This is the algebra generated by the almost periodic multipliers, the Fourier multipliers which are homogeneous near ∞, and the operators with relatively compact image in the Π_∞-factor. Here we show that the usual theorem for scalar singular integral operators on a compact manifold generalizes. In particular, the index is zero for $n > 1$ and is determined by the difference of the mean motions of the symbol when $n = 1$.

We can characterize the elements A of $\mathcal{A}_0$ in several different ways. One interesting way is that the Fourier transform $E(e_\lambda A)$ is always the uniform limit of Fourier multipliers corresponding to the sum of an L^1 function and a homogeneous continuous one.

We close this section by giving an intrinsic meaning of the index in the C^*-algebra $\mathcal{A}$. We also show that index zero together with one-sided invertibility is equivalent to invertibility. Since elements of $\mathcal{A}_0$, when $n>1$, always have zero index, invertibility of $A \in \mathcal{A}_0$ is equivalent to an apriori estimate for A or A^*.

§ 5 contains a discussion of the systems case.

1. Notation

For the sake of brevity, we have not indicated the domain of function spaces on $\mathbf{R}^n$. CAP thus denotes the continuous almost periodic functions on $\mathbf{R}^n$. HC denotes the functions homogeneous of degree 0 whose restriction to the unit sphere S^{n-1} is continuous. L_c^∞ denotes the bounded measurable functions with compact support and C_c^∞ the infinitely differentiable ones. For $\varphi \in L^\infty$, M_φ denotes the corresponding multiplication operator: $M_\varphi u = \varphi u$, $u \in L^2$. $\tilde{M}_\varphi$ denotes the conjugate of M_φ with the Fourier transform $\mathcal{F}$: $\tilde{M}_\varphi = \mathcal{F}^{-1} M_\varphi \mathcal{F}$. For $\lambda \in \mathbf{R}^n$, $e_\lambda = M_\varphi$ where $\varphi(x) = e^{i\lambda \cdot x}$, $x \in \mathbf{R}^n$. $\tilde{e}_\lambda = T_{-\lambda}$, translation by $-\lambda$. Thus, $T_\lambda u(x) = u(x-\lambda)$, $x \in \mathbf{R}_n$. When $A \in \mathcal{B}(L^2)$, the bounded operators on L^2, set $\tau_\lambda(A) = T_{-\lambda} A T_\lambda$ and $\varepsilon_\lambda(A) = e_{-\lambda} A e_\lambda$, $\lambda \in \mathbf{R}^n$.

2. The symbol map

PROPOSITION 2.1. *Let $\mathcal{A}$ be a C^*-algebra and H a Hilbert space. If $\{\varrho_\alpha\}$ is any net of representations of $\mathcal{A}$ on H, then*

$$\mathcal{A}_\varrho = \{A \in \mathcal{A}\colon \varrho_\alpha(A) \text{ and } \varrho_\alpha(A)^* \text{ converge strongly}\}$$

is a C^*-algebra and

$$\sigma\colon A \to \lim \varrho_\alpha(A) \tag{2.1}$$

is a representation of $\mathcal{A}_\varrho$ on H.

Proof. Since representations are norm decreasing and multiplication is strongly continuous on bounded sets, $\mathcal{A}_\varrho$ is a *-algebra. Since operator norm is strongly lower semicontinuous, (2.1) is a norm decreasing *-homomorphism. It follows that $\mathcal{A}_\varrho$ is closed.

Remark 2.2. The strong topology can be replaced by other topologies. For example, the ultra-strong, the Mackey topology induced by the trace class (pre-dual of $\mathcal{B}(H)$), and the topology of uniform convergence on compact sets in H all induce the same topology on norm bounded sets in $\mathcal{B}(H)$ and hence lead to the same algebra for a given net.

Much of our analysis will concern the behavior at ∞ of semi-groups of automorphisms applied to the standard representation M of L^∞. Recall that M is a faithful representation. We shall use L^∞_s to denote L^∞ equipped with the locally convex topology induced by the strong operator topology via M.

PROPOSITION 2.3. *On norm bounded sets in L^∞, L^∞_s and L^p_{loc}, $1 \leq p < \infty$, each induce the topology of convergence in measure on compact sets in $\mathbf{R}^n$.*

Consider a bounded operator A on L^2 such that for every $\lambda \in S^{n-1}$,

$$\sigma_\lambda(A) = \lim_{t\to\infty} \varepsilon_{t\lambda}(A) \tag{2.2}$$

exists in the strong operator topology. This defines an extension of the function $\lambda \mapsto \varepsilon_\lambda(A)$ to the radial compactification of $\mathbf{R}^n$. Since $\varepsilon_\lambda(A)$ is strongly continuous in λ, the extended function is strongly continuous iff the above limit is uniform. We shall think of $\sigma_\lambda(A)$ as defining an operator valued function $\sigma(A)$ on $\mathbf{R}^n \setminus \{0\}$ which is homogeneous of degree 0 in λ. Then the uniformity of (2.2) is equivalent to

$$\lim_{|\lambda|\to\infty} \|\varepsilon_\lambda(A)u - \sigma_\lambda(A)u\| = 0, \qquad u \in L^2. \tag{2.3}$$

PROPOSITION 2.4. *The set $\mathcal{H}$ of bounded operators A on $L^2(\mathbf{R}^n)$ such that (2.2) exists uniformly for both A and A^* is a C^*-algebra. σ defines a $*$-homomorphism of $\mathcal{H}$ into $C(S^{n-1}, M(L^\infty)_s)$, the continuous functions from S^{n-1} to $M(L^\infty)_s$.*

Proof. If $A \in \mathcal{H}$, then by the above remarks, $\sigma(A) \in C(S^{n-1}, \mathcal{B}(L^2)_s)$. If also $B \in \mathcal{H}$ and $\lambda \in \mathbf{R}^n \setminus \{0\}$,

$$\varepsilon_\lambda(BA) - \sigma_\lambda(B)\,\sigma_\lambda(A) = [\varepsilon_\lambda(B) - \sigma_\lambda(B)]\,\sigma_\lambda(A) + \varepsilon_\lambda(A)\,[\varepsilon_\lambda(A) - \sigma_\lambda(A)].$$

Since $\varepsilon_\lambda(B) - \sigma_\lambda(B)$ is bounded in norm by $2\|B\|$ for every $\lambda \neq 0$, it follows that for any compact set $\mathcal{K}$ in L^2,

$$\sup_{v\in\mathcal{K}} \|[\varepsilon_\lambda(B) - \sigma_\lambda(B)]v\| \to 0 \text{ as } |\lambda| \to \infty.$$

In particular, for each $u \in L^2$, take

$$\mathcal{K} = \{\sigma_\lambda(A)\,u : \lambda \in S^{n-1}\}.$$

By the continuity of $\sigma(A)$, $\mathcal{K}$ is compact and it follows that $\varepsilon_\lambda(BA) - \sigma_\lambda(B)\,\sigma_\lambda(A) \to 0$ as $|\lambda| \to \infty$. Thus $\mathcal{H}$ is a $*$-algebra and it is easily seen to be closed in norm. Finally observe that for $\lambda, \mu \in S^{n-1}$, $u \in L^2$,

$$\varepsilon_\mu(\sigma_\lambda(A))\,u = \lim_{t\to\infty} \varepsilon_{t\lambda+\mu}(A)\,u = \sigma_\lambda(A)\,u,$$

since $|t\lambda + \mu| \geq t - 1$ and tends to ∞ with t. Thus $\sigma_\lambda(A)$ commutes with e_μ, i.e., $\sigma_\lambda(A) \in M(L^\infty)$.

Remark 2.5. We shall identify $\sigma(A)$ with its corresponding element of $C_b(S^{n-1}, L_s^\infty)$, the norm bounded functions from S^{n-1} to L^∞ which are continuous when L^∞ has the strong topology induced by the representation M. We shall see in Theorem 2.14 that σ is surjective.

It is clear that every multiplication operator M_φ is in $\mathcal{H}$ with $\sigma_\lambda(M_\varphi)=\varphi$ for every $\lambda \neq 0$. To describe the convolution operators in $\mathcal{H}$ we introduce the following.

Definition 2.6. L_0^∞ is the set of those ψ in L^∞ such that for each compact K in $\mathbf{R}^n$,

$$\int_K |\psi(x+y)|\,dx \to 0 \quad \text{as} \quad |y| \to \infty.$$

Our interest in this set centers on the fact that for $\psi \in L^\infty$ and $\lambda \in \mathbf{R}^n$, $\tau_\lambda(M_\psi)$ is multiplication by $T_{-\lambda}\psi$ and $\tau_\lambda(M_\psi)^\sim = \varepsilon_\lambda(\tilde{M}_\psi)$. Applying Proposition 2.3, we have that $\psi \in L_0^\infty$ iff $\tilde{M}_\psi \in \mathcal{H}$ and $\sigma(\tilde{M}_\psi)=0$. Thus, L_0^∞ is closed in L^∞ and is easily seen to be a *-ideal in L^∞. More importantly, it provides a direct summand for the convolution operators in $\mathcal{H}$.

PROPOSITION 2.7. *Let $\psi \in L^\infty$. $\tilde{M}_\psi \in \mathcal{H}$ iff $\psi \in HC + L_0^\infty$. Moreover, for $\lambda \in S^{n-1}$, $\sigma_\lambda(\tilde{M}_\psi) = \psi(\lambda)$ when $\psi \in HC$ and $=0$ when $\psi \in L_0^\infty$. Thus the sum is direct.*

Proof. When $\psi \in HC$,

$$\psi(x+t\lambda) - \psi(\lambda) = \psi(t^{-1}x+\lambda) - \psi(\lambda) \to 0 \quad \text{as} \quad t \to \infty,$$

and the convergence is uniform for compact sets of x and λ ($\lambda \neq 0$). By Proposition 2.3 and the above remark, $\tilde{M}_\psi \in \mathcal{H}$ and $\sigma_\lambda(\tilde{M}_\psi) = \psi(\lambda)$. Also by the above remark, $\tilde{M}_\psi \in \mathcal{H}$ when $\psi \in L_0^\infty$ with $\sigma(\tilde{M}_\psi)=0$. Conversely, if $\tilde{M}_\psi \in \mathcal{H}$, then $\tau_\mu(\tilde{M}_\psi) = \tilde{M}_\psi$ for every μ and since $\tau_\mu \varepsilon_\lambda = \varepsilon_\lambda \tau_\mu$, $\sigma_\lambda(\tilde{M}_\psi)$ is a translation invariant function on $\mathbf{R}^n$ for every λ. Thus $\sigma_\lambda(\tilde{M}_\psi)$ is a constant say $\omega(\lambda)$. Defining $\psi_0 = \psi - \omega$ we see that $\varepsilon_\lambda(\psi_0) \to 0$ strongly as $|\lambda| \to \infty$, and by continuity of ω and above remark, $\psi \in HC + L_0^\infty$.

A word of caution about L_0^∞ is in order. Although it contains L_c^∞ (the compactly supported functions in L^∞) and $L^1 \cap L^\infty$, the norm closures of these are ideals in L^∞ distinct from L_0^∞ and each other. (Look at characteristic functions of sets which are thin but nontrivial near ∞.)

Another important subspace of $\mathcal{H}$ is the compact operators. Indeed, if A has rank 1 and is given by $f \otimes g$, then $\varepsilon_\lambda(A) = (e_{-\lambda} f) \otimes (e_{-\lambda} g)$ which clearly tends to 0 strongly as $|\lambda| \to \infty$. Thus, not only are the compact operators in $\mathcal{H}$ but they are in the ideal $\ker \sigma$ (of course). An immediate consequence is that $\mathcal{H}$ is irreducible.

From these facts it follows that $\mathcal{H}$ contains all operators of the form $M_\varphi S M_\psi$ where φ and ψ are compactly supported and S is a singular integral operator (see Seeley [17], [18]). Also, the symbol agrees with the usual one by virtue of Proposition 2.7.

A word of caution about the localization properties of $\mathcal{H}$ is in order. It is obvious that $A\in\mathcal{H}$ and $\varphi\in L^\infty$ implies $M_\varphi A\in\mathcal{H}$. But there are bounded operators A on L^2 such that $A\notin\mathcal{H}$ while $M_\varphi A\in\mathcal{H}$ for every $\varphi\in C_c^\infty$. Fourier transform is such an operator. Thus, for some purposes it may be desirable to enlarge the algebra $\mathcal{H}$, but that will not concern us here since that is really a question of what one wishes to accept as the kernel of σ.

What will be important is that if for a given function on $\mathbf{R}^n\times\mathbf{R}^n$ one can construct a family of operators which locally have this function as a symbol, then there is an operator in $\mathcal{H}$ with this function as its symbol. Precisely,

PROPOSITION 2.8. *Let s be a function on $\mathbf{R}^n\times\mathbf{R}^n$ such that $s(\cdot, t\lambda)=s(\cdot,\lambda)$, $t>0$. Let $\{K_j\}_{j=1}^\infty$ be a measurable partition of $\mathbf{R}^n$. For each j, let χ_j be the characteristic function of K_j. Further, assume that there exists a bounded operator A_j on $L^2(K_j)$ such that*

$$\varepsilon_\lambda(A_j)-M_{s(\cdot,\lambda)}\to 0 \quad as \quad |\lambda|\to\infty$$

strongly in $L^2(K_j)$. Finally, assume that $\|A_j\|$ is bounded independent of j. Then

$$A=\sum\chi_jA_j\chi_j$$

defines a bounded operator on $L^2(\mathbf{R}^n)$ which is in $\mathcal{H}$, and $\sigma(A)=s$.

Proof. $\{\chi_j\}_{j=1}^\infty$ is a sequence of mutually orthogonal projections whose sum is 1. Thus A is bounded. Also, when $u\in L^2(\mathbf{R}^n)$ and $\chi_ju=0$ for all but a finite number of j,

$$\|\varepsilon_\lambda(A)u-s(\cdot,\lambda)\,u\|^2=\sum\|\chi_j\varepsilon_\lambda(A_j)\,\chi_ju-s(\cdot,\lambda)\,\chi_ju\|^2\to 0, \quad \text{as} \quad |\lambda|\to\infty,$$

since the sum is finite. But, the set of such u is dense in $L^2(\mathbf{R}^n)$. Thus, $A\in\mathcal{H}$ and $\sigma(A)=s$.

Remark 2.9. Observe that the map

$$\{A_j\}\to A$$

is linear and positive in the sense that if $A_j\geqslant 0$ in $L^2(K_j)$ for all j then $A\geqslant 0$.

Another class of operators in $\mathcal{H}$ are the operators of negative order. More precisely, let Λ be the operator such that $\tilde{\Lambda}$ is multiplication by $(1+|\xi|^2)^{1/2}$. We shall say that a linear map A from C_c^∞ to its dual has order r if for all $j\in\mathbf{Z}$, $\Lambda^{-j-r}A\Lambda^j$ extends to a bounded operator in L^2.

PROPOSITION 2.10. *If A has negative order, then $A\in\ker\sigma\subset\mathcal{H}$.*

Proof. Let A have order $r<0$. Then

$$\|\varepsilon_\lambda(A)\,u\|=\|\varepsilon_\lambda(A\Lambda^{-r})\,\varepsilon_\lambda(\Lambda^r)\,u\|\leqslant\|A\Lambda^{-r}\|\,\|\varepsilon_\lambda(\Lambda^r)\,u\|\to 0$$

as $|\lambda| \to \infty$ since $\varepsilon_\lambda(\Lambda^r)^\sim$ is multiplication by $(1+(\xi+x)^2)^r$, and this converges to 0 uniformly on compact sets.

Remark 2.11. In Proposition 2.10 it is sufficient to assume that $A\Lambda^\delta$ and $\Lambda^\delta A$ are bounded in L^2 for some $\delta > 0$.

We continue our discussion of the algebra $\mathcal{H}$ by showing that certain of the pseudo-differential operators considered by Hörmander in [11] are in $\mathcal{H}$. In particular, let $0 \leq \delta < \varrho \leq 1$ and let $s \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ be such that

$$(1+|\xi|)^{-\delta|\alpha|+\varrho|\beta|} D_x^\alpha D_\xi^\beta s(x, \xi) \tag{2.4}$$

is bounded on $\mathbf{R}^n \times \mathbf{R}^n$ for each α, β and there is a function s_∞ such that

$$s(\cdot, t\xi) - s_\infty(\cdot, \xi) \to 0 \quad \text{in} \quad L^1_{\text{loc}} \quad \text{as} \quad t \to \infty \tag{2.5}$$

uniformly for $|\xi| = 1$. By Theorem 3.5 of [11], the operator defined by

$$\Psi_s u(x) = (2\pi)^{-n} \int s(x, \xi)\, e^{ix\cdot\xi}\, \hat{u}(\xi)\, d\xi, \quad x \in \mathbf{R}^n, \quad u \in C_c^\infty, \tag{2.6}$$

extends to a bounded operator on $L^2(\mathbf{R}^n)$. Further, it is shown in the proof of that theorem that when x is not in the support of u,

$$\Psi_s u(x) = \int K(x, x-y)\, u(y)\, dy$$

where $K(x, z)$ is C^∞ for $z \neq 0$ and satisfies an estimate of the form

$$|K(x, z)| \leq ck(z),\ x \in \mathbf{R}^n,\ z \in \mathbf{R}^n \setminus 0,$$

with k smooth and integrable on the complement of every neighborhood of 0. Thus, if u has support in compact set L and L' is a set whose distance from L is positive, then

$$\varepsilon_\lambda(\Psi_s)\, u(x) = \int_L K(x, x-y)\, e^{i\lambda\cdot(y-x)}\, u(y)\, dy$$

converges to 0 as $|\lambda| \to \infty$ for each $x \in L'$ and

$$|\varepsilon_\lambda(\Psi_s) u(x)| \leq C \int_L k(x-y)\, |u(y)|\, dy, \quad x \in L', \quad \text{all } \lambda.$$

Since the right hand side of this last inequality is in L^2 on L', $\varepsilon_\lambda(\Psi_s)\, u \to 0$ in L^2 on L'. Thus, to show that $\Psi_s \in \mathcal{H}$, it is enough to study the behavior of $\varepsilon_\lambda(\Psi_s)\, u$ on the support of u for each $u \in C_c^\infty$. We may then apply Theorem 3.6 of [11] to obtain

$$\left\| \varepsilon_\lambda(\Psi_s)u - t(\cdot, \lambda)u + i\sum \frac{\partial s}{\partial y_j}(\cdot, \lambda)\frac{\partial u}{\partial x_j}\right\| \leqslant C\,|\lambda|^{-\varrho}\,\|u\|_1$$

where $\|\cdot\|$ is the L^2 norm on the support of u and $\|\cdot\|_1$ is the L^2 norm of the gradient. Since

$$\left|\frac{\partial s}{\partial \lambda_j}\right| \leqslant C(1+|\lambda|)^{-\varrho}$$

by assumption, $\varepsilon_\lambda(\Psi_s) \to s_\infty(\cdot, \lambda)$ strongly as $|\lambda| \to \infty$, as desired.

Actually, when it comes to defining pseudo-differential operators which lie in $\mathcal{H}$, much weaker assumptions on the symbol will suffice. Moreover, in view of Proposition 2.8, it is enough to construct such operators locally.

PROPOSITION 2.12. *Let s be a measurable function on $\mathbf{R}^n \times \mathbf{R}^n$ such that every derivate s' of order $\leqslant 1 + n/2$ in the first variables satisfies*

$$\sup_\xi \int_K |s'(x, \xi)|\, dx < \infty \tag{2.7}$$

for every compact set K. Moreover, assume that s has a symbol s_∞ in the sense that $s_\infty(\cdot, \lambda)$ is homogeneous of degree 0 and

$$s(x, \xi + \lambda) - s_\infty(x, \lambda) \to 0$$

as $|\lambda| \to \infty$ for a.e. (x, ξ). Then Ψ_s is bounded in $L^2(K)$ and has s_∞ as its symbol for every compact K.

Proof. Let $\theta \in C_c^\infty$ with $\theta = 1$ near K. Then $\theta(x)\, s(x, \xi)$ satisfies the hypotheses of Kohn-Nirenberg [13], Remark 3.1. Thus $\Psi_{\theta s}$ is a bounded operator on $L^2(\mathbf{R}^n)$. Now, if $v_\lambda = \varepsilon_\lambda(\Psi_\theta)\, u$, then

$$\hat{v}_\lambda(\eta) = \int d\xi \int dx\, e^{ix\cdot(\xi-\eta)}\theta(x)\, s(x, \xi+\lambda)\, \hat{u}(\xi).$$

By assumption,
$$\int dx\, e^{ix\cdot(\xi-\eta)}\theta(x)\,[s(x, \xi+\lambda) - s_\infty(x, \lambda)] \to 0$$

as $|\lambda| \to \infty$ for a.e. (ξ, η). Moreover, by the Kohn-Nirenberg remark, there exists $\omega \in L^1$ such that

$$\left|\int dx\, e^{ix\cdot(\xi-\eta)}\theta(x)\, s(x, \xi+\lambda)\right| \leqslant \omega(\xi - \eta).$$

Therefore, if $u \in L^2$, v_λ is dominated independent of λ by an L^2 function. We may therefore use pointwise limits. It follows that

$$\|\varepsilon_\lambda(\Psi_{\theta s})\, u - \theta s(\cdot, \lambda)\, u\| \to 0 \quad \text{as} \quad |\lambda| \to \infty$$

for $u \in C_c^\infty$ and hence for $u \in L^2$. If $u \in L^2(K)$, then extend by zero to obtain a function u_1 in $L^2(\mathbf{R}^n)$. As an operator in $L^2(K)$, Ψ_s is given by the restriction of $\Psi_s u_1$ to K and hence the restriction of $\Psi_{\theta s} u_1$ to K. The result now follows.

Remark 2.13. It is clear from the proof (due to the choice of θ) that the norm of Ψ_s in $L^2(K)$ depends on the estimates (2.7) for a larger K. Actually, it is not difficult to show that for each $\delta > 0$, there is a constant $C_\delta > 0$ such that if K_δ is the set of points whose distance from K is at most δ, then the norm of Ψ_s in $L^2(K)$ is dominated by C_δ times the maximum of (2.7) taken over K_δ. Thus, if one has estimates of (2.7) which are uniform when K runs through a collection of balls of fixed radius, then one can combine this Proposition with 2.8 to get an element of $\mathcal{H}$ with s as its symbol.

Applying the above constructions we can now prove the surjectivity of σ.

THEOREM 2.14. *σ maps $\mathcal{H}$ onto $C_b(S^{n-1}, L_s^\infty)$.*

Proof. Given $s \in C_b(S^{n-1}, L_s^\infty)$, we shall construct an operator S in $\mathcal{H}$ with s as its symbol following a method suggested by Hörmander. By Proposition 2.8 it suffices to assume that s has compact support. Extend s as a homogeneous function of degree 0. Choose a non-negative φ in $C_c^\infty(\mathbf{R}^n)$ such that $\int \varphi(x)\, dx = 1$ and φ has support in the unit ball. Let $0 < \varepsilon < 1/4$ and let ϱ denote the function $(1 + |\lambda|^2)^\varepsilon$. Define

$$a(x, \lambda) = \int dy \int d\mu\, \varrho^{-n} \varphi(\varrho(x-y))\, \varphi\left(\frac{\lambda - \mu}{\varrho^2}\right) s(y, \mu),$$

and

$$b(x, \lambda) = \int d\mu\, \varrho^{-2n} \varphi\left(\frac{\lambda - \mu}{\varrho^2}\right) s(x, \mu).$$

Then

$$\int dx\, |b(x, \lambda) - s(x, \lambda)| \leq \int d\mu\, \varphi\left(\frac{\lambda - \mu}{\varrho^2}\right) \varrho^{-2n} \int |s(x, \mu) - s(x, \lambda)|\, dx$$

converges to zero as $|\lambda| \to \infty$ by the continuity of s and the fact that when $|\lambda - \mu| \leq \varrho$, $\mu/|\mu| \to \lambda/|\lambda|$ as $|\lambda| \to \infty$. Hence, it follows that b is continuous on the radial compactification of $\mathbf{R}^n$ with values in L^1. In particular, $\{b(\cdot, \lambda) : \lambda \in \mathbf{R}^n\}$ is totally bounded in L^1. Since

$$a(\cdot, \lambda) = \varphi_\varrho * b(\cdot, \lambda)$$

where $\varphi_\varrho(x) = \varrho^n \varphi(\varrho x)$, and $\varphi_\varrho *$ converges to the identity strongly in L^1, $a(\cdot, \lambda) - s(\cdot, \lambda) \to 0$ in L^1 as $|\lambda| \to \infty$. Now, a is a C^∞ function on $\mathbf{R}^n \times \mathbf{R}^n$ which vanishes off a set of the form $K \times \mathbf{R}^n$, K compact. Moreover, if we differentiate,

$$|D_x^\alpha D_\lambda^\beta a(x,\lambda)| \leqslant \sup|s| \left(\int |\varphi^{(\alpha)}|\right)\left(\int |\varphi^{(\beta)}|\right) \varrho^{|\alpha|-2|\beta|}$$

and hence, for each multi-indices α, β,

$$(1+|\lambda|)^{4\varepsilon|\beta|-2\varepsilon|\alpha|} D_x^\alpha D_\lambda^\beta a(x,\lambda)$$

is bounded on $\mathbf{R}^n$. Therefore a satisfies (2.4) and (2.5) with s_∞ equal to the s of this proof.

Therefore $\Psi_a \in \mathcal{H}$ and $\sigma(\Psi_a) = s$.

Remark 2.15. The map $s \mapsto a$ considered above is a positive linear map of $C(S^{n-1}, L_s^\infty)$ into an order zero symbol class of the Hörmander type, and we have shown that the restrictions of a to spheres define elements of $C(S^{n-1}, L_s^\infty)$ which converge to s as the radii tend to ∞.

One of the undesirable features of the map Ψ from symbols to operators is that it is not positive. For a positive linear map we shall use the Friedrichs construction ([9], § I–12) which is valid for systems as well. The construction is possible under a variety of assumptions about smoothness and behavior at ∞ of the symbols. Combined with remarks 2.9 and 2.15, this construction leads to many positive linear right inverses for the symbol map. Even in the almost periodic case where localization is not possible and smoothing is more delicate, the Friedrichs construction leads to a positive right inverse (see § 3). Therefore, we shall devote the remainder of this section to the construction and discuss some of its basic properties.

First, let $\theta \in C_c^\infty$ be a positive function supported in the unit ball of $\mathbf{R}^n$ with $\int \theta^2 = 1$. Let $\omega(\xi) = (1+|\xi|^2)^{-1/4}$ and

$$q(\zeta, \xi) = \theta((\zeta - \xi)\,\omega(\xi))\,\omega(\xi)^{n/2}.$$

LEMMA 2.16.

$$\int d\zeta\, q(\zeta,\xi)\, q(\zeta,\eta)$$

is bounded independent of ξ and η.

Proof. The integrand is supported by

$$\{\zeta\colon |\zeta - \xi| \leqslant 1/\omega(\xi) \quad\text{and}\quad |\zeta-\eta| \leqslant 1/\omega(\eta)\}.$$

This set is empty unless $\qquad |\xi - \eta| \leqslant 1/\omega(\xi) + 1/\omega(\eta).$

For such ξ, η, $\qquad |\xi| - 1/\omega(\xi) \leqslant |\eta| + 1/\omega(\eta).$

Thus, for $|\xi|$ sufficiently large, $|\xi| \leqslant 2|\eta|$. Similarly, for $|\eta|$ sufficiently large, $|\eta| \leqslant 2|\xi|$.

Since the integral is continuous in ξ and η, it is enough to prove boundedness as $|\xi|$ and $|\eta|\to\infty$ with $|\xi|\,|\eta|^{-1}$ bounded. Using a change of variables, we must consider

$$\int d\zeta\,\theta(\zeta)\,\theta([\omega(\xi)^{-1}\zeta+\xi-\eta]\,\omega(\eta))\,\omega(\eta)^{n/2}\,\omega(\xi)^{-n/2}$$

which is bounded as desired.

Next, define the operator

$$Q_{x,\zeta}(u)=\int d\xi\,q(\zeta,\xi)\,e^{ix\cdot\xi}\,\hat{u}(\xi),\qquad \hat{u}\in C_c^\infty,\quad x,\zeta\in\mathbf{R}^n.$$

For such u, $Q_{x,\zeta}(u)$ is C^∞ in x,ζ and compactly supported in ζ. Using the identity

$$\frac{\partial}{\partial\xi_j}(e^{ix\cdot\xi})=ix_je^{ix\cdot\xi}$$

repeatedly, it is easily seen that every polynomial in x times $Q_{x,\zeta}(u)$ is bounded. In particular, we may define the operator F_a (for $a\in L^\infty(\mathbf{R}^n\times\mathbf{R}^n)$) by

$$\langle F_au,v\rangle=\int dx\int d\zeta\,a(x,\zeta)\,Q_{x,\zeta}(u)\,\overline{Q_{x,\zeta}(v)}.$$

It is clear that $a\mapsto F_a$ is a positive linear map.

PROPOSITION 2.17. *F_a extends to a bounded operator on $L^2(\mathbf{R}^n)$ for each $a\in L^\infty$.*

Proof. Since for real a

$$e^{-\alpha x^2}a(x,\xi)\leqslant e^{-\alpha x^2}\|a\|_{L^\infty}$$

and $e^{-\alpha x^2}\uparrow 1$ as $\alpha\downarrow 0$, it is enough to assume that $a(x,\xi)=e^{-\alpha x^2}=b(x)$ and the bound is independent of α. But then

$$\langle F_au,v\rangle=\int d\xi\int d\eta\int d\zeta\,\hat{b}(\eta-\xi)\,q(\zeta,\xi)\,q(\zeta,\eta)\,\hat{u}(\xi)\,\overline{\hat{v}(\eta)}$$

and by Lemma 2.16,
$$|\langle F_au,v\rangle|\leqslant C\left(\int|\hat{b}|\right)\|u\|\,\|v\|$$

where C depends only on q. Thus

$$\|F_a\|\leqslant C\int|\hat{b}|=C\int\hat{b}=Cb(0)=C$$

as desired.

Remark 2.18. Taking $a=1$, we see that Q is a bounded operator from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{2n})$, and $F_a=Q^*M_aQ$, where M_a is the multiplication operator defined by the function a on $\mathbf{R}^{2n}=T^*(\mathbf{R}^n)$. In the next theorem we shall see that F_a is a pesudo-differential operator whose principal symbol is the same as that of Ψ_a. Thus Q provides a specific spatial transformation of pseudo-differential operators into multiplication operators, modulo operators of lower order. We also note in passing that an alternate approach to the definition of F_a can be based on the fact that Q is in fact an isometry from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{2n})$ (but is not unitary).

The following is a special case of the Lax-Nirenberg Theorem as given in [9], § I-12.

THEOREM 2.19. *Let a satisfy* (2.4) *and* (2.5) *with $\varrho=1$ and $\delta=0$. Further assume that $a(x,\xi)=0$ for x outside of a compact set. If a is self-adjoint matrix valued, then*

$$\Lambda^{1/2}[2F_a-\Psi_a-\Psi_a^*]\,\Lambda^{1/2}$$

defines a bounded operator in $L^2(\mathbf{R}^n)$.

COROLLARY 2.20. *With a as above, $F_a\in\mathcal{H}$ and $\sigma(F_a)=a$.*

Proof. See Proposition 2.10 and Theorem 2.14.

LEMMA 2.21. *If $a\in L^\infty$ and has compact support, then F_a is a Hilbert-Schmidt operator.*

Proof. Let $a^\#(\xi,\zeta)$ denote the Fourier transform of $a(x,\zeta)$ with respect to x. Then $a^\#(\xi,\zeta)$ is bounded and supported in a compact set of ζ. Observe that

$$\langle F_a u, v\rangle=\int d\xi\int d\eta\int d\zeta\, a^\#(\eta-\xi,\zeta)\,q(\zeta,\xi)\,q(\zeta,\eta)\,\hat{u}(\xi)\,\overline{\hat{v}(\eta)}.$$

Since as $|\xi|\to\infty$, the support of $q(\zeta,\xi)$ in ζ lies exterior to the ball of radius $|\xi|/2$,

$$\int d\zeta\, a^\#(\eta-\xi,\zeta)\,q(\zeta,\xi)\,q(\zeta\;\eta)$$

is a bounded, compactly supported function. Thus, $\hat{F}_a$ is an integral operator with L^2 kernel.

THEOREM 2.22. *Let a satisfy* (2.4) *and* (2.5) *with $\varrho=1$ and $\delta=0$. Further assume that $a=a_\infty+a_0$ where $a_\infty(x,\xi)$ is independent of x and every polynomial in x times a_0 satisfies* (2.4) *with $\varrho=1$ and $\delta=0$. If a is self-adjoint matrix valued then*

$$\Lambda^{1/2}[2F_a-\Psi_a-\Psi_a^*]\,\Lambda^{1/2}$$

defines a bounded operator in L^2.

COROLLARY 2.23. *With a as above, $F_a \in \mathcal{H}$ with $\sigma(F_a) = a$.*

Proof. See Propositions 2.10 and 2.12.

COROLLARY 2.24. *If $a \in \mathbf{C} \otimes HC$, then $F_a \in \mathcal{H}$ with $\sigma(F_a) = a$.*

Proof. For such a, $\Psi_a = \tilde{M}_a$ and result follows from Proposition 2.7.

LEMMA 2.25. *If $a(x, \xi) = e^{i\mu\cdot x}$, then $F_a \in \mathcal{H}$ and $\sigma(F_a) = a$.*

Proof. A straightforward computation shows that

$$\widehat{F_a u}(\eta) = \hat{u}(\eta + \mu) \int d\zeta \, q(\zeta, \eta + \mu) \, q(\zeta, \eta).$$

Thus, $\varepsilon_\lambda(F_a)$ amounts to multiplication of the shifted Fourier transform by

$$\int d\zeta \, q(\zeta, \eta + \mu + \lambda) \, q(\zeta, \eta + \lambda).$$

Since $\omega(\eta + \mu + \lambda)/\omega(\eta + \lambda) \to 1$ as $|\lambda| \to \infty$ for each η, a change of variables leads to the fact that the above converges to

$$\int d\zeta \, \theta(\zeta)^2 = 1$$

for each η. Hence, $F_a \in \mathcal{H}$ and $\sigma_\lambda(F_a) = e_\mu$ for every λ.

3. Almost periodic operators

Definition 3.1. Let $\mathcal{C}$ be a C^*-algebra and $\mathcal{G}$ a subgroup of the automorphism group of $\mathcal{C}$. $B \in \mathcal{C}$ is said to be almost periodic relative to $\mathcal{G}$ if the function

$$\alpha \mapsto \alpha(B)$$

from $\mathcal{G}$ to $\mathcal{C}$ is almost periodic.

Given a C^*-algebra $\mathcal{C}$ and group $\mathcal{G}$ of automorphisms on $\mathcal{C}$, we have

PROPOSITION 3.2. *The following are equivalent:*

(i) *B is almost periodic relative to $\mathcal{G}$,*
(ii) *$\{\alpha(B): \alpha \in \mathcal{G}\}$ is totally bounded in $\mathcal{C}$,*
(iii) *B^* is almost periodic relative to $\mathcal{G}$.*

Proof. Since automorphisms of $\mathcal{C}$ are isometries, we have for each $\alpha, \beta, \gamma \in \mathcal{G}$ and $B \in \mathcal{C}$,

$$\|\alpha(\beta(B)) - \alpha(\gamma(b))\| = \|\beta(B) - \gamma(B)\| = \|\beta(B^*) - \gamma(B^*)\|.$$

These are independent of α and hence give the same pseudo-metric on $\mathcal{G}$ as their supremum over α. Thus, total boundedness is the same for each.

PROPOSITION 3.3. *The set of $B \in \mathcal{L}$ which are almost periodic relative to $\mathcal{G}$ is a closed *-subalgebra of $\mathcal{L}$.*

Proof. It is clear from (ii) above that this set is a closed linear subspace and is *-invariant by (iii). If B and C are almost periodic relative to $\mathcal{G}$, then for $\alpha, \beta \in \mathcal{G}$,

$$\|\alpha(BC)-\beta(BC)\| \leq \|\alpha(B)-\beta(B)\|\,\|C\| + \|B\|\,\|\alpha(C)-\beta(C)\|,$$

and it follows that BC is almost periodic relative to $\mathcal{G}$.

In the case of singular integral operators we shall use the notion of almost periodic relative to the translation group $\{\tau_\mu : \mu \in \mathbf{R}^n\}$ since M_φ is almost periodic in this sense iff φ is a. p. in the usual sense, and $\tilde{M}_\psi$ commutes with translations. Let $\mathcal{A}$ be the C^*-algebra of almost periodic operators in $\mathcal{H}$. Observe that $\tau_\mu(A)$ is strongly continuous in μ and, when $A \in \mathcal{A}$, lies in a norm compact set of operators. Thus, $\mu \mapsto \tau_\mu(A)$ is in fact norm continuous.

PROPOSITION 3.4. *For each $\mu, \lambda \in \mathbf{R}^n$, $\varepsilon_\lambda \tau_\mu = \tau_\mu \varepsilon_\lambda$ and for each $A \in \mathcal{A}$, $\tau_\mu(\varepsilon_\lambda(A)) \in \mathcal{A}$.*

Proof. $\varepsilon_\lambda \tau_\mu = \tau_\mu \varepsilon_\lambda$ by straightforward verification. If $A \in \mathcal{A}$, then $\{\tau_\mu(A)\}_{\mu \in \mathbf{R}^n}$ is compact and hence $\varepsilon_\lambda(\tau_\mu(A))$ has a strong limit as $|\lambda| \to \infty$ (radially) uniformly in μ. Thus $\tau_\mu(A) \in \mathcal{H}$ and a fortiori in $\mathcal{A}$. Since $e_\lambda \in \mathcal{A}$ for every λ, the result follows.

Observe that like $\mathcal{H}$, $\mathcal{A}$ is an irreducible algebra of operators on $L^2(\mathbf{R}^n)$, even though it does not contain a non-zero compact operator. Indeed, $\mathcal{A}$ contains all $\tilde{M}_\psi$ where $\psi \in L^\infty$ and is compactly supported, and thus the weak closure of $\mathcal{A}$ contains all $\tilde{M}_\psi$. Similarly, the weak closure of $\mathcal{A}$, contains all (almost) periodic multipliers, and hence all multipliers. It follows that $\mathcal{A}$ is weakly dense in $\mathcal{B}(L^2(\mathbf{R}^n))$ and hence is irreducible. Using the irreducibility and the fact that $\mathcal{A}$ has a faithful Π_∞ representation (see § 4), one can show that $\mathcal{A}$ contains no non-zero compact operators. One can also see this more directly since if $\mathcal{A}$ contained a non-zero compact operator, it would contain all of them. In particular, it would contain an operator of rank one of the form $A = f \otimes g$ where f has compact support. For such an operator

$$\|\tau_\lambda(A) - A\| = \|A\|, \ |\lambda| \text{ sufficiently large,}$$

and hence $\{\tau_\lambda(A) : \lambda \in \mathbf{R}^n\}$ is not totally bounded in norm. Thus $\mathcal{A}$ contains no non-zero compact operator. [The simplest proof of this fact follows from Propositions 3.12 and 3.13 applied to the ideal of compact operators in $\mathcal{A}$.]

On the other hand, the operators in $\ker \sigma \cap \mathcal{A}$ play the role in $\mathcal{A}$ that the compact operators play in the C^*-algebra generated by the singular integral operators on a compact manifold. We shall discuss this in section 4.

PROPOSITION 3.5. *If $A \in \mathcal{A}$, then for each λ, $\sigma_\lambda(A) \in$ CAP. Moreover, the continuity of $\sigma_\lambda(A)$ (x) in x is uniform on $S^{n-1} \times \mathbf{R}^n$.*

Proof. By assumption, $\{\tau_\mu(A)\}$ is totally bounded in norm. The strong topology is then the same as the norm topology on $\{\tau_\mu(A)\}$, and hence, $\mu \mapsto \tau_\mu(A)$ is uniformly norm continuous on $\mathbf{R}^n$. Since $\varepsilon_\lambda \circ \tau_\mu = \tau_\mu \circ \varepsilon_\lambda$, one has $\sigma_\lambda \circ \tau_\mu = \tau_\mu \circ \sigma_\lambda$ for every λ and μ. Thus

$$\|\tau_\mu(\sigma_\lambda(A)) - \tau_\nu(\sigma_\lambda(A))\| \leqslant \|\tau_\mu(A) - \tau_\nu(A)\| \tag{3.1}$$

and it follows that $\tau_\mu(\sigma_\lambda(A))$ is norm continuous and almost periodic in μ uniformly in λ. Since for each λ, $\sigma_\lambda(A)$ is a multiplication operator, we may choose a $\varphi_\lambda \in$ CAP such that $\sigma_\lambda(A) = M_{\varphi_\lambda}$. Inequality (3.1) then implies the uniformity of the continuity in x of $\varphi_\lambda(x)$.

PROPOSITION 3.6. *$\sigma(\mathcal{A})$ contains as a dense subspace in the uniform topology those bounded functions f on $S^{n-1} \times \mathbf{R}^n$ such that*

(i) *for each $\lambda \in S^{n-1}$, $f_\lambda \in C^\infty$,*
(ii) *each derivative of f_λ is continuous and almost periodic uniformly in λ,*
(iii) *each derivative of f_λ is continuous as a function of λ from S^{n-1} to L^1_{loc}.*

Proof. Let $\varrho \in C_c^\infty$ with $\varrho \geqslant 0$ and $\int \varrho = 1$. Define

$$\varrho_t(x) = t^n \varrho(tx), \quad x \in \mathbf{R}^n, \quad t > 0.$$

Then for $f \in \sigma(\mathcal{A})$, $\varrho_t * f$ has the above properties and lies in $\sigma(\mathcal{A})$. Indeed, (i) is immediate. If D is a derivative, then

$$T_\mu(D(\varrho_t^t * f_\lambda)) = (D\varrho_t) * T_\mu f_\lambda$$

and the right hand side defines a continuous linear operator on the bounded measurable functions f_λ with sup norm. Thus, the uniform continuity and almost periodicity of f_λ implies that of $D(\varrho * f_\lambda)$, i.e., (ii) holds. For (iii) we simply use the fact that $\varphi \mapsto \varrho * \varphi$ is continuous from L^1_{loc} to L^1_{loc}. Finally, we have

$$|f_\lambda(x) - \varrho_t * f_\lambda(x)| = \left| \int_{\mathbf{R}^n} t^n \varrho(ty)\, [f_\lambda(x) - f_\lambda(x-y)]\, dy \right|$$

$$\leqslant \sup_{y \in K} |f_\lambda(x) - f_\lambda(x - t^{-1}y)|, \quad x \in \mathbf{R}^n,$$

and hence $\varrho_t * f_\lambda \to f_\lambda$ uniformly as $t \to \infty$ by the uniform continuity of f.

LEMMA 3.4. *Let X be a compact space and a a bounded function on $\mathbf{R}^n \times X$. Assume that $x \mapsto a(\cdot, x)$ is continuous from X to L^1_{loc} and that the set of translates $\{a(\cdot, -\lambda, \cdot)\}$ is totally bounded in the space of bounded functions. Then a is uniformly continuous and belongs to* CAP$\otimes C(X)$.

Proof. The closure (with respect to sup norm) of the set of translates is compact. Further, $\{a(\cdot, x)\}_{x\in X}$ is compact in L^1_{loc}. Thus translation is continuous uniformly on this set. Since the topology of convergence in L^1_{loc} uniformly on X is a coarser Hausforff topology than that of uniform convergence on $\mathbf{R}^n \times X$, $\lambda \mapsto a(\cdot - \lambda, \cdot)$ is continuous from $\mathbf{R}^n$ to the bounded functions. Let $\varepsilon > 0$ be given. Then there is a finite set $\lambda_1, \ldots, \lambda_m$ in $\mathbf{R}^n$ such that

$$U_j = \{\lambda : \sup_x |a(\lambda, x) - a(\lambda_j, x)| < \varepsilon\}$$

form an open cover of $\mathbf{R}^n$. Choosing a compact set $K_j \subset U_j$ and averaging, we have

$$\left| \frac{1}{|K_j|} \int_{K_j} a(\lambda, x)\, d\lambda - a(\lambda_j, x) \right| < \varepsilon, \quad x \in X.$$

Therefore,

$$|a(\lambda, x) - a(\lambda, x')| \leqslant 4\varepsilon + \max_j \frac{1}{|K_j|} \left| \int_{K_j} [a(\lambda, x) - a(\lambda, x')]\, d\lambda \right|$$

and hence, by the assumed continuity of $a(\cdot, x)$,

$$\sup_\lambda |a(\lambda, x) - a(\lambda, x')| \to 0$$

as $x \to x'$. We have already shown the uniform continuity in λ. Thus a is uniformly continuous and since $a(\cdot, x) \in \mathrm{CAP}$ for each x, $a \in \mathrm{CAP} \otimes C(X)$.

COROLLARY 3.8. *If $A \in \mathcal{A}$, then $\sigma(A) \in \mathrm{CAP} \otimes C(S^{n-1})$.*

PROPOSITION 3.9. *Let a be a bounded measurable function on $\mathbf{R}^n \times \mathbf{R}^n$ whose translates $a(x - \lambda, \xi)$ are totally bounded in $L^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. Then F_a is almost periodic relative to $\{\tau_\mu\}$.*

Proof. It is immediate that

$$Q_{x,\xi}(T_\mu u) = Q_{x-\mu,\xi}(u)$$

and hence

$$T_{-\mu} F_a T_\mu = F_{T_\mu a}$$

where

$$T_\mu a(x, \xi) = a(x - \mu, \xi).$$

Since $a \mapsto F_a$ is continuous on L^∞ the result follows.

THEOREM 3.10. *σ is a *-homomorphism of $\mathcal{A}$ onto $\mathrm{CAP} \otimes HC$, and F is a positive linear right inverse.*

One of the distinguishing features of the algebra $\mathcal{A}$ is that the usual localization methods are not applicable. Indeed, if $\varphi \in C_c^\infty$ and A is any bounded operator on L^2, then $AM_\varphi \in \mathcal{A}$

iff $A\varphi=0$ since $\|\tau_\mu(AM_\varphi)-AM_\varphi\|=\|AM_\varphi\|$ when $|\mu|$ is sufficiently large. This is not surprising, since almost periodicity is not a local notion.

As concerns the pseudo-differential operators in $\mathcal{A}$, observe that the suprema of (2.4) define a countable collection of semi-norms. The functions a for which (2.4) are bounded form a Frechet space with the corresponding topology. The map $a\mapsto\Psi_a$ is continuous for this topology. Thus, if the translates $\{a(\cdot+\mu,\cdot)\}$ are totally bounded in this topology, $\Psi_a\in\mathcal{A}$.

Remark 3.11. It is enough to require the total boundedness relative to a finite number of the semi-norms which gives the continuity of Ψ. It does seem necessary, though, to use the differentiability in both variables. This is in contrast to the standard requirements of the original theory of Kohn-Nirenberg where it can be assumed that sufficiently many x derivatives are globally integrable in x uniformly in ξ. Although such assumptions give $\Psi_a\in\mathcal{A}$, the desired almost periodicity makes these assumptions useless.

In addition to the symbol on $\mathcal{A}$, there is another natural positive linear map obtained by averaging the function $\tau_\mu(A)$. More precisely, when $A\in\mathcal{A}$, $\mu\mapsto\tau_\mu(A)$ being almost periodic extends to a (norm) continuous map of the Bohr compactification $\mathbf{R}^n_B$ to $\mathcal{A}$. The integral of this function with respect to Haar measure defines an element of $\mathcal{A}$. Explicitly,

$$E(A)=\lim_{R\to\infty}\frac{1}{(2R)^n}\int_{[-R,R]^n}\tau_\mu(A)\,d\mu.$$

By the translation invariance of Haar measure, $E(\tau_\lambda(A))=E(A)=\tau_\lambda(E(A))$ for every λ. It is clear that E is a norm-decreasing positive linear map.

Now, when $A\in\mathcal{A}$ and $\tau_\mu(A)=A$ for every μ, $E(AB)=AE(B)$ for every $B\in\mathcal{A}$. On the other hand, the image of E consists of operators in $\mathcal{A}$ invariant under τ_μ. Since $\tilde{M}(L^\infty)$ is the set of all translation invariant operators on $L^2(\mathbf{R}^n)$, we have by Proposition 2.7:

PROPOSITION 3.12. *If $\mathcal{M}=\tilde{M}(HC+L_0^\infty)$, then $\mathcal{M}$ is a subalgebra of $\mathcal{A}$ and E is a positive $\mathcal{M}$-linear map of $\mathcal{A}$ onto $\mathcal{M}$.*

Another important feature of $\mathcal{A}$ is that $M(L^\infty)\cap\mathcal{A}=M(\mathrm{CAP})$. Further, when $\varphi\in\mathrm{CAP}$, $E(M_\varphi)$ is the usual Haar integral, which we also denote by E. From Proposition 3.4 one has $\varepsilon_\lambda(E(A))=E(\varepsilon_\lambda(A))$, $A\in\mathcal{A}$, $\lambda\in\mathbf{R}^n$, and it readily follows that

$$E(\sigma_\lambda(A))=\sigma_\lambda(E(A)),\quad \lambda\in\mathbf{R}^n,\quad A\in\mathcal{A}.$$

We may summarize these facts by the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\sigma} & \mathrm{CAP} \otimes (S^{n-1}) \\ E \big\downarrow & & \big\downarrow E \otimes id \\ \mathcal{M} & \xrightarrow{\sigma} & C(S^{n-1}) \end{array} \tag{3.2}$$

We shall say that a set $\mathcal{S}$ in $\mathcal{A}$ is translation invariant if $\tau_\mu(S) \in \mathcal{S}$, when $S \in \mathcal{S}$, $\mu \in \mathbf{R}^n$.

PROPOSITION 3.13. *If $\mathcal{A}_1$ is any translation invariant C^*-subalgebra of $\mathcal{A}$ and $\mathcal{J}$ is any closed, translation invariant *-ideal in $\mathcal{A}_1$, then for positive semidefinite $A \in \mathcal{A}_1$, $E(A) \in \mathcal{J}$ iff $A \in \mathcal{J}$.*

Proof. By translation invariance, $\{\tau_\mu\}$ defines an automorphism group on $\mathcal{A}/\mathcal{J}$, and every element is almost periodic relative to this group. If E_1 is the corresponding average. We want to show that for $[A] \geqslant 0$, the class of A in the quotient algebra, $E_1([A]) = 0$ iff $[A] = 0$. For this, we faithfully represent the quotient algebra on some Hilbert space and observe that only trivial positive semidefinite operator can have zero average. Thus, $A \geqslant 0$ and $E(A) \in \mathcal{J}$ implies that $A \in \mathcal{J}$.

It is an immediate corollary (assuming translation and *-invariance) that every closed ideal in every closed subalgebra is generated by its intersection with $\mathcal{M}$.

Example 3.14. Let $\mathcal{A}_1$ be the subalgebra generated by $\{e_\lambda\}_{\lambda \in \mathbf{R}^n}$ and those $\tilde{M}_\varphi$ where $\varphi \in C^\infty(\mathbf{R}^n)$ and φ has (uniform) radial limits at ∞. $\ker \sigma \cap \mathcal{A}_1$ is then such an ideal and hence is generated by $\ker \sigma \cap \mathcal{A}_1 \cap \mathcal{M}$. It is not too difficult to see that this is precisely $\tilde{M}C_0(\mathbf{R}^n)$, where C_0 denotes the functions vanishing at ∞. One can also consider the algebra generated by $\{e_\lambda\}_{\lambda \in \mathbf{R}^n}$ and $\tilde{M}(HC)$. This gives a larger set of translation invariant operators of the form $\tilde{M}_\varphi$ where φ is bounded and has suitable discontinuities. The generator of the relative kernel of σ i.e., $\ker \sigma \cap \mathcal{A}$, in this case is the set of all such functions which vanish at ∞. In both cases the relative kernel of σ is the commutator ideal [14]. Also, the relative kernel has a rather simple form.

If we add $\ker \sigma$ to these algebras, we generate all of $\mathcal{A}$. Indeed,

PROPOSITION 3.15. *Let $\mathcal{D}$ be a subset of $\mathcal{A}$ such that $\sigma(\mathcal{D})$ is dense in* $\mathrm{CAP} \otimes C(S^{n-1})$. *Then* $\ker \sigma + \mathcal{D}$ *is dense in* $\mathcal{A}$.

Proof. σ induces an isometry of $\mathcal{A}/\ker \sigma$ onto $\mathrm{CAP} \otimes C(S^{n-1})$.

Examples 31.6. Take $\mathcal{D}$ to be the linear span of $e_\lambda \tilde{M}_\varphi$ where $\lambda \in \mathbf{R}^n$ and φ runs over a set of functions in $L_0^\infty + HC$ with φ_∞ dense in $C(S^{n-1})$. Other candidates for $\mathcal{D}$ are the ranges of Ψ and F. In the latter case, $\ker \sigma + \mathcal{D} = \mathcal{A}$.

Unfortunately, we know very little about $\ker \sigma$ except for the following

THEOREM 3.17. *$\ker\sigma$ is the commutator ideal of $\mathcal{A}$.*

Proof. The commutator ideal is translation invariant and hence is generated (as an ideal) by its image $\mathcal{J}$ under E. If $\ker\sigma$ is not the commutator ideal, then $\mathcal{J}$ must be a proper subspace of $E(\ker\sigma)=\tilde{M}(L_0^\infty)$. For subalgebras of $\mathcal{A}$, the commutator ideals are smaller and hence are mapped by E into subspaces of $\mathcal{J}$. But, the algebra $\mathcal{A}_1$ generated by $\tilde{M}(L_0^\infty)$ and e_λ has $\mathcal{A}_1\cap\ker\sigma$ as its commutator ideal (see [14]), and $E(\mathcal{A}_1\cap\ \ker\sigma)\supset E(\tilde{M}(L_0^\infty))=\tilde{M}(L_0^\infty)$.

4. Representation

We shall consider representations in the II_∞ factor given by a group-measure space construction of Murray and von Neumann. In particular, we shall let $\mathcal{U}$ be the W^* algebra on $L^2(\mathbf{R}^n)\otimes l^2(\mathbf{R}^n)$ generated by the groups $\{e_\lambda\otimes T_\lambda: \lambda\in\mathbf{R}^n\}$ and $\{T_\lambda\otimes I: \lambda\in\mathbf{R}^n\}$. Observe that the second set generates $\{\tilde{M}_\varphi\otimes I: \varphi\in L^\infty\}$ and $\{e_\lambda\}$ is a unitary group acting ergodically on $\tilde{M}(L^\infty)$. For the basic facts concerning such an algebra see Dixmier [7], page 130 ff.

Since
$$T_\lambda e_\mu=e^{-i\mu\cdot\lambda}e_\mu T_\lambda, \tag{4.1}$$

one can show that the commutant $\mathcal{U}'$ of $\mathcal{U}$ is generated by the groups $\{T_\lambda\otimes e_\lambda: \lambda\in\mathbf{R}^n\}$ and $\{I\otimes T_\lambda: \lambda\in\mathbf{R}^n\}$. This facilitates the study of algebras represented in $\mathcal{U}$. However, it will be more convenient for our purposes to change the representation space to $L^2(\mathbf{R}^n)\otimes L^2(\mathbf{R}^n_B)$ by means of the unitary map $I\times\mathfrak{F}$ where $\mathfrak{F}$ is the Fourier transform on $l^2(\mathbf{R}^n)$. Thus, if $f\in l^2(\mathbf{R}^n)$,
$$\mathfrak{F}f=\sum_\lambda f(\lambda)\,e_\lambda.$$

It is immediate that $\mathfrak{F}T_\mu=e_\mu\mathfrak{F}$ and $\mathfrak{F}e_\mu=T_{-\mu}\mathfrak{F}$. It follows that the image of $\mathcal{U}$ is generated by
$$\{e_\lambda\otimes e_\lambda: \lambda\in\mathbf{R}^n\}\cup\{T_\lambda\otimes I: \lambda\in\mathbf{R}^n\}$$

and that the image of $\mathcal{U}'$ is generated by
$$\{T_\lambda\otimes T_{-\lambda}: \lambda\in\mathbf{R}^n\}\cup\{I\otimes e_\lambda: \lambda\in\mathbf{R}^n\}.$$

We shall actually use $\mathcal{U}$ to denote this representation.

The advantage of this representation lies in the fact that the commutant of $\{I\otimes e_\lambda\}$ consists of the bounded measurable functions from $\mathbf{R}^n_B$ into $\mathcal{B}(L^2(\mathbf{R}^n))$ and a fortiori the continuous almost periodic functions from $\mathbf{R}^n$ to $\mathcal{B}(L^2(\mathbf{R}^n))$ equipped with the weak operator topology. Moreover, conjugation with $T_\lambda\otimes T_{-\lambda}$ is simply $\tau_\lambda\otimes\tau_{-\lambda}$ on $\mathcal{B}(L^2(\mathbf{R}^n))\otimes M(L^\infty(R^n_B))$. Thus, the almost periodic functions with values in $\mathcal{B}(L^2(\mathbf{R}^n))$ which have a representation which commutes with $T_\lambda\otimes T_{-\lambda}$ are exactly the functions $\mu\mapsto B_\mu$ such that

$$\tau_\lambda(B_{\mu-\lambda}) = B_\mu, \quad \text{all } \lambda, \mu$$

i.e.,

$$B_\mu = \tau_\mu(B_0), \quad \mu \in \mathbf{R}^n.$$

Thus $\mathcal{A}$ has a natural representation ϱ in $\mathcal{U}$ obtained by mapping $A \in \mathcal{A}$ to the function $\mu \mapsto \tau_\mu(A)$ considered as an element of $\mathcal{B}(L^2(\mathbf{R}^n)) \otimes M(L^\infty(\mathbf{R}^n_B))$. It is obvious that this representation is faithful.

It is clear that ϱ extends as an isometry on the linear space $\mathcal{AP}_w$ of all operators A on $L^2(\mathbf{R}^n)$ which are weakly almost periodic with respect to τ. An important subalgebra of this is $\mathcal{AP}$, the strong-* τ-almost periodic operators. In a sense, this is the "largest" C^*-algebra in $\mathcal{B}(L^2(\mathbf{R}^n))$ to which ϱ extends naturally as a representation. An easy computation gives

Proposition 4.1. *If $B \in \mathcal{AP}_w$ and $f \in L^2(\mathbf{R}^n) \otimes C(\mathbf{R}^n_B)$, then $f \in \ker \varrho(B)$ iff $T_\mu f(\mu) \in \ker B$ for every $\mu \in \mathbf{R}^n$.*

Remark 4.2. In general this does not determine the kernel of $\varrho(B)$.

Next let us observe that the symbol on $\mathcal{A}$ has a natural analogue on $\varrho(\mathcal{A})$. Indeed

$$\varrho \circ \varepsilon_\lambda = (\varepsilon_\lambda \otimes I) \circ \varrho$$

and by the proof of Proposition 3.4, if $A \in \mathcal{A}$, then

$$(\varepsilon_\lambda \otimes I)\,(\varrho(A)) - \varrho(\sigma_\lambda(A)) \to 0$$

strongly as $|\lambda| \to \infty$. Defining σ_λ to be the radial limit of $\varepsilon_\lambda \otimes I$, we have $\sigma \circ \varrho = \varrho \circ \sigma$. Note that the symbols on the representation are functions on $\mathbf{R}^n \times \mathbf{R}^n_B \times S^{n-1}$ which are invariant under translations in the first two factors of the form $(\mu, -\mu)$, $\mu \in \mathbf{R}^n$.

In a similar fashion we see that the expectation E also has an analogue on $\mathcal{U}$. Indeed, if J denotes the Haar integral on $\mathbf{R}^n_B$, then since $\mathcal{U} \subset \mathcal{B}(L^2(\mathbf{R}^n)) \otimes M(L^\infty(\mathbf{R}^n))$, $I \otimes (J \circ M^{-1})$ defines a positive linear map on $\mathcal{U}$, which we shall also denote by E. Since J is translation invariant, $E \circ (I \otimes \tau_\mu) = E$ for every μ. When $A \in \mathcal{AP}_w$ $(E\varrho(A))$ is precisely the constant function on $\mathbf{R}^n_B$ whose value is the average of $\tau_\mu(A)$, i.e. $E(\varrho(A)) = \varrho(E(A))$. Also, E is $\tilde{M}(L^\infty) \otimes \mathbf{C}$-linear on $\mathcal{U}$.

The importance of E on $\mathcal{U}$ centers on the fact that it allows one to lift a trace on $\tilde{M}(L^\infty)^+$ (invariant under ε) to one on $\mathcal{U}^+$. We shall take

$$\operatorname{tr}(\tilde{M}_\psi) = \int_{\mathbf{R}^n} \psi(x)\,dx, \quad \psi \geqslant 0, \quad \psi \in L^\infty.$$

Then $(\operatorname{tr} \otimes I) \circ E$ is a trace on $\mathcal{U}^+$, which we shall also denote by tr. Up to a positive scale factor, this is the Murray–von Neumann trace on $\mathcal{U}$, and is unique. In fact we have

THEOREM 4.3. *If ϱ_1 is a faithful representation of $\mathcal{A}$ in a II_∞ factor $\mathcal{U}_1$ with trace ω, then $\omega\circ\varrho_1=\mathrm{tr}$ on $\mathcal{A}$ up to normalization and there is an isomorphism ν of $\mathcal{U}$ into $\mathcal{U}_1$ such that $\omega\circ\nu$ is a positive multiple of* tr.

Proof. $\omega\circ\varrho_1\circ\tilde{M}$ is a faithful, normal, semi-finite trace on $(L^1\cap L^\infty)^+$. Since $e_\lambda\in\mathcal{A}$, $\omega\circ\varrho_1$ is invariant under ε, i.e., $\omega\circ\varrho_1\circ\tilde{M}$ is translation invariant. Thus, $\omega\circ\varrho_1$ is a positive multiple of tr on $\tilde{M}(L^1\cap L^\infty)^+$, which we take to be 1. Now, it is easy to see that for each $\lambda\in\mathbf{R}^n$ and $A\in\ker\sigma$, $AT_\lambda\in\mathcal{A}$ and hence

$$\omega\circ\varrho_1(T_{-\lambda}A^*AT_\lambda)=\omega\circ\varrho_1(AA^*)=\omega\circ\varrho_1(A^*A).$$

Thus, $\omega\circ\varrho_1$ is translation invariant on $(\ker\sigma)^+$. Since for $\varphi\in(L^1\cap L^\infty)$, $\omega\circ\varrho_1(\tilde{M}_\varphi A)$ is then (norm) continuous and translation invariant as a function of A, we may integrate the composite of this with $\tau_\mu(A)$ over $\mathbf{R}^n_B$ to obtain

$$\omega\circ\varrho_1(\tilde{M}_\varphi A)=\omega\circ\varrho_1(\tilde{M}_\varphi E(A))=\mathrm{tr}\,(\tilde{M}_\varphi A),\quad A\in\mathcal{A}.$$

It follows by normality that $\omega\circ\varrho_1=\mathrm{tr}$ on $\mathcal{A}$. Using the left regular representation we have that $\mathcal{U}$ is isomorphic to the weak closure of

$$\{\varrho_1(A)\colon A\in\mathcal{A},\ \mathrm{tr}\,(A^*A)<\infty\}$$

in $\mathcal{U}_1$ with identification of traces (see [8], § 6.6).

We note in passing that the above proof can be adapted to any translation invariant subalgebra of $\mathcal{A}$ containing all elements of the form $e_\lambda\tilde{M}_\varphi$, $\lambda\in\mathbf{R}^n$, $\varphi\in C(\mathbf{R}^n)$ with compact support.

We shall let $\mathcal{K}$ denote the norm closed ideal generated by the positive elements of $\mathcal{U}$ with finite trace. The elements of $\mathcal{K}$ are called relatively compact operators.

PROPOSITION 4.4. *As a closed *-ideal in $\varrho(\mathcal{A})$, $\mathcal{K}\cap\varrho(\mathcal{A})$ is generated by*

$$E(\mathcal{K})=\tilde{M}(\overline{L^1\cap L^\infty})\otimes\mathbf{C}. \tag{4.2}$$

(Recall that $\overline{L^1\cap L^\infty}$ is the norm closure of $L^1\cap L^\infty$ in L^∞.)

Proof. The set $\{A\in\mathcal{U}^+\colon \mathrm{tr}\,(A)<\infty\}$ is translation invariant and is mapped by E onto $[\tilde{M}(L^1\cap L^\infty)\otimes\mathbf{C}]^+$. Thus, $\mathcal{K}$ is translation invariant and (4.2) holds. Since (4.2) is contained in $\varrho(\mathcal{A})$, we may apply the proof of Proposition 3.13 to show that $E(\mathcal{K})$ is a set of generators in $\varrho(\mathcal{A})$.

Remark 4.5. Propositions 2.7 and 3.13 show that in $\varrho(\mathcal{A})$, $\ker\sigma$ is generated by $\tilde{M}(L^\infty_0)\otimes\mathbf{C}$. However, as remarked after 2.7, this is strictly larger than $E(\mathcal{K})$. Hence, $\mathcal{K}\cap\varrho(\mathcal{A})\nsupseteq\ker\sigma$. By Theorem 3.17, $\varrho(\mathcal{A})/\mathcal{K}$ is not Abelian.

In the following, we shall study elements of C^*-algebras $\mathcal{L}$ which are regular modulo some closed $*$-ideal $\mathcal{J}$ in $\mathcal{L}$. We shall call these Fredholm (relative to $(\mathcal{L}, \mathcal{J})$) and denote the set of such by Fred $(\mathcal{L}, \mathcal{J})$. When $\mathcal{L}=\mathcal{U}$ and $\mathcal{J}=\mathcal{K}$ or $\mathcal{L}$ is a C^*-subalgebra of $\mathcal{A}$ and $\mathcal{J}=\mathcal{L}\cap\varrho^{-1}(\mathcal{K})$, we shall write Fred $(\mathcal{L})$.

PROPOSITION 4.6. *Let $\mathcal{L}_j$ be a C^*-algebra with a closed $*$-ideal $\mathcal{J}_j$, $j=1,2$. If η is a morphism of $\mathcal{L}_1$ into $\mathcal{L}_2$ such that $\eta(\mathcal{J}_1)\subset\mathcal{J}_2$, then*

$$\text{Fred}\,(\mathcal{L}_1, \mathcal{J}_1)\subset\text{Fred}\,(\mathcal{L}_1, \eta^{-1}(\mathcal{J}_2))=\eta^{-1}(\text{Fred}\,(\mathcal{L}_2, \mathcal{J}_2)).$$

Further, we have equality iff $\mathcal{J}_1=\eta^{-1}(\mathcal{J}_2)$.

Proof. By assumption, there is a morphism $\tilde{\eta}$ such that

$$\begin{array}{ccc} \mathcal{L}_1 & \xrightarrow{\ \eta\ } & \mathcal{L}_2 \\ \downarrow & & \downarrow \\ \mathcal{L}_1/\mathcal{J}_1 & \xrightarrow{\ \tilde{\eta}\ } & \mathcal{L}_2/\mathcal{J}_2 \end{array}$$

Thus, we may assume $\mathcal{J}_1=\mathcal{J}_2=0$. The result is then a standard fact about regular elements in C^*-algebras.

We shall now apply Breuer's Fredholm theory for II_∞ factors [2, 3]. Indeed, the relative Fredholm operators in a II_∞ factor are those which are regular modulo the relatively compact ideal. For $\mathcal{U}$, this is Fred $(\mathcal{U})$. On Fred $(\mathcal{U})$, one has a real valued *analytic index* given by

$$i_{\text{an}}(A)=\text{tr}\,(N_A)-\text{tr}\,(N_{A^*}),\quad A\in\text{Fred}\,(\mathcal{U}),$$

where N_A denotes the null projection of A. The basic properties of i_{an} are

(i) i_{an} is locally constant relative to the norm topology,
(ii) $i_{\text{an}}(A+K)=i_{\text{an}}(A)$, $A\in\text{Fred}\,(\mathcal{U})$, $K\in\mathcal{K}$,
(iii) $i_{\text{an}}(AB)=i_{\text{an}}(A)+i_{\text{an}}(B)$, $A, B\in\text{Fred}\,(\mathcal{U})$,
(iv) $i_{\text{an}}(A^*)=-i_{\text{an}}(A)$, $A\in\text{Fred}\,(\mathcal{U})$,
(v) $i_{\text{an}}(A)=0$ when A is invertible.

It follows immediately that i_{an} is determined by a group homomorphism on the connected components of the group of regular elements of $\mathcal{U}/\mathcal{K}$ and a fortiori of $\varrho(\mathcal{A})/\mathcal{K}$. However, we do not know a set of generators for this latter group and hence are unable to determine i_{an} on Fred $(\varrho(\mathcal{A}))$ in terms of $\mathcal{A}$. The central difficulty is an incomplete understanding of ker σ, which by Theorem 3.17 is the commutator ideal in $\mathcal{A}$.

There are, however, subalgebras of $\mathcal{A}$ for which we can determine i_{an}. Let $\mathcal{A}_{00}$ be the closed *-subalgebra of $\mathcal{A}$ generated by $\{e_\lambda: \lambda \in \mathbf{R}^n\}$, $\tilde{M}(L^1 \cap L^\infty)$, and $\tilde{M}(HC)$.

PROPOSITION 4.7. *$\mathcal{A}_{00}$ is the closed linear span of*

$$\{e_\lambda \tilde{M}_\varphi: \lambda \in \mathbf{R}^n, \varphi \in L^1 \cap L^\infty + HC\}.$$

Proof: $L^1 \cap L^\infty$ is an ideal in L^∞. Thus, $L^1 \cap L^\infty + HC$ is an algebra. Using Proposition 2.7 and the fact that $L^1 \cap L^\infty \subset L_0^\infty$, we know that $\overline{L^1 \cap L^\infty} + HC$ is the closure of $L^1 \cap L^\infty + HC$ in L^∞. Since

$$\tilde{M}_\varphi e_\lambda = e_\lambda \tilde{M}_\varphi + e_\lambda \tilde{M}_\psi$$

where ψ is the translate of φ by $-\lambda$, it is enough to show that $\tilde{M}_\psi \in \overline{L^1 \cap L^\infty} + HC$. This is trivial when $\varphi \in L^1 \cap L^\infty$. Otherwise we use

LEMMA 4.8. *If $\varphi \in HC$ and $\lambda \in \mathbf{R}^n$, then*

$$T_\lambda \varphi - \varphi \in \overline{L_c^\infty} \subset \overline{L^1 \cap L^\infty}$$

where L_c^∞ denotes the set of all compactly supported L^∞ functions.

Proof: If $x \in \mathbf{R}^n$.

$$\varphi(x+\lambda) - \varphi(x) = \varphi\left(\frac{x+\lambda}{|x|}\right) - \varphi\left(\frac{x}{|x|}\right).$$

Thus, for $r > 0$,

$$\sup_{|x| \geqslant r} |\varphi(x+\lambda) - \varphi(x)| \leqslant \sup\{|\varphi(y+z) - \varphi(y)|: |y| = 1, |z| \leqslant r^{-1}\},$$

and the right hand side tends to zero as $r \to \infty$. Since $\overline{L_c^\infty}$ is the set of L^∞ functions whose essential supremum outside compact sets tends to zero as the sets increase, we are through.

It is immediate that $\mathcal{A}_{00}$ is translation invariant and hence by Proposition 3.13, $\ker \sigma \cap \mathcal{A}_{00}$ is generated as a closed *-ideal by $E(\ker \sigma \cap \mathcal{A}_{00})$, and by diagram (3.2) this is $\ker \sigma \cap E(\mathcal{A}_{00})$. Since E is continuous and $\mathcal{M}$-linear,

$$E(\mathcal{A}_{00}) = \overline{L^1 \cap L^\infty} + HC.$$

By Proposition 2.7, $\ker \sigma \cap E(\mathcal{A}_{00}) = L^1 \cap L^\infty$. Combining these facts and applying Lemma 4.7 in the same fashion as we did in Proposition 4.6, we have

PROPOSITION 4.9. $\ker \sigma \cap \mathcal{A}_{00}$ *is the closed linear span of*

$$\{e_\lambda \tilde{M}_\varphi: \lambda \in \mathbf{R}^n, \varphi \in \overline{L^1 \cap L^\infty}\}.$$

Remark 4.10. We may replace $L^1 \cap L^\infty$ in Propositions 4.7 and 4.8 by any ideal in L^∞ whose closure lies between $\overline{L_c^\infty}$ and L_0^∞. In fact, if we replace HC by an algebra of continuous functions which equal functions in HC outside of the unit ball, then similar results hold using an ideal of bounded continuous functions such that the closure of the ideal contains the compactly supported functions and is contained in L_0^∞. This gives the structure of some of the algebras in example 3.14. Our interest in $\mathcal{A}_{00}$ lies in the fact that of all such algebras, it is the largest algebra satisfying $\varrho(\mathcal{A}_{00} \cap \ker \sigma) \subset \mathcal{K}$.

We now define $\mathcal{A}_0 = \mathcal{A}_{00} + \varrho^{-1}(\mathcal{K})$. Since $\varrho^{-1}(\mathcal{K})$ is an ideal in $\mathcal{A}$, $\mathcal{A}_0$ is an algebra. It is immediate that

$$\operatorname{span}\{e_\lambda \tilde{M}_\varphi : \varphi \in HC, \lambda \in \mathbf{R}^n\} + \varrho^{-1}(\mathcal{K})$$

is dense in $\mathcal{A}_0$ and that $\varrho^{-1}(\mathcal{K}) = \mathcal{A}_0 \cap \ker \sigma$. This implies that

$$\operatorname{Fred}(\varrho(\mathcal{A}_0)) = \varrho\,(\operatorname{Fred}(\mathcal{A}_0, \ker \sigma)),$$

i.e., the Fredholm elements of $\varrho(\mathcal{A}_0)$ are those with invertible symbol (sometimes called the elliptic operators). In particular we may determine i_{an} in terms of the symbol. When $n = 1$, it is given in terms of the mean motion (of the symbol) defined by

$$m(\varphi) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d(\arg \varphi),$$

$\varphi \in \text{CAP}$ and regular.

THEOREM 4.11. *If* $n \geqslant 2$, $i_{\text{an}} = 0$ *on* Fred $(\varrho(\mathcal{A}_0))$. *If* $n = 1$ *and* $A \in \operatorname{Fred}(\varrho(\mathcal{A}_0))$,

$$-i_{\text{an}}(A) = m(\sigma_{+1}(A)) - m(\sigma_{-1}(A)).$$

Proof. As already remarked, i_{an} is given via σ by a group homomorphism on the group of connected components of invertibles in $\text{CAP} \otimes C(S^{n-1})$. By [15], this group is canonically isomorphic to $H^1(\mathbf{R}_B^n \times S^{n-1})$, the first Čech cohomology group with integer coefficients. Since

$$H^1(\mathbf{R}_B^n \times S^{n-1}) = \begin{cases} \mathbf{R} + \mathbf{R}, & n = 1 \\ \mathbf{R}^n + Z, & n = 2, \\ \mathbf{R}^n, & n > 2 \end{cases}$$

and the mean motion on the regular elements of CAP and the winding number on the regular elements of $C(S^1)$ are homotopy invariants, we can give a set of generators of the connected components of invertibles in $\text{CAP} \otimes C(S^{n-1})$ and it is sufficient to compute i_{an} for any operator in $\mathcal{A}_0$ whose symbol is a generator. When $n > 2$, $\{e_\lambda : \lambda \in \mathbf{R}^n\}$ is a set of generators. These are the symbols of themselves and hence have index 0 by (v). When

$n=2$, the sets $\{e_\lambda, \lambda \in \mathbf{R}^2\}$ and $\{\text{id}\}$, where id is the identity map of S^1 to S^1 as a subset of $\mathbf{C}$, give a set of generators. Again e_λ is the symbol of itself and has index 0. Extending id to an element φ in HC, we see that $\sigma(\tilde{M}_\varphi)=\text{id}$. But $\tilde{M}_\varphi$ is invertible and hence has index 0. Thus $i_{\text{an}}=0$ when $n=2$. For the case where $n=1$, symbols are pairs of CAP functions $(\sigma_{-1}, \sigma_{+1})$ and a set of generators is (e_λ, e_λ) and $(e_{-\lambda}, e_\lambda)$ for $\lambda \in R$. Since e_λ has symbol (e_λ, e_λ), such pairs must give index zero. For an operator in $\mathcal{A}_0$ with symbol $(e_{-\lambda}, e_\lambda)$ let $P_\pm = \tilde{M}_{\chi_\pm}$ where $\chi_\pm$ is the characteristic function of $\pm R_+$; then for $A=P_- e_{-\lambda} P_- + P_+ e_\lambda P_+$, we see that $\sigma(A)=(e_{-\lambda}, e_\lambda)$. A is easily seen to be a partial isometry and it follows that

$$N_{\varrho(A)} = I - \varrho(A)^* \varrho(A) = \varrho(I - A^*A).$$

Now letting $\chi\{a, b\}$ denote the characteristic function of the interval (a, b), we see by direct computation that

$$N_{\varrho(A)} = \tilde{M}_{\chi\{\min(\lambda,0),\max(-\lambda,0)\}} \otimes I$$

$$N_{\varrho(A^*)} = \tilde{M}_{\chi\{\min(-\lambda,0),\max(\lambda,0)\}} \otimes I$$

and hence

$$-i_{\text{an}}(\varrho(A)) = 2\lambda.$$

Since the difference of mean motions is a group homomorphism, and

$$m(e_\lambda) - m(e_{-\lambda}) = 2\lambda$$

the result follows.

Remark 4.12. The case $n=1$ is essentially that of [5] and is included here for completeness.

Although we can only determine the index on $\mathcal{A}_0$, this is enough to cover most basic examples. We shall now proceed to show that the algebras of Example 3.14 are all contained in $\mathcal{A}_0$. This will follow from 4.6, 4.7, 4.8, 4.12 and 4.13.

PROPOSITION 4.13. *The Friedrichs map F maps* $\text{CAP} \otimes C(S^{n-1})$ *into* $\mathcal{A}_0$.

Proof. It is enough to consider F_a when $a = e_\lambda \otimes \varphi$, $\lambda \in \mathbf{R}^n$, $\varphi \in HC$. As in Lemma 2.25, $F_a = \tilde{M}_\psi e_\lambda$ where

$$\psi(\eta) = \int d\zeta\, q(\zeta, \mu + \lambda)\, q(\zeta, \eta)\, \varphi(\zeta),$$

and we have identified φ with its corresponding element of HC. This will give the desired result provided that we can show $\psi - \varphi \in \overline{L_c^\infty}$. Now,

$$\psi(\eta) - \varphi(\eta) = \int d\zeta\, q(\zeta, \eta+\lambda)\, q(\zeta, \eta)\, [\varphi(\zeta) - \varphi(\eta)] + \varphi(\eta) \left[\int d\zeta\, q(\zeta, \eta+\lambda)\, q(\zeta, \eta) - 1\right]$$

and we know that the second term tends to 0 as $|\eta| \to \infty$ (see 2.25). As observed in the proof of Lemma 2.16, the integrand of the first term is supported in $\{\zeta \in \mathbf{R}^n : |\zeta - \eta| \leqslant \omega(\eta)^{-1}\}$ and thus if C is a bound from 2.16, the first term is bounded by

$$C \sup \{|\varphi(\zeta) - \varphi(\eta)| : |\zeta - \eta| \leqslant \omega(\eta)^{-1}\}.$$

Since $\varphi \in HC$ and $\omega(\eta)^{-1}$ is $0(|\eta|^{1/2})$, this bound tends to zero as $|\eta| \to \infty$, as desired.

THEOREM 4.14. *Let $A \in \mathcal{A}$. $A \in \mathcal{A}_0$ iff*

$$E(e_\lambda A) \in \tilde{\mathcal{M}}(\overline{L^1 \cap L^\infty} + HC), \quad \lambda \in \mathbf{R}^n.$$

Proof. The result is immediate when $A \in \mathcal{A}_0$ since $\tilde{M}(\overline{L^1 \cap L^\infty})$ generates $\varrho^{-1}(\mathcal{K})$ as an ideal in $\mathcal{A}$ and E is $\mathcal{M}$-linear. Conversely, if A satisfies this condition, then so does $A - F_{\sigma(A)}$ since $F_{\sigma(A)} \in \mathcal{A}_0$. Thus it suffices to assume that $A \in \ker \sigma$. Then $E(e_\lambda A) \in \ker \sigma$ for every λ and by Proposition 2.7, $E(e_\lambda A) \in \tilde{M}(\overline{L^1 \cap L^\infty})$, $\lambda \in \mathbf{R}^n$. Consider the algebra $\mathcal{A}/\varrho^{-1}(\mathcal{K})$. Since $\varrho^{-1}(\mathcal{K})$ is translation invariant, the automorphism group τ induces one on $\mathcal{A}/\varrho^{-1}(\mathcal{K})$ relative to which each element is almost periodic. Since $\tau_\mu(e_\lambda A) = e^{i\lambda\cdot\mu} e_\lambda \tau_\mu(A)$, the assumption amounts to the statement that as an element of $\mathcal{A}/\varrho^{-1}(\mathcal{K})$, $e^{i\lambda\cdot\mu}\tau_\mu(A)$ has zero Haar integral with respect to μ on $\mathbf{R}^n$. By the uniqueness of the Fourier transform on $\mathbf{R}^n_B$, $\tau_\mu(A) = 0$, mod $\varrho^{-1}(\mathcal{K})$. In particular, $A \in \varrho^{-1}(\mathcal{K}) \subset \mathcal{A}_0$.

COROLLARY 4.15. *If for some $\varepsilon > 0$, $A\Lambda^\varepsilon \in \mathcal{A}$, then $\varrho(A) \in \mathcal{K}$.*

Proof. Λ^ε is translation invariant. Hence $A \in \mathcal{A}$ and for every $\lambda \in \mathbf{R}^n$, $E(e_\lambda A\Lambda^\varepsilon) = E(e_\lambda A)\,\Lambda^\varepsilon$. Since $\Lambda^{-\varepsilon} \in \tilde{M}(\overline{L^\infty})$, $E(e_\lambda A) \in \varrho^{-1}(\mathcal{K})$ for every $\lambda \in \mathbf{R}^n$.

We would now like to give an intrinsic meaning to the index for the algebra $\mathcal{A}_0$.

THEOREM 4.16. *Let φ_n be a decreasing sequence of real valued continuous functions such that for each n,*

(i) $\varphi_n(x) = 1$, $x \leqslant 0$,

(ii) $\varphi_n(x) = 0$, $x \geqslant 1/n$.

If $A \in \mathcal{A}_0$ and $\sigma(A)$ is regular, then

$$\operatorname{tr}(N_{\varrho(A)}) = \lim_{n\to\infty} \operatorname{tr} \varphi_n(A^*A).$$

In particular $\varphi_n(A^*A)$ and $\varphi_n(AA^*)$ eventually have finite trace and

$$\mathrm{n}(A) = \lim_{n\to\infty} \operatorname{tr}[\varphi_n(A^*A) - \varphi_n(AA^*)].$$

Proof. For any self-adjoint operator S in $\mathcal{U}$, $\varphi_n(S) \downarrow N_S$ as $n \to \infty$. Moreover, if $A \in \mathcal{A}$ and is self-adjoint, then $\varphi_n(A) = \varphi_n(\varrho(A))$. Suppose A is an elliptic operator in $\mathcal{A}$. Then so are A^*A and AA^*. If φ is a continuous function on $\mathbf{R}$, then $\sigma(\varphi(A^*A)) = \varphi(\sigma(A^*A))$. Since $\sigma(A^*A)$ is invertible, $\varphi(\sigma(A^*A)) = 0$ when φ is supported sufficiently near 0. Thus, since $\ker \sigma \cap \mathcal{A}_0 = \varrho^{-1}(\mathcal{K})$,

$$\varphi(\varrho(A^*A)) \in \mathcal{K}, \ \varphi \in C(\mathbf{R}), \ \operatorname{supp} \varphi \text{ near } 0.$$

Since each positive operator which is dominated by an operator in $\mathcal{K}$ is also in $\mathcal{K}$, the same is true for each Borel measurable function φ supported near 0. In particular, the spectral projections of $\varrho(A^*A)$ near 0 are in $\mathcal{K}$. But the projections in $\mathcal{K}$ all have finite trace. Therefore

$$\operatorname{tr} \varphi(A^*A) = \operatorname{tr} \varphi(\varrho(A^*A)) < \infty$$

for $\varphi \in C(\mathbf{R})$, $\varphi \geqslant 0$, φ supported sufficently near 0. In particular, $\operatorname{tr} \varphi_n(A^*A)$ is eventually finite and hence converges to $\operatorname{tr}(N_{A^*A})$. Since $N_{A^*A} = N_A$, the result follows.

COROLLARY 4.17. *If ϱ is any faithful Π_∞-factor representation of $\mathcal{A}_0$ then the analytic index is given by Theorem* 4.11 *up to normalization.*

Proof. Theorems 4.3 and 4.16.

COROLLARY 4.18. *If $A \in \mathcal{A}_0$ is elliptic and has closed range, then N_A and $N_{A^*} \in \mathcal{A}_0$ and*

$$i_{\mathrm{an}}(A) = \operatorname{tr} N_A - \operatorname{tr} N_{A^*}.$$

Proof. If A has closed range, then A^*A has an isolated eigenvalue at 0. Thus $\varphi_n(A^*A) = N_A$ for n sufficiently large.

COROLLARY 4.19. *If $A \in \mathcal{A}_0$ is elliptic and $i_{\mathrm{an}}(A) = 0$, then A has a one sided inverse iff A is invertible. In particular, when $n > 1$, invertibility of an elliptic $A \in \mathcal{A}_0$ is equivalent to the existence of a $c > 0$ such that*

$$\|Au\| \geqslant c\|u\|, \quad u \in L^2(\mathbf{R}^n).$$

5. Remarks on the "systems" case

For systems, let $\mathcal{A}_k = \mathcal{A} \otimes M_k$ where M_k is the algebra of $k \times k$ complex matrices and $\mathcal{A}$ is the algebra of almost-periodic operators on $L^2(\mathbf{R}^n)$ described in section 3. We also consider $(\mathcal{A}_0)_k = \mathcal{A}_0 \otimes M_k$ for $\mathcal{A}_0$ the sub-algebra of $\mathcal{A}$ described in section 4. Clearly $\mathcal{A}_k$ and $(\mathcal{A}_0)_k$ act on the Hilbert space of k-tuples of functions in $L^2(\mathbf{R}^n)$. We can construct a symbol homomorphism $\sigma_k = \sigma \otimes 1_k$ mapping $\mathcal{A}_k$ into $\mathrm{CAP} \otimes C(S^{n-1}) \otimes M_k$ with $\ker \sigma_k =$

$\ker \sigma \otimes M_k$. We can also extend the representation ϱ to $\mathcal{A}_k$. First, we form the $\prod_\infty$ factor [7] $\mathcal{U}_k = \mathcal{U} \otimes M_k$; then $\varrho_k = \varrho \otimes I_k$ is a faithful representation of $\mathcal{A}_k$ in $\mathcal{U}_k$. The trace ideal in $\mathcal{U}_k$ is just $\mathcal{K}_k = \mathcal{K} \otimes M_k$.

It is easy to see, as in section 4, that $\varrho_k^{-1}(\mathcal{K}_k) \subsetneqq \ker \sigma_k$, and so we consider only $(\mathcal{A}_0)_k$. We now get

$$\varrho_k^{-1}(\mathcal{K}_k) \cap (\mathcal{A}_0)_k = \ker \sigma_k \cap (\mathcal{A}_0)_k.$$

For A in Fred $\{\varrho_k[(\mathcal{A}_0)_k]\}$, the analytic index $i_{\mathrm{an}}(A)$ can be computed in terms of $\sigma(A)$. Recently, D. Schaeffer [16] found

$$i_{\mathrm{an}}(A) = -\{m(\det \sigma_+(A)) - m(\det \sigma_{-1}(A))\}$$

for $n=1$ and arbitrary k. Here, det is the determinant function on Gl_k and $\sigma_\pm(A)$ are the values of $\sigma(A)$ on the unit sphere S^0.

The formula for $n>1$ is completely analogous to the ordinary integer valued index for elliptic operators on hypersurfaces [1, p. 601]. That is, the cohomology of Gl_k is an exterior algebra generated by elements $h_j \in H^{2j-1}(Gl_k)$. Then

$$i_{\mathrm{an}}(A) = \frac{(-1)^{k+n-1}}{(n-1)!} \int_{\mathbf{R}^n} \int_{S^{n-1}} \sigma(A)^* h_n.$$

Here, $\int_{\mathbf{R}^n}$ is to be interpreted as the almost periodic mean, i.e., as $\int_{\mathbf{R}^n_B}$.

The proof of this formula will appear elsewhere. It follows from general functorial properties once it is known for $n=1$ and $n=2$. For $n=2$, the formula is verified by a direct (and long) computation involving the residues of the zeta function for A^*A [18].

Finally, we remark that the analytic index for systems is also unique, with the proof a slight variation of the one given in the previous section for $k=1$.

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