

C^* -EXTREME ENTANGLEMENT BREAKING MAPS ON OPERATOR SYSTEMS

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ABSTRACT. Let \mathcal{E} denote the set of all unital entanglement breaking (UEB) linear maps defined on an operator system $\mathcal{S} \subset M_d$ and, mapping into M_n . As it turns out, the set \mathcal{E} is not only convex in the classical sense but also in a quantum sense, namely it is C^* -convex. The main objective of this article is to describe the C^* -extreme points of this set \mathcal{E} . By observing that every EB map defined on the operator system \mathcal{S} dilates to a positive map with commutative range and also extends to an EB map on M_d , We show that the C^* -extreme points of the set \mathcal{E} are precisely the UEB maps that are maximal in the sense of Arveson ([A] and [A69]) and that they are also exactly the linear extreme points of the set \mathcal{E} with commutative range. We also determine their explicit structure, thereby obtaining operator system generalizations of the analogous structure theorem and the Krein-Milman type theorem given in [BDMS]. As a consequence, we show that C^* -extreme (UEB) maps in \mathcal{E} extend to C^* -extreme UEB maps on the full algebra. Finally, we obtain an improved version of the main result in [BDMS], which contains various characterizations of C^* -extreme UEB maps between the algebras M_d and M_n .

1. INTRODUCTION

The notions of positivity and convexity are fundamental to Mathematical analysis and in particular, to the theory of C^* -algebras. Among positive maps between C^* -algebras, the ones that are completely so, are of considerable interest. The study of completely positive maps was initiated by Stinespring and Arveson in the seminal papers [S],

2020 *Mathematics Subject Classification.* 81P40, 47L07 (Primary); 15B48, 81R15, 81P42, 81P45 (Secondary).

Key words and phrases. Operator systems, Entanglement breaking maps, Maximal, Dilation, Extension, Mapping cone, C^* -convexity, C^* -extreme, Extremal, Krein-Milman, Schmidt number, EB rank, Choi rank.

* Supported by the grant MTR/2018/000113 from the Department of Science and Technology (DST), Govt. of India.

¹ Supported by the fellowship 0203/16(8)/2018-R&D-II from the National Board for Higher Mathematics (NBHM), Govt. of India.

[A69] and [A72]. Among various results of significant importance in [A69], of particular interest to us is an abstract characterization of the (linear) extreme points of the convex set of completely positive maps between a C^* -algebra \mathcal{A} and $B(H)$ for some Hilbert space H , in terms of the (minimal) Stinespring dilation. As important as classical convexity is, it still has some limitations in the non-commutative setting. Two "non-commutative" convexity notions that have gathered significant attention recently are matrix-convexity and C^* -convexity, the former introduced and studied by Webster and Winkler in [W] and [WW] and the latter by Hopenwasser, Loebel, Moore and Paulsen in [LP] and [HMP]. Although these notions appear to be similar, they are vastly different as was pointed out by Farenick in [F]. Farenick and Morenz also obtained a complete characterization of C^* -extreme unital completely positive maps between C^* -algebras in [FM93] and [FM97]. Further contributions on this and related topics can be found in [BBK], [BK].

Our main objects of focus in this article are entanglement breaking (EB) maps. These maps have drawn considerable attention recently and are particularly sought after in quantum information theory. In the finite dimensional setting, an important aspect of the set of unital entanglement breaking (UEB) maps is that it is C^* -convex. This warrants the study of C^* -extreme UEB maps. A complete description of such maps between matrix algebras was obtained in [BDMS]. The purpose of this article is four fold. Firstly, we obtain the explicit structure of UEB maps defined on a finite-dimensional operator system that are maximal with respect to the dilation order (see [A]) on the set of UEB maps (See Theorem 1.10). As a consequence we show that every UEB map on a finite dimensional operator system dilates to a maximal UEB map (See Theorem 1.11). Secondly, we show that UEB maps on a finite-dimensional operator system that are maximal with respect to the dilation order on the set of UEB maps are precisely the C^* -extreme UEB maps and that they are also exactly the linear extreme UEB maps with commutative range (See Theorem 1.12). As a consequence, we obtain an operator system generalization of a characterization of C^* -extreme UEB maps as well as the Krein-Milman type theorem for UEB maps given in [BDMS]. (See Corollary 1.14). It is to be noted that the partial order used for obtaining the abstract characterization of C^* -extreme UEB maps in [BDMS] is the usual partial order (and not the dilation order) on the set of UEB maps, i.e., for UEB maps Φ and Ψ , $\Phi \leq \Psi$ if and only if $\Psi - \Phi$ is a UEB map. Thirdly, we show that a C^* -extreme UEB map defined on a finite dimensional

operator system extends to a C^* -extreme UEB map on the full algebra (See Corollary 1.13). Finally, we obtain an improved version of the main result Theorem 5.3 in [BDMS] (See Theorem 6.2). To help us prove the above mentioned assertions, we make use of the following key observations namely, an EB map defined on a finite dimensional operator system dilates to a positive map with commutative range (See Theorem 1.7) and also has an EB extension to the full algebra (See Theorem 1.5).

Before we explicitly state our main observations and results, we introduce some notations and definitions. Throughout this article, L and K will denote separable Complex Hilbert spaces and $B(L)$ will denote the C^* -algebra of bounded linear maps defined on L . An operator system $\mathcal{M} \subset B(L)$ is a self adjoint subspace containing the identity operator I_L . A linear map $\Phi : \mathcal{M} \rightarrow B(K)$ is said to be **unital** if $\Phi(I_L) = I_K$, **positive** if $\Phi(A)$ is a positive operator in $B(K)$ (in this case we write $\Phi(A) \succeq 0$ or $\Phi(A) \in B(K)^+$), whenever A is a positive operator in \mathcal{M} . A unital positive linear functional on \mathcal{M} is called a **state**. The notation M_k will denote the space of all $k \times k$ Complex matrices and $M_k(\mathcal{M})$, the operator system of all $k \times k$ matrices with entries from \mathcal{M} . For a given linear map $\Phi : \mathcal{M} \rightarrow B(K)$ and $k \in \mathbb{N}$, the k^{th} -**ampliation** $\Phi_k : M_k(\mathcal{M}) \rightarrow M_k(B(K))$ is defined by $\Phi_k([A_{i,j}]) = [\Phi(A_{i,j})]$. Under the identification of $M_k(\mathcal{M})$ with $M_k \otimes \mathcal{M}$, one sees that $\Phi_k := \mathbf{i}_k \otimes \Phi$, where \mathbf{i}_k is the identity operator on M_k . More specifically, $\Phi_k : M_k \otimes \mathcal{M} \rightarrow M_k \otimes B(K)$ is the linear map determined by $\Phi_k(X \otimes A) = X \otimes \Phi(A)$. The linear map $\Phi : \mathcal{M} \rightarrow B(K)$ is said to be **completely positive (CP)** if Φ_k is positive for all $k \in \mathbb{N}$.

Next we define the entanglement breaking property of linear maps. There are a number of competing definitions for the notion of an entanglement breaking map $\Phi : \mathcal{M} \rightarrow B(K)$ that agree with the usual notion in the case that $\mathcal{M} = B(L)$ and K is a finite dimensional Hilbert space. In any case, they all reference the cone of (separable) matrices,

$$M_k^+ \otimes B(K)^+ := \left\{ \sum_{m=1}^{\ell} A_m \otimes B_m : \ell \in \mathbb{N}, A_m \in M_k^+, B_m \in B(K)^+ \right\}.$$

Since we are mainly interested in the case where $\mathcal{M} \subset B(L)$ is an operator system and, K and L are finite dimensional Hilbert spaces, inspired by the various notions of separability introduced in [CH], we will say that the linear map $\Phi : \mathcal{M} \rightarrow B(K)$ is **entanglement breaking (EB)** if $(\mathbf{i}_k \otimes \Phi)(X) \in \overline{M_k^+ \otimes B(K)^+}$, for all $k \in \mathbb{N}$ and $X \in (M_k \otimes \mathcal{M})^+$, where the closure is with respect to the norm topology on $B(\mathbb{C}^k \otimes K)$. Evidently, when K is finite dimensional, $M_k^+ \otimes B(K)^+$ is

already norm closed. The abbreviations **UCP** and **UEB** will be used for "unital completely positive" and "unital entanglement breaking" respectively. The collection of all UEB maps mapping \mathcal{M} to $B(K)$ will be denoted by $\text{UEB}(\mathcal{M}, B(K))$. Finally, since we work mainly in the finite dimensional setting, we make the following notational conventions.

Convention 1.1. *Throughout $E = \mathbb{C}^d$ and $H = \mathbb{C}^n$ and $\mathcal{S} \subset B(E)$ is an operator system.*

Given a linear map $\Phi : \mathcal{S} \rightarrow B(H)$, the **dual functional** $s_\Phi : B(H) \otimes \mathcal{S} \rightarrow \mathbb{C}$ associated to it, is the linear mapping determined by

$$s_\Phi(X \otimes A) = \text{trace}(\Phi(A)X^t),$$

where t is the transpose operator in $B(H)$ induced by a fixed orthonormal basis of H . It is easily seen that this definition of s_Φ is independent of the choice of the orthonormal basis and hence of the transpose operator induced by it and that the correspondence $\Phi \mapsto s_\Phi$ is bijective. Please see [St], [B] for more details.

The **Choi matrix** C_Φ associated with the linear map $\Phi : B(E) \rightarrow B(H)$ is defined as $[\Phi(e_i e_j^*)]_{i,j=1}^d = \sum_{i,j=1}^d e_i e_j^* \otimes \Phi(e_i e_j^*) \in B(E) \otimes B(H)$, where $\{e_1, \dots, e_d\}$ is the standard orthonormal basis of E . One of the many significant applications of the Choi-matrix is the following well-known characterization of CP maps due to Choi.

Theorem 1.2. ([C, Theorem 1], [St, Theorem 4.1.8]) *Let $\Phi : B(E) \rightarrow B(H)$ be a linear map. The following statements are equivalent.*

- (i) Φ is CP.
- (ii) $\Phi(X) = \sum_{k=1}^{\ell} V_k^* X V_k$, for some linear maps $V_k : H \rightarrow E$.
- (iii) $C_\Phi \in (B(E) \otimes B(H))^+$.

The formula for Φ given in statement (ii) above is called a **Choi-Kraus decomposition** of Φ and the matrices coefficients V_k are called **Choi-Kraus operators/coefficients**. The minimum number of Choi-Kraus operators required to represent Φ in the form of a Choi-Kraus decomposition is known as the **Choi-rank of Φ** . The **Schmidt rank of the vector** $\xi \in E \otimes H$, denoted by $SR(\xi)$, is the smallest natural number k such that $\xi = \sum_{i=1}^k x_i \otimes y_i \in E \otimes H$. Given a completely positive map $\Phi : B(E) \rightarrow B(H)$, by Theorem 1.2 the Choi matrix $C_\Phi \in (B(E) \otimes B(H))^+$ and hence has a spectral decomposition of the form $C_\Phi = \sum_{i=1}^m \xi_i \xi_i^*$, where $\xi_i \in E \otimes H$. Let Π_Φ denote the set of all spectral decompositions of C_Φ . The **Schmidt number of the Choi matrix** $C_\Phi \in (B(E) \otimes B(H))^+$ is denoted by $SN(C_\Phi)$ and is defined

as

$$SN(C_\Phi) := \min_{\Gamma \in \Pi_\Phi} \left\{ \max_{1 \leq j \leq m} \left\{ SR(\xi_j) \mid \Gamma = \sum_{i=1}^m \xi_i \xi_i^* \in \Pi_\Phi \right\} \right\}.$$

See [TH] for more details.

In this article, our main focus will be on EB maps. The following are a few well-known characterizations of EB maps that will be used in the sequel.

Theorem 1.3. ([HSR, Theorem 4], [TH]) *Let $\Phi : B(E) \rightarrow B(H)$ be a linear map. The following statements are equivalent.*

- (i) Φ is EB.
- (ii) $\Phi(X) = \sum_{j=1}^m \phi_j(X)R_j$, where the ϕ_j 's are states defined on $B(E)$ and the R_j 's are positive operators in $B(H)$.
- (iii) $\Phi(X) = \sum_{k=1}^\ell V_k^* X V_k$, where each $V_k : H \rightarrow E$ is a linear map of rank one.
- (iv) $C_\Phi \in B(E)^+ \otimes B(H)^+$. (i.e., C_Φ is separable.)
- (v) $SN(C_\Phi) = 1$.

The formula for Φ given in statement (ii) of the above theorem is called a **Holevo form of Φ** . The **EB-rank of Φ** is defined as the minimum number of Choi-Kraus operators of rank one, required to represent Φ as in statement (iii) of Theorem 1.3. Please see [PPPR] for more details.

"Extremal" UEB maps are of utmost importance to us. Let $\mathcal{S} \subset B(E)$ be an operator system. Here we mainly consider two notions of extreme points of $UEB(\mathcal{S}, B(H))$. The UEB map $\Phi : \mathcal{S} \rightarrow B(H)$ is said to be a **linear extreme point** of $UEB(\mathcal{S}, B(H))$ if

$$\Phi = \sum_{i=1}^k t_i \Phi_i$$

for some UEB maps $\Phi_i : \mathcal{S} \rightarrow B(H)$ and $t_i \in (0, 1)$ satisfying $\sum_{i=1}^k t_i = 1$, then $\Phi_i = \Phi$ for all i . Linear extreme UCP maps between C^* -algebras were characterized by Arveson in [A69].

The notion of C^* -convexity and C^* -extreme points were first introduced and studied in [LP] and [HMP] respectively. C^* -extreme UCP maps were extensively studied in [FM93], [FM97], [FZ] and [Z]. In [BDMS], the authors consider C^* -extreme UEB maps between matrix algebras. Along the same lines, we define C^* -extreme points in $UEB(\mathcal{S}, B(H))$.

Let $\Phi_1, \Phi_2, \dots, \Phi_k \in UEB(\mathcal{S}, B(H))$. A UEB map $\Phi : \mathcal{S} \rightarrow B(H)$ is said to be a **C^* -convex combination** of the UEB maps Φ_i , if there

exists $T_1, \dots, T_k \in B(H)$ such that $\sum_{i=1}^k T_i^* T_i = I_H$ and $\Phi(X) = \sum_{i=1}^k T_i^* \Phi_i(X) T_i$ for every $X \in \mathcal{S}$. The T_i 's will be referred to as the **coefficients** of this C^* -convex combination. If the coefficients, i.e., the T_i 's are positive, then this C^* -convex combination will be called a **positive C^* -convex combination**.

A UEB map $\Phi : \mathcal{S} \rightarrow B(H)$ is a **proper C^* -convex combination** of the UEB maps Φ_i , $1 \leq i \leq k$, if there exists invertible operators $T_i \in B(H)$ such that $\sum_{i=1}^k T_i^* T_i = I_H$ and

$$\Phi(X) = \sum_{i=1}^k T_i^* \Phi_i(X) T_i, \quad (1.1)$$

for all $X \in \mathcal{S}$. This C^* -convex combination is **trivial** if each Φ_i is unitarily equivalent to Φ ; that is, there exist unitary operators U_i such that $\Phi_i(X) = U_i^* \Phi(X) U_i$. The UEB map $\Phi : \mathcal{S} \rightarrow B(H)$ is said to be a **C^* -extreme point** of $\text{UEB}(\mathcal{S}, B(H))$ if, every representation of Φ as a proper C^* -convex combination is trivial.

Remark 1.4. *Given our finite dimensionality assumptions, it is a standard observation that C^* -extreme UEB maps from $\mathcal{S} \subset B(E)$ mapping into $B(H)$ are also linear extreme UEB maps. Please see [Z, Theorem 2.2.2] for a proof of this fact for UCP maps. However, not every linear extreme UEB map is C^* -extreme, as an example in [HSR] shows. See also example 5.7 in [BDMS].*

With the above given definitions and remarks, we proceed to state our main observations and results.

1.1. EB maps - Extension and Structure. A well-known extension theorem for CP maps due to Arveson says that every CP map defined on an operator system in a C^* -algebra mapping into $B(K)$ for some Hilbert space K , has a CP extension (See [P, Theorem 7.5]). Our first main observation in this article is the following analogous extension theorem for EB maps.

Theorem 1.5. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\phi : \mathcal{S} \rightarrow B(H)$ be an EB map. There exists an EB map $\Phi : B(E) \rightarrow B(H)$ such that $\Phi|_{\mathcal{S}} = \phi$.*

A proof of Theorem 1.5 is given in Section 2. As an immediate consequence of the above theorem, one obtains the following operator system version of Theorem 1.3.

Corollary 1.6. *The equivalence of statements (i), (ii) and (iii) of Theorem 1.3 holds even for UEB maps defined on operator systems. More*

precisely, if $\mathcal{S} \subset B(E)$ is an operator system and $\Phi : \mathcal{S} \rightarrow B(H)$ is a linear map, then the following statements are equivalent.

- (i) Φ is EB.
- (ii) Φ can be written in the Holevo form, i.e., $\Phi(X) = \sum_{j=1}^m \phi_j(X)R_j$, where the ϕ_j 's are states defined on the operator system \mathcal{S} and the R_j 's are positive operators in $B(H)$.
- (iii) Φ has a Choi-Kraus decomposition with rank-one Choi-Kraus coefficients, i.e., there exists linear maps $V_k : H \rightarrow E$ of rank one such that $\Phi(X) = \sum_{k=1}^{\ell} V_k^* X V_k$, for all $X \in \mathcal{S}$.

It is a well-known result due to Stinespring ([P, Theorem 4.1]) that a UCP map defined on a C*-algebra \mathcal{A} mapping into $B(K)$ for some Hilbert space K , dilates to a representation. In this article we point out a similar structure theorem for EB maps (for our finite dimensional setting). The following characterization of an EB map in terms of a dilation, is our second main observation in this article.

Theorem 1.7. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\Phi : \mathcal{S} \rightarrow B(H)$ a linear map. The following statements are equivalent.*

- (i) Φ is a (unital) EB map.
- (ii) $\Phi(X) = V^* \Gamma(X) V$ for all $X \in \mathcal{S}$, where $V : H \rightarrow K$ is an isometry for some finite dimensional Hilbert space K and $\Gamma : \mathcal{S} \rightarrow B(K)$ is a (unital) positive map with commutative range.
- (iii) Φ is the compression of a (unital) EB map with commutative range contained in $B(K)$ for some finite dimensional Hilbert space K .

A proof of Theorem 1.7 is given in Section 3. It relies mainly on the observation that an entanglement breaking map between matrix algebras factors via the commutative C*-algebra ℓ_k^∞ for some k . See [KMP] and [JKPP] for more details. We anticipate that the above structure theorem for "EB" maps being compressions of "positive" maps with commutative range should hold for a much more general setting than is considered here (for instance, for strongly entanglement breaking maps in the infinite dimensional setting (See [LD])).

Remark 1.8. *The above theorem can also be deduced from the well-known fact that a Positive Operator Valued Measure (POVM) dilates to a Projection Valued Measure (PVM) (see [Y]), which in turn can be deduced from Naimark's dilation Theorem ([P, Theorem 4.6]).*

1.2. Maximal UEB maps. In [A] and [A72], Arveson defined the notion of a maximal UCP dilation of a given UCP map on an operator system. He also showed that maximal UCP dilations always exist and

that such maps are precisely the ones that satisfy a unique extension property. Following [A] and [A72], here we consider the UEB analog of maximal dilations (for our finite dimensional setting.) Let $\mathcal{S} \subset B(E)$ be an operator system and let $\Phi : \mathcal{S} \rightarrow B(H)$ be a UEB map. A linear map $\Psi : \mathcal{S} \rightarrow B(K)$ is said to be a **UEB dilation** of Φ if Ψ is a UEB map, K is a separable Hilbert space, and there exists an isometry $V : H \rightarrow K$ such that

$$\Phi(X) = V^*\Psi(X)V, \quad (1.2)$$

for all $X \in \mathcal{S}$. In this case we write $\Phi \leq \Psi$. Note that if H is identified with VH , then " \leq " is a partial order on the set of all UEB maps defined on \mathcal{S} . The UEB dilation Ψ of Φ is said to be **trivial** if

$$\Psi(X)V(x) = V(\Phi(X)(x)), \quad (1.3)$$

for all $X \in \mathcal{S}$ and $x \in H$. The UEB map Φ is said to be **maximal** if every UEB dilation of Φ is trivial.

Remark 1.9. *The following observations are immediate from the above definition of maximality for UEB maps.*

- (i) *Since VV^* is the projection onto $\text{Range}(V)$, it follows that the UEB dilation Ψ is trivial if and only if VH is an invariant subspace for $\Psi(X)$ for all $X \in \mathcal{S}$. Also, since \mathcal{S} is an operator system and $\Psi(X^*) = \Psi(X)^*$ for all $X \in \mathcal{S}$, it follows that the UEB dilation Ψ of Φ is trivial if and only if VH is a reducing subspace for $\Psi(X)$ for every $X \in \mathcal{S}$.*
- (ii) *Since H is finite dimensional and $V : H \rightarrow K$ is an isometry, there is no loss of generality in assuming K to be finite dimensional. (See Lemma 4.4)*

The following characterization of maximal UEB maps is one of our main results in this article.

Theorem 1.10. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\Phi : \mathcal{S} \rightarrow B(H)$ be a UEB map. The following statements are equivalent.*

- (i) *Φ is maximal.*
- (ii) *Φ has the form*

$$\Phi(X) = \sum_{i=1}^k \phi_i(X)P_i, \quad (1.4)$$

for all $X \in \mathcal{S}$, where the ϕ_i 's are distinct linear extremal states on \mathcal{S} , $k \leq n$ and the P_i 's are mutually orthogonal projections in $B(H)$ such that $\sum_{i=1}^k P_i = I_H$.

A proof of Theorem 1.10 is given in Section 4. Using Theorems 1.7 and 1.10, we also observe the following.

Theorem 1.11. *Let $\mathcal{S} \subset B(E)$ be an operator system. Every UEB map $\Phi : \mathcal{S} \rightarrow B(H)$ dilates to a maximal UEB map $\Psi : \mathcal{S} \rightarrow B(K)$ with $\dim(K) < \infty$.*

A proof of Theorem 1.11 can be found in Section 4.

1.3. C^* -extreme UEB maps. In [BDMS], the authors obtain various characterizations of C^* -extreme UEB maps between matrix algebras. A primary objective of ours is to characterize C^* -extreme UEB maps defined on finite dimensional operator systems. One of our main results in this article along these lines is the following.

Theorem 1.12. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\Phi : \mathcal{S} \rightarrow B(H)$ a UEB map. Then the following statements are equivalent.*

- (i) Φ is a maximal UEB map.
- (ii) Φ is a C^* -extreme point of $UEB(\mathcal{S}, B(H))$.
- (iii) Φ is a linear extreme point of $UEB(\mathcal{S}, B(H))$ with commutative range.

A proof of Theorem 1.12 is given in Section 5. Combined with Theorem 1.10, the equivalence (i) \iff (ii) of Theorem 1.12 is an operator system generalization of the equivalence (i) \iff (v) of [BDMS, Theorem 5.3], which says that $\Phi \in UEB(B(E), B(H))$ is C^* -extreme if and only if it has the form given in equation (1.4).

After establishing a characterization as given above, it is only natural to ask whether C^* -extreme UEB maps defined on operator systems in $B(E)$ extend to C^* -extreme UEB maps on the whole algebra. This is still only a partially answered question for C^* -extreme UCP maps. Please see [Z] for more details and some results on this problem. Here, as an application of Theorems 1.10 and 1.12, we obtain the following extension result.

Corollary 1.13. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\Phi : \mathcal{S} \rightarrow B(H)$ be a C^* -extreme UEB map. There exists a C^* -extreme UEB map $\Psi : B(E) \rightarrow B(H)$ such that $\Psi|_{\mathcal{S}} = \Phi$.*

A proof of Corollary 1.13 is given in Section 5.

In [BDMS], a Krein-Milman type theorem was established for the compact convex set $UEB(B(E), B(H))$. To be precise, it was shown

that the set $UEB(B(E), B(H))$ equals the C^* -convex hull of its C^* -extreme points. As another application of our main results, we obtain the following operator system analog of this result.

Corollary 1.14. *$UEB(\mathcal{S}, B(H))$ equals the C^* -convex hull of its C^* -extreme points.*

A proof of Corollary 1.14 is given in Section 5.

2. EXTENSIONS OF EB MAPS

In this section we give a proof of Theorem 1.5. We first include some necessary definitions adapted to our finite dimensional setting. Please see [St, Chapter 5] for more details.

Let M and L be finite dimensional Hilbert spaces. Let $\mathcal{P}(M)$ denote the cone of all positive linear maps from $B(M)$ to $B(M)$. A closed convex cone $\mathcal{C} \subset \mathcal{P}(M)$ is said to be a **mapping cone** if for each nonzero $A \in B(M)^+$, there exists a $\varphi \in \mathcal{C}$ such that $\varphi(A) \neq 0$ and

$$\phi \circ \sigma \circ \psi \in \mathcal{C},$$

for all $\sigma \in \mathcal{C}$ and CP maps $\phi, \psi : B(M) \rightarrow B(M)$. The mapping cone \mathcal{C} is said to be **symmetric** if $\phi \in \mathcal{C}$ implies both ϕ^* and $t \circ \phi \circ t$ are in \mathcal{C} , where t is the transpose operator in $B(M)$ induced by a fixed orthonormal basis of M and $\phi^* : B(M) \rightarrow B(M)$ is **the adjoint of ϕ** with respect to the Hilbert-Schmidt inner product on $B(M)$, i.e., ϕ^* is determined by

$$\langle \phi(A), B \rangle = \langle A, \phi^*(B) \rangle,$$

for all $A, B \in B(M)$, where $\langle C, D \rangle := \text{trace}(D^*C)$.

Remark 2.1. *Typical examples of mapping cones that are symmetric are $UCP(B(M))$ and $EB(B(M))$.*

Suppose that $\mathcal{S} \subset B(L)$ is an operator system and $\mathcal{C} \subset \mathcal{P}(M)$ is a mapping cone. A linear map $\phi : \mathcal{S} \rightarrow B(M)$ is said to be **\mathcal{C} -positive** if the corresponding dual functional $s_\phi : B(M) \otimes \mathcal{S} \rightarrow \mathbb{C}$ takes positive values on the cone

$$P(\mathcal{S}, \mathcal{C}) := \{Y \in B(M) \otimes \mathcal{S} : Y = Y^*, (\alpha \otimes \mathbf{i})(Y) \succeq 0 \text{ for all } \alpha \text{ in } \mathcal{C}\},$$

where \mathbf{i} is the identity map on \mathcal{S} .

Lemma 2.2. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\phi : \mathcal{S} \rightarrow B(H)$ be an EB map. If \mathcal{C} denotes the mapping cone $EB(B(H))$, then ϕ is \mathcal{C} -positive.*

Proof. Since ϕ is an EB map, it follows from [St09, Proposition 1(ii)] that the dual functional s_ϕ takes positive values on the cone K , where $K := \{X \in (B(H) \otimes \mathcal{S}) : (\omega \otimes \mathbf{i})(X) \succeq 0 \text{ for all states } \omega \text{ on } B(H)\}$.

Thus it suffices to show that $P(\mathcal{S}, \mathcal{C}) \subseteq K$. To prove this statement, let ω be a state defined on $B(H)$. Suppose also that $Y = \sum_i A_i \otimes B_i \in P(\mathcal{S}, \mathcal{C})$ and $Z \in B(H)^+$ is a fixed matrix of rank one. Define the linear map $\gamma : B(H) \rightarrow B(H)$ by $\gamma(X) = \omega(X)Z$. Observe that γ has the Holevo form (See Theorem 1.3) and hence $\gamma \in \mathcal{C}$. Since $(\alpha \otimes \mathbf{i})(Y) \succeq 0$ for all $\alpha \in \mathcal{C}$, it follows that

$$(\gamma \otimes \mathbf{i})(Y) = \sum_i \omega(A_i)Z \otimes B_i = Z \otimes \left[\sum_i \omega(A_i)B_i \right] \succeq 0$$

and therefore $\sum_i \omega(A_i)B_i = (\omega \otimes \mathbf{i})(Y) \succeq 0$. Thus $Y \in K$ and the proof is complete. \square

Lemma 2.3. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\phi : \mathcal{S} \rightarrow B(H)$ be an EB map. Given $\epsilon > 0$, there exists an EB map $\Phi : B(E) \rightarrow B(H)$ such that $\|\Phi(A) - \phi(A)\| < \epsilon\|A\|$ for all $A \in \mathcal{S}$.*

Proof. Let $\mathcal{C} = \text{EB}(B(H))$. Since $\phi : \mathcal{S} \rightarrow B(H)$ is an EB map, it follows from Lemma 2.2 that ϕ is \mathcal{C} -positive. By observing that \mathcal{C} is a symmetric mapping cone, it follows from [St, Theorem 5.1.13] that there exists a sequence $\phi_j := \sum_{i=1}^{r_j} \alpha_i^{(j)} \circ \psi_i^{(j)} \in \mathcal{C}$ with $\alpha_i^{(j)} \in \mathcal{C}$ and CP maps $\psi_i^{(j)} : \mathcal{S} \rightarrow B(H)$ such that $\phi_j \rightarrow \phi$ in norm, in $B(H)$. Note that norm topology on $B(H)$ coincides with the BW-topology due to the finite dimensionality of $B(H)$. By the Arveson extension theorem [P, Theorem 7.5], there exist CP maps $\Psi_i^{(j)} : B(E) \rightarrow B(H)$ such that $\Psi_i^{(j)}|_{\mathcal{S}} = \psi_i^{(j)}$. Define $\Phi_j := \sum_{i=1}^{r_j} \alpha_i^{(j)} \circ \Psi_i^{(j)}$. It follows that $\Phi_j : B(E) \rightarrow B(H)$ is an EB map for all j . Let $A \in \mathcal{S}$ be arbitrary. Since $\phi_j \rightarrow \phi$, it follows that there exists an $j_0 \in \mathbb{N}$ such that $\|\phi_j(A) - \phi(A)\| < \epsilon\|A\|$ for all $j \geq j_0$. Thus for all $j \geq j_0$, one obtains

$$\begin{aligned} \|\Phi_j(A) - \phi(A)\| &= \left\| \sum_{i=1}^{r_j} \left(\alpha_i^{(j)} \circ \Psi_i^{(j)} \right) (A) - \phi(A) \right\| \\ &= \left\| \sum_{i=1}^{r_j} \left(\alpha_i^{(j)} \circ \psi_i^{(j)} \right) (A) - \phi(A) \right\| \\ &= \|\phi_j(A) - \phi(A)\| \\ &< \epsilon\|A\|. \end{aligned}$$

Defining $\Phi := \Phi_{j_0}$ completes the proof. \square

Remark 2.4. Recall that $E = \mathbb{C}^d$. Due to the convexity of the mapping cone \mathcal{C} and the finite dimensionality of $B(E)$, Caratheodory's theorem [DD, Theorem 16.1.8] implies that the map ϕ_j in the above proof can be written as $\phi_j = \sum_{i=1}^r \alpha_i^{(j)} \circ \Psi_i^{(j)}$, where $r = (2d)^2 + 1$ (is independent of j).

Proof of Theorem 1.5. Let $j \in \mathbb{N}$ be arbitrary. By Lemma 2.3, there exists an EB map $\Phi_j : B(E) \rightarrow B(H)$ such that $\|\Phi_j(A) - \phi(A)\| < \frac{1}{j}\|A\|$ for all $A \in \mathcal{S}$. Note that $\|\Phi_j\| = \|\Phi_j(I_E)\|$ and

$$\begin{aligned} \|\Phi_j(I_E)\| &\leq \|\Phi_j(I_E) - \phi(I_E)\| + \|\phi(I_E)\| \\ &< \frac{1}{j} + \|\phi(I_E)\|. \end{aligned}$$

Thus the sequence $\{\Phi_j\}_{j \in \mathbb{N}}$ is bounded. Consider the set

$$\mathcal{K} := \{\Gamma : B(E) \rightarrow B(H) : \Gamma \text{ is EB and } \|\Gamma\| \leq 1 + \|\phi(I_E)\|\}.$$

Observe that \mathcal{K} is compact and $\{\Phi_j\}_{j \in \mathbb{N}}$ is a sequence in \mathcal{K} . By the compactness of \mathcal{K} , there exists a subsequence $\{\Phi_{j_k}\}_{k \in \mathbb{N}}$ of $\{\Phi_j\}_{j \in \mathbb{N}}$ and an EB map $\Phi \in \mathcal{K}$ such that $\Phi_{j_k} \rightarrow \Phi$ as $k \rightarrow \infty$. The proof is complete by observing that $\Phi|_{\mathcal{S}} = \phi$. □

Remark 2.5. It is to be noted that Theorem 1.5 does not necessarily follow from the extension theorem [St, Theorem 5.2.3]. It was remarked in [St18] that the conclusion of [St, Theorem 5.2.3] holds only for Real operator systems and that this is due to the fact that Krein's Extension Theorem ([St, Theorem A.3.1]) which is formulated only for Real spaces, was incorrectly applied to Complex spaces, while proving [St, Theorem 5.2.3].

3. DILATIONS OF UEB MAPS

In this section, we prove Theorem 1.7. The proof relies on a crucial observation from [RJP, Lemma 3.1]. We begin with the following remark.

Remark 3.1. Let $\Psi : B(E) \rightarrow B(H)$ be a (not necessarily unital) positive map with commutative range. The argument in the proof of [RJP, Lemma 3.1] can easily be adapted to conclude that Ψ is EB.

Lemma 3.2. Let K be a finite dimensional Hilbert space and $\mathcal{S} \subset B(E)$ be an operator system. If $\Gamma : \mathcal{S} \rightarrow B(K)$ is a positive map with commutative range, then Γ is EB.

Proof. By [HJRW, Corollary 2], there exists a commutative C^* -algebra $\mathcal{B} \subset B(K)$ containing $\text{Range}(\Gamma)$ and a CP map $\Theta : B(E) \rightarrow \mathcal{B} \subset B(K)$ such that $\Theta|_{\mathcal{S}} = \Gamma$. It follows from Remark 3.1 that Θ is EB. Since Γ is the restriction of the EB map Θ to \mathcal{S} , it is also EB. \square

Lemma 3.3. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\Phi : \mathcal{S} \rightarrow B(H)$ be an EB map. There exists a commutative C^* -algebra \mathcal{A} , a finite dimensional Hilbert space K , an isometry $V : H \rightarrow K$, a unital $*$ -algebra homomorphism $\pi : \mathcal{A} \rightarrow B(K)$ and a positive map $\eta : \mathcal{S} \rightarrow \mathcal{A}$ such that $\Phi(X) = V^*(\pi \circ \eta)(X)V$, for all $X \in \mathcal{S}$.*

Proof. By Theorem 1.5, there exists an EB map $\Psi : B(E) \rightarrow B(H)$ such that $\Psi|_{\mathcal{S}} = \Phi$. It follows from Theorem 1.3 and Corollary 1.6, Φ can be written in Holevo form, i.e., $\Phi(X) = \sum_{j=1}^m \phi_j(X)R_j$ for all $X \in \mathcal{S}$, where the ϕ_j 's are states on \mathcal{S} and the R_j 's are positive operators in $B(H)$. Without loss of generality, assume that $\|\Phi\| \leq 1$.

Case(i) - Φ is non-unital: Let $\mathcal{A} = \ell_{m+1}^\infty$. Define the linear maps $\gamma : \mathcal{A} \rightarrow B(H)$ and $\eta : \mathcal{S} \rightarrow \mathcal{A}$ by $\gamma(x_1, \dots, x_{m+1}) = \sum_{j=1}^{m+1} x_j R_j$ and $\eta(X) = (\phi_1(X), \dots, \phi_m(X), 0)$, where $R_{m+1} = I_H - \sum_{j=1}^m R_j = I_H - \Phi(I_E) \in B(H)^+$. Note that γ and η are positive maps with γ also being unital. In fact, by [P, Theorem 3.9], γ is a UCP map since the domain of γ is a commutative C^* -algebra. By Stinespring's dilation theorem [P, Theorem 4.1], there exists a finite dimensional Hilbert space K , an isometry $V : H \rightarrow K$ and a unital $*$ -algebra homomorphism $\pi : \mathcal{A} \rightarrow B(K)$ such that $V^*\pi(\cdot)V = \gamma(\cdot)$.

Case(ii) - Φ is unital: Let $\mathcal{A} = \ell_m^\infty$. Define the linear maps $\gamma : \mathcal{A} \rightarrow B(H)$ and $\eta : \mathcal{S} \rightarrow \mathcal{A}$ by $\gamma(x_1, \dots, x_m) = \sum_{j=1}^m x_j R_j$ and $\eta(X) = (\phi_1(X), \dots, \phi_m(X))$. Note that both γ and η are unital positive maps. Arguing as above, one obtains a finite dimensional Hilbert space K , an isometry $V : H \rightarrow K$ and a unital $*$ -algebra homomorphism $\pi : \mathcal{A} \rightarrow B(K)$ such that $V^*\pi(\cdot)V = \gamma(\cdot)$.

In both cases above, observe that $\Phi = \gamma \circ \eta$. It follows that $\Phi(X) = V^*(\pi \circ \eta)(X)V$, for all $X \in \mathcal{S}$. \square

Proof of Theorem 1.7. To prove (i) \implies (ii), observe that by Lemma 3.3, there exists a commutative C^* -algebra \mathcal{A} , a finite dimensional Hilbert space K , an isometry $V : H \rightarrow K$, a unital $*$ -algebra homomorphism $\pi : \mathcal{A} \rightarrow B(K)$ and a positive map $\eta : \mathcal{S} \rightarrow \mathcal{A}$ such that $\Phi(X) = V^*(\pi \circ \eta)(X)V$ for all $X \in \mathcal{S}$. Define $\Gamma := \pi \circ \eta$. Since π is a $*$ -algebra homomorphism and η is a positive map with commutative range, it follows that Γ is a positive map with commutative range that dilates Φ . Finally, observe that if Φ is unital, then Γ is too.

The implication (ii) \implies (iii) follows directly from Lemma 3.2.

To prove (iii) \implies (i), let $\Phi(X) = V^*\Gamma(X)V$ for all $X \in \mathcal{S}$, where $V : H \rightarrow K$ is an isometry for some finite dimensional Hilbert space K and $\Gamma : \mathcal{S} \rightarrow B(K)$ is an EB map with commutative range. It follows from Theorem 1.3 and Corollary 1.6 that $\Gamma(X) = \sum_{j=1}^{\ell} \psi_j(X)R_j$ for some states ψ_j on \mathcal{S} and positive operators R_j in $B(K)$. Observe that for each $X \in \mathcal{S}$,

$$\Phi(X) = V^* \left(\sum_{j=1}^{\ell} \psi_j(X)R_j \right) V = \sum_{j=1}^{\ell} \psi_j(X)(V^*R_jV),$$

which is again a map in the Holevo form and hence is EB. Finally, if Γ is unital, then Φ is too. \square

4. MAXIMAL UEB DILATIONS

In this section we prove Theorems 1.10 and 1.11. The proof of Theorem 1.10 makes use of the following Lemmas, which contain some key properties of maximal UEB maps defined on operator systems.

Lemma 4.1. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\Phi : \mathcal{S} \rightarrow B(H)$ be a UEB map. If Φ has commutative range, then*

$$\Phi(X) = \sum_{i=1}^k \phi_i(X)P_i, \quad (4.1)$$

where the ϕ_i 's are distinct states on \mathcal{S} and the P_i 's are mutually orthogonal projections in $B(H)$ such that $\sum_{i=1}^k P_i = I_H$.

Proof. Let $\mathcal{A} = C^*(\text{Range}(\Phi)) \subset B(H)$. There exists a unital $*$ -algebra isomorphism $\pi : \mathcal{A} \rightarrow \ell_k^\infty$ for some k . Consider $\pi \circ \Phi : \mathcal{S} \rightarrow \ell_k^\infty$. For each $X \in \mathcal{S}$, $\pi \circ \Phi(X) = \sum_{i=1}^k \lambda_{i,X}e_i$, where $\lambda_{i,X}$'s are scalars (depending on X) and $\{e_1, \dots, e_k\}$ is the standard basis of ℓ_k^∞ . For $1 \leq i \leq k$, define $\phi_i : \mathcal{S} \rightarrow \mathbb{C}$ via $\phi_i(X) = \lambda_{i,X}$. Since $\pi \circ \Phi$ is a unital positive map, it follows that ϕ_i 's are states on \mathcal{S} . Indeed

$$\Phi(X) = \sum_{i=1}^k \phi_i(X)P_i,$$

where $P_i = \pi^{-1}(\{e_i\})$. Note that the P_i 's are mutually orthogonal projections in $B(H)$ such that $\sum_{i=1}^k P_i = I_H$. Without loss of generality, one can assume that the ϕ_i 's are distinct states on \mathcal{S} because if two ϕ_i 's are the same, we can sum the corresponding projections, i.e., the P_i 's together, and obtain a single projection. \square

Lemma 4.2. *Let $\mathcal{S} \subset B(E)$ be an operator system. If $\Phi : \mathcal{S} \rightarrow B(H)$ is a maximal UEB map, then $\text{Range}(\Phi)$ is commutative.*

Proof. Since Φ is a UEB map, it follows from part (iii) of Theorem 1.7 that there exists a finite dimensional Hilbert space K , a UEB map $\Gamma : \mathcal{S} \rightarrow B(K)$ with commutative range and an isometry $V : H \rightarrow K$ such that $V^*\Gamma(X)V = \Phi(X)$, for all $X \in \mathcal{S}$. Since Φ is a maximal UEB map, Γ is a trivial UEB dilation of Φ . Thus it follows from Remark 1.9 that the subspace VH is an invariant subspace for $\Gamma(X)$ for all $X \in \mathcal{S}$. Since VV^* is the projection of K onto $\text{Range}(V)$, it follows that

$$\begin{aligned} \Phi(X)\Phi(Y)(a) &= V^*\Gamma(X)V V^*\Gamma(Y)V(a) \\ &= V^*\Gamma(X)\Gamma(Y)V(a) \\ &= V^*\Gamma(Y)\Gamma(X)V(a) \\ &= V^*\Gamma(Y)V V^*\Gamma(X)V(a) \\ &= \Phi(Y)\Phi(X)(a), \quad \text{for all } a \in H. \quad \square \end{aligned}$$

Lemma 4.3. *Let $\mathcal{S} \subset B(E)$ be an operator system and let $\Phi : \mathcal{S} \rightarrow B(H)$ be a UEB map defined by $\Phi(X) = \sum_{i=1}^k \phi_i(X)P_i$, where ϕ_i are states defined on \mathcal{S} and $P_i \in B(H)$ are mutually orthogonal projections satisfying $\sum_{i=1}^k P_i = I_H$. If Φ is a maximal UEB map, then each ϕ_i is a linear extremal state.*

Proof. Fix $i \in \{1, 2, \dots, k\}$. Suppose that σ, τ are states on \mathcal{S} such that

$$\phi_i = t\sigma + (1-t)\tau,$$

for some $t \in (0, 1)$. It suffices to show that $\sigma = \tau$. Let $F = \text{Range}(P_i) \subset H$ and $G = F^\perp \subset H$. Let $L := F \oplus F \oplus G$. Define $V : H \rightarrow L$ by

$$V(x+y) = (\sqrt{t}x, \sqrt{(1-t)}x, y),$$

for all $x \in F$ and $y \in G$. It is easily seen that V is an isometry. Let $\Psi : \mathcal{S} \rightarrow B(L)$ be defined by

$$\Psi(X)(x, y, z) = \left(\sigma(X)x, \tau(X)y, \sum_{j \neq i} \phi_j(X)P_j(z) \right),$$

for all $X \in \mathcal{S}, x, y \in F$ and $z \in G$. Observe that Ψ is unital. Define the coordinate projections Q_1, Q_2, Q_3 on L by $Q_1(x, y, z) = (x, 0, 0)$, $Q_2(x, y, z) = (0, y, 0)$ and $Q_3(x, y, z) = (0, 0, z)$. One sees that $\Psi(X) = \sigma(X)Q_1 + \tau(X)Q_2 + \left(\sum_{j \neq i} \phi_j(X)P_j \right) Q_3$, which is in Holevo form for Ψ . By Corollary 1.6, it follows that Ψ is a UEB map. Observe

that

$$\begin{aligned}
\langle V^*(u, v, w), (x + y) \rangle &= \langle (u, v, w), V(x + y) \rangle \\
&= \langle (u, v, w), (\sqrt{t}x, \sqrt{(1-t)}x, y) \rangle \\
&= \sqrt{t}\langle u, x \rangle + \sqrt{(1-t)}\langle v, x \rangle + \langle w, y \rangle \\
&= \langle \sqrt{t}u + \sqrt{(1-t)}v, x \rangle + \langle w, y \rangle \\
&= \langle \sqrt{t}u + \sqrt{(1-t)}v + w, x + y \rangle,
\end{aligned}$$

for all $u, v, x \in F$ and $w, y \in G$. Thus V^* is defined by

$$V^*(u, v, w) := \sqrt{t}u + \sqrt{(1-t)}v + w.$$

For each $X \in \mathcal{S}$, $x \in F$ and $y \in G$, it follows that

$$\begin{aligned}
V^*\Psi(X)V(x + y) &= V^*\Psi(X)(\sqrt{t}x, \sqrt{(1-t)}x, y) \\
&= V^*\left(\sqrt{t}\sigma(X)x, \sqrt{(1-t)}\tau(X)x, \sum_{j \neq i} \phi_j(X)P_j(y)\right) \\
&= t\sigma(X)x + (1-t)\tau(X)x + \sum_{j \neq i} \phi_j(X)P_j(y) \\
&= \phi_i(X)x + \sum_{j \neq i} \phi_j(X)P_j(y) \\
&= \sum_{j=1}^k \phi_j(X)P_j(x + y) \\
&= \Phi(X)(x + y).
\end{aligned}$$

Hence Ψ is a UEB dilation of Φ . By hypothesis, $\Psi(X)VH \subset VH$ for all $X \in \mathcal{S}$. Choosing $0 \neq x \in F$, it follows that

$$\begin{aligned}
(\sqrt{t}\sigma(X)x, \sqrt{(1-t)}\tau(X)x, 0) &= \Psi(X)V(x) \\
&= V(z + w) = (\sqrt{t}z, \sqrt{(1-t)}z, w)
\end{aligned}$$

for some $z \in F$ and $w \in G$. Hence $\sigma(X)x = \tau(X)x$ for each $X \in \mathcal{S}$. This in turn implies that $\sigma = \tau$ and the proof is complete. \square

Lemma 4.4. *Let $\mathcal{S} \subset B(E)$ be an operator system and let $\Phi : \mathcal{S} \rightarrow B(H)$ be a UEB map. If every UEB dilation $\Psi : \mathcal{S} \rightarrow B(L)$ of Φ is trivial whenever L is finite dimensional, then Φ is a maximal UEB map.*

Proof. Let $\Psi : \mathcal{S} \rightarrow B(K)$ be a UEB dilation of Φ where K is an infinite dimensional separable Hilbert space. Let the isometry $V : H \rightarrow K$

be such that $V^*\Psi(X)V = \Phi(X)$ for all $X \in \mathcal{S}$. Let $\{e_1, \dots, e_n\}$ denote an orthonormal basis for VH . Extend it to an orthonormal basis $\{e_i : i \in \mathbb{N}\}$ for the Hilbert space K . For each $m \geq n$, let $K_m := \text{span}\{e_1, \dots, e_m\}$, $W_m : K_m \rightarrow K$ denote the inclusion map and P_m denote the orthogonal projection of K onto K_m . Observe that $P_m = W_m^*$. Define $V_m = P_m \circ V$ and $\Psi_m : \mathcal{S} \rightarrow B(K_m)$ to be the compression of Ψ to K_m , i.e., $\Psi_m(X) = P_m\Psi(X)|_{K_m} = W_m^*\Psi(X)W_m$. Let $k \in \mathbb{N}$ and $Z \in (M_k \otimes \mathcal{S})^+$. Since Ψ is an EB map, $(\mathbf{i}_k \otimes \Psi)(Z) \in \overline{M_k^+ \otimes B(K_m)^+}$. It follows that $(\mathbf{i}_k \otimes \Psi_m)(Z) = (\mathbf{i}_k \otimes W_m)^*(\mathbf{i}_k \otimes \Psi)(Z)(\mathbf{i}_k \otimes W_m) \in \overline{M_k^+ \otimes B(K_m)^+} = M_k^+ \otimes B(K_m)^+$. Thus $\Psi_m : \mathcal{S} \rightarrow B(K_m)$ is a UEB map. Moreover, V_m is an isometry and $V_m^*\Psi_m(X)V_m = \Phi(X)$ for all $X \in \mathcal{S}$, i.e., Ψ_m is a UEB dilation of Φ . Let $a \in H$ and $X \in \mathcal{S}$. Take $V_m = P_m \circ V$, where V is defined as above. Since K_m is finite dimensional, by hypothesis, it follows that Ψ_m is a trivial UEB dilation of Φ . By Remark 1.9, one gets that $\Psi_m(X)V_mH \subset V_mH = VH$. Finally, for $x \in H$,

$$\begin{aligned} \Psi(X)(Vx) &= \lim_{m \rightarrow \infty} P_m\Psi(X)P_m(Vx) = \lim_{m \rightarrow \infty} W_m^*\Psi(X)W_mP_m(Vx) \\ &= \lim_{m \rightarrow \infty} \Psi_m(X)V_m(x) \in VH. \end{aligned} \quad \square$$

Remark 4.5. *It is to be noted that the definition of an EB map adopted here in terms of separability in the norm-closure sense, is not the usual way it is defined in the literature. If one works with a weak*-continuous CP map (or equivalently, a normal CP map), say $\Phi : B(L) \rightarrow B(M)$, then there is a way of defining an EB map that is along the lines of the usual definition namely: $\Phi : B(L) \rightarrow B(M)$ is EB if for every $k \in \mathbb{N}$, $\Psi : \mathfrak{T}(M) \rightarrow \mathfrak{T}(L)$ maps positive matrices in $M_k \otimes \mathfrak{T}(M)$ to separable matrices in $M_k \otimes \mathfrak{T}(L)$, where $\mathfrak{T}(L)$ denotes the space of trace-class operators on the Hilbert space L , and Ψ is the unique map whose (Banach space) adjoint Ψ^* equals Φ . This certainly is a generalization of the usual definition of an EB map from the finite to the infinite dimensional setting. But since here we mainly work on unital maps defined only on operator systems, defining the "entanglement breaking" property using this type of duality becomes a challenge.*

We would like to emphasize that the above remark is only relevant when either of the Hilbert spaces L or M is infinite dimensional and so does not impact the results here.

Proof of Theorem 1.10. (i) \implies (ii): Since Φ is a maximal UEB map, it follows from Lemma 4.2 that Φ has commutative range. By applying Lemma 4.1, one gets that $\Phi(X) = \sum_{i=1}^k \phi_i(X)P_i$, where the ϕ_i 's are distinct states defined on \mathcal{S} and $P_i \in B(H)$ are mutually orthogonal

projections satisfying $\sum_{i=1}^k P_i = I_H$. That the ϕ_i 's are linear extremal states on \mathcal{S} follows from an application of Lemma 4.3.

To prove the implication (ii) \implies (i), let Ψ be an arbitrary UEB dilation of Φ ; that is, $\Psi : \mathcal{S} \rightarrow B(K)$ is a UEB map such that there exists an isometry $V : H \rightarrow K$ satisfying $\Phi(X) = V^*\Psi(X)V$, for all $X \in \mathcal{S}$. By Lemma 4.4, one can assume that K is finite dimensional. It suffices to show that $\Psi(X)Vx \in VH$ for all $x \in H$ and $X \in \mathcal{S}$. If $x = 0$, then there is nothing to prove. Suppose that $x \neq 0$. We first consider the case $x \in \text{Range}(P_j)$ for some j . For such an x , it follows that $\Phi(X)x = \phi_j(X)x$ and hence $V^*\Psi(X)Vx = \phi_j(X)x$.

Let a Holevo form (see Theorem 1.3 and Corollary 1.6) for Ψ be given by $\Psi(X) = \sum_{i=1}^k \psi_i(X)R_i$, where the ψ_i 's are distinct states and the R_i 's are positive matrices with $\sum_{i=1}^k R_i = I_K$. Observe that

$$\begin{aligned} \phi_j(X)\|x\|^2 &= \langle \phi_j(X)x, x \rangle = \langle \Phi(X)x, x \rangle = \left\langle \left(\sum_{i=1}^k V^*\psi_i(X)R_iV \right) x, x \right\rangle \\ &= \sum_{i=1}^k \psi_i(X) \langle R_iVx, Vx \rangle. \end{aligned} \quad (4.2)$$

Thus, $\phi_j(X) = \sum_{i=1}^k \alpha_i \psi_i(X)$, where $\alpha_i = \frac{1}{\|x\|^2} \langle R_iVx, Vx \rangle \geq 0$. Note that α_i 's are independent of X . Also, since V is an isometry and $\sum_{i=1}^k R_i = I_K$, it follows that

$$\sum_{i=1}^k \alpha_i = \frac{1}{\|x\|^2} \sum_{i=1}^k \langle R_iVx, Vx \rangle = \frac{1}{\|x\|^2} \left\langle \left(\sum_{i=1}^k R_i \right) Vx, Vx \right\rangle = 1. \quad (4.3)$$

Combining equations (4.2) and (4.3), it follows that ϕ_j is a convex combination of the ψ_i 's. Since the ψ_i 's are distinct and ϕ_j is a linear extremal state, it follows that there exists an index i_0 such that $\alpha_{i_0} = 1$, $\phi_j = \psi_{i_0}$ and $\alpha_i = \frac{1}{\|x\|^2} \langle R_iVx, Vx \rangle = 0$ for all $i \neq i_0$. Since the R_i 's are positive, it follows that $R_iVx = 0$ for all $i \neq i_0$, implying $Vx = \sum_{i=1}^k R_iVx = R_{i_0}Vx$. Thus

$$\Psi(X)Vx = \psi_{i_0}(X)R_{i_0}Vx = \psi_{i_0}(X)Vx \in VH.$$

Now using the fact that the P_i 's are mutually orthogonal projections satisfying $\sum_{i=1}^k P_i = I_H$ and $\bigoplus_{i=1}^k \text{Range}(P_i) = H$, it follows that $\Psi(X)Vx \in V(H)$ for all $x \in H$, and the proof is complete. \square

Our next task is to prove Theorem 1.11, before which we prove the following crucial observation concerning a Holevo form of a UEB map.

Lemma 4.6. *Let $\mathcal{S} \subset B(E)$ be an operator system. If $\Phi : \mathcal{S} \rightarrow B(H)$ is a UEB map, then $\Phi(X) = \sum_{j=1}^k \phi_j(X)S_j$ for each $X \in \mathcal{S}$, where ϕ_j are distinct linear extremal states on \mathcal{S} and S_j are positive operators in $B(H)$ satisfying $\sum_{j=1}^k S_j = I_H$.*

Proof. Let a Holevo form for Φ be given by

$$\Phi(X) = \sum_{i=1}^r \phi_i(X)R_i,$$

where the ϕ_i 's are states on \mathcal{S} and the R_i 's are positive operators in $B(H)$ such that $\sum_{i=1}^r R_i = I_H$. Observe that the set of all states on \mathcal{S} is a compact and convex subset of the dual of \mathcal{S} . A well known consequence of Caratheodory's theorem (See [DD, Theorem 16.1.8] and [DD, Corollary 16.1.9]) and the Krein-Milman theorem is that in a finite dimensional topological vector space, a compact convex set \mathcal{C} equals the convex hull of its extreme points. Using this fact here, each ϕ_i can be written as a convex combination of linear extremal states, i.e.,

$$\phi_i(X) = t_{i,1}\varphi_{i,1}(X) + \cdots + t_{i,\ell_i}\varphi_{i,\ell_i}(X),$$

where $t_{i,j} \in [0, 1]$ with $\sum_{j=1}^{\ell_i} t_{i,j} = 1$ and the $\varphi_{i,j}$'s are linear extremal states on \mathcal{S} . It follows that

$$\Phi(X) = \sum_{i=1}^r \sum_{j=1}^{\ell_i} t_{i,j}\varphi_{i,j}(X)R_i = \sum_{i=1}^r \sum_{j=1}^{\ell_i} \varphi_{i,j}(X)(t_{i,j}R_i),$$

for all $X \in \mathcal{S}$. Note that by combining suitable terms in the above sum, it can be rewritten in the desired form. This completes the proof. \square

Proof of Theorem 1.11. Using Theorem 1.5 and the Holevo form (see Theorem 1.3 and Corollary 1.6) for EB maps, we have

$$\Phi(X) = \sum_{i=1}^r \phi_i(X)R_i, \tag{4.4}$$

for all $X \in \mathcal{S}$, where the ϕ_i 's are distinct states on \mathcal{S} and the R_i 's are positive operators in $B(H)$ such that $\sum_{i=1}^r R_i = I_H$. By Lemma 4.6, there is no loss of generality in assuming that the ϕ_i 's in equation (4.4) are distinct linear extremal states on \mathcal{S} . By Lemma 3.3, there exists a finite dimensional Hilbert space K , an isometry $V : H \rightarrow K$, $r \in \mathbb{N}$, a unital $*$ -algebra homomorphism $\pi : \ell_r^\infty \rightarrow B(K)$ and a positive map $\eta : \mathcal{S} \rightarrow \ell_r^\infty$ such that

$$\Phi(X) = V^*(\pi \circ \eta)(X)V,$$

for all $X \in \mathcal{S}$. As observed in the proof of Lemma 3.3, the unitality of Φ implies the unitality of η . Note that $\pi(x_1, \dots, x_r) = \sum_{i=1}^r x_i \pi(e_i)$, where $\{e_1, \dots, e_r\}$ is the standard basis of ℓ_r^∞ . Also $(\pi \circ \eta)(X) = \pi(\sum_{i=1}^r \phi_i(X) e_i) = \sum_{i=1}^r \phi_i(X) \pi(e_i)$, for all $X \in \mathcal{S}$. Since π is a unital $*$ -algebra homomorphism, it follows that the $\pi(e_i)$'s are mutually orthogonal projections and $\sum_{i=1}^r \pi(e_i) = I_K$. Define $\Psi := \pi \circ \eta$. It follows from Theorem 1.10 that Ψ is maximal. This completes the proof. \square

5. C^* -EXTREME UEB MAPS ON OPERATOR SYSTEMS

In this section we prove Theorem 1.12 and Corollary 1.13. The proofs use techniques from [FM93], [FM97], [FZ] and [Z]. We also need the following Lemmas, the first of which contains an equivalent definition of a C^* -extreme UEB map and the second one contains a description of $\text{UEB}(\mathcal{S}, B(H))$ in terms of C^* -convex combinations.

Lemma 5.1. *Let $\mathcal{S} \subset B(E)$ be an operator system and $\Phi : \mathcal{S} \rightarrow B(H)$ be a UEB map. The following statements are equivalent.*

- (i) Φ is a C^* -extreme point of $\text{UEB}(\mathcal{S}, B(H))$.
- (ii) If $\Phi = \sum_{i=1}^2 T_i^* \Phi_i T_i$, for some invertible operators $T_1, T_2 \in B(H)$ satisfying $T_1^* T_1 + T_2^* T_2 = I_H$, then there exist unitaries $U_1, U_2 \in B(H)$ such that $\Phi_i(X) = U_i^* \Phi(X) U_i$, $i = 1, 2$.

Lemma 5.1 is essentially the operator system version of [BDMS, Proposition 3.2]. The proof given there works equally well for the operator system setting too.

Lemma 5.2. *Let $\mathcal{S} \subset B(E)$ be an operator system, \mathcal{K} denote the set $\{\Psi : \mathcal{S} \rightarrow B(H) : \Psi(X) = g(X) I_H, g \text{ is a linear extremal state on } \mathcal{S}\}$, \mathcal{E} denote the C^* -convex hull of \mathcal{K} , i.e., the set of all C^* -convex combinations of elements of \mathcal{K} and \mathcal{E}_+ denote the set of all positive C^* -convex combinations of elements of \mathcal{K} . The following statements hold.*

- (i) $\mathcal{E}_+ = \mathcal{E} = \text{UEB}(\mathcal{S}, B(H))$.
- (ii) If $\Gamma \in \mathcal{E}$ is given by $\Gamma(X) = \sum_{i=1}^m T_i^* (g_i(X) I_H) T_i \in \mathcal{E}$, then $\Gamma \in \mathcal{E}_+$ and there exists $B_i \in B(H)^+$ such that $\sum_{i=1}^m B_i^2 = I_H$ and

$$\Gamma(X) = \sum_{i=1}^m B_i (g_i(X) I_H) B_i,$$

for all $X \in \mathcal{S}$.

Proof. Evidently, $\mathcal{E}_+ \subset \mathcal{E} \subset \text{UEB}(\mathcal{S}, B(H))$. To complete the proof of part (i), it suffices to show that $\text{UEB}(\mathcal{S}, B(H)) \subset \mathcal{E}_+$. To this end, let

$\Phi \in \text{UEB}(\mathcal{S}, B(H))$. By Lemma 4.6, it follows that for each $X \in \mathcal{S}$, $\Phi(X) = \sum_{j=1}^k \phi_j(X)S_j$, where ϕ_j are distinct linear extremal states on \mathcal{S} and S_j are positive operators in $B(H)$ satisfying $\sum_{j=1}^k S_j = I_H$. Rewrite

$$\Phi(X) = \sum_{j=1}^k \phi_j(X)S_j = \sum_{j=1}^k \sqrt{S_j}(\phi_j(X)I_H)\sqrt{S_j}.$$

Thus $\Phi \in \mathcal{E}_+$. To prove part (ii), write T_i in its polar decomposition, i.e., $T_i = U_i B_i$, where U_i is a unitary operator and $B_i = \sqrt{T_i^* T_i} \in B(H)^+$. Note that $\sum_{i=1}^m B_i^2 = \sum_{i=1}^m T_i^* T_i = I_H$. By part (i), it follows that $\Gamma \in \mathcal{E}_+$. Also, for each $X \in \mathcal{S}$,

$$\begin{aligned} \Gamma(X) &= \sum_{i=1}^m T_i^*(g_i(X)I_H)T_i = \sum_{i=1}^m B_i U_i^*(g_i(X)I_H)U_i B_i \\ &= \sum_{i=1}^m B_i(g_i(X)I_H)B_i. \end{aligned} \quad \square$$

Proof of Theorem 1.12. (i) \Rightarrow (ii) : Using the alternate definition of a C^* -extreme UEB map given in Lemma 5.1, let $\Phi(X) = T_1^* \Phi_1(X)T_1 + T_2^* \Phi_2(X)T_2$ for all $X \in \mathcal{S}$, where $\Phi_1, \Phi_2 : \mathcal{S} \rightarrow B(H)$ are UEB maps and $T_1, T_2 \in B(H)$ are invertible operators such that $\sum_{i=1}^2 T_i^* T_i = I_H$. It suffices to show that Φ_1 and Φ_2 are unitarily equivalent to Φ .

Define an isometry $V : H \rightarrow H \oplus H$ via $V(x) = (T_1(x), T_2(x))$ and a linear map $\Psi : \mathcal{S} \rightarrow B(H \oplus H)$ by

$$\Psi(X) = \begin{bmatrix} \Phi_1(X) & 0 \\ 0 & \Phi_2(X) \end{bmatrix},$$

for all $X \in \mathcal{S}$. Observe that Ψ being a direct sum of UEB maps, is a UEB map. In fact, it is a UEB dilation of Φ since $V^* \Psi(X)V = \Phi(X)$ for all $X \in \mathcal{S}$. By hypothesis, it follows that Ψ is a trivial UEB dilation of Φ , that is, $\Psi(X)V = V\Phi(X)$ for all $X \in \mathcal{S}$. Equivalently,

$$\begin{bmatrix} \Phi_1(X) & 0 \\ 0 & \Phi_2(X) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \Phi(X)$$

for all $X \in \mathcal{S}$. Consequently, we have

$$\Phi_1(X) = T_1 \Phi(X) T_1^{-1} \quad \text{and} \quad \Phi_2(X) = T_2 \Phi(X) T_2^{-1}$$

for all $X \in \mathcal{S}$.

Using the polar decomposition of the invertible operator T_i^* , $i = 1, 2$, there exists a unitary operator $W_i \in B(H)$ and a positive invertible

operator $P_i \in B(H)$ given by $\sqrt{T_i T_i^*}$ such that $T_i = P_i W_i$, $i = 1, 2$. Hence

$$\Phi(X) = W_i^* P_i^{-1} \Phi_i(X) P_i W_i, \quad (5.1)$$

for $i = 1, 2$. Since Ψ is a trivial UEB dilation of Φ , it follows that

$$\Psi(X) V V^* = V V^* \Psi(X), \quad (5.2)$$

for all $X \in \mathcal{S}$, (See Remark 1.9). Applying the definition of Ψ and $V V^*$ in Equation (5.2), one obtains

$$\Phi_i(X) T_i T_i^* = T_i T_i^* \Phi_i(X),$$

for $i = 1, 2$. Note that $T_i T_i^*$ is a positive operator and $\Phi_i(X)$ commutes with $T_i T_i^*$. Hence

$$\Phi_i(X) P_i = P_i \Phi_i(X), \quad (5.3)$$

for each $X \in \mathcal{S}$ and $i = 1, 2$.

It follows from equations (5.1) and (5.3) that

$$\Phi(X) = W_i^* P_i^{-1} P_i \Phi_i(X) W_i = W_i^* \Phi_i(X) W_i,$$

for each $X \in \mathcal{S}$ and $i = 1, 2$. Since W_1 and W_2 are unitaries, the proof is complete.

The following proof of the implication (ii) \Rightarrow (i) is adapted from [FM93, Theorem 4.1]. Recall the notations \mathcal{K} , \mathcal{E} and \mathcal{E}_+ from Lemma 5.2. Let $\Phi : \mathcal{S} \rightarrow B(H)$ be a C^* -extreme UEB map. Using Lemma 5.2, write $\Phi(X) = \sum_{i=1}^m T_i^* \Psi_i(X) T_i$, where $\Psi_i(X) = g_i(X) I_H$ for all $X \in \mathcal{S}$, the g_i 's are linear extremal states on \mathcal{S} , and $\sum_{i=1}^m T_i^* T_i = I_H$. By part (ii) of Lemma 5.2, there is no loss of generality in assuming that the T_i 's are positive. Let m be the least number of coefficients required to represent Φ as a positive C^* -convex combination, i.e., as a C^* -convex combination of elements of \mathcal{K} with positive coefficients (or equivalently to represent Φ as an element of \mathcal{E}_+ .) If $m = 1$, then there is nothing to prove, due to Theorem 1.10. So assume $m \geq 2$. Note that there must exist an i such that $\|T_i\| = 1$. Otherwise $\|T_i\| < 1$ for all i and, by modifying Technique-A in [FM93] to our current setting, one can rewrite Φ as a proper C^* -convex combination of some $\Gamma_j \in \mathcal{E}$ with the additional property that each Γ_j is a C^* -convex combination of less than m of the Ψ_i 's. Since Φ is C^* -extreme, each Γ_j is unitarily equivalent to Φ . An application of part (ii) of Lemma 5.2 implies that Φ is a positive C^* -convex combination of fewer than m of the Ψ_i 's, which contradicts the minimality of m . Thus at least one of the T_i 's has unit norm. Without loss of generality, assume that $\|T_1\| = 1$. Due to the unitality of Φ , note also that $\sum_{i=1}^m T_i^2 = I_H$. Since $T_1 \geq 0$ and

$\|T_1\| = 1$, it follows that there exists a unitary $U \in B(H)$ such that

$$U^*T_1U = \begin{pmatrix} I & 0 \\ 0 & Y_1 \end{pmatrix}$$

where Y_1 is a diagonal matrix satisfying $Y_1 \geq 0$, $\|Y_1\| < 1$ and I is the identity operator of suitable size. Since $T_i \geq 0$ for all $i \geq 2$ and $\sum_{i=1}^m T_i^2 = I_H$,

$$U^*T_iU = \begin{pmatrix} 0 & \\ & Y_i \end{pmatrix}, \text{ with } Y_i \geq 0 \text{ for all } i \geq 2.$$

Let $W_i = U^*T_iU$ for each $i \in \{1, 2, \dots, m\}$. Observe that $W_i \geq 0$ and

$$U^*\Phi(X)U = \sum_{i=1}^m W_i(U^*\Psi_i(X)U)W_i = \sum_{i=1}^m W_i\Psi_i(X)W_i. \quad (5.4)$$

Since $\sum_{i=1}^m W_i^2 = I_H$, it follows that $\sum_{i=1}^m Y_i^2 = I$. Since $\|Y_1\| < 1$, $I - Y_1^2$ invertible. Also since

$$\sum_{i \geq 2} \begin{pmatrix} 0 & 0 \\ 0 & Y_i^2 \end{pmatrix} = \sum_{i \geq 2} W_i^2 = I_H - W_1^2 = \begin{pmatrix} 0 & 0 \\ 0 & I - Y_1^2 \end{pmatrix},$$

it follows that $\sum_{i \geq 2} Y_i^2$ is invertible. Adapting Technique-C from [FM93] to our setting and applying it here allows us to write $\sum_{i \geq 2} W_i\Psi_i(X)W_i$ as a single term $T_0^*\Gamma_0(X)T_0$ for some $\Gamma_0 \in \mathcal{E}$. Note that T_0 need not be positive. By using the polar decomposition $T_0 = U_0W_0$ where U_0 is unitary and W_0 is a positive operator, observe that $T_0^*\Gamma_0(X)T_0 = W_0\Psi_0(X)W_0$, for some $\Psi_0 \in \mathcal{E}$. Thus

$$\sum_{i \geq 2} W_i\Psi_i(X)W_i = W_0\Psi_0(X)W_0 \quad (5.5)$$

for some positive operator W_0 and $\Psi_0 \in \mathcal{E}$.

Indeed $W_0^2 + W_1^2 = I_H$ and $W_0 = \begin{pmatrix} 0 & 0 \\ 0 & Y_0 \end{pmatrix}$ for some positive operator Y_0 . Recall the positive matrix Y_1 and rewrite it as $Y_1 = \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \end{pmatrix}$ where Z_1 is a positive invertible matrix with $\|Z_1\| < 1$. Since $Y_0^2 + Y_1^2 = I$, $Y_0 = \begin{pmatrix} Z_0 & 0 \\ 0 & I \end{pmatrix}$ for some matrix Z_0 . Thus

$$W_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad W_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Z_0 & 0 \\ 0 & 0 & I \end{pmatrix},$$

where Z_1, Z_0 are positive matrices. Since $W_0^2 + W_1^2 = I_H$, it must be the case that $Z_0^2 + Z_1^2 = I$, which in turn implies that Z_0 is invertible, since

$\|Z_1\| < 1$. Recall that Z_1 is also invertible. It follows from equations (5.4) and (5.5) and the fact that $\Psi_1 \in \mathcal{K}$ that

$$\begin{aligned} U^*\Phi(X)U &= W_0\Psi_0(X)W_0 + W_1\Psi_1(X)W_1 \\ &= \begin{pmatrix} I & 0 & 0 \\ 0 & Z_0 & 0 \\ 0 & 0 & I \end{pmatrix} \Theta(X) \begin{pmatrix} I & 0 & 0 \\ 0 & Z_0 & 0 \\ 0 & 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Psi_1(X) \begin{pmatrix} 0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (5.6)$$

where

$$\Theta(X) = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Psi_1(X) \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \Psi_0(X) \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \quad (5.7)$$

$\in \mathcal{E}$.

By the invertibility of Z_0 , it is immediate that $Y_0 = \begin{pmatrix} Z_0 & 0 \\ 0 & I \end{pmatrix}$ is also invertible. Applying Technique-B from [FM93] allows for writing $U^*\Phi U$ as a proper C^* -convex combination of Θ and some other $\Gamma \in \mathcal{E}$. By hypothesis, Φ is a C^* -extreme UEB map. Hence so is $U^*\Phi U$. It follows that Φ is unitarily equivalent to Θ . Let Q_1 and Q_1^\perp denote the projections

$\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ respectively. Indeed, with respect to the decomposition $H = \text{Range}(Q_1) \oplus \text{Range}(Q_1^\perp)$,

$$\Theta(X) = Q_1\Psi_1(X)|_{\text{Range}(Q_1)} \oplus Q_1^\perp\Psi_0(X)|_{\text{Range}(Q_1^\perp)}, \quad (5.8)$$

for all $X \in \mathcal{S}$. Since $Q_1^\perp\Psi_0(X)|_{\text{Range}(Q_1^\perp)} \in \text{UEB}(\mathcal{S}, B(\text{Range}(Q_1^\perp)))$, one can write

$$Q_1^\perp\Psi_0(X)|_{\text{Range}(Q_1^\perp)} := \sum_{i=1}^k S_i^* \Gamma_i(X) S_i,$$

where $\Gamma_i \in \text{UEB}(\mathcal{S}, B(\text{Range}(Q_1^\perp)))$ are given by $\Gamma_i(X) = \gamma_i(X)I_{\text{Range}(Q_1^\perp)}$, γ_i are linear extremal states defined on \mathcal{S} , $S_i \in B(\text{Range}(Q_1^\perp))$ satisfy $\sum_{i=1}^k S_i^* S_i = I_{\text{Range}(Q_1^\perp)}$. Since $\Theta \in \text{UEB}(\mathcal{S}, B(H))$ is a C^* -extreme map, it follows from an argument similar to the one on page 770 in [FM93], that each direct summand of Θ in equation (5.8) is also a C^* -extreme UEB map. In particular, the UEB map $Q_1^\perp\Psi_0(X)|_{\text{Range}(Q_1^\perp)}$ is a C^* -extreme point of $\text{UEB}(\mathcal{S}, B(\text{Range}(Q_1^\perp)))$. As before, there is no loss of generality in assuming that $\|S_1\| = 1$. We can now repeat

the arguments from before to the current set up and conclude (by taking appropriate direct sums with the zero operator) that there exists a projection Q_2 in $B(H)$ such that

$$\Theta(X) = Q_1\Psi_1(X)Q_1 + Q_2\Delta_1(X)Q_2 + \text{an EB map,}$$

where Q_1 and Q_2 are mutually orthogonal projections in $B(H)$, $\Delta_1(X) = \gamma_1(X)I_H$ and γ_1 is a linear extremal state on \mathcal{S} . This process has to end after a finite number of steps, due to our finite dimensionality assumptions. This along with the fact that Θ is unitarily equivalent to Φ implies that $\Phi(X) = \sum_{i=1}^{\ell} \phi_i(X)P_i$, for all $X \in \mathcal{S}$, where the ϕ_i 's are linear extremal states on \mathcal{S} and the P_i 's are mutually orthogonal projections in $B(H)$ satisfying $\sum_{i=1}^{\ell} P_i = I_H$. An application of Theorem 1.10 implies that Φ is a maximal UEB map and the proof is complete.

To prove the implication (ii) \implies (iii), observe that since Φ is a C^* -extreme UEB map, by the (proved) equivalence of statements (i) and (ii) and by an application of Lemma 4.2, Φ has commutative range. That Φ is a linear extreme UEB map follows from an easy and direct adaptation of Theorem 2.2.2 in [Z] to UEB maps.

Finally, to prove the implication (iii) \implies (i), observe that since Φ is a UEB map with commutative range, Lemma 4.1 implies that

$$\Phi(X) = \sum_{i=1}^k \phi_i(X)P_i,$$

for some $k \in \mathbb{N}$, where the ϕ_i 's are distinct states on \mathcal{S} and the P_i 's are mutually orthogonal projections in $B(H)$ satisfying $\sum_{i=1}^k P_i = I_H$. By appealing to Theorem 1.10, it suffices to show that ϕ_i is a linear extremal state for each i . Fix $i \in \{1, \dots, k\}$. Let σ and τ be states defined on \mathcal{S} such that

$$\phi_i = t\sigma + (1-t)\tau,$$

for some $t \in (0, 1)$. It is enough to show that $\sigma = \tau$. Observe that

$$\Phi(X) = t \left(\sigma(X)P_i + \sum_{j \neq i} \phi_j(X)P_j \right) + (1-t) \left(\tau(X)P_i + \sum_{j \neq i} \phi_j(X)P_j \right),$$

for all $X \in \mathcal{S}$. Define the linear maps $\Psi, \Gamma : \mathcal{S} \rightarrow B(H)$ by

$$\Psi(X) = \sigma(X)P_i + \sum_{j \neq i} \phi_j(X)P_j \text{ and } \Gamma(X) = \tau(X)P_i + \sum_{j \neq i} \phi_j(X)P_j.$$

Observe that $\Psi, \Gamma \in \text{UEB}(\mathcal{S}, B(H))$ and $\Phi(X) = t\Psi(X) + (1-t)\Gamma(X)$, for all $X \in \mathcal{S}$. From the assumption that Φ is linear extreme, it

follows that $\Psi = \Gamma$. This in turn implies that $\sigma = \tau$ and the proof is complete. \square

Remark 5.3. *The following remarks concern the above proof of Theorem 1.12.*

- (i) *The proof of the implication (i) \Rightarrow (ii) in the above theorem works equally well for UCP maps or even just unital positive maps.*
- (ii) *In the proof of the implication (ii) \implies (i), it is assumed that T_1 is not invertible and hence the appearance of the zero block in the definition of Y_1 . If the operator T_1 is invertible, even then the proof works just fine by letting $Y_1 = Z_1$, $Y_0 = Z_0$ and by deleting the last row and column in the coefficient matrices occurring in equations (5.6) and (5.7).*

Proof of Corollary 1.13. Since Φ is a C^* -extreme point in $\text{UEB}(\mathcal{S}, B(H))$, it follows from Theorems 1.12 and 1.10 that $\Phi(X) = \sum_{i=1}^{\ell} \phi_i(X)P_i$, where the ϕ_i 's are linear extremal states on \mathcal{S} and the P_i 's are orthogonal projections in $B(H)$ such that $\sum_{i=1}^{\ell} P_i = I_H$. By [Z, Proposition 1.2.4] there exist linear extremal states $\psi_i : B(E) \rightarrow \mathbb{C}$ such that $\psi_i|_{\mathcal{S}} = \phi_i$, for each i . Define $\Psi : B(E) \rightarrow B(H)$ by $\Psi(X) = \sum_{i=1}^{\ell} \psi_i(X)P_i$. Observe that $\Psi|_{\mathcal{S}} = \Phi$. It follows from Theorems 1.10 and 1.12 that Ψ is a C^* -extreme UEB extension of Φ . \square

The following example shows that the converse of Corollary 1.13 is not true in general, i.e., the restriction of a C^* -extreme UEB map on an operator system need not always be C^* -extreme. Recall that for states defined on operator systems, C^* -extremality coincides with linear extremality. Let $\mathcal{S} \subset M_2(\mathbb{C})$ denote the operator system

$$\mathcal{S} := \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in M_2(\mathbb{C}) : a, b, c \in \mathbb{C} \right\}.$$

Define $\phi : \mathcal{S} \rightarrow \mathbb{C}$ by $\phi(X) = \text{trace}(XE)$ for all $X \in \mathcal{S}$, where $E = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$. Then ϕ is a state on \mathcal{S} . In particular, ϕ is a UEB map. Define the (distinct) states $\phi_1, \phi_2 : \mathcal{S} \rightarrow \mathbb{C}$ by $\phi_i(X) = \text{trace}(XF_i)$, for all $X \in \mathcal{S}$, $i = 1, 2$, where $F_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ and $F_2 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$.

It follows that

$$\begin{aligned} \phi(X) &= \text{trace} \left(X \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \right) = \frac{1}{2} \text{trace}(X(F_1 + F_2)) \\ &= \frac{1}{2} (\phi_1(X) + \phi_2(X)). \end{aligned}$$

This in turn implies that ϕ is not a linear extremal state and hence not a C^* -extreme state on \mathcal{S} . Let $\psi : M_2 \rightarrow \mathbb{C}$ be the state defined by $\psi(Y) = \text{trace}(YE_{11})$, where $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since E_{11} is a projection of rank one, it is well-known that ψ has to be a linear extremal state (See [St] and [Wa]) or equivalently a C^* -extremal state on M_2 . Finally, observe that $\psi|_{\mathcal{S}} = \phi$.

We end this section with a proof of Corollary 1.14, which is a Krein-Milman type theorem for UEB maps.

Proof of Corollary 1.14. Recall the notations \mathcal{E} and \mathcal{K} from Lemma 5.2. It follows from Lemma 5.2 that $\mathcal{E} = \text{UEB}(\mathcal{S}, B(H))$. By Theorems 1.10 and 1.12, it follows that each element of \mathcal{K} is a C^* -extreme UEB map in $\text{UEB}(\mathcal{S}, B(H))$. This completes the proof. \square

6. C^* -EXTREME UEB MAPS BETWEEN MATRIX ALGEBRAS

This section contains an improved version of [BDMS, Theorem 5.3], which includes various characterizations of a C^* -extreme UEB map between matrix algebras (See Theorem 6.2). Recall the convention 1.1 and the definitions of the Choi-rank, EB rank, Schmidt rank and Schmidt number from Section 1. For a positive matrix $X = \sum_{i=1}^{\ell} A_i \otimes B_i \in B(E \otimes H)$, **the partial trace of X with respect to the first coordinate**, is denoted by $\text{trace}_1(X)$ and is defined as

$$\text{trace}_1(X) = \sum_{i=1}^{\ell} \text{trace}(A_i)B_i \in B(H).$$

The following lemma is a variant of [HSR, Lemma 8].

Lemma 6.1. *Let $\Psi : M_d \rightarrow M_n$ be an EB map and $C_{\Psi} = \sum_{j=1}^m \xi_j \xi_j^*$ with $SR(\xi_j) = 1$ for each $j \in \{1, 2, \dots, n\}$. If the Choi-rank of $\Psi = n$ and $\text{trace}_1(C_{\Psi}) = \Psi(I_d) \in M_n$ is invertible, then $m \geq n$ and there exists $\gamma_1, \dots, \gamma_n \in \mathbb{C}^d \otimes \mathbb{C}^n$ with $SR(\gamma_k) = 1$ such that $C_{\Psi} = \sum_{k=1}^n \gamma_k \gamma_k^*$.*

Proof. Since the Choi-rank of Ψ equals the rank of the Choi matrix C_{Ψ} (See [Wa] and [St]), it is clear that $m \geq n$. Suppose that $m > n$ and $C_{\Psi} = \sum_{j=1}^m \xi_j \xi_j^*$ where $SR(\xi_j) = 1$ for all $1 \leq j \leq m$. For each

$1 \leq j \leq m$, let $\xi_j = x_j \otimes y_j$ with $\|x_j\| = 1$. By hypothesis, $\Psi(I_d) = \text{trace}_1(C_\Psi) = \sum_{j=1}^m \text{trace}(x_j x_j^*) y_j y_j^* = \sum_{j=1}^m y_j y_j^*$ is invertible. This implies that $\text{Span}\{y_1, \dots, y_m\} = \mathbb{C}^n$. Without loss of generality, assume that $\{y_1, \dots, y_n\}$ is a basis for $\mathbb{C}^n = \text{Range}(\text{trace}_1(C_\Psi))$. Note that $B = \{\xi_j : 1 \leq j \leq n\}$ is linearly independent. By the Douglas range inclusion property ([D, Theorem 1]) and due to the hypothesis that the rank of $C_\Psi = n$, it follows that $\{\xi_j : 1 \leq j \leq m\} \subset \text{Range}(C_\Psi)$ and that B is in fact a basis for $\text{Range}(C_\Psi)$. Consider the sum $\sum_{j=1}^{n+1} \xi_j \xi_j^*$. There exists α_j such that $\xi_{n+1} = \sum_{j=1}^n \alpha_j \xi_j$. It follows that $x_j = \lambda_j x_{n+1}$, for some scalars λ_j , whenever $\alpha_j \neq 0$, $1 \leq j \leq n$. Thus $|\lambda_j| = 1$ (since the x_j 's are unit vectors) and

$$x_{n+1} \otimes y_{n+1} = \xi_{n+1} = \sum_{j:\alpha_j \neq 0} \alpha_j \xi_j = \sum_{j:\alpha_j \neq 0} x_{n+1} \otimes (\lambda_j \alpha_j) y_j.$$

Consequently, $y_{n+1} = \sum_{j:\alpha_j \neq 0} (\lambda_j \alpha_j) y_j$, and

$$\begin{aligned} \sum_{j=1}^{n+1} \xi_j \xi_j^* &= x_{n+1} x_{n+1}^* \otimes \left(\left(\sum_{j:\alpha_j \neq 0} y_j y_j^* \right) + y_{n+1} y_{n+1}^* \right) \\ &+ \sum_{j:\alpha_j = 0} x_j x_j^* \otimes y_j y_j^*. \end{aligned} \quad (6.1)$$

Let r denote the cardinality of the set $\{j : \alpha_j = 0\}$. Since $\{y_j : \alpha_j \neq 0\}$ is linearly independent and $y_{n+1} \in \text{span}\{y_j : \alpha_j \neq 0\}$,

$$\sum_{j:\alpha_j \neq 0} y_j y_j^* + y_{n+1} y_{n+1}^* = \sum_{k=1}^{n-r} w_k w_k^*,$$

for some non-zero vectors $w_k \in \text{span}\{y_j : \alpha_j \neq 0\} \subseteq \mathbb{C}^n$. Thus,

$$\sum_{j=1}^{n+1} \xi_j \xi_j^* = \sum_{k=1}^{n-r} \eta_k \eta_k^* + \sum_{j:\alpha_j = 0} \xi_j \xi_j^*,$$

where $\eta_k = x_{n+1} \otimes w_k$. Note that the right hand side is a sum consisting of exactly n summands that are vectors in $\mathbb{C}^d \otimes \mathbb{C}^n$ with Schmidt rank equal to one. Repeat the above process, by adding the $(n+2)^{\text{th}}$ term of C_Ψ to $\sum_{j=1}^{n+1} \xi_j \xi_j^*$. Continuing this way, after a finite number of iterations, one can conclude that C_Ψ is of the desired form, i.e., $C_\Psi = \sum_{k=1}^n \gamma_k \gamma_k^*$ with $\gamma_k \in \mathbb{C}^d \otimes \mathbb{C}^n$ and $SR(\gamma_k) = 1$. This completes the proof. \square

Theorem 6.2. *Let $\Phi : M_d \rightarrow M_n$ be a UEB map. The following statements are equivalent.*

- (i) Φ is a C^* -extreme UEB map.
- (ii) Φ is a linear extreme UEB map with commutative range.
- (iii) Φ is a maximal UEB map.
- (iv) The map Φ has the form $\Phi(X) = \sum_{i=1}^{\ell} \phi_i(X)P_i$, where the ϕ_i 's are distinct linear extremal states defined on M_d and the P_i 's are mutually orthogonal projections in M_n satisfying $\sum_{i=1}^{\ell} P_i = I_n$.
- (v) The Choi-rank of Φ is n .
- (vi) If $C_{\Phi} = \sum_{j=1}^m \xi_j \xi_j^*$ with $SR(\xi_j) = 1$ for each $j \in \{1, 2, \dots, m\}$, then $m \geq n$ and there exists $\gamma_1, \dots, \gamma_n \in \mathbb{C}^d \otimes \mathbb{C}^n$ with $SR(\gamma_j) = 1$ such that $C_{\Phi} = \sum_{j=1}^n \gamma_j \gamma_j^*$.
- (vii) If $\Phi(X) = \sum_{i=1}^k V_i^* X V_i$ and rank of V_i is one for each $i \in \{1, 2, \dots, k\}$, then $k \geq n$ and there exists matrices W_1, \dots, W_n of rank one such that $\Phi(X) = \sum_{i=1}^n W_i^* X W_i$.
- (viii) The EB-rank of Φ is n .

Proof. The equivalence of (i), (ii) and (iii) follows by letting $\mathcal{S} = B(E)$ in Theorem 1.12

The equivalence (iii) \iff (iv) follows by letting $\mathcal{S} = B(E)$ in Theorem 1.10.

To prove (iv) \implies (v), rewrite $\Phi(X) = \sum_{k=1}^n \psi_k(X)w_k w_k^*$, where $\psi_k \in \{\phi_i : 1 \leq i \leq \ell\}$ and $\{w_k : 1 \leq k \leq n\}$ is an orthonormal set of vectors satisfying $\sum_{k=1}^n w_k w_k^* = I_n$. For each $k \in \{1, 2, \dots, n\}$, Observe that $\psi_k(X) = \text{trace}(X u_k u_k^*) = \langle X u_k, u_k \rangle$ for some unit vector u_k . Recall the Choi matrix $C_{\Phi} \in M_d \otimes M_n$. It follows that

$$\begin{aligned} C_{\Phi} &= \sum_{i,j=1}^d e_i e_j^* \otimes \Phi(e_i e_j^*) = \sum_{k=1}^n \sum_{i,j=1}^d e_i e_j^* \otimes (u_k^* e_i e_j^* u_k) w_k w_k^* \\ &= \sum_{k=1}^n \left(\sum_{i=1}^d (u_k^* e_i) e_i \otimes w_k \right) \left(\sum_{i=1}^d (e_i^* u_k) e_i^* \otimes w_k^* \right) = \sum_{k=1}^n z_k z_k^*, \end{aligned}$$

where $z_k = \sum_{i=1}^d (u_k^* e_i) e_i \otimes w_k$. Since C_{Φ} is a sum of rank-one operators, the rank of C_{Φ} , i.e., the Choi-rank of Φ is at most n . Also since C_{Φ} is a sum of the positive rank-one operators $z_k z_k^*$, the Douglas range inclusion property ([D, Theorem 1]) implies that $\{z_k : 1 \leq k \leq n\}$ is contained in $\text{Range}(C_{\Phi})$. By the orthonormality of $\{z_k : 1 \leq k \leq n\} \subset \text{Range}(C_{\Phi})$ it follows that $\text{Choi-rank}(\phi) \geq n$. Thus in fact, the Choi-rank of Φ equals n .

Since Φ is unital, the implication (v) \implies (vi) follows from the observation that $\text{trace}_1(C_\Phi) = I_n$ and a direct application of Lemma 6.1.

The implication (vi) \implies (vii), is a direct consequence of the well-known fact (See [St, Proposition 4.1.6] and [Bh, Theorem 3.1.1] for instance), that every spectral decomposition of C_Φ as a sum of ℓ rank one positive matrices $\xi\xi^*$ with $SR(\xi) = 1$, yields a Choi-Kraus decomposition of Φ with exactly ℓ Choi-Kraus coefficients, each having rank one (and vice-versa).

The implication (vii) \implies (viii) is immediate from the definition of the EB-rank of Φ .

The below proof of the implication (viii) \implies (iii) is essentially the same as that given in [BDMS]. We include it here for the sake of completeness. Let $\Phi(X) = \sum_{j=1}^n (v_j u_j^*) X (u_j v_j^*)$, where $\{u_1, \dots, u_n\} \subset \mathbb{C}^d$ are unit vectors and $\{v_1, \dots, v_n\} \subset \mathbb{C}^n$. It suffices to show that $\{v_1, \dots, v_n\}$ forms an orthonormal basis for \mathbb{C}^n or equivalently, the matrix $W := (v_1 | v_2 | \dots | v_n) \in M_n$ is unitary. This easily follows from the fact that $I_n = \Phi(I_d) = \sum_{j=1}^n v_j v_j^* = WW^*$. Let $\{u_{j_k} : 1 \leq k \leq \ell\}$ be the distinct unit vectors among u_1, \dots, u_n . For $1 \leq k \leq \ell$, define the states ϕ_k on M_d by $\phi_k(X) = \text{trace}(X u_{j_k})$. Rewrite Φ such that

$$\Phi(X) = \sum_{j=1}^n (v_j u_j^*) X (u_j v_j^*) = \sum_{k=1}^{\ell} \phi_k(X) P_k.$$

It is easily seen that the P_k 's are mutually orthogonal projections satisfying $\sum_{k=1}^{\ell} P_k = I_n$. This completes the proof. \square

Acknowledgements: The authors thank Professor Scott McCullough (UF) for several discussions and many helpful suggestions, Mr. Devedra Repana (IITM) for a clarification/suggestion on Lemma 6.1 and also for pointing us to some important references and Mr. Chinmay Ajay Tamhankar (IITM) for his comments/suggestions on Theorem 1.10.

REFERENCES

- [A] Arveson, William B., *Notes on the Unique Extension Property*, <https://math.berkeley.edu/~arveson/Dvi/unExt.pdf> 1, 2, 7, 8
- [A69] Arveson, W.B., *Subalgebras of C^* -algebras*, Acta Math. 123 (1969), 141-224. 1, 2, 5
- [A72] W.B. Arveson. *Subalgebras of C^* -algebras II*, Acta Math., 128 (1972). 271-308. 2, 7, 8
- [B] Balasubramanian, Sriram., *A note on decomposable maps on operator systems*, New York J. Math. 26 (2020) 790–798. 4

- [BBK] Banerjee, T., Bhat, B.R. and Kumar, M., *C^* -extreme points of positive operator valued measures and unital completely positive maps*, Commun. Math. Phys, 388(3), pp.1235-1280, 2021. [2](#)
- [BK] Bhat, B.R. and Kumar, M., *C^* -extreme maps and nests*, J. Funct. Anal., 282(8), p.109397, 2022. [2](#)
- [Bh] R. Bhatia, *Positive Definite Matrices*, Princeton Series in Applied Mathematics Princeton University Press, Princeton, NJ, USA, (2007). [30](#)
- [BDMS] Bhat, B. V., Devendra, R., Mallick, N., and Sumesh, K., *C^* -extreme points of entanglement breaking maps*, Rev. Math. Phys. 35 (2023), no. 3, Paper No. 2350005, 17 pp. [1](#), [2](#), [3](#), [5](#), [6](#), [9](#), [20](#), [27](#), [30](#)
- [C] Choi, M.D., *Completely positive linear maps on complex matrices*, Linear Algebra Appl. 10 (1975), 285–290. [4](#)
- [CH] Chai, J.F., Hou, J.C., *Separability for Positive Operators on Tensor Product of Hilbert Spaces*, Acta. Math. Sin.-English Ser. 37, 893–910 (2021). [3](#)
- [DD] Kenneth R. Davidson and Allan P. Donsig, *Real analysis and applications, Theory in practice*, Undergraduate Texts in Mathematics, Springer, New York, 2010. [12](#), [19](#)
- [D] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc., 17:413–415, 1966. [28](#), [29](#)
- [F] Farenick, D. R., *Extremal matrix states on operator systems*, J. London Math. Soc. (2) 61 (2000), no. 3, 885–892. [2](#)
- [FM93] Farenick, D. R. and Morenz, P.B., *C^* -extreme points of some compact C^* -convex sets* Proc. Amer. Math. Soc., 118(3), pp.765-775,1993. [2](#), [5](#), [20](#), [22](#), [23](#), [24](#)
- [FM97] Farenick, D. R., Morenz, P., *C^* -extreme points in the generalised state spaces of a C^* -algebra*, Trans.of the Amer. Math. Soc., 349(5), 1997, pp.1725-1748. [2](#), [5](#), [20](#)
- [FZ] Farenick, D. R. and Zhou, H. *The structure of C^* -extreme points in spaces of completely positive linear maps on C^* -algebras*, Proc. Amer. Math. Soc. 126 (1998), no. 5, 1467–1477. [5](#), [20](#)
- [HJRW] Heinosaari, Jivulescu, Reeb, Wolf, *Extending quantum operations*, Journal of Mathematical Physics 53, 102208 (2012). [13](#)
- [HMP] Hopenwasser, A., Moore, R. L. and Paulsen, V. I., *C^* -Extreme Points*. Trans. of the Amer. Math. Soc., 266(1), 291–307 (1981). [2](#), [5](#)
- [HSR] Horodecki, M., Shor, P. and Ruskai, M., *Entanglement breaking channels*, Rev. Math. Phys 15, 629–641 (2003). [5](#), [6](#), [27](#)
- [JKPP] N. Johnston, D. Kribs, V. Paulsen, and R. Pereira, *Minimal and maximal operator spaces and operator systems in entanglement theory*, J. Funct. Anal. 260 (2011), no. 8, 2407–2423. [7](#)
- [KMP] Kennedy, Matthew; Manor, Nicholas A.; Paulsen, Vern I, *Composition of PPT maps*, Quantum Inf. Comput. 18 (2018), no. 5-6, 472–480. [7](#)
- [LD] Li, Yuan; Du, Hong-Ke, *Interpolations of entanglement breaking channels and equivalent conditions for completely positive maps*, J. Funct. Anal. 268 (2015), no. 11, 3566–3599. [7](#)
- [LP] Loebel, Richard I.; Paulsen, Vern I, *Some remarks on C^* -convexity*, Linear Algebra Appl. 35 (1981), 63–78. [2](#), [5](#)
- [P] Paulsen. *Completely bounded maps and operator algebras* Cambridge university press, 2002. [6](#), [7](#), [11](#), [13](#)

- [PPPR] Pandey, S.K., Paulsen, V.I., Prakash, J. and Rahaman, M., *Entanglement breaking rank and the existence of SIC POVMs*. J. Math. Phys, 61(4), p.042203, 2020. [5](#)
- [RJP] Rahaman, M., Jaques, Paulsen, V., *Eventually entanglement breaking maps*, Journal of Mathematical Physics 59.6 (2018): 062201. [12](#)
- [S] W.F. Stinespring, *Positive functions on C^* -algebras*, Proc. Amer. Math. Soc.. 6 (1955), 211-216. [1](#)
- [St] Størmer, Erling. *Positive linear maps of Operator algebras*, Springer, ISBN 978-3-642-34369-8. [4](#), [10](#), [11](#), [12](#), [27](#), [30](#)
- [St09] Størmer, Erling. *Separable states and positive maps II*, Mathematica Scandinavica (2009): 188-198. [11](#)
- [St18] Størmer, E. *Extension of positive maps*, arXiv preprint arXiv:1805.05148, 2018. [12](#)
- [TH] Terhal, B.M. and Horodecki, P., *Schmidt number for density matrices*, Phys. Rev. A, 61(4), p.040301, 2000. [5](#)
- [Wa] Watrous, J. *The Theory of Quantum Information*, Cambridge University Press, 2018, doi:10.1017/9781316848142. [27](#)
- [W] S. Winkler, *Matrix convexity*, PhD thesis, University of California, Los Angeles, 1996. [2](#)
- [WW] C. Webster and S. Winkler, *The Krein-Milman theorem in operator convexity*, Trans. Amer. Math. Soc. 351 (1999), no. 1, 307–322. [2](#)
- [Y] Yuan, R., *A Brief Introduction to POVM Measurement in Quantum Communications*, <https://arxiv.org/ftp/arxiv/papers/2201/2201.07968.pdf> [7](#)
- [Z] Zhuo, H., 1998. *C^* -extreme points in spaces of completely positive maps*, PhD Thesis, Faculty of Graduate Studies and Research, University of Regina. [5](#), [6](#), [9](#), [20](#), [25](#), [26](#)

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