# $C^\infty$ SCALING ASYMPTOTICS FOR THE SPECTRAL PROJECTOR OF THE LAPLACIAN

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Abstract. This article concerns new off-diagonal estimates on the remainder and its derivatives in the pointwise Weyl law on a compact $n$-dimensional Riemannian manifold. As an application, we prove that near any non self-focal point, the scaling limit of the spectral projector of the Laplacian onto frequency windows of constant size is a normalized Bessel function depending only on $n$.

## 0. Introduction

Let $(M, g)$ be a compact, smooth, Riemannian manifold without boundary. We assume throughout that the dimension of $M$ is $n \geq 2$ and write $\Delta_g$ for the non-negative Laplace-Beltrami operator. Denote the spectrum of $\Delta_g$ by

$$0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots \nearrow \infty.$$

This article concerns the behavior of the Schwarz kernel of the projection operators

$$E_I : L^2(M) \to \bigoplus_{\lambda_j \in I} \ker(\Delta_g - \lambda_j^2),$$

where $I \subset [0, \infty)$. Given an orthonormal basis $\{\varphi_j\}_{j=1}^\infty$ of $L^2(M, g)$ consisting of real-valued eigenfunctions,

$$\Delta_g \varphi_j = \lambda_j^2 \varphi_j \qquad \text{and} \qquad \|\varphi_j\|_{L^2} = 1, \tag{1}$$

the Schwarz kernel of $E_I$ is

$$E_I(x, y) = \sum_{\lambda_j \in I} \varphi_j(x)\varphi_j(y). \tag{2}$$

The study of $E_{[0,\lambda]}(x, y)$ as $\lambda \to \infty$ has a long history, especially when $x = y$. For instance, it has been studied notably in [7, 8, 9, 10] for its close relation to the asymptotics of the spectral counting function

$$\#\{j : \lambda_j \leq \lambda\} = \int_M E_{[0,\lambda]}(x, x) dv_g(x), \tag{3}$$

where $dv_g$ is the Riemannian volume form. An important result, going back to Hörmander [8, Thm 4.4], is the pointwise Weyl law (see also [4, 18]), which says that there exists $\varepsilon > 0$ so that if the Riemannian distance $d_g(x, y)$ between $x$ and $y$ is less than $\varepsilon$, then

$$E_{[0,\lambda]}(x, y) = \frac{1}{(2\pi)^n} \int_{|\xi|_{g_y} < \lambda} e^{i\langle \exp_y^{-1}(x), \xi\rangle} \frac{d\xi}{\sqrt{|g_y|}} + R(x, y, \lambda). \tag{4}$$

The integral in (4) is over the cotangent fiber $T_y^*M$ and the integration measure is the quotient of the symplectic form $d\xi \wedge dy$ by the Riemannian volume form $dv_g = \sqrt{|g_y|}dy$. In Hörmander's original theorem, the phase function $\langle \exp_y^{-1}(x), \xi \rangle$ is replaced by any so-called adapted phase function and one still obtains that

$$\sup_{d_g(x,y)<\varepsilon} \left| \nabla_x^j \nabla_y^k R(x,y,\lambda) \right| = O(\lambda^{n-1+j+k}) \tag{5}$$

as $\lambda \to \infty$, where $\nabla$ denotes covariant differentiation. The estimate (5) for $j = k = 0$ is already in [8, Thm 4.4], while the general case follows from the wave kernel method (e.g. as in §4 of [16] see also [3, Thm 3.1]).

Our main technical result, Theorem 2, shows that the remainder estimate (5) for $R(x,y,\lambda)$ can be improved from $O(\lambda^{n-1+j+k})$ to $o(\lambda^{n-1+j+k})$ under the assumption that $x$ and $y$ are near a non self-focal point (defined below). This paper is a continuation of [4] where the authors proved Theorem 2 for $j = k = 0$. An application of our improved remainder estimates is Theorem 1, which shows that we can compute the scaling limit of $E_{(\lambda,\lambda+1]}(x,y)$ and its derivatives near a non self-focal point as $\lambda \to \infty$.

**Definition 1.** A point $x \in M$ is *non self-focal* if the loopset

$$\mathcal{L}_x := \{\xi \in S_x^*M : \exists\, t > 0 \text{ with } \exp_x(t\xi) = x\}$$

has measure 0 with respect to the natural measure on $T_x^*M$ induced by $g$. Note that $\mathcal{L}_x$ can be dense in $S_x^*M$ while still having measure 0 (e.g. for points on a flat torus).

**Theorem 1.** *Let $(M,g)$ be a compact, smooth, Riemannian manifold of dimension $n \geq 2$, with no boundary. Suppose $x_0 \in M$ is a non self-focal point and consider a non-negative function $r_\lambda$ satisfying $r_\lambda = o(\lambda)$ as $\lambda \to \infty$. Define the rescaled kernel*

$$E^{x_0}_{(\lambda,\lambda+1]}(u,v) := \lambda^{-(n-1)} E_{(\lambda,\lambda+1]}\left(\exp_{x_0}\left(\frac{u}{\lambda}\right),\ \exp_{x_0}\left(\frac{v}{\lambda}\right)\right).$$

*Then, for all $k, j \geq 0$,*

$$\sup_{|u|,|v| \leq r_\lambda} \left| \partial_u^j \partial_v^k \left( E^{x_0}_{(\lambda,\lambda+1]}(u,v) - \frac{1}{(2\pi)^n} \int_{S^*_{x_0}M} e^{i\langle u-v,\omega\rangle} d\omega \right) \right| = o(1)$$

*as $\lambda \to \infty$. The inner product in the integral over the unit sphere $S^*_{x_0}M$ is with respect to the flat metric $g(x_0)$ and $d\omega$ is the hypersurface measure on $S^*_{x_0}M$ induced by $g(x_0)$.*

**Remark 1.** Theorem 1 holds for $\Pi_{(\lambda,\lambda+\delta]}$ with arbitrary fixed $\delta > 0$. The difference is that the limiting kernel is multiplied by $\delta$ and the rate of convergence in the $o(1)$ term depends on $\delta$.

**Remark 2.** One can replace the shrinking ball $B(x_0, r_\lambda)$ in Theorem 1 by a compact set $S \subset M$ in which for any $x, y \in S$ the measure of the set of geodesics joining $x$ and $y$ is zero (see Remark 3 after Theorem 2).

Theorem 1 follows from Theorem 2 by combining (9) with the relation $E_{(\lambda,\lambda+1]} = E_{[0,\lambda+1]} - E_{[0,\lambda]}$. In normal coordinates at $x_0$, Theorem 1 shows that the scaling limit of $E^{x_0}_{(\lambda,\lambda+1]}$ in the $C^\infty$ topology is

$$E_1^{\mathbb{R}^n}(u,v) = \frac{1}{(2\pi)^n} \int_{S^{n-1}} e^{i\langle u-v,\omega\rangle} d\omega,$$

which is the kernel of the frequency 1 spectral projector for the flat Laplacian on $\mathbb{R}^n$. Theorem 1 can therefore be applied to studying the local behavior of random waves on $(M,g)$. More precisely, a frequency $\lambda$ monochromatic random wave $\varphi_\lambda$ on $(M,g)$ is a Gaussian random linear combination

$$\varphi_\lambda = \sum_{\lambda_j \in (\lambda, \lambda+1]} a_j \varphi_j \qquad a_j \sim N(0,1) \text{ i.i.d},$$

of eigenfunctions with frequencies in $\lambda_j \in (\lambda, \lambda+1]$. In this context, random waves were first introduced by Zelditch in [20]. Since the Gaussian field $\varphi_\lambda$ is centered, its law is determined by its covariance function, which is precisely $E_{(\lambda,\lambda+1]}(x,y)$. In the language of Nazarov-Sodin [11] (cf [6, 14]), the estimate (6) means that frenquency $\lambda$ monochromatic random waves on $(M,g)$ have frequeny 1 random waves on $\mathbb{R}^n$ as their translation invariant local limits at every non self-focal point. This point of view is taken up in the forthcoming article [5].

**Theorem 2.** *Let $(M,g)$ be a compact, smooth, Riemannian manifold of dimension $n \geq 2$, with no boundary. Let $K \subseteq M$ be the set of all non self-focal points in $M$. Then for all $k, j \geq 0$ and all $\varepsilon > 0$ there is a neighborhood $\mathcal{U} = \mathcal{U}(\varepsilon, k, j)$ of $K$ and constants $\Lambda = \Lambda(\varepsilon, k, j)$ and $C = C(\varepsilon, k, j)$ for which*

$$\|R(x,y,\lambda)\|_{C_x^k(\mathcal{U}) \times C_y^j(\mathcal{U})} \leq \varepsilon \lambda^{n-1+j+k} + C\lambda^{n-2+j+k} \tag{6}$$

*for all $\lambda > \Lambda$. Hence, if $x_0 \in K$ and $\mathcal{U}_\lambda$ is any sequence of sets containing $x_0$ with diameter tending to $0$ as $\lambda \to \infty$, then*

$$\|R(x,y,\lambda)\|_{C_x^k(\mathcal{U}_\lambda) \times C_y^j(\mathcal{U}_\lambda)} = o(\lambda^{n-1+j+k}). \tag{7}$$

**Remark 3.** One can consider more generally any compact $S \subseteq M$ such that all $x, y \in S$ are mutually non-focal, whic means

$$\mathcal{L}_{x,y} := \{\xi \in S_x^* M : \exists\, t > 0 \text{ with } \exp_x(t\xi) = y\}$$

has measure zero. Then, combining [12, Thm 3.3] with Theorem 2, for every $\varepsilon > 0$, there exists a neighborhood $\mathcal{U} = \mathcal{U}(\varepsilon, j)$ of $S$ and constants $\Lambda = \Lambda(\varepsilon, j, S)$ and $C = C(\varepsilon, j, S)$ such that

$$\sup_{x,y \in S} \left| \nabla_x^j \nabla_y^j R(x,y,\lambda) \right| \leq \varepsilon \lambda^{n-1+2j} + C\lambda^{n-2+2j}.$$

We believe that this statement is true even when the number of derivatives in $x, y$ is not the same but do no take this issue up here.

Our proof of Theorem 2 relies heavily on the argument for Theorem 1 in [4], which treated the case $j = k = 0$. That result was in turn was based on the work of Sogge-Zelditch [18, 19], who studied $j = k = 0$ and $x = y$. This last situation was also studied (independently and significantly before [4, 18, 19]) by Safarov in [12] (cf [13]) using a somewhat different method. The case $j = k = 1$ and $x = y$ is essentially Proposition 2.3 in [20]. We refer the reader to the introduction of [4] for more bground on estimates like (6).

## 1. Proof of Theorem 2

Let $x_0$ be a non-self focal point. Let $I, J$ be multi-indices and set

$$\Omega := |I| + |J|.$$

We abbreviate

$$E_\lambda \equiv E_{[0,\lambda]}.$$

Using that

$$\int_{S^{n-1}} e^{i\langle u,w\rangle} dw = (2\pi)^{n/2} J_{\frac{n-2}{2}}(|u|)|u|^{-\frac{n-2}{2}} \tag{8}$$

for all $u \in \mathbb{R}^n$, we have

$$\frac{1}{(2\pi)^n} \int_{|\xi|_{g_y}<\lambda} e^{i\langle \exp_y^{-1}(x),\, \xi\rangle} \frac{d\xi}{\sqrt{|g_y|}} = \int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^{\frac{n}{2}}} \left( \frac{J_{\frac{n-2}{2}}\left(\mu d_g(x,y)\right)}{\left(\mu d_g(x,y)\right)^{\frac{n-2}{2}}} \right) d\mu. \tag{9}$$

Choose coordinates around $x_0$. We seek to show that there exists a constant $c > 0$ so that for every $\varepsilon > 0$ there is an open neighborhood $\mathcal{U}_\varepsilon$ of $x_0$ and a constant $c_\varepsilon$ so that we have

$$\sup_{x,y\in\mathcal{U}_\varepsilon} \left| \partial_x^I \partial_y^J E_\lambda(x,y) - \int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^{\frac{n}{2}}} \partial_x^I \partial_y^J \left( \frac{J_{\frac{n-2}{2}}\left(\mu d_g(x,y)\right)}{\left(\mu d_g(x,y)\right)^{\frac{n-2}{2}}} \right) d\mu \right| \leq c\,\varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}. \tag{10}$$

Let $\rho \in \mathcal{S}(\mathbb{R})$ satisfy $\operatorname{supp}(\hat\rho) \subseteq (-\operatorname{inj}(M,g), \operatorname{inj}(M,g))$ and

$$\hat\rho(t) = 1 \qquad \text{for all} \qquad |t| < \tfrac{1}{2}\operatorname{inj}(M,g). \tag{11}$$

We prove (10) by first showing that it holds for the convolved measure $\rho * \partial_x^I \partial_y^J E_\lambda(x,y)$ and then estimating the difference $\left|\rho * \partial_x^I \partial_y^J E_\lambda(x,y) - \partial_x^I \partial_y^J E_\lambda(x,y)\right|$ in the following two propositions.

**Proposition 3.** *Let $x_0$ be a non-self focal point. Let $I, J$ be multi-indices and set $\Omega = |I| + |J|$. There exists a constant $c$ so that for every $\varepsilon > 0$ there exist an open neighborhood $\mathcal{U}_\varepsilon$ of $x_0$ and a constant $c_\varepsilon$ so that we have*

$$\left| \rho * \partial_x^I \partial_y^J E_\lambda(x,y) - \int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^{\frac{n}{2}}} \partial_x^I \partial_y^J \left( \frac{J_{\frac{n-2}{2}}\left(\mu d_g(x,y)\right)}{\left(\mu d_g(x,y)\right)^{\frac{n-2}{2}}} \right) d\mu \right| \leq c\,\varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega},$$

*for all $x, y \in \mathcal{U}_\varepsilon$.*

**Proposition 4.** *Let $x_0$ be a non-self focal point. There exists a constant $c$ so that for every $\varepsilon > 0$ there exist an open neighborhood $\mathcal{U}_\varepsilon$ of $x_0$ and a constant $c_\varepsilon$ so that for all multi-indices $I, J$ we have*

$$\sup_{x,y\in\mathcal{U}_\varepsilon} \left|\rho * \partial_x^I \partial_y^J E_\lambda(x,y) - \partial_x^I \partial_y^J E_\lambda(x,y)\right| \leq c\,\varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}.$$

The proof of Proposition 4 hinges on the fact that $x_0$ is a non self-focal point. Indeed, for each $\varepsilon > 0$, Lemma 15 in [4] (which is a generalization of Lemma 3.1 in

[18]) yields the existence of a neighborhood $\mathcal{O}_\varepsilon$ of $x_0$, a function $\psi_\varepsilon \in C_c^\infty(M)$ and operators $B_\varepsilon, C_\varepsilon \in \Psi^0(M)$ supported in $\mathcal{O}_\varepsilon$ satisfying both:

$$\bullet\ \mathrm{supp}(\psi_\varepsilon) \subset \mathcal{O}_\varepsilon \ \text{ and } \ \psi_\varepsilon = 1 \ \text{ on a neighborhood of} \, x_0, \tag{12}$$

$$\bullet\ B_\varepsilon + C_\varepsilon = \psi_\varepsilon^2. \tag{13}$$

The operator $B_\varepsilon$ is built so that it is microlocally supported on the set of cotangent directions that generate geodesic loops at $x_0$. Since $x_0$ is non self-focal, the construction can be carried so that the principal symbol $b_0(x,\xi)$ satisfies $\|b_0(x,\cdot)\|_{L^2(B_x^*M)} \leq \varepsilon$ for all $x \in M$. The operator $C_\varepsilon$ is built so that $U(t)C_\varepsilon^*$ is a smoothing operator for $\frac{1}{2}\mathrm{inj}(M,g) < |t| < \frac{1}{\varepsilon}$. In addition, the principal symbols of $B_\varepsilon$ and $C_\varepsilon$ are real valued and their sub-principal symbols vanish in a neighborhood of $x_0$ (when regarded as operators acting on half-densities).

In what follows we use the construction above to decompose $E_\lambda$, up to an $O(\lambda^{-\infty})$ error, as

$$E_\lambda(x,y) = E_\lambda B_\varepsilon^*(x,y) + E_\lambda C_\varepsilon^*(x,y) \tag{14}$$

for all $x, y$ sufficiently close to $x_0$. This decomposition is valid since $\psi_\varepsilon \equiv 1$ near $x_0$.

1.1. **Proof of Proposition 3.** The proof of Proposition 3 consists of writing

$$\rho * \partial_x^I \partial_y^J E_\lambda(x,y) = \int_0^\lambda \partial_\mu(\rho * \partial_x^I \partial_y^J E_\mu(x,y))\, d\mu,$$

and on finding an estimate for $\partial_\mu(\rho * \partial_x^I \partial_y^J E_\mu(x,y))$. Such an estimate is given in Lemma 5, which is stated for the more general case $\partial_\mu(\rho * \partial_x^I \partial_y^J E_\mu Q^*(x,y))$ with $Q \in \{Id, B_\varepsilon, C_\varepsilon\}$ that is needed in the proof of Proposition 4.

**Lemma 5.** *Let $(M,g)$ be a compact, smooth, Riemannian manifold of dimension $n \geq 2$, with no boundary. Let $Q \in \{Id, B_\varepsilon, C_\varepsilon\}$ have principal symbol $D_0^Q$. Consider $\rho$ as in* (11)*, and define*

$$\Omega = |I| + |J|\,.$$

*Then, for all $x, y \in M$ with $d_g(x,y) \leq \frac{1}{2}\,\mathrm{inj}(M,g)$, all multi-indices $I, J$, and all $\mu \geq 1$, we have*

$$\begin{aligned}
&\partial_\mu(\rho * \partial_x^I \partial_y^J E_\mu Q^*)(x,y) \\
&= \frac{\mu^{n-1}}{(2\pi)^n} \partial_x^I \partial_y^J \left( \int_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x), \omega\rangle_{g_y}} \left( D_0^Q(y,\omega) + \mu^{-1} D_{-1}^Q(y,\omega) \right) \frac{d\omega}{\sqrt{|g_y|}} \right) \\
&+ W_{I,J}(x,y,\mu).
\end{aligned} \tag{15}$$

*Here, $d\omega$ is the Euclidean surface measure on $S_y^*M$, and $D_{-1}^Q$ is a homogeneous symbol of order $-1$. The latter satisfy*

$$D_{-1}^{B_\varepsilon}(y,\cdot) + D_{-1}^{C_\varepsilon}(y,\cdot) = 0 \qquad \forall\, y \in \mathcal{O}_\varepsilon, \tag{16}$$

*where $\mathcal{O}_\varepsilon$ is as in* (12)*. Moreover, there exists $C > 0$ so that for every $\varepsilon > 0$*

$$\sup_{x,y \in \mathcal{O}_\varepsilon} \left| \int_{S_y^*M} e^{i\langle \exp_y^{-1}(x), \omega\rangle_{g_y}} D_{-1}^Q(y,\omega) \frac{d\omega}{\sqrt{|g_y|}} \right| \leq C\, \varepsilon. \tag{17}$$

*Finally, $W_{I,J}$ is a smooth function in $(x,y)$ for which there exists $C>0$ such that for all $x,y$ satisfying $d_g(x,y)\leq \frac{1}{2}\operatorname{inj}(M,g)$ and all $\mu>0$*

$$|W_{I,J}(x,y,\mu)|\leq C\mu^{n-2+\Omega}\left(d_g(x,y)+(1+\mu)^{-1}\right). \tag{18}$$

**Remark 4.** Note that Lemma 5 does not assume that $x,y$ are near an non self-focal point.

**Remark 5.** We note that Lemma 5 is valid for more general operators $Q$. Indeed, if $Q\in\Psi^k(M)$ has vanishing subprincipal symbol (when regarded as an operator acting on half-densities), then (15) holds with $D_0^Q(y,\omega)$ substituted by $\mu^k D_k^Q(y,\omega)$ and with $\mu^{-1}D_{-1}^Q(y,\omega)$ substituted by $\mu^{k-1}D_{k-1}^Q(y,\omega)$. Here, $D_k^Q$ is the principal symbol of $Q$ and $D_{k-1}^Q$ is a homogeneous polynomial of degree $k-1$. In this setting, the error term satisfies $|W_{I,J}(x,y,\mu)|\leq C\mu^{n+k-2+\Omega}\left(d_g(x,y)+(1+\mu)^{-1}\right)$.

*Proof of Lemma 5.* We use that

$$\partial_\mu(\rho * EQ^*)(x,y,\lambda)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{it\lambda}\hat{\rho}(t)\,U(t)Q^*(x,y)dt, \tag{19}$$

where $Q\in\Psi(M)$ is any pseudo-differential operator and $U(t)=e^{-it\sqrt{\Delta_g}}$ is the half-wave propagator. The argument from here is identical to that of [4, Proposition 12], which relies on a parametrix for the half-wave propagator for which the kernel can be controlled to high accuracy when $x$ and $y$ are close to the diagonal. The main corrections to the proof of [4, Proposition 12] are that $\partial_x^I\partial_y^J$ gives an $O(\mu^{n-3+\Omega})$ error in equations (54) and (60), and gives an $O(\mu^{n-1})$ error in (59). We must also take into account that $\partial_x\Theta(x,y)^{1/2}$ and $\partial_y\Theta(x,y)^{1/2}$ are both $O(d_g(x,y))$. $\square$

*Proof of Proposition 3.* Following the technique for proving [4, Proposition 7], we obtain Proposition 3 by applying Lemma 5 to $Q=Id$ (this gives $D_0^{Id}=1$ and $D_{-1}^{Id}=0$) and integrating the expression in (15) from $\mu=0$ to $\mu=\lambda$. One needs to choose $\mathcal{U}_\varepsilon$ so that its diameter is smaller than $\varepsilon$, since this makes $\int_0^\lambda W_{I,J}(x,y,\mu)d\mu = O(\varepsilon\lambda^{n-1+\Omega}+\lambda^{n-2+\Omega})$ as needed. One also uses identity (9) to obtain the exact statement in Proposition 3. $\square$

1.2. **Proof of Proposition 4.** As in (14),

$$E_\lambda(x,y)=E_\lambda B_\varepsilon^*(x,y)+E_\lambda C_\varepsilon^*(x,y)+O\left(\lambda^{-\infty}\right)$$

for all $x,y$ sufficiently close to $x_0$. Proposition 4 therefore reduces to showing that there exist a constant $c$ independent of $\varepsilon$, a constant $c_\varepsilon=c_\varepsilon(I,J,x_0)$, and a neighborhood $\mathcal{U}_\varepsilon$ of $x_0$ such that

$$\sup_{x,y\in\mathcal{U}_\varepsilon}\left|\partial_x^I\partial_y^J E_\lambda B_\varepsilon^*(x,y)-\rho*\partial_x^I\partial_y^J E_\lambda B_\varepsilon^*(x,y)\right|\leq c\,\varepsilon\lambda^{n-1+\Omega}+c_\varepsilon\lambda^{n-2+\Omega}, \tag{20}$$

and

$$\sup_{x,y\in\mathcal{U}_\varepsilon}\left|\partial_x^I\partial_y^J E_\lambda C_\varepsilon^*(x,y)-\rho*\partial_x^I\partial_y^J E_\lambda C_\varepsilon^*(x,y)\right|\leq c\,\varepsilon\lambda^{n-1+\Omega}+c_\varepsilon\lambda^{n-2+\Omega}. \tag{21}$$

Our proofs of (20) and (21) use that these estimates hold on diagonal when $|I|=|J|=0$ (i.e. no derivatives are involved). This is the content of the following result, which was proved in [18] for $Q = Id$. Its proof extends without modification to general $Q \in \Psi^0(M)$.

**Lemma 6** (Theorem 1.2 and Proposition 2.2 in [18])**.** *Let $Q \in \Psi^0(M)$ have real-valued principal symbol $q$. Fix a non-self focal point $x_0 \in M$ and write $\sigma_{sub}(QQ^*)$ for the subprincipal symbol of $QQ^*$ (acting on half-densities). Then, there exists $c > 0$ so that for every $\varepsilon > 0$ there exist a neighborhood $\mathcal{O}_\varepsilon$ and a constant $C_\varepsilon$ making*

$$QE_\lambda Q^*(x,x) = (2\pi)^{-n} \int_{|\xi|_{g_x} < \lambda} \left(|q(x,\xi)|^2 + \sigma_{sub}(QQ^*)(x,\xi)\right) \frac{d\xi}{\sqrt{|g_x|}} + R_Q(x,\lambda),$$

*with*

$$\sup_{x \in \mathcal{U}} |R_Q(x,\lambda)| \le c\,\varepsilon\lambda^{n-1} + C_\varepsilon \lambda^{n-2}$$

*for all $\lambda \ge 1$.*

We prove relation (20) in Section 1.2.1 and relation (21) in Section 1.2.2.

1.2.1. *Proof of relation* (20). Define

$$g_{I,J}(x,y,\lambda) := \partial_x^I \partial_y^J E_\lambda B_\varepsilon^*(x,y) - \rho * \partial_x^I \partial_y^J E_\lambda B_\varepsilon^*(x,y).$$

Note that $g_{I,J}(x,y,\cdot)$ is a piecewise continuous function. We aim to find $c, c_\varepsilon$ and $\mathcal{U}_\varepsilon$ so that $x_0 \in \mathcal{U}_\varepsilon$ and

$$\sup_{x,y \in \mathcal{U}_\varepsilon} |g_{I,J}(x,y,\lambda)| \le c\,\varepsilon\lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}. \tag{22}$$

By [4, Lemma 17], which is a Tauberian Theorem for non-monotone functions, relation (22) reduces to checking the following two conditions:

- $$\mathcal{F}_{\lambda\to t}(g_{I,J})(x,y,t) = 0 \quad \text{for all} \quad |t| < \frac{1}{2}\,\mathrm{inj}(M,g), \tag{23}$$
- $$\sup_{x,y \in \mathcal{U}_\varepsilon} \sup_{s \in [0,1]} |g_{I,J}(x,y,\lambda+s) - g_{I,J}(x,y,\lambda)| \le c\,\varepsilon\lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}. \tag{24}$$

By construction, $\mathcal{F}_{\lambda\to t}(\partial_\lambda g_{I,J})(x,y,t) = (1-\hat\rho(t))\partial_x^I\partial_y^J U(t) B_\varepsilon^*(x,y) = 0$ for all $|t| < \frac{1}{2}\,\mathrm{inj}(M,g)$. Hence, since $\mathcal{F}_{\lambda\to t}(g_{I,J})$ is continuous at $t = 0$, we have (23). To prove (24) we write

$$\sup_{s\in[0,1]} |g_{I,J}(x,y,\lambda+s) - g_{I,J}(x,y,\lambda)| \le \sup_{s\in[0,1]} \left|\partial_x^I\partial_y^J E_{(\lambda,\lambda+s]} B_\varepsilon^*(x,y)\right| + \sup_{s\in[0,1]} \left|\rho * \partial_x^I\partial_y^J E_{(\lambda,\lambda+s]} B_\varepsilon^*(x,y)\right|. \tag{25}$$

The second term in (25) is bounded above by the right hand side of (24) by Lemma 5. To bound the first term, use Cauchy-Schwartz to get

$$\begin{aligned}\sup_{s\in[0,1]} \left|\partial_x^I\partial_y^J [E_{(\lambda,\lambda+s]} B_\varepsilon^*(x,y)]\right| &= \sup_{s\in[0,1]} \Big|\sum_{\lambda_j \in (\lambda,\lambda+1]} \partial_x^I \varphi_j(x) \cdot \partial_y^J B_\varepsilon \varphi_j(y)\Big| \\ &\le \sum_{\lambda_j\in(\lambda,\lambda+1]} \left|\left[\left(B_\varepsilon \partial_y^J + [\partial_y^J, B_\varepsilon]\right)\varphi_j(y)\right]\right| \cdot \left|\partial_x^I \varphi_j(x)\right|.\end{aligned}$$

Write $b_0$ for the principal symbol of $B_\varepsilon$. By construction, for all $y$ in a neighborhood of $x_0$, we have $\partial_y b_0(y,\xi) = 0$. Therefore, $\sigma_{|J|-1}\left([\partial_y^J, B_\varepsilon]\right) = i^{|J|}\{\xi^J, b_0(y,\xi)\} = 0$ and we conclude that $[\partial_y^J, B_\varepsilon] \in \Psi^{|J|-2}$. Thus, by the usual pointwise Weyl Law (e.g. [19, Equation (2.31)]),

$$\sup_{s\in[0,1]} \left|\partial_x^I \partial_y^J [E_{(\lambda,\lambda+s]} B_\varepsilon^*(x,y)]\right| \leq \sum_{\lambda_j \in (\lambda,\lambda+1]} \left|B_\varepsilon \partial_y^J \varphi_j(y)\right| \cdot \left|\partial_x^I \varphi_j(x)\right| + O(\lambda^{n-3+\Omega})$$

Next, define for each multi-index $K \in \mathbb{N}^n$ the order zero pseudo-differential operator

$$P_K := \partial^K \Delta_g^{-|K|/2}.$$

Using Cauchy-Schwarz and that $\partial^K \varphi_j = \lambda_j^{|K|} P_K \varphi_j$, we find

$$\begin{aligned}\sum_{\lambda_j\in(\lambda,\lambda+1]} &\left|B_\varepsilon \partial_y^J \varphi_j(y)\right| \cdot \left|\partial_x^I \varphi_j(x)\right| \\ &\leq (\lambda+1)^\Omega [(B_\varepsilon P_J) E_{(\lambda,\lambda+1]} (B_\varepsilon P_J)^*(y,y)]^{\frac{1}{2}} [P_I E_{(\lambda,\lambda+1]} P_I^*(x,x)]^{\frac{1}{2}}.\end{aligned}$$

Again using the pointwise Weyl Law (see [19, Equation (2.31)]), we have $[P_I E_{(\lambda,\lambda+1]} P_I^*(x,x)]^{\frac{1}{2}}$ is $O(\lambda^{\frac{n-1}{2}})$. Next, since according to the construction of $B_\varepsilon$ we have

$$\sup_{x\in\mathcal{U}_\varepsilon} \|b_0(x,\cdot)\|_{L^2(B_x^*M)} \leq \varepsilon$$

and $\partial_x b_0(x,\xi) = 0$ for $x$ in a neighborhood $\mathcal{U}_\varepsilon$ of $x_0$, we conclude that

$$\sup_{x\in\mathcal{U}_\varepsilon} \|\sigma_{sub}(B_\varepsilon P_J (B_\varepsilon P_J)^*)(x,\cdot)\|_{L^2(B_x^*M)} \leq \varepsilon^2.$$

Proposition 6 therefore shows that there exists $c > 0$ making

$$\sup_{x,y\in\mathcal{U}_\varepsilon} \left|(B_\varepsilon P_J) E_{(\lambda,\lambda+1]} (B_\varepsilon P_J)^*(y,y)\right|^{\frac{1}{2}} \leq c\varepsilon\lambda^{\frac{n-1}{2}}. \tag{26}$$

This proves (24), which together with (23) allows us to conclude (22).

1.2.2. *Proof of relation* (21). Write

$$\partial_x^I \partial_y^J E_\lambda C_\varepsilon^*(x,y) = \sum_{\lambda_j\leq\lambda} \lambda_j^\Omega \left(P_I \varphi_j(x)\right) \cdot \left(C_\varepsilon P_J \varphi_j(y)\right) + \sum_{\lambda_j\leq\lambda} \lambda_j^{|I|} \left(P_I \varphi_j(x)\right) \cdot \left([\partial^J, C_\varepsilon]\varphi_j(y)\right). \tag{27}$$

As before, $[\partial^J, C_\varepsilon] \in \Psi^{|J|-2}$. Hence, by the usual pointwise Weyl law, the second term in (27) and its convolution with $\rho$ are both $O(\lambda^{n-2+\Omega})$. Hence,

$$\begin{aligned}\sup_{x,y\in\mathcal{U}_\varepsilon} \left|\partial_x^I \partial_y^J E_\lambda C_\varepsilon^*(x,y) - \rho * \partial_x^I \partial_y^J E_\lambda C_\varepsilon^*(x,y)\right| = \sup_{x,y\in\mathcal{U}_\varepsilon} &\left|V(x,y,\lambda) - \rho * V(x,y,\lambda)\right| \\ &+ O\left(\lambda^{n+\Omega-2}\right),\end{aligned}$$

where we have set

$$V(x,y,\lambda) := \partial^I E_\lambda (C_\varepsilon \partial^J)^*(x,y) = \sum_{\lambda_j\leq\lambda} \lambda_j^\Omega \left(P_I \varphi_j(x)\right) \cdot \left(C_\varepsilon P_J \varphi_j(y)\right).$$

Define

$$\alpha_{I,J}(x,y,\lambda) := V(x,y,\lambda) + \frac{1}{2}\sum_{\lambda_j\le\lambda} \lambda_j^{\Omega}\left(\left|P_I\varphi_{\lambda_j}(x)\right|^2 + \left|C_\varepsilon P_J\varphi_{\lambda_j}(y)\right|^2\right) \tag{28}$$

$$\beta_{I,J}(x,y,\lambda) := \rho * V(x,y,\lambda) + \frac{1}{2}\sum_{\lambda_j\le\lambda} \lambda_j^{\Omega}\left(\left|P_I\varphi_{\lambda_j}(x)\right|^2 + \left|C_\varepsilon P_J\varphi_{\lambda_j}(y)\right|^2\right). \tag{29}$$

By construction, $\alpha_{I,J}(x,y,\cdot)$ is a monotone function of $\lambda$ for $x,y$ fixed, and $\alpha_{I,J}(x,y,\lambda) - \beta_{I,J}(x,y,\lambda) = V(x,y,\lambda) - \rho * V(x,y,\lambda)$. So we aim to show that

$$\sup_{x,y\in\mathcal{U}_\varepsilon} |\alpha_{I,J}(x,y,\lambda) - \beta_{I,J}(x,y,\lambda)| \le c\,\varepsilon\lambda^{n-1+\Omega} + c_\varepsilon\lambda^{n-2+\Omega}. \tag{30}$$

We control the difference in (30) applying a Tauberian theorem for monotone functions [4, Lemma 16]. To apply it we need to show the following:

- There exists $c > 0$ and $c_\varepsilon > 0$ making

$$\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |\partial_\mu \beta_{I,J}(x,y,\mu)|\, d\mu \le c\varepsilon\lambda^{n-1+\Omega} + c_\varepsilon\lambda^{n-2+\Omega}. \tag{31}$$

- For all $N$ there exists $M_{\varepsilon,N}$ so that for all $\lambda > 0$

$$|\partial_\lambda\left(\alpha_{I,J}(x,y,\cdot) - \beta_{I,J}(x,y,\cdot)\right) * \phi_\varepsilon(\mu)| \le M_{\varepsilon,N}\,(1+|\lambda|)^{-N}. \tag{32}$$

In equation (32) we have set $\phi_\varepsilon(\lambda) := \frac{1}{\varepsilon}\phi\left(\frac{\lambda}{\varepsilon}\right)$ for some $\phi \in \mathcal{S}(\mathbb{R})$ chosen so that $\operatorname{supp}\hat{\phi} \subseteq (-1,1)$ and $\hat{\phi}(0) = 1$.

Relation (31) follows after applying Lemma 6 to the piece of the integral corresponding to the second term in (29) and from applying Lemma 5 together with Remark 5 to $\rho * V = \rho * \partial^I E_\lambda Q^*$, where $Q := C_\varepsilon \partial^J$ has vanishing subprincipal symbol.

To verify (32) note that $\operatorname{supp}(1-\widehat{\rho}) \subseteq \{t : |t| \ge \operatorname{inj}(M,g)/2\}$ and $\operatorname{supp}(\widehat{\phi_\varepsilon}) \subseteq \{t : |t| \le \frac{1}{\varepsilon}\}$. Observe that

$$\partial_\lambda\Big(\alpha_{I,J}(x,y,\cdot) - \beta_{I,J}(x,y,\cdot)\Big) * \phi_\varepsilon\,(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}\Big((1-\hat{\rho}(t))\,\hat{\phi}_\varepsilon(t)\partial^I U(t)(\partial^J C_\varepsilon)^*(x,y)\Big)(\lambda).$$

By construction $U(t)C_\varepsilon^*$ is a smoothing operator for $\frac{1}{2}\operatorname{inj}(M,g) < |t| < \frac{1}{\varepsilon}$. Thus, so is $\partial^I U(t)\left(\partial^J C_\varepsilon\right)^*$ which implies (32). This concludes the proof of relation (21). □

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