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C^*-EXTREME POINTS IN THE GENERALISED STATE SPACES OF A C^*-ALGEBRA

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Abstract. In this paper we study the space $S_H(A)$ of unital completely positive linear maps from a C^*-algebra A to the algebra $B(H)$ of continuous linear operators on a complex Hilbert space H. The state space of A, in this notation, is $S_{\mathbb{C}}(A)$. The main focus of our study concerns noncommutative convexity. Specifically, we examine the C^*-extreme points of the C^*-convex space $S_H(A)$. General properties of C^*-extreme points are discussed and a complete description of the set of C^*-extreme points is given in each of the following cases: (i) the cases $S_{\mathbb{C}^2}(A)$, where A is arbitrary ; (ii) the cases $S_{\mathbb{C}^r}(A)$, where A is commutative; (iii) the cases $S_{\mathbb{C}^r}(M_n)$, where M_n is the C^*-algebra of $n \times n$ complex matrices. An analogue of the Krein-Milman theorem will also be established.

Introduction

Since their introduction in the paper [16] of Stinespring, there has been continued interest in completely positive maps on C^*-algebras and their uses in the theory of operator algebras and mathematical physics. In the seminal work [1] of Arveson, where a substantial development of the theory and applications of completely positive maps is presented, considerable attention is paid to the affine structure of the convex cone formed by these maps. The convexity properties of this cone were studied even further in several subsequent papers [3], [6], [7], [11], [15], [18], [19], and the present paper continues along these lines. We shall concentrate on a particular set of completely positive maps and follow Smith and Ward [15] in calling this set a "generalised state space." Specifically, given a unital C^*-algebra A and a fixed Hilbert space H, the generalised state space $S_H(A)$ of A is to be the set of all unital completely positive maps of A into $B(H)$, the algebra of bounded linear operators on H. In this notation, the usual state space of A is denoted by $S_{\mathbb{C}}(A)$, and so it is appropriate to view the generalised state spaces of a C^*-algebra as a "quantization" of its ordinary state space.

The term noncommutative convexity refers to any one of the various forms of convexity in which operator-valued convex coefficients are assumed, and C^*-convexity is one of these forms. Two recent and very closely related notions, both of which are

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relevant to the present paper in connection with completely positive maps and duality, are those of CP-convexity in generalised quasi-state spaces [9] and quantum-convexity in matrix-ordered spaces [6]. In contrast to previous works [7], [8], [10], [13], [14], where the study of C^*-convex sets has been carried out in the context of C^*-algebras themselves rather than within their state spaces, the principal objective of the present paper is to describe the structure of those completely positive maps that are C^*-extreme points among all elements in the generalised state space.

To begin with, let us establish what is meant by a C^*-convex space of completely positive maps and by the C^*-extreme points of these spaces. Observe that any C^*-convex combination of n completely positive maps $\varphi_i : A \to B(H)$, which by definition is an operator-valued convex combination of the form

$$\sum_{i=1}^{n} t_i^* \varphi_i(\cdot) t_i \ , \quad \text{where each } t_i \in B(H) \ \text{ and } \sum_{i=1}^{n} t_i^* t_i = 1 \ ,$$

produces yet another a completely positive map $A \to B(H)$; moreover, if each φ_i is unital, then so is $\sum_{i=1}^{n} t_i^* \varphi_i t_i$. Being closed under the formation of C^*-convex combinations of their elements, generalised state spaces are, therefore, C^*-convex sets. The natural topology on $S_H(A)$ is the bounded-weak-topology, and with respect to this topology, $S_H(A)$ is a compact space [1]. A generalised state $\varphi \in S_H(A)$ is a (linear) extreme point of $S_H(A)$ if, whenever

$$\varphi = \sum_{i=1}^{n} \lambda_i \varphi_i \quad \text{such that } \varphi_i \in S_H(A), \ \lambda_i \in (0,1) \subset \mathbb{R} \ \text{ for all } i, \ \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \ ,$$

then $\varphi_i = \varphi$ for all i. Following the ideas of Loebl and Paulsen [13], an element $\varphi \in S_H(A)$ is a C^*-extreme point of $S_H(A)$ if, whenever

$$\varphi = \sum_{i=1}^{n} t_i^* \varphi_i t_i \quad \text{such that } \varphi_i \in S_H(A), \ t_i^{-1} \ \text{ exists for all } i, \ \text{ and } \sum_{i=1}^{n} t_i^* t_i = 1 \ ,$$

then φ_i is unitarily equivalent to φ for all i. (What is meant by saying that φ is unitarily equivalent to ψ is that there is a unitary $u \in B(H)$ for which $\varphi(\cdot) = u^* \psi(\cdot) u$; the unitary equivalence of two generalised states is denoted by $\varphi \sim \psi$.) A C^*-convex combination of completely positive maps for which every C^*-convex coefficient t_i is invertible will be called *proper*, in analogy to the usual notion of a proper convex combination (where all the convex coefficients are nonzero). We will denote the group of operators invertible in $B(H)$ by $GL(H)$ and the sets of extreme and C^*-extreme points of $S_H(A)$ by $\text{ext}S_H(A)$ and C^*-$\text{ext}S_H(A)$.

With the 1-dimensional Hilbert space $H = \mathbb{C}$, the C^*-extreme points of the state space are exactly the same as the extreme points, because in this particular case there is no difference in the notions "convexity" and "C^*-convexity." With Hilbert spaces of higher finite dimension, we will prove that every C^*-extreme point of $S_H(A)$ is an extreme point as well. However we will demonstrate further that the set of C^*-extreme points is smaller, has more structure, yet is still sufficiently large to generate a C^*-convex subset that is dense in $S_H(A)$. When H is infinite-dimensional, much less is known; it has not even been determined, for example, whether C^*-extreme points are necessarily extreme.

In the course of this paper we will prove the following theorem, which illustrates the sort of structure that C^*-extreme points enjoy. More detailed results will be described in later sections of the paper.

Theorem. *Suppose that H has finite dimension and that $\varphi \in S_H(A)$ is a C^*-extreme point. Then φ is block-diagonal and the diagonal blocks are pure maps; that is, there is a decomposition $H = H_1 \oplus \cdots \oplus H_k$, pure unital completely positive maps $\varphi_i : A \to B(H_i)$, and a unitary $u \in B(H)$ such that with respect to this decomposition of H, $u^*\varphi(x)u = \varphi_1(x) \oplus \varphi_2(x) \oplus \cdots \oplus \varphi_k(x)$, for every $x \in A$. Furthermore, if A is commutative, then φ is C^*-extreme in the generalised state space $S_H(A)$ if and only if φ is a unital $*$-homomorphism.*

It is worthwhile to remark on one aspect of the theorem above. If A is a commutative C^*-algebra, it is known from the work of Arveson [1] that $S_H(A)$ has extreme points that are not multiplicative; it is, therefore, rather striking how closely the C^*-extreme points of $S_H(A)$ reflect the extremal states of A (given that the extremal states of A are the $*$-homomorphisms $A \to \mathbb{C}$). Thus, the theorem above lends weight to the view that $S_H(A)$ really is a quantized state space, and that C^*-extreme points are the appropriate extreme points for this space.

The organisation of this paper is as follows. Some sufficient conditions for a completely positive map to be C^*-extreme are presented in §1. In §2 we make a close study of $S_H(A)$ for finite-dimensional H, obtain the main decomposition theorem, and apply this result to a number of special cases, including the case of commutative C^*-algebras. A method for constructing C^*-extreme points from a single pure completely positive map is developed in §3. This construction is the basis for the proof of a generalised Krein-Milman theorem, which shows that there are enough C^*-extreme points to determine $S_H(A)$, and for the classification of the C^*-extremal generalised states on full matrix algebras given in §4. The final section of this paper describes how our work stands in relation to the work of other authors on this subject.

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2. Extreme points and C^*-extreme points: some sufficient conditions

By the theorem of Stinespring [16], every completely positive map $\varphi : A \to B(H)$ has a decomposition of the form $\varphi(\cdot) = v^*\pi(\cdot)v$, where π is a $*$-representation of A on a Hilbert space H_π and $v : H \to H_\pi$ is a bounded linear operator. (The operator v is an isometry if φ is unital.) Moreover, all minimal decompositions, that is, those decompositions for which the closed linear span $[\pi(A)v(H)]$ of $\pi(A)v(H)$ is dense in H_π, are unitarily equivalent.

There is a natural partial order on completely positive maps whereby $\psi \leq \varphi$, for a pair of completely positive maps ψ and φ, if $\varphi - \psi$ is completely positive. The notion of a pure completely positive map is suggested by that of a pure state: a completely positive map φ is pure if the only completely positive maps ψ for which $\psi \leq \varphi$ are those of the form $\psi = \lambda\varphi$ for some $\lambda \in [0, 1] \subset \mathbb{R}$. An important tool for the analysis of completely positive maps is the following theorem of Arveson.

A Radon-Nikodym-type Theorem ([1, 1.4.2])**.** *In the cone of completely positive maps $A \to B(H)$, suppose that φ has a minimal decomposition $\varphi = v^*\pi v$. Then $\psi \leq \varphi$ if and only if there exists a* (*uniquely determined*) *positive contraction*

h in the commutant $\pi(A)'$ of $\pi(A)$ such that $\psi(x) = v^*h\pi(x)v$ for every $x \in A$. Moreover, φ is pure if and only if π is irreducible.

A complicating factor is that, unlike the state space, generalised state spaces need not possess pure elements. Take, for example, a commutative C^*-algebra A. Irreducible representations of A take place on Hilbert spaces of dimension 1; if H is of dimension greater than 1, then there are no isometric maps of H into the Hilbert space on which the irreducible representation of A acts and therefore there are no elements of $S_H(A)$ that are pure.

Arveson solved a number of extremal problems concerning the cone of completely positive maps. In particular, for the compact convex subset $S_H(A)$ of this cone he obtained a complete and very useful characterisation of the set $\mathrm{ext}S_H(A)$ of extreme points of $S_H(A)$.

Extreme Point Theorem ([1, 1.4.6]). *Suppose that $\varphi \in S_H(A)$ has a minimal decomposition $\varphi = v^*\pi v$ and let $\mathcal{C} : \pi(A)' \to B(v(H))$ be the map $\mathcal{C}(z) = pz_{|_{v(H)}}$, where $p \in B(H_\pi)$ is the projection with range $v(H)$. Then φ is an extreme point of $S_H(A)$ if and only if $\mathcal{C}$ is an injection.*

The condition that $\mathcal{C}$ be injective (in the theorem above) is equivalent to saying that "the range of v is a faithful subspace of the von Neumann algebra $\pi(A)'$" [1].

In the present paper, we shall make extensive use of the following consequence of the Extreme Point Theorem: if H has finite dimension and if $\varphi = v^*\pi v$ is an extreme point of $S_H(A)$, then $\pi(A)'$ has finite dimension. In such cases, the representation π has a decomposition as a direct sum of finitely many irreducible representations of A.

The first proposition describes the manner in which extreme points and C^*-extreme points are related. It is not known whether the conclusion of Proposition 1.1 remains true if H is taken to be a Hilbert space of infinite dimension.

Proposition 1.1. *Suppose that φ is a C^*-extreme point of $S_H(A)$. Denote the ideal of compact operators on H by $K(H)$. If for some $x \in A$, $\varphi(x) \in K(H) + \mathbb{C}1$ and is irreducible, or if $\varphi(x) \in K(H) + \mathbb{C}1$ for every $x \in A$, then φ is an extreme point of $S_H(A)$. In particular, if H has finite dimension, then $C^*\text{-ext}\, S_H(A) \subset \mathrm{ext}S_H(A)$.*

Proof. We begin with the latter assertions. If $\varphi(A) \subset K(H) + \mathbb{C}1$, then $\varphi(x)$ is an extreme point of the convex hull of its unitary orbit [12] for every $x \in A$. Thus, if the C^*-extreme point φ is expressed as a proper convex combination $\varphi = \mu\psi + (1-\mu)\theta$, then $\varphi \sim \psi \sim \theta$ implies, pointwise, that $\varphi(x) \sim \psi(x) \sim \theta(x)$, and so $\varphi(x)$ is a proper convex combination of elements from its unitary orbit; hence, $\varphi(x) = \psi(x) = \theta(x)$, for every $x \in A$. In the case that H has finite dimension, $B(H) = K(H)$ and so $C^*\text{-ext}S_H(A) \subset \ \mathrm{ext}S_H(A)$.

To establish the first assertion, assume that φ is a C^*-extreme point with the property that $\varphi(x) \in K(H) + \mathbb{C}1$ and is irreducible. Suppose that φ is expressed as a proper convex combination $\varphi = \mu\psi + (1-\mu)\theta$ with $\psi, \theta \in S_H(A)$. Then there exist unitaries $u, w \in B(H)$ such that $\psi = u^*\varphi u$ and $\theta = w^*\varphi w$. Hence, $\varphi(x)$ is a convex combination of operators unitarily equivalent to it. By the arguments of the first paragraph, we have that $u^*\varphi(x)u = \varphi(x)$. Therefore, $\varphi(x)$ commutes with u and hence with the spectral projections for u. But $\varphi(x)$ is irreducible and so u must be a scalar multiple of the identity, whence $\psi = \varphi$ and similarly $\theta = \varphi$. □

We continue now with some examples of C^*-extreme points. An "inflation" of a pure state is a map $A \to B(H)$ of the form $x \mapsto \psi(x)1$, where $\psi : A \to \mathbb{C}$ is a pure state. Condition (3) below is motivated by the idea of $v(H)$ being faithful for $\pi(A)'$ in the Extreme Point Theorem; however "invariance" is a much stronger property.

Proposition 1.2. *Each of the following conditions is sufficient for a unital completely positive map $\varphi : A \to B(H)$ to be C^*-extreme and extreme in $S_H(A)$:*

(1) *that φ be the inflation of a pure state;*
(2) *that φ be multiplicative;*
(3) *that $v(H)$ be invariant for the commutant $\pi(A)'$, where $\varphi = v^*\pi v$ is a minimal decomposition of φ;*
(4) *that φ be pure.*

Moreover, it follows from condition (1) *and from the existence of pure states that every generalised state space has C^*-extreme points.*

Proof. 1. Suppose that ψ is a pure state and let $\varphi : A \to B(H)$ be its inflation. Suppose that $\varphi = \sum_{i=1}^k t_i^*\varphi_i t_i$ is a representation of φ as a proper C^*-convex combination of $\varphi_i \in S_H(A)$. Each unit vector $\xi \in H$ induces k states ψ_i^ξ on A via the definition $\psi_i^\xi(x) = ||t_i\xi||^{-2}(\varphi_i(x)t_i\xi, t_i\xi)$, and ψ is a proper convex combination of the ψ_i^ξ:

$$\psi(x) = (\varphi(x)\xi, \xi) = \sum_{i=1}^k ||t_i\xi||^2 \psi_i^\xi(x)\ , \qquad x \in A.$$

But as ψ is pure, we must have $\psi = \psi_i^\xi$ for every i. We will prove that $\varphi_i = \varphi$ for every i. Fix i and let $x \in A$ be arbitrary. If $\eta \in H$ is any unit vector, then there exists a unit vector $\xi \in H$ so that $\eta = ||t_i\xi||^{-1}t_i\xi$. Hence

$$(\varphi(x)\eta, \eta) = \psi(x) = \psi_i^\xi(x) = (\varphi_i(x)\eta, \eta).$$

As this is true for all unit vectors $\eta \in H$, we have $\varphi_i(x) = \varphi(x)$. Thus, φ is C^*-extreme and, evidently, extreme in $S_H(A)$.

2. Suppose that φ is multiplicative and that $\varphi = \sum_i t_i^*\varphi_i t_i$ expresses φ as a proper C^*-convex combination of elements of $S_H(A)$. Because φ is a representation of A on H that is nondegenerate, the minimal Stinespring decomposition of φ is trivial. Therefore, the inequality $t_i^*\varphi_i t_i \leq \varphi$ implies that there exist positive contractions $h_i \in \pi(A)'$ such that $t_i^*\varphi_i t_i = h_i\varphi$. Thus, $1 = \varphi_i(1) = \varphi(1)$ and $\varphi_i = (h_i^{\frac{1}{2}}t_i^{-1})^*\varphi(h_i^{\frac{1}{2}}t_i^{-1})$ imply that $\varphi_i \sim \varphi$ for every i. It is already known that multiplicative maps are extreme points [17].

3. Assume that φ satisfies condition (3) and that $\varphi = \sum_i t_i^*\varphi_i t_i$ is a representation of φ as a proper C^*-convex combination of $\varphi_i \in S_H(A)$. From $t_i^*\varphi_i t_i \leq \varphi$ and the Radon-Nikodym theorem, there exist positive contractions $h_i \in \pi(A)'$ such that $t_i^*\varphi_i t_i = v^*h_i\pi v$. Let $s_i = v^*h_i^{\frac{1}{2}}v$. Because $v(H)$ is invariant for $\pi(A)'$, the projection vv^* commutes with $\pi(A)'$. This means, in particular, that the operators s_i satisfy $vs_i = h_i^{\frac{1}{2}}v$. In addition,

$$s_i = (s_i^2)^{\frac{1}{2}} = (v^*h_iv)^{\frac{1}{2}} = (t_i^*t_i)^{\frac{1}{2}} \in GL(H).$$

Thus,

$$t_i^*\varphi_i t_i = v^*h_i^{\frac{1}{2}}\pi h_i^{\frac{1}{2}}v = s_i^*v^*\pi v s_i = s_i^*\varphi s_i$$

and

$$\varphi_i = (s_i t_i^{-1})^* \varphi (s_i t_i^{-1}) \ , \quad \varphi = (t_i s_i^{-1})^* \varphi_i (t_i s_i^{-1}) \ .$$

All that remains to prove is that $s_i t_i^{-1}$ is unitary. From $1 = \varphi(1) = \varphi_i(1)$ we have that $s_i t_i^{-1}$ is an isometry and that $1 = (t_i s_i^{-1})^*(t_i s_i^{-1})$. In passing to inverses in this last equation we obtain $(s_i t_i^{-1})(s_i t_i^{-1})^* = 1$ and therefore $s_i t_i^{-1}$ is unitary.

To prove that φ is an extreme point of $S_H(A)$, we need only show that $v(H)$ is a faithful subspace for the commutant $\pi(A)'$ (i.e., to show that the compression map $\mathcal{C}$ is injective — see the Extreme Point Theorem). Now since the range of v is cyclic for $\pi(A)$, it is also cyclic for the double commutant $\pi(A)''$; hence, $v(H)$ is separating for $\pi(A)'$ (i.e., $z \in \pi(A)'$ is zero whenever the compression of z^*z to $v(H)$ is zero). To show that $\mathcal{C}$ is injective, assume that the compression of $z \in \pi(A)'$ to $v(H)$ is zero. Because $v(H)$ reduces $\pi(A)'$, the compressions of z^* and, consequently, z^*z to $v(H)$ are zero. Hence $z = 0$ from the fact that $v(H)$ is separating for $\pi(A)'$ and so $\mathcal{C}$ is injective.

4. If φ is pure and if the minimal decomposition of φ is given by $\varphi = v^*\pi v$, then π is irreducible and $[v(H)]$ is plainly invariant for $\pi(A)' \cong_* \mathbb{C}$. Thus, the pure map φ fulfills condition (3) above. □

Example 1. With two examples we demonstrate here, first of all, how a map as in (3) in Proposition 1.2 arises and, secondly, how a direct sum of pure maps can fail to be C^*-extreme.

On the full matrix algebra M_n, each of the n maps $\varphi_i : M_n \to \mathbb{C}$ that sends $x \in M_n$ to the i-th diagonal element x_{ii} of x is an extreme point of the state space of M_n. In fact these states are vector states $\varphi_i(x) = (x\xi_i, \xi_i)$, where $\xi_1, \xi_2, \ldots, \xi_n$ denote the canonical orthonormal basis vectors of $\mathbb{C}^n$. Consider the C^*-algebra $A = M_2 \oplus M_1$ and the generalised state $\varphi : A \to M_2$ given by

$$\varphi\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \oplus \begin{pmatrix} x_{33} \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} x_{11}+x_{33} & x_{11}-x_{33} \\ x_{11}-x_{33} & x_{11}+x_{33} \end{pmatrix} .$$

The minimal Stinespring decomposition of φ is $\varphi(x) = v^*\pi(x)v$, where π is the inclusion identity $\pi(x) = x$ of $A \to B(\mathbb{C}^3)$ and where the isometry $v : \mathbb{C}^2 \to \mathbb{C}^3$ is

$$v = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} .$$

A matrix calculation verifies that the projection $p = vv^*$ onto the range of v commutes with the algebra $\pi(A)'$, which means that the range of v is invariant for $\pi(A)'$ and so, by Proposition 1.2, φ is a C^*-extreme point of the generalised state space $S_{\mathbb{C}^2}(A)$.

Consider now this same example from a second point of view. The range of φ is a commutative manifold, unitarily equivalent to a 2-dimensional diagonal algebra. More precisely, for every $x \in A$,

$$\varphi(x) = \frac{1}{2}\begin{pmatrix} x_{11}+x_{33} & x_{11}-x_{33} \\ x_{11}-x_{33} & x_{11}+x_{33} \end{pmatrix} = u^*\begin{pmatrix} x_{11} & 0 \\ 0 & x_{33} \end{pmatrix} u \, ,$$

where $u \in M_2$ is the unitary $u = 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Thus, φ is (unitarily equivalent to) a direct sum of two pure states on A, namely $x \mapsto x_{11}$ and $x \mapsto x_{33}$. What makes such a direct sum C^*-extreme in the state space $S_{\mathbb{C}^2}(A)$ is that these two pure states arise as compressions of "nonequivalent" irreducible representations of A, namely the irreducible representations $\pi_1(x) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ and $\pi_2(x) = x_{33}$ (which are clearly nonequivalent).

In general, however, a direct sum of pure maps or states need not be C^*-extreme or even extreme. Consider, for example, the following direct sum of pure states: the map $\varphi : M_2 \to M_2$ defined by

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \longmapsto \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix} .$$

For every $x \in M_2$,

$$\begin{aligned} \varphi(x) = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_{11} & -x_{12} \\ -x_{21} & x_{22} \end{pmatrix} \\ &= \frac{1}{2}\mathrm{id}(x) + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{2}\mathrm{id}(x) + \frac{1}{2}\theta(x) , \end{aligned}$$

where

$$\theta(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

As φ is a nontrivial convex combination of id and θ, φ is neither extreme nor C^*-extreme in the space $S_{\mathbb{C}^2}(M_2)$. □

The example above motivates the following definition and the proposition that follows. The proposition confirms that the simplest way to construct C^*-extreme points as direct sums of individual pure maps is through the use of mutually disjoint summands.

Definition. If $\varphi_i \in S_{H_i}(A)$ is pure and has minimal decomposition $\varphi_i = v_i^* \pi_i v_i$ for $1 \leq i \leq k$, then we say that $\varphi_1, \ldots, \varphi_k$ are disjoint if the (irreducible) representations $\pi_1, \ldots, \pi_k$ of A are mutually nonequivalent.

Proposition 1.3. *If $\varphi_i \in S_{H_i}(A)$ is pure for $1 \leq i \leq k$, and if $\varphi_1, \ldots, \varphi_k$ are disjoint, then $\varphi_1 \oplus \cdots \oplus \varphi_k$ is a C^*-extreme point of $S_H(A)$, where $H = \sum_i^{\oplus} H_i$.*

Proof. Let $\varphi = \sum_i^{\oplus} \varphi_i$ so that $\varphi \in S_H(A)$. In $B(H)$, let $q_1, \ldots, q_k$ denote the projections $q_j : H \to H_j$. Suppose that $\varphi = v_i^* \pi_i v_i$ denotes the minimal decomposition of each pure map $\varphi_i \in S_{H_i}(A)$ (so $v_i^* v_i$ =identity on H_i). Let $H_\pi = \sum_i^{\oplus} H_{\pi_i}$ and let $v : H \to H_\pi$ be the isometry defined by $v\xi = \sum_i^{\oplus} v_i q_i \xi, \quad \xi \in H$. Finally, let $\pi = \pi_1 \oplus \cdots \oplus \pi_k$, a representation of A on H_π. Because the representations $\pi_1, \ldots, \pi_k$ are mutually disjoint, the decomposition of φ as $v^* \pi v$ is a minimal one (for further details, see the proof of [1, 1.4.9]) and, moreover, $\pi(A)' \cong_* \mathbb{C}^k$. Clearly $v(H)$ is invariant for $\pi(A)'$ and so by Proposition 1.2 it follows that φ is a C^*-extreme point of $S_H(A)$. □

2. Structure of C^*-extreme points

We prove below that for finite-dimensional H the C^*-extreme points of $S_H(A)$ have a block-diagonal structure.

Theorem 2.1. *If H has finite dimension, and if φ is a C^*-extreme point of $S_H(A)$, then φ is the direct sum of pure completely positive maps. Specifically, there exist finitely many subspaces $H_i \subset H$ and generalised states $\varphi_i \in S_{H_i}(A)$ such that*

(1) *φ_i is pure for each i, and*
(2) *$H = \sum_i^{\oplus} H_i$ and $\varphi \sim \sum_i^{\oplus} \varphi_i$.*

Proof. Because C^*-ext $S_H(A) \subset$ ext $S_H(A)$, it follows from the extreme point theorem that π has a finite decomposition: that is, there exist finitely many irreducible representations $\pi_1, \cdots, \pi_k$ of A on subspaces $L_i \subset H_\pi$ such that $H_\pi = \sum_i^{\oplus} L_i$ and $\pi = \sum_i^{\oplus} \pi_i$. Let v_i denote the operator that projects $v\xi$ into L_i for every $\xi \in H$. Express each v_i in its polar form: $v_i = w_i a_i$ for some positive operator $a_i \in B(H)$ and partial isometry $w_i : H \to H_\pi$ such that $w_i(H) \subset L_i$ and $\ker w_i = \ker a_i$. If we set $\psi_i = w_i^* \pi_i w_i$, then we may write φ as a C^*-convex combination of the pure completely positive maps ψ_i via

$$\varphi = \sum_{i=1}^{k} a_i \psi_i a_i, \quad \sum_{i=1}^{k} a_i^2 = 1 \ .$$

By the minimality of the decomposition $\varphi = v^* \pi v$, the integer k is the least possible of all of the integers j for which φ can be expressed as a C^*-convex combination of elements of $\{\psi_i : 1 \leq i \leq k\}$ using j summands. At this point, we now adapt the techniques of [8, 4.1] for manipulating matrix-valued C^*-convex coefficients to the case at hand. That the arguments used in the proof of Theorem 4.1 of [8] apply here is owing to the fact that the manipulations performed involve only the coefficients a_i — not the summands ψ_i — with the possible exception of an alteration of some of the ψ_i by a unitary equivalence transformation, which we allow. The methods used in [8, 4.1] and applied here show that for every real number $\lambda \in [0, 1]$,

$$\varphi \sim t(\lambda)^* \theta'' t(\lambda) + s(\lambda)^* \theta_\lambda s(\lambda),$$

where

(i) $\theta'', \theta_\lambda \in S_H(A)$,
(ii) $t(\lambda)^* t(\lambda) + s(\lambda)^* s(\lambda) = 1$,
(iii) $t(\lambda), s(\lambda) \in GL(H)$ for every $\lambda \in (0, 1)$,
(iv) θ_λ is a C^*-convex combination of ψ_1 and ψ_0, where ψ_0 is a C^*-convex combination of the ψ_j, for $j \geq 2$, and where the coefficients used to obtain θ_λ are continuous functions of λ,
(v) $\theta_1 = q_1 \psi_1 q_1 + \sum_{i \geq 2} s_i^* \psi_i s_i$, where $q_1^* = q_1 = q_1^2, q_1 + \sum_{i \geq 2} s_i^* s_i = 1$, and $q_1(H) \perp s_i(H)$ for all $i \geq 2$.

Having expressed φ as a proper C^*-convex combination of θ'' and θ_λ for those $\lambda \in (0, 1)$, it follows that there exist unitaries $u_\lambda \in B(H)$, indexed by $\lambda \in (0, 1)$, such that $\varphi = u_\lambda^* \theta_\lambda u_\lambda$. Suppose now that $\epsilon > 0$. Because θ_λ depends continuously on λ, there exists $\lambda \in (0, 1)$ such that $||\theta_\lambda - \theta_1|| < \epsilon$. Hence, for every $x \in A$ of norm $||x|| \leq 1$,

$$\begin{aligned} ||\varphi(x) - u_\lambda^* \theta_1(x) u_\lambda|| &= ||\varphi(x) - u_\lambda^* \theta_\lambda(x) u_\lambda + u_\lambda^* \theta_\lambda(x) u_\lambda - u_\lambda^* \theta_1(x) u_\lambda|| \\ &= ||u_\lambda^* (\theta_\lambda(x) - \theta_1(x)) u_\lambda|| \\ &\leq ||\theta_\lambda - \theta_1|| \, ||x|| < \epsilon \ . \end{aligned}$$

Thus, $\varphi = \lim_{\lambda\to 1} u_\lambda^* \theta_1 u_\lambda$ and so by the compactness of the unitary group, there exists a unitary $u \in B(H)$ satisfying $\varphi = u^*\theta_1 u$. Consequently,

$$\varphi \sim \theta_1 = q_1\psi_1 q_1 + \sum_{i\geq 2} s_i^*\psi_i s_i.$$

As in the proof of Theorem 4.1 in [8], an exhaustion of this process leads to

$$\varphi \sim \sum_{i=1}^{k} q_i\psi_i q_i,$$

where the q_i's are mutually orthogonal projections summing to the identity. This gives a "block diagonal form" for φ: let $H_i = q_i(H)$, so that $H = \sum_i^{\oplus} H_i$, and observe that from $1 = \varphi(1)$, the compressions $q_i\psi_i(\cdot)_{|H_i}$ must also be unital. In defining $\varphi_i \in S_{H_i}(A)$ to be

$$\varphi_i(x) = q_i w_i^* \pi_i(x) w_{i|_{H_i}} \ ,$$

we have that φ_i is pure – because π_i is irreducible – and that $\varphi \sim \sum_i^{\oplus} \varphi_i$. □

At this point a few applications of Theorem 2.1 are within easy reach. Our first use of the theorem gives a complete characterisation of the C^*-extreme points of $S_H(A)$ in the case where A is abelian and H is finite-dimensional. The theorem below includes, for the sake of comparison, Arveson's description of the extreme points of this particular space.

Proposition 2.2. *Suppose that X is a compact Hausdorff space and that H has finite dimension. Every extreme point φ of $S_H(C(X))$ has the form*

$$\varphi(f) = \sum_{j=1}^{k} f(x_j)h_j \ , \qquad f \in C(X) \ ,$$

where $x_1, ..., x_n \in X$ are distinct, the operators $h_j \geq 0$ sum to 1, and $\{h_j(H)\}_j$ is a weakly independent family of subspaces. Every C^*-extreme point φ has the form of an extreme point, but with the stronger property that the positive operators $h_1, ..., h_n$ are projections (with mutually orthogonal ranges); i.e., there exists a unitary $u \in B(H)$ such that for all $f \in C(X)$,

$$u^*\varphi(f)u = \begin{pmatrix} f(x_1)1_{n_1} & & & \\ & f(x_2)1_{n_2} & & \\ & & \ddots & \\ & & & f(x_k)1_{n_k} \end{pmatrix}.$$

Moreover, φ is a C^*-extreme point if and only if φ is a unital $*$-homomorphism.

Proof. The statement concerning the extreme points is taken from [1, 1.4.10] of Arveson.

Unital $*$-homomorphisms $A \to B(H)$ are C^*-extreme in $S_H(A)$, independent of any specific assumptions on A or H (Proposition 2).

Conversely, if $\varphi \in C^*\text{-ext}S_H(C(X))$, then φ is unitarily equivalent to a direct sum of pure completely positive unital maps φ_j. Because $C(X)$ is commutative, these pure maps must be characters on $C(X)$ and so they have the form $\varphi_j(f) = f(x_j)$ for some $x_j \in X$. If some characters appear more than once, collect them (i.e., inflate them) so that only distinct x_j are left. The result is that φ has the stated form. Plainly, φ is a unital $*$-homomorphism. □

With Theorem 2.2 in hand, the following example shows that Theorem 2.1 does not hold if H has infinite dimension.

Example 2. There exists a commutative C^*-algebra A and an infinite-dimensional Hilbert space H such that $S_H(A)$ has C^*-extreme points that are not multiplicative. In particular, consider the commutative C^*-algebra $C(\mathbb{T})$ of 2π-periodic continuous functions $\mathbb{R} \to \mathbb{C}$. If m denotes normalised Lebesgue measure on the unit circle $\mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$, then let π be the representation of $C(\mathbb{T})$ on the Hilbert space $L^2(\mathbb{T}, m)$ that sends $f \in C(\mathbb{T})$ to the operator M_f of multiplication by f. Let $H = H^2(\mathbb{T}, m)$, the Hardy space of $\mathbb{T}$; so H is the closed subspace of $L^2(\mathbb{T}, m)$ that is generated by the analytic trigonometric polynomials (an analytic trigonmetric polynomial is a linear combination of the functions $f_n(\theta) = e^{in\theta}$ for $n \geq 0$). Now let $v : H \to L^2(\mathbb{T}, m)$ be the inclusion map. The completely positive map $\varphi : C(\mathbb{T}) \to B(H)$ given by

$$\varphi(f) = p_H M_f|_H \ ,$$

where p_H is the projection $L^2(\mathbb{T}, m) \to H$, has $v^*\pi v$ as its minimal decomposition. Arveson has shown in [1, p.164] that φ is an extreme point of $S_H(C(\mathbb{T}))$. We prove here that φ is in fact a C^*-extreme point.

Suppose that φ is a proper C^*-convex combination of $\varphi_i \in S_H(C(\mathbb{T}))$. Let z denote the function $z(\theta) = e^{i\theta}$. Then $\varphi(z)$ is the unilateral shift operator on the Hardy space and is expressed as a proper C^*-convex combination of contractions $\varphi_i(z)$. But isometries are C^*-extreme points of the closed unit ball of operators on Hilbert space [10] and so $\varphi_i(z) \sim \varphi(z)$ for each i. Each $\varphi_i(z)$ is, therefore, an isometry; hence,

$$\varphi_i(z^*)\varphi_i(z) = 1 = \varphi_i(1) = \varphi_i(z^*z) \ .$$

The above equation is a case of equality in the Schwarz inequality and thus from [4, 4.3] we have for each i that $\pi_i(z)v_i = v_i\varphi_i(z)$, where $\varphi_i = v_i^*\pi_i v_i$ denotes a minimal decomposition. Indeed it follows that $\pi_i(q)v_i = v_i\varphi_i(q)$ for every analytic trigonometric polynomial q. Therefore, if $u_i \in B(H)$ denotes the unitary that implements the equivalence $\varphi_i(z) \sim \varphi(z)$, then for every i and every analytic trigonometric polynomial q, $\varphi_i(q) = u_i^*\varphi(q)u_i$. Finally, because the system $\{p + q^* : p, q$ are analytic trigonometric polynomials$\}$ is uniformly dense in $C(\mathbb{T})$, by the Weierstrass Approximation Theorem, we have that $\varphi_i(f) = u_i^*\varphi(f)u_i$ for each i and every $f \in C(\mathbb{T})$. That is, φ is a C^*-extreme point of $S_H(C(\mathbb{T}))$. That φ is not multiplicative follows from the fact that M_z is unitary but the unilateral shift operator $\varphi(z)$ on the Hardy space is not. □

Theorem 2.1 leads also to a description of the C^*-extremal states in $S_{\mathbb{C}^2}(A)$.

Theorem 2.3. *A unital completely positive map $\varphi : A \to M_2$ is a C^*-extreme point of $S_{\mathbb{C}^2}(A)$ if and only if φ is*

a. *a pure map,*
b. *an inflated pure state, or*
c. *unitarily equivalent to the direct sum of two disjoint pure states.*

Proof. Propositions 1.2 and 1.3 show that the three types of maps described are all C^*-extreme points. Conversely, assume that φ is a C^*-extreme point of $S_{\mathbb{C}^2}(A)$ but that φ is not one of a,b,c. Because statement (a) is not true of φ, then by Theorem 2.1 above, $\varphi \sim \varphi_1 \oplus \varphi_2$ for some pure states φ_1, φ_2 on A. Because statement (c) is not true, φ_1 and φ_2 must be compressions of unitarily equivalent representations; without loss of generality, we may assume that there is a single irreducible representation π of A on a Hilbert space H_π and unit vectors $\xi_1, \xi_2 \in H_\pi$, cyclic for $\pi(A)$, such that, for each i, $\varphi_i(x) = (\pi(x)\xi_i, \xi_i)$ for every $x \in A$. Because (b) is not true, ξ_1 and ξ_2 must be linearly independent. Hence, the mappings $w_1, w_2 : \mathbb{C}^2 \to H_\pi$ given by

$$w_1(\alpha_1 e_1 + \alpha_2 e_2) = 2^{-1/2}(\alpha_1\xi_1 + \alpha_2\xi_2)$$

and

$$w_2(\alpha_1 e_1 + \alpha_2 e_2) = 2^{-1/2}(\alpha_1\xi_1 - \alpha_2\xi_2)$$

are one-to-one and satisfy $w_1^*\pi w_1 + w_2^*\pi w_2 = \varphi_1 \oplus \varphi_2$. Let $w_i = v_i a_i$ be the polar decomposition of each w_i; here, v_i is an isometry and $a_i \in M_2$ is positive and invertible. From $a_1^2 + a_2^2 = 1$ and

$$\sum_{i=1}^{2} a_i(v_i^*\pi v_i)a_i = \varphi_1 \oplus \varphi_2$$

it follows, as $\varphi_1 \oplus \varphi_2$ is a C^*-extreme point, that $\varphi_1 \oplus \varphi_2 \sim v_1^*\pi v_1 \sim v_2^*\pi v_2$; that is, $\varphi_1 \oplus \varphi_2$ is pure, which it is clearly not. In light of this contradiction, φ must be one of a,b,c. $\square$

Example 3. $S_{\mathbb{C}^2}(M_2)$. There is only one irreducible representation of M_2 (up to unitary equivalence) and so by Theorem 2.2 a unital completely positive map $\varphi : M_2 \to M_2$ is a C^*-extreme point of $S_{\mathbb{C}^2}(M_2)$ if and only if φ is of the form $\varphi(x) = u^*xu$, for some unitary $u \in M_2$, or is the inflation of a vector state on M_2. $\square$

Example 4. $S_{\mathbb{C}^2}(K(H) + \mathbb{C}1)$, where $K(H)$ is the set of compact operators on a separable infinite-dimensional Hilbert space H. For this C^*-algebra, there are only two distinct (up to unitary equivalence) irreducible representations of A, and so it is straightforward to determine the C^*-extreme points of $\varphi \in S_{\mathbb{C}^2}(A)$. They are, by Theorem 2.2,

a. (*pure maps*): all compressions $x \mapsto \begin{pmatrix} (x\xi,\xi) & (x\eta,\xi) \\ (x\xi,\eta) & (x\eta,\eta) \end{pmatrix}$, where $\xi, \eta \in H$ are orthogonal unit vectors;

b. (*inflations of pure states*): all maps $x \mapsto \begin{pmatrix} (x\xi,\xi) & 0 \\ 0 & (x\xi,\xi) \end{pmatrix}$, where $\xi \in H$ is a unit vector, and the $*$-homomorphism $\alpha 1 + k \mapsto \alpha 1_2$, where $\alpha \in \mathbb{C}$ and $k \in K(H)$; and

c. (*direct sums of disjoint pure maps*): all maps $\alpha 1 + k \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & (x\xi,\xi) \end{pmatrix}$, where $\xi \in H$ is a unit vector and $x = \alpha 1 + k$.

$\square$

3. The direct sum of nested pure maps

From Proposition 1.3 it is known that a direct sum of disjoint pure maps is one way of constructing new C^*-extreme points. In this section we indicate a much different way of obtaining C^*-extreme from pure maps, and from this construction we will obtain a characterisation of C^*-$\mathrm{ext}S_{\mathbb{C}^n}(M_n)$ (carried out in Section 4) and an analogue of the Krein-Milman theorem.

Suppose that ν denotes either a positive integer, countable ∞, or some larger cardinal number, and let $l^2(\nu)$ denote the sequence space $\mathbb{C}^\nu$ if ν is a positive integer, the space $l^2(\mathbb{N})$ if $\nu = \infty$, and the space $l^2(\Omega)$ if Ω is a set of cardinality ν strictly larger than the cardinality of $\mathbb{N}$. If π is an irreducible representation of A on a Hilbert space H_π, then we may assume (via unitary isomorphism) that H_π is the sequence space $l^2(\nu)$ for some ν. Let $\{e_j\}_{j=1}^{\nu}$ denote the standard orthonormal basis of $l^2(\nu)$ and let M_ν denote the C^*-algebra of $\nu \times \nu$ matrices that represent bounded operators on $l^2(\nu)$ with respect to the standard orthonormal basis. For a positive integer $m \leq \nu$, let $\tau_{\nu,m} : M_\nu \to M_m$ be the generalised (pure) state that sends each $x \in M_\nu$ to its leading $m \times m$ principal submatrix: explicitly, if $x \in M_\nu$, then $\tau_{\nu,m}(x) = q_{\nu,m}^* x q_{\nu,m}$, where $q_{\nu,m} : \mathbb{C}^m \to l^2(\nu)$ is the isometry

$$q_{\nu,m} = \big(e_1, e_2, \ldots, e_m\big)$$

and where again $\{e_j\}_{j=1}^{\nu}$ denotes the standard orthonormal basis of $l^2(\nu)$.

Now let A be a unital C^*-algebra and let π be a fixed irreducible representation of A on the Hilbert space $l^2(\nu)$. Suppose that $\nu \geq n_1 \geq n_2 \geq \cdots \geq n_k$ ($n_1, \ldots, n_k$ are positive integers) and $r = n_1 + \cdots + n_k$, and let $\varphi : A \to M_r$ be the generalised state

$$\varphi(x) = \tau_{\nu,n_1} \circ \pi(x) \oplus \tau_{\nu,n_2} \circ \pi(x) \oplus \cdots \oplus \tau_{\nu,n_k} \circ \pi(x) \ , \ \text{for } x \in A \ .$$

The minimal Stinespring decomposition of φ has the form $\varphi = w^* \pi_k w$, where π_k is the representation of A on $l^2(\nu) \otimes \mathbb{C}^k$ given by $\pi_k(x) = \pi(x) \oplus \cdots \oplus \pi(x)$ (k-times), and where $w : \mathbb{C}^r \to l^2(\nu) \otimes \mathbb{C}^k$ is the isometry

$$w = \begin{pmatrix} q_{\nu,n_1} & & & \\ & q_{\nu,n_2} & & \\ & & \ddots & \\ & & & q_{\nu,n_k} \end{pmatrix} .$$

The commutant $\pi_k(A)'$ is $*$-isomorphic to M_k; however we shall need to be more precise: if $h \in \pi_k(A)'$, then $h = (h_{ij} 1_\nu)_{1 \leq i,j \leq k}$, where each $h_{ij} \in \mathbb{C}$. (In contrast, an element of the commutant of $M_2 \oplus u^* M_2 u$, where $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, is of the form $\begin{pmatrix} \alpha 1_2 & \beta u \\ \gamma u & \delta 1_2 \end{pmatrix}$.) As $B(H_{\pi_k})$ is precisely $M_\nu \otimes M_k$, it will sometimes be convenient to write elements $h \in \pi_k(A)'$ as $1_\nu \otimes h'$, where $h' = (h_{ij}) \in M_k$. From our information concerning the structure of w and $\pi_k(A)'$, a straightforward matrix computation shows that if $h \in \pi_k(A)'$, then

$$w^* h w = \big(h_{ij} q_{\nu,n_i}^* q_{\nu,n_j}\big)_{1 \leq i,j \leq k} \ .$$

Observe that each $q_{\nu,n_i}^* q_{\nu,n_j}$ is an $n_i \times n_j$ matrix with 1's on the diagonal and zeros elsewhere. The $r \times r$ matrix $\big(q_{\nu,n_i}^* q_{\nu,n_j}\big)_{1 \leq i,j \leq k}$ is precisely $Q^* Q$, where Q is the $\nu \times r$ matrix

$$Q = \big(e_1, e_2, \ldots, e_{n_1}, e_1, e_2, \ldots, e_{n_2}, \ldots, e_1, e_2, \ldots, e_{n_k}\big) \ .$$

For each $1 \leq i \leq \nu$, let ρ_i be the number of nonzero entries on row i of Q. The nonnegative integers ρ_i satisfy $k = \rho_1 \geq \rho_2 \geq \cdots \geq 0$ and $\sum_i \rho_i = r$. Let $\rho_1, \ldots, \rho_\mu$ be the positive integers in the chain $\rho_1, \rho_2, \ldots$; these will be called the *commutant compression-indices* of φ. Observe that the commutant compression-indices of φ

are determined from the *compression-indices* $n_1, \ldots, n_k$ of φ, and that there are only a finite number of them.

Lastly, if σ is an element of the symmetric group Ξ_r of permutations on r letters, and if $\mathcal{L}$, $\mathcal{K}$ are linear subspaces of M_r, then the notation $\mathcal{L} \sim_\sigma \mathcal{K}$ is to mean that $\mathcal{K} = \{p_\sigma^* a p_\sigma \ : \ a \in \mathcal{L}\}$, where p_σ is the unitary in M_r that permutes the standard orthonormal basis $\{\xi_1, \ldots, \xi_r\}$ of $\mathbb{C}^r$ into the basis $\{\xi_{\sigma(1)}, \ldots, \xi_{\sigma(r)}\}$.

Lemma 3.1. *With the notation established above, let $\rho_1, \ldots, \rho_\mu$ be the commutant compression-indices of $\varphi = w^*\pi_k w$. Then there is a permutation $\sigma \in \Xi_r$ such that*

$$\begin{aligned} w^*\big(\pi_k(A)'\big)w &\sim_\sigma \tau_{k,\rho_1}(M_k) \oplus \tau_{k,\rho_2}(M_k) \oplus \cdots \oplus \tau_{k,\rho_\mu}(M_k) \\ &= M_k \oplus \tau_{k,\rho_2}(M_k) \oplus \cdots \oplus \tau_{k,\rho_\mu}(M_k) \ . \end{aligned}$$

Moreover, if $h = 1_n \otimes h' \in \pi_k(A)'$, then $w^*hw \sim_\sigma h' \oplus \tau_{k,\rho_2}(h') \oplus \cdots \oplus \tau_{k,\rho_\mu}(h')$.

Proof. Let $\mathcal{BG} = (\mathcal{V}, \mathcal{E})$ be the bipartite graph affiliated with the $\nu \times r$ $(0,1)$-matrix Q. The vertex set $\mathcal{V}$ of $\mathcal{BG}$ is a disjoint union $\mathcal{V} = \mathcal{X} \cup \mathcal{Y}$, where $\mathcal{X}$ has μ vertices α_i (recall μ is the number of nonzero rows of Q) and $\mathcal{Y}$ has r vertices β_j (as Q has r nonzero columns), and there is an edge $\{\alpha_i, \beta_j\} \in \mathcal{E}$ if and only if the (i,j)-entry of Q is nonzero. So each β_i is connected through α_i to all the β_j for which the (i,j)-entry of Q is nonzero. As each column of Q has exactly one nonzero entry, we therefore obtain a decomposition of $\mathcal{BG}$ into its connected components $\mathcal{BG}_1, \ldots, \mathcal{BG}_\mu$, where each $\mathcal{BG}_i$ is a bipartite graph with vertices and edges $\{\alpha_i, \beta_j\}$ and where the j's are the column positions of the nonzero entries on row i of Q. Hence, the degree of the vertex α_i in $\mathcal{BG}$ is precisely the degree of α_i in $\mathcal{BG}_i$ and is equal to ρ_i, the number of nonzero entries on row i of Q.

Given that Q is a $(0,1)$-matrix with precisely one nonzero entry in each of its columns, the connected components of $\mathcal{BG}$ correspond to connected components of the graph $\mathcal{G}$ of the symmetric matrix Q^*Q. Because Q^*Q is an $r \times r$ matrix, the graph $\mathcal{G}$ has r-vertices β_j with the property that β_l is connected to β_m in $\mathcal{G}$ if and only if β_l is connected to β_m (through some unique α_i) in the bipartite graph $\mathcal{BG}$. Thus $\mathcal{G}$ has μ connected components $\mathcal{G}_1, \ldots, \mathcal{G}_\mu$, each with ρ_i vertices, and so standard graph/matrix theory says that Q^*Q is permutationally-equivalent to a direct sum of μ matrices, each of size $\rho_i \times \rho_i$. The permutation $\sigma \in \Xi_r$ is determined as follows. The first ρ_1 integers in $\{1, 2, \ldots, r\}$ are sent by σ to the indices of the vertices β_j connected to α_1 in $\mathcal{BG}$: to be precise, if $1 = j_1 < j_2 < \cdots < j_{\rho_1}$ are the indices of the vertices β_j connected to α_1 (i.e. the nonzero entries on row 1 of Q occur in column positions $j_1, \ldots, j_{\rho_1}$), then $\sigma(m) = j_m$ for $1 \leq m \leq \rho_1$. The next ρ_2 integers in $\{1, 2, \ldots, r\}$ are sent by σ, in a similar fashion, to the indices of the vertices β_j connected to α_2 in $\mathcal{BG}$. Continue in this manner until the final ρ_μ integers in $\{1, 2, \ldots, r\}$ are sent by σ to the indices of the vertices β_j connected to α_μ in $\mathcal{BG}$.

Now if $h \in \pi_k(A)'$, then the combinatorial structures (i.e. the graph structures) of $Q^*Q = (q_{\nu,n_i}^* q_{\nu,n_j})_{1 \leq i,j \leq k}$ and $w^*hw = (h_{ij} q_{\nu,n_i}^* q_{\nu,n_j})_{1 \leq i,j \leq k}$ are the same, and hence $p_\sigma^*(w^*hw)p_\sigma$, like $p_\sigma^* Q^*Q p_\sigma$, is a direct sum of μ matrices, each summand being a $\rho_i \times \rho_i$ matrix. Observe that along any row of w^*hw the entries are zero together with the entries coming from h: namely, $h_{m1}, h_{m2}, \ldots, h_{m\rho_i}$ for some m and ρ_i. By the definition of σ, the first connected component $\mathcal{G}_1$ of $\mathcal{G}$ has $\rho_1 = k$ vertices and this component determines the first direct summand, a $k \times k$ matrix, in $p_\sigma^*(w^*hw)p_\sigma$: take the $k \times k$ principal submatrix of w^*hw determined by the k rows of Q^*Q that have k nonzero entries, together with the corresponding k columns. The relevant entries in each of these k rows are $h_{m1}, h_{m2}, \ldots, h_{mk}$, $1 \le m \le k$. In other words, the first summand is the $k \times k$ matrix h'. The second connected component $\mathcal{G}_2$ of $\mathcal{G}$ has ρ_2 vertices; the associated $\rho_2 \times \rho_2$ direct summand of $p_\sigma^*(w^*hw)p_\sigma$ is the $\rho_2 \times \rho_2$ principal submatrix of w^*hw determined by the ρ_2 rows of Q^*Q that have ρ_2 nonzero entries, together with the corresponding columns. The relevant entries in each of these rows are $h_{m1}, h_{m2}, \ldots, h_{m\rho_2}$, $1 \le m \le \rho_2$. Thus, the second direct summand of $p_\sigma^*(w^*hw)p_\sigma$ is $\tau_{k,\rho_2}(h')$, the $\rho_2 \times \rho_2$ leading principal submatrix of h'. By continuing with these arguments for each of the connected components of $\mathcal{G}$, we thereby have established the claim asserted in the statement of the lemma. $\square$

Lemma 3.1 is perhaps best understood by considering an example. Let $A = M_5$ and suppose that $\varphi : M_5 \to M_8$ is given by

$$\begin{pmatrix} x_{11} & \ldots & x_{15} \\ \vdots & & \vdots \\ x_{51} & \ldots & x_{55} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & \ldots & x_{13} \\ \vdots & & \vdots \\ x_{31} & \ldots & x_{33} \end{pmatrix} \oplus \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \oplus \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \oplus \begin{pmatrix} x_{11} \end{pmatrix}.$$

That is, π is the identity and $\varphi(x) = \tau_{5,3}(x) \oplus \tau_{5,2}(x) \oplus \tau_{5,2}(x) \oplus \tau_{5,1}(x)$ for all $x \in M_5$. In this case $w : \mathbb{C}^8 \to \mathbb{C}^{20}$, $\pi_4(M_5)'$ is $*$-isomorphic to M_4, and Q is given by

$$Q = \begin{pmatrix} q_{5,3} & q_{5,2} & q_{5,2} & q_{5,1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The bipartite graph of Q has 3 connected components (as Q has 3 nonzero rows) and the permutation $\sigma \in \Xi_8$ is determined from Q (or its graph):

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 4 & 6 & 8 & 2 & 5 & 7 & 3 \end{pmatrix}.$$

The commutant compression-indices are also read off from Q: they are $\rho_1 = 4$, $\rho_2 = 3$, and $\rho_3 = 1$. Direct computation shows that for $h' \in M_4$ and $h = 1_5 \otimes h' \in \pi_4(M_5)'$,

$$w^*hw = \begin{pmatrix} h_{11} & & & h_{12} & & h_{13} & & h_{14} \\ & h_{11} & & & h_{12} & & h_{13} & \\ & & h_{11} & & & & & \\ h_{21} & & & h_{22} & & h_{23} & & h_{24} \\ & h_{21} & & & h_{22} & & h_{22} & \\ h_{31} & & & h_{32} & & h_{33} & & h_{34} \\ & h_{31} & & & h_{32} & & h_{33} & \\ h_{41} & & & h_{42} & & h_{43} & & h_{44} \end{pmatrix}$$

$$\sim_\sigma \begin{pmatrix} \begin{pmatrix} h_{11} & \ldots & h_{14} \\ \vdots & & \vdots \\ h_{41} & \ldots & h_{44} \end{pmatrix} & & \\ & \begin{pmatrix} h_{11} & \ldots & h_{13} \\ \vdots & & \vdots \\ h_{31} & \ldots & h_{33} \end{pmatrix} & \\ & & \begin{pmatrix} h_{11} \end{pmatrix} \end{pmatrix}.$$

From Lemma 3.1 it is evident that the range of the isometry w is faithful for commutant $\pi_k(A)'$, and so we have the following immediate consequence: the map φ is an extreme point of $S_{\mathbb{C}^r}(A)$. It is a consequence of Theorem 3.3 that φ is actually C^*-extreme in $S_{\mathbb{C}^r}(A)$.

Definition. If $\varphi : A \to B(H)$ and $\psi : A \to B(K)$ are unital completely positive maps, then we say that ψ is a compression of φ if there is an isometry $w : K \to H$ for which $\psi(x) = w^*\varphi(x)w$ for all $x \in A$.

Observe that if, in the definition above, φ is pure with Stinespring decomposition $\varphi = v^*\pi v$, then ψ is also pure and of the form $\psi = a^*\pi a$, where a is an isometry $K \to B(H_\pi)$ whose range is included within the range of v (equivalently, $aa^* \leq vv^*$). It is also easy to see that the sentence "ψ is unitarily equivalent to a compression of φ" is equivalent to "ψ is a compression of φ."

Lemma 3.2. *Suppose that π is an irreducible representation of A on the Hilbert space $l^2(\nu)$ and that $n_1 \geq n_2 \geq \cdots \geq n_k$ are positive integers with $\nu \geq n_1$. If $\varphi_i : A \to M_{n_i}$ is a pure unital completely positive map for each i, and if φ_1 is a compression of π, then each φ_{i+1} is a compression of φ_i if and only if there exist unitaries $u \in M_\nu$ and $u_i \in M_{n_i}$ such that $\varphi_i(x) = u_i^*\tau_{\nu,n_i}(u^*\pi(x)u)u_i$ for all i and all $x \in A$.*

Proof. Suppose that φ_{i+1} is a compression of φ_i for all $1 \leq i < n$. Therefore isometries $w_i : \mathbb{C}^{n_{i+1}} \to \mathbb{C}^{n_i}$ exist and satisfy $\varphi_{i+1} = w_i^*\varphi_i w_i$ for each i. As φ_1 is a compression of π, there is an isometry $v : \mathbb{C}^{n_1} \to l^2(\nu)$ such that $\varphi_1 = v^*\pi v$. Let $u \in M_\nu$ be any unitary of the form $u = [v\ *]$; that is, $u \in M_\nu$ is any unitary whose first n_1 columns are given by the n_1 columns of v. Hence $\varphi_1(x) = \tau_{\nu,n_1}(u^*\pi(x)u)$ for every $x \in A$. Starting with $u_1 = 1_{n_1}$, we construct the unitaries $u_i \in M_{n_i}$ inductively. If u_i has been constructed so as to satisfy $\varphi_i(x) = u_i^*\tau_{\nu,n_i}(u^*\pi(x)u)u_i$ for all $x \in A$, then let $u_{i+1} = q_{n_i,n_{i+1}}^* u_i w_i$. For every $x \in A$,

$$
\begin{aligned}
u_{i+1}^*\tau_{\nu,n_{i+1}}\big(u^*\pi(x)u\big)u_{i+1} &= w_i^*u_i^*q_{n_{i+1},n_i}q_{\nu,n_{i+1}}^*(u^*\pi(x)u)q_{\nu,n_{i+1}}q_{n_i,n_{i+1}}^*u_iw_i \\
&= w_i^*u_i^*q_{\nu,n_i}^*(u^*\pi(x)u)q_{\nu,n_i}u_iw_i \\
&= w_i^*u_i^*\tau_{\nu,n_i}(u^*\pi(x)u)u_iw_i \\
&= w_i^*\varphi_i(x)w_i \\
&= \varphi_{i+1}(x)\ .
\end{aligned}
$$

Conversely, if there exist unitaries $u \in M_\nu$, $u_i \in M_{n_i}$ such that for all i and all $x \in A$ we have $\varphi_i(x) = u_i^*\tau_{\nu,n_i}(u^*\pi(x)u)u_i$, then we seek isometries $w_i : \mathbb{C}^{n_{i+1}} \to \mathbb{C}^{n_i}$ that satisfy $\varphi_{i+1} = w_i^*\varphi_i w_i$ for all i. It is easily verified that the isometries $w_i = u_i^*q_{n_i,n_{i+1}}u_{i+1}$ have this property. □

Theorem 3.3. *Suppose that A is a unital C^*-algebra and that $\varphi_i : A \to M_{n_i}$ is a pure unital completely positive map for each $i = 1, 2, ..., k$. Let $r = \sum_i n_i$ and $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_k$. If φ_{i+1} is a compression of φ_i for each $1 \leq i < k$, then φ is a C^*-extreme point of $S_{\mathbb{C}^r}(A)$. In particular the generalised state $\varphi : M_n \to M_r$ given by $\varphi = \tau_{n,n_1} \oplus \tau_{n,n_2} \oplus \cdots \oplus \tau_{n,n_k}$ is a C^*-extreme point of $S_{\mathbb{C}^r}(M_n)$.*

Proof. The hypotheses that φ_1 is pure and φ_{i+1} is a compression of φ_i for each $1 \leq i < k$ is to say that each φ_i has the form $\varphi_i = v_i^*\pi v_i$, where each $v_i : \mathbb{C}^{n_i} \to H_\pi$ is an isometry and the projections $v_iv_i^*$ satisfy $v_iv_i^* \geq v_{i+1}v_{i+1}^*$, $i = 1, \ldots, k$. Here, π is an irreducible representation of A on a Hilbert space $H_\pi = l^2(\nu)$. By Lemma

3.2, there exist unitaries $u \in M_\nu$, $u_i \in M_{n_i}$ such that for all i and all $x \in A$ we have $\varphi_i(x) = u_i^* \tau_{\nu,n_i}(u^*\pi(x)u)u_i$. Hence φ is unitarily equivalent, via the unitary $\sum_i^\oplus u_i$, to the map

$$(\dagger) \qquad x \mapsto w^* \begin{pmatrix} \pi(u^*xu) & & & \\ & \pi(u^*xu) & & \\ & & \ddots & \\ & & & \pi(u^*xu) \end{pmatrix} w \quad \forall\, x \in A\,,$$

where $w : \mathbb{C}^r \to l^2(\nu) \otimes \mathbb{C}^k$ is the isometry

$$w = \begin{pmatrix} q_{\nu,n_1} & & & \\ & q_{\nu,n_2} & & \\ & & \ddots & \\ & & & q_{\nu,n_k} \end{pmatrix}.$$

For simplicity, let us denote the map in ($\dagger$) by φ once again. Likewise, we may replace the irreducible representation π by the irreducible representation defined by $\varrho(x) = \pi(u^*xu)$ for $x \in A$ and then replace π_k by ϱ_k. Because the commutants $\varrho_k(A)'$ and $\pi_k(A)'$ are exactly the same, we simply denote this new representation ϱ by π once again.

Suppose now that $\varphi = t^*\theta_1 t + s^*\theta_1 s$, where $\theta_1, \theta_2 \in S_{\mathbb{C}^r}(M_n)$ and s and t are invertible C^*-convex coefficients. By passing to the polar decomposition of t and s and then by absorbing the unitary parts of these decompositions into θ_1 and θ_2, we may assume without loss of generality that t and s are positive and invertible. As $t\theta_1 t \le \varphi = w^*\pi_k w$, Arveson's Radon-Nikodym Theorem has $t\theta_1 t = w^* h\pi_k w$ for some unique positive contraction $h \in \pi_k(A)'$. Thus,

$$\theta_1 = (h^{1/2}wt^{-1})^*\pi_k(h^{1/2}wt^{-1})$$

and $h^{1/2}wt^{-1}$ is an isometry. To prove that φ is a C^*-extreme point, we shall find a unitary $v \in \pi_k(A)'$ and a unitary $u \in M_r$ satisfying

$$(*) \qquad vw = h^{1/2}wt^{-1}u\,.$$

The relation above leads to

$$\begin{aligned} \varphi = w^*\pi_k w &= u^*(h^{1/2}wt^{-1})^*v^*\pi_k v(h^{1/2}wt^{-1})u \\ &= u^*(h^{1/2}wt^{-1})^*\pi_k v^* v(h^{1/2}wt^{-1})u \\ &= u^*(h^{1/2}wt^{-1})^*\pi_k(h^{1/2}wt^{-1})u \\ &= u^*\theta_1 u \end{aligned}$$

and $\theta_1 \sim \varphi$. Similarly, $\theta_2 \sim \varphi$, which proves that φ is a C^*-extreme point. The remainder of the proof is devoted to finding the unitaries u and v satisfying equation ($*$).

It will be convenient to introduce a change of basis. Denote the standard orthonormal bases of $\mathbb{C}^r$ and $l^2(\nu) \otimes \mathbb{C}^k$ by

$$\begin{aligned} \mathcal{B}_r &= \{\xi_1, \ldots, \xi_r\}, \\ \mathcal{B}_{\nu k} &= \{\eta_1 \otimes \psi_1, \eta_2 \otimes \psi_1, \ldots; \eta_1 \otimes \psi_2, \eta_2 \otimes \psi_2, \ldots; \eta_1 \otimes \psi_k, \eta_2 \otimes \psi_k, \ldots\}\,. \end{aligned}$$

(Here, $\{\eta_i\}_{i=1}^{\nu}$ and $\{\psi_1, \ldots, \psi_k\}$ are the standard orthonormal bases of $l^2(\nu)$ and $\mathbb{C}^k$.) The representation of the operators w,h, and t thus far has been with respect to the standard bases $\mathcal{B}_r$ and $\mathcal{B}_{\nu k}$. Consider now the following orthonormal bases (obtained from $\mathcal{B}_r$ and $\mathcal{B}_{\nu k}$ by permuting the orderings):

$$\begin{aligned}\tilde{\mathcal{B}}_r &= \{\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)}\},\\ \tilde{\mathcal{B}}_{\nu k} &= \{\eta_1 \otimes \psi_1, \eta_1 \otimes \psi_2, \dots, \eta_1 \otimes \psi_k, \eta_2 \otimes \psi_1, \dots, \eta_2 \otimes \psi_k, \eta_3 \otimes \psi_1, \dots\},\end{aligned}$$

where $\sigma \in \Xi_r$ is the permutation determined by φ as described in Lemma 3.1. Let $p_\sigma \in M_r$ and $z \in M_\nu \otimes M_k$ be the (unitary) transition matrices from $\mathcal{B}_r$ to $\tilde{\mathcal{B}}_r$ and from $\mathcal{B}_{\nu k}$ to $\tilde{\mathcal{B}}_{\nu k}$. With respect to the orthonormal bases $\tilde{\mathcal{B}}_r$ and $\tilde{\mathcal{B}}_{\nu k}$, the isometry $h^{1/2} w t^{-1}$ is represented by the matrix

$$z^{-1} h^{1/2} w t^{-1} p_\sigma \ ,$$

which in turn can be written as

$$z^{-1} h^{1/2} z z^{-1} w p_\sigma p_\sigma^{-1} t^{-1} p_\sigma \ ,$$

where

$$z^{-1} h^{1/2} z = \sum_i^{\nu}{}^{\oplus} h' = \begin{pmatrix} h' & & & \\ & h' & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \in B(l^2(\nu) \otimes \mathbb{C}^k) \ ,$$

$$z^{-1} w p_\sigma = \begin{pmatrix} 1_k & & & \\ & a_{\rho_2} & & \\ & & \ddots & \\ & & & a_{\rho_\mu} \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \in B(\mathbb{C}^r, l^2(\nu) \otimes \mathbb{C}^k) \ ,$$

$$p_\sigma^{-1} t^{-1} p_\sigma = \begin{pmatrix} h'^{-\frac{1}{2}} & & & \\ & (a_{\rho_2}^* h' a_{\rho_2})^{-\frac{1}{2}} & & \\ & & \ddots & \\ & & & (a_{\rho_\mu}^* h' a_{\rho_\mu})^{-\frac{1}{2}} \end{pmatrix} \in M_r \ .$$

Here, $k = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_\mu > 0$ are the commutant compression-indices and each a_{ρ_j} is the canonical $k \times \rho_j$ isometry whose ρ_j columns are given by the first ρ_j standard orthonormal basis vectors of $\mathbb{C}^k$. The rows of zeros at the bottom of the matrix $z^{-1} w p_\sigma$ are $\nu - \mu$ in number and appear only if $\mu \neq \nu$ (which happens most frequently). The product of $z^{-1} w p_\sigma$ and $p_\sigma^{-1} t^{-1} p_\sigma$ produces the matrix

$$\begin{pmatrix} h'^{-\frac{1}{2}} & & & \\ & \begin{pmatrix} (a_{\rho_1}^* h' a_{\rho_1})^{-\frac{1}{2}} \\ 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} (a_{\rho_\mu}^* h' a_{\rho_\mu})^{-\frac{1}{2}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

and therefore $z^{-1}h^{1/2}zz^{-1}wp_\sigma p_\sigma^{-1}t^{-1}p_\sigma$ is

$$\begin{pmatrix} h'^{\frac{1}{2}} & & & & \\ & h'^{\frac{1}{2}} & & & \\ & & h'^{\frac{1}{2}} & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} h'^{-\frac{1}{2}} & & & \\ & \begin{pmatrix} (a_{\rho_1}^* h' a_{\rho_1})^{-\frac{1}{2}} \\ 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} (a_{\rho_\mu}^* h' a_{\rho_\mu})^{-\frac{1}{2}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} b_{\rho_1} & & & \\ & b_{\rho_2} & & \\ & & \ddots & \\ & & & b_{\rho_\mu} \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{pmatrix},$$

where

$$b_{\rho_j} = h'^{1/2} \begin{pmatrix} (a_{\rho_j}^* h' a_{\rho_j})^{-1/2} \\ 0 \end{pmatrix}.$$

We now invoke Lemma 3.4 (stated and proved after the proof of the present theorem), whose results are as follows: each $k \times \rho_j$ matrix b_{ρ_j} is an isometry into $\mathbb{C}^k$ and for every $1 \leq j \leq \mu$ there are unitaries $u_{\rho_j} \in M_{\rho_j}$ satisfying $b_{\rho_j} u_{\rho_j} = u_k a_{\rho_j}$.

Thus,

$$v(z^{-1}wp_\sigma) = \begin{pmatrix} u_k & & & \\ & u_k & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 1_k & & & \\ & a_{\rho_2} & & \\ & & \ddots & \\ & & & a_{\rho_\mu} \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} b_{\rho_1} & & & \\ & b_{\rho_2} & & \\ & & \ddots & \\ & & & b_{\rho_\mu} \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} u_k & & & \\ & u_{\rho_2} & & \\ & & \ddots & \\ & & & u_{\rho_\mu} \end{pmatrix}$$

$$= (z^{-1}wp_\sigma)u \ ,$$

where $u \in M_r$ and $v \in \pi_k(A)'$ are the unitaries (with respect to the orthonormal bases $\tilde{\mathcal{B}}_r$ and $\tilde{\mathcal{B}}_{\nu k}$) given by

$$u = \begin{pmatrix} u_k & & & \\ & u_{\rho_2} & & \\ & & \ddots & \\ & & & u_{\rho_\mu} \end{pmatrix} \text{ and } v = \begin{pmatrix} u_k & & & \\ & u_k & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} .$$

This completes the construction of the unitaries $u \in M_r$ and $v \in \pi_k(A)'$ satisfying equation $(*)$. □

Lemma 3.4. *Let $\{e_1, \ldots, e_k\}$ denote the standard orthonormal basis of $\mathbb{C}^k$, and for each $1 \le i \le k$, let $g_i : \mathbb{C}^i \to \mathbb{C}^k$ be the isometry $g_i = (e_1, e_2, \ldots, e_i)$. Suppose that $h \in M_k$ is a positive invertible contraction. Then*

(1) $h_i = h^{\frac{1}{2}} \begin{pmatrix} (g_i^* h g_i)^{\frac{-1}{2}} \\ 0 \end{pmatrix}$ *is an isometry $\mathbb{C}^i \to \mathbb{C}^k$;*
(2) *there exist unitaries $u_i \in M_i$, for each $1 \le i < n$, such that $h_i u_i = h_{i+1} u_{{i+1}_{|rang_i}}$ for each i.*

Proof. A straightforward matrix computation shows that each h_i is an isometry. To prove that unitaries $u_i \in M_i$ exist with the claimed properties, we will construct each unitary in sequence, beginning with $u_1 = 1 \in M_1$. Suppose now that such unitaries $u_1, \ldots, u_m$ have been found; we indicate below how to obtain u_{m+1} satisfying

$$h_m u_m = h_{m+1} u_{{m+1}_{|\operatorname{ran} g_m}} \ .$$

The equation above is, in matricial form, simply

$$h^{\frac{1}{2}}\begin{pmatrix}(g_m^*hg_m)^{\frac{-1}{2}}\\ 0\end{pmatrix}u_m = h^{\frac{1}{2}}\begin{pmatrix}(g_{m+1}^*hg_{m+1})^{\frac{-1}{2}}\\ 0\end{pmatrix}u_{m+1_{|\mathrm{ran}g_m}}\ ,$$

which is satisfied, because h is invertible, if and only if

$$(\dagger)\qquad \begin{pmatrix}(g_m^*hg_m)^{\frac{-1}{2}}\\ 0\end{pmatrix}u_m = \begin{pmatrix}(g_{m+1}^*hg_{m+1})^{\frac{-1}{2}}\\ 0\end{pmatrix}u_{m+1_{|\mathrm{ran}g_m}}\ .$$

Now if $(\dagger)$ holds, then

$$(\ddagger)\qquad \begin{pmatrix}(g_{m+1}^*hg_{m+1})^{\frac{1}{2}} & 0\end{pmatrix}\begin{pmatrix}(g_m^*hg_m)^{\frac{-1}{2}}\\ 0\end{pmatrix}u_m = u_{m+1_{|\mathrm{ran}g_m}}\ .$$

The left-hand side of $(\ddagger)$ is a $k\times m$ matrix, which is easily seen to be, moreover, an isometry. So let u_{m+1} be any unitary matrix that has its first i columns given by the i columns on the left-hand side of $(\ddagger)$. All that remains is to verify that $(\dagger)$ is satisfied with the choice of u_{m+1}. Well this is true because if $p_i \in M_k$ is the canonical projection with range $\mathrm{ran}g_i$, $1\le i\le k$, then $p_i = g_ig_i^*$; so we see that equation $(\ddagger)$ implies equation $(\dagger)$ by simply multiplying $(\ddagger)$ by $\begin{pmatrix}(g_{m+1}^*hg_{m+1})^{\frac{-1}{2}} & 0\end{pmatrix}$ to get $(\dagger)$. Finally, observe that

$$u_k = h_ku_k = \begin{pmatrix}h_{k-1}u_{k-1} & *\end{pmatrix} = \begin{pmatrix}h_{k-2}u_{k-2} & * & *\end{pmatrix} = \cdots = \begin{pmatrix}h_1u_1 & * & * & \dots & *\end{pmatrix}.$$

Thus, for every i the equation $u_kg_i = h_iu_i$ holds. □

As an application of the construction in Theorem 3.3, we now prove an analogue, for C^*-convexity in generalised state spaces, of the usual Krein-Milman theorem. We shall need the following fact, which is a direct consequence of Theorem 3.3: if $\varphi : A \to M_n$ is a pure unital completely positive map, and if ψ is a pure state on A of the form $\psi(x) = (\varphi(x)\xi,\xi)$ for some unit vector $\xi\in\mathbb{C}^n$, then $\varphi\oplus\psi 1_k$ is a C^*-extreme point of $S_{\mathbb{C}^{n+k}}(A)$ for every positive integer k.

Theorem 3.5. *For every unital C^*-algebra A, the C^*-convex hull of the set of C^*-extreme points of $S_{\mathbb{C}^r}(A)$ is a dense subset, with respect to the bounded-weak-topology, of $S_{\mathbb{C}^r}(A)$.*

Proof. By the Krein-Milman theorem, $S_H(A)$ is the BW-closure of the convex hull of the extreme points of $S_H(A)$. Thus, we need only represent every extreme point as a C^*-convex combination of C^*-extreme points.

Assume that φ is an extreme point of $S_H(A)$ and that $\varphi = v^*\pi v$ is a minimal decomposition of φ. If π is irreducible, then φ is pure and we are through; otherwise, decompose π further so that it is a direct sum of irreducible representations π_j on Hilbert spaces K_j. Using that H is finite-dimensional and that φ is an extreme point, the number of representations π_j must be finite, by the Extreme Point theorem. Thus, H_π is a finite direct sum of the spaces K_j; the isometry v is a finite sum of contractions $v_j = q_jv$, where q_j is projection mapping H_π onto the subspace K_j; and φ itself is represented by the finite sum $\sum_j v_j^*\pi_jv_j$.

In $B(H)$, let $a_j = v_j^*v_j$ and let p_j be the projection mapping H onto the range of the positive operator a_j. Let b_j be the positive contraction that acts as $a_j^{-\frac{1}{2}}$ on the range of a_j and as 0 on the kernel of a_j. Thus, the range of b_j is that of a_j and

p_j, and $b_j a_j b_j = p_j$. Select any pure state θ_j on A that is a compression of the pure generalised state $b_j v_j^* \pi_j v_j b_j$. We now construct the following generalised state:

$$\omega_j(x) = b_j v_j^* \pi_j(x) v_j b_j + \theta_j(x)(1 - p_j), \quad \text{for all} \quad x \in A \ .$$

So ω_j, being the direct sum of a unital pure completely positive map and an inflation of a pure state that is a compression of this map, is a C^*-extreme point of $S_H(A)$ by Theorem 3.3. Finally, for every $x \in A$,

$$\varphi(x) = \sum_j v_j^* \pi_j(x) v_j = \sum_j a_j^{\frac{1}{2}} \omega_j(x) a_j^{\frac{1}{2}} \quad \text{and} \quad \sum_j a_j = 1 \ ,$$

which completes the proof. □

4. C^*-extremal generalised states on matrix algebras

An important special case in the theory of completely positive maps occurs with maps between matrix algebras. In [5] M.-D. Choi obtained a useful and elegant description of the extreme points in the generalised state spaces $S_{\mathbb{C}^r}(M_n)$, and his methods make no recourse to Arveson's extreme point theorem. In this section we shall determine all of the C^*-extreme points of $S_{\mathbb{C}^r}(M_n)$. It is interesting to note that our methods, in contrast to Choi's, rely quite extensively on Arveson's approach to extreme points (in that the Radon-Nikodym Theorem is one of the main constituents in the proof of Theorem 3.3).

Theorem 4.1. *A unital completely positive map $\varphi : M_n \to M_r$ is a C^*-extreme point of $S_{\mathbb{C}^r}(M_n)$ if and only if there exist positive integers $n_1 \geq n_2 \geq \cdots \geq n_k$ with $n \geq n_1$ and $\sum_i n_i = r$, and pure unital completely positive maps $\varphi_i : M_n \to M_{n_i}$ such that*

(1) *φ_{i+1} is a compression of φ_i for each $1 \leq i < k$, and*
(2) *φ is unitarily equivalent to the direct sum $\sum_i^{\oplus} \varphi_i$.*

Proof. The sufficiency of the hypotheses is proved by Theorem 3.3; in this case we note that every irreducible representation π of M_n is unitarily equivalent to the identity map. We now prove that the stated conditions are necessary in order for φ to be a C^*-extreme point. From Theorem 2.1 one knows that $\varphi \sim \sum_i^{\oplus} \varphi_i$ for some sequence of pure generalised states $\varphi_i : M_n \to M_{n_i}$ with $n \geq n_1$ and $\sum_i n_i = r$. In fact we may reduce the proof to the case where $k = 2$ for the following reason: φ is a C^*-extreme point only if $\varphi_i \oplus \varphi_{i+1}$ is a C^*-extreme point of $S_{\mathbb{C}^{n_i + n_{i+1}}}(M_n)$ for each $1 \leq i < k$. So in the case $k = 2$, we argue below that $\varphi_1 \oplus \varphi_2$ is a C^*-extreme point of $S_{\mathbb{C}^r}(M_n)$ only when φ_2 is a compression of φ_1.

Assume, on the contrary, that φ_2 is not a compression of φ_1. We shall prove that this assumption leads to a contradiction of the fact that $\varphi = \varphi_1 \oplus \varphi_2$ is a C^*-extreme point of $S_{\mathbb{C}^r}(M_n)$, where $r = n_1 + n_2$. Let $v_i : \mathbb{C}^{n_i} \to \mathbb{C}^n$ be isometric and such that $\varphi(x) = v_1^* x v_1 \oplus v_2^* x v_2$ for every $x \in M_n$. We now consider new orthonormal bases of $\mathbb{C}^n$, $\mathbb{C}^{n_1}$, and $\mathbb{C}^{n_2}$. Let E_1 be an orthonormal basis for $v_1(\mathbb{C}^{n_1}) \cap v_2(\mathbb{C}^{n_2})$. (We allow for the possibility that $v_1(\mathbb{C}^{n_1}) \cap v_2(\mathbb{C}^{n_2})$ is $\{0\}$; this causes no difficulties.) Extend E_1 by E_2 so that $E_1 \cup E_2$ is an orthonormal basis for $v_1(\mathbb{C}^{n_1})$. Finally, extend $E_1 \cup E_2$ by E_3 to obtain an orthonormal basis $E = E_1 \cup E_2 \cup E_3$ for $\mathbb{C}^n$. In $\mathbb{C}^{n_1}$, let $F_1 = v_1^{-1}(E_1)$, $F_2 = v_1^{-1}(E_2)$, and $F = F_1 \cup F_2$, an orthonormal basis. In $\mathbb{C}^{n_2}$, let $G_1 = v_2^{-1}(E_1)$ and extend this by G_2 to a basis G. Note that $G_2 \neq \phi$ because φ_2 is not a compression of φ_1.

With respect to the bases $E = (E_1, E_2, E_3)$ for $\mathbb{C}^n$ and $(F, G) = (F_1, F_2, G_1, G_2)$ of $\mathbb{C}^r = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$, we find that the isometries have the representations

$$v_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & q_2 \end{pmatrix},$$

where $q_1^* q_1 + q_2^* q_2 = 1$ and where q_2 is one-to-one (because $v_2(G_2) \cap v_1(F)$ is empty). Thus $1 - q_1^* q_1 = q_2^* q_2$ is invertible, and hence so are

$$Q_1 = \begin{pmatrix} 1 & & & \\ & 1 & & q_1 \\ & & 1 & \\ & q_1^* & & 1 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 1 & & & \\ & 1 & & -q_1 \\ & & 1 & \\ & -q_1^* & & 1 \end{pmatrix} \in M_r.$$

Let

$$w_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & q_1 \\ 0 & 0 & 0 & q_2 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & q_1 \\ 0 & 0 & 0 & q_2 \end{pmatrix},$$

and let $\psi_1(\cdot) = (Q_1^{-\frac{1}{2}}/\sqrt{2})(w_1^*(\cdot)w_1 + w_2^*(\cdot)w_2)(Q_1^{-\frac{1}{2}}/\sqrt{2})$. Similarly, let

$$w_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -q_1 \\ 0 & 0 & 0 & -q_2 \end{pmatrix} \quad \text{and} \quad w_4 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -q_1 \\ 0 & 0 & 0 & -q_2 \end{pmatrix},$$

and let $\psi_2(\cdot) = (Q_2^{-\frac{1}{2}}/\sqrt{2})(w_3^*(\cdot)w_3 + w_4^*(\cdot)w_4)(Q_2^{-\frac{1}{2}}/\sqrt{2})$.

Now $\psi_1,\ \psi_2 \in S_{\mathbb{C}^r}(M_n)$, and

$$\varphi(\cdot) = (Q_1^{\frac{1}{2}}/\sqrt{2})^* \psi_1(\cdot)(Q_1^{\frac{1}{2}}/\sqrt{2}) + (Q_2^{\frac{1}{2}}/\sqrt{2})^* \psi_2(\cdot)(Q_2^{\frac{1}{2}}/\sqrt{2})$$

expresses φ as a proper C^*-convex combination of ψ_1 and ψ_2 (because $Q_1^{\frac{1}{2}}/\sqrt{2}$ and $Q_2^{\frac{1}{2}}/\sqrt{2}$ are invertible and $(Q_1 + Q_2)/2 = 1$). Since φ is C^*-extreme, we must have $\varphi \sim \psi_1 \sim \psi_2$. We can see that this fails, however, by choosing $x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \neq 0 \in M_n$. Thus φ cannot be C^*-extreme and this contradiction completes the proof. □

A Markov map on M_n is a trace preserving unital completely positive map $M_n \to M_n$. Markov maps have been studied recently in [3] and [11]. The results of these papers demonstrate that there are many (nonequivalent) Markov maps that are extreme points of $S_{\mathbb{C}^n}(M_n)$. However, by Theorem 4.1, a Markov map φ is a C^*-extreme point of $S_{\mathbb{C}^n}(M_n)$ if and only if φ is unitarily equivalent to the identity map on M_n.

5. Remarks

This paper represents a natural continuation of our study of C^*-convex sets in matrix algebras [7], [8], [14]. One reason for considering C^*-convexity and complete positivity together is that the operation of forming C^*-convex combinations in a C^*-algebra is a completely positive operation. Our use here of noncommutative convexity in describing properties of spaces of completely positive maps is somewhat related to the earlier works [9] and [15]. In [15], Smith and Ward describe the faces of $S_H(A)$ in the cases where H has finite dimension. Their result, briefly, is that all faces of $S_{\mathbb{C}^n}(A)$ are determined, not necessarily uniquely, by projections in $A''\otimes M_n$, where A'' denotes the enveloping von Neumann algebra of A; the faces F of $S_{\mathbb{C}^n}(A)$ that are C^*-convex are in a one-to-one correspondence with the projections of A'' (and the faces of $S(A)$). However from the point of view of noncommutative convexity, the "C^*-convex faces" of [15] are a hybrid of two different notions: they are faces in the usual sense and convex in the general sense (i.e. C^*-convex). In independent works, Fujimoto and the second author have introduced notions of a face for noncommutative convexity. The idea of [14] is more aptly suited to the theme of the present paper, for it is based on C^*-convexity rather than Fujimoto's CP-convexity, which is different. Following [14], a C^*-face F of $S_H(A)$ is a subset having the property that if $\varphi \in F$ and if φ is expressed as a proper C^*-convex combination of $\varphi_j \in S_H(A)$, then necessarily each $\varphi_j \in F$. Although C^*-faces need not be convex (so they are not true faces), they do lend themselves well to standard arguments based on Zorn's Lemma, as in [14]. The C^*-faces of $S_H(A)$ have not yet been seriously studied. Through personal correspondence, we have learned from I. Fujimoto of his notions of extreme point and faces in the context of CP-convexity. These notions are substantially different from the corresponding ones in C^*-convexity. Indeed, the main results of the present paper appear to have no analogue in Fujimoto's program of CP-convexity for at least one reason: the (usually large) dimensions of the Hilbert spaces required by CP-convexity are dictated by the particular C^*-algebra A at hand, whereas in our work here we allow the Hilbert spaces to have arbitrary dimension (often small). More recently, Effros and Winkler [6] have put forth some very interesting ideas concerning a quantization of the geometric form of the classical Hahn-Banach extension theorem. In [6], the generalised state spaces $S_{\mathbb{C}^r}(M_n)$ appear as a matrix-convex system.

In another direction, Tsui [18] has studied how well Kadison's characterisation of pure states extends to more general settings. Recall that the left-kernel of a completely positive map φ is the set $L_\varphi = \{x \in A : \varphi(x^*x) = 0\}$. If φ is a state on A, then Kadison showed that φ is pure if and only if the kernel of φ is $L_\varphi + L_\varphi^*$. Tsui has proved the following.

Theorem (Tsui [18, 2.2]). *If H has finite dimension and if $\varphi \in S_H(A)$ is pure, then $L_\varphi + L_\varphi^* = \ker\varphi$. The converse is false.*

From the results in this paper, we see that, in case H is finite-dimensional and $A = M_n$, that Kadison's left kernel property is satisfied by every C^*-extreme point. In fact, even when A is arbitrary, all the C^*-extreme points we have found so far also satisfy this property. The obvious question, then, is whether this characterises the set of C^*-extreme points. The answer is no: with the non-C^*-extreme generalised state $\varphi : M_n \to M_n$ defined by $\varphi(x) = \frac{1}{2}\mathrm{diag}(x) + \frac{1}{2}x$, one has that $\ker\varphi = L_\varphi = \{0\}$.

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